Existence of complete Lyapunov functions for semiflows on separable metric spaces

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Abstract

The aim of this short note is to show how to construct a complete Lyapunov function of a semiflow by using a complete Lyapunov function of its time-one map. As a byproduct we assure the existence of complete Lyapunov functions for semiflows on separable metric spaces.

1 Complete Lyapunov functions

Let $\phi^t : X \to X$ be a semiflow on a metric space X and denote by $\mathcal{R}_C(\phi^t)$ its chain recurrent set. A complete Lyapunov function for the semiflow ϕ^t is a continuous function $L : X \to [0,1]$ satisfying the following requirements:

1. The function $t \mapsto L(\phi^t(x))$ is constant for each $x \in \mathcal{R}_C(\phi^t)$ and is decreasing for each $x \in X - \mathcal{R}_C(\phi^t)$.

2. The set $L(\mathcal{R}_C(\phi^t))$ is nowhere dense in $[0,1]$.

3. If $c \in L(\mathcal{R}_C(\phi^t))$, then $L^{-1}(c)$ is a chain transitive component of $\mathcal{R}_C(\phi^t)$.

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A complete Lyapunov function for a continuous map $T : X \to X$ is a continuous function $L : X \to [0,1]$ satisfying the above requirements, changing the function $t \mapsto L(\phi^t(x))$ by the function $n \mapsto L(T^n(x))$ and changing the chain recurrent set of ϕ^t by the chain recurrent set of T, denoted by $\mathcal{R}_C(T)$.

The existence of a complete Lyapunov function for flows on compact metric spaces was originally proved by Conley in [2]. After this, in the reference [3], Franks proved the existence of a complete Lyapunov function for maps on compact metric spaces, which was extended to locally compact and then to separable metric spaces by Hurley respectively in [4] and [6]. In some places (see e.g. [1, 7, 8]), these existence results are referred as the Fundamental Theorem of Dynamical Systems.

The aim of this short note is to show how to construct a complete Lyapunov function of a semiflow by using a complete Lyapunov function of its time-one map. As a byproduct we assure the existence of complete Lyapunov functions for semiflows on separable metric spaces.

Theorem 1.1 *Let ϕ^t be a semiflow on X and $\ell : X \to [0,1]$ be a complete Lyapunov function for its time-one map ϕ^1. Thus the map $L : X \to [0,1]$ given by*

$$L(x) = \int_0^1 \ell(\phi^t(x))\,dt$$

is a complete Lyapunov function for the semiflow ϕ^t. In particular, if X is a separable metric space, a complete Lyapunov function always exists for any given semiflow.

Proof: We first note that $L(x)$ is well defined, for each $x \in X$, since the function $t \mapsto \ell(\phi^t(x))$ is continuous. It is straightforward to show that L is continuous at any given $x \in X$. Indeed, given $\varepsilon > 0$, using the continuity of the semiflow and of ℓ, for each $t \in [0,1]$, there exist open neighborhoods U_t and V_t, respectively, of t and of x such that

$$|\ell(\phi^s(y)) - \ell(\phi^t(x))| < \frac{\varepsilon}{2},$$

for every $s \in U_t$ and $y \in V_t$. Since $\{U_t : t \in [0,1]\}$ is an open cover of the compact set $[0,1]$, there exists a finite subcover $\{U_{t_1}, \ldots, U_{t_n}\}$ and thus $V = V_{t_1} \cap \cdots \cap V_{t_n}$ is an open neighborhood of x. Hence, for every $t \in [0,1]$ and $y \in V$, we have that $t \in U_{t_k}$ and $y \in V_{t_k}$, for some k. Therefore, for every $t \in [0,1]$ and every $y \in V$, we have that

$$|\ell(\phi^t(y)) - \ell(\phi^t(x))| \le |\ell(\phi^t(y)) - \ell(\phi^{t_k}(x))| + |\ell(\phi^{t_k}(x)) - \ell(\phi^t(x))| < \varepsilon$$

and thus

$$|L(y) - L(x)| \leq \int_0^1 |\ell(\phi^t(y)) - \ell(\phi^t(x))| \, dt < \varepsilon,$$

showing that L is continuous at x.

Now we need to prove the three above requirements. We start by using reference [5] to remember that $\mathcal{R}_C(\phi^t) = \mathcal{R}_C(\phi^1)$, i.e., the chain recurrent sets of both the semiflow and its time-one map coincide. In order to verify the first requirement, for any $s \in (0, 1]$, let us consider

$$L(\phi^s(x)) = \int_0^1 \ell(\phi^t(\phi^s(x))) \, dt = \int_0^1 \ell(\phi^{t+s}(x)) \, dt = \int_s^{1+s} \ell(\phi^t(x)) \, dt.$$

Thus we have that

$$\begin{aligned} L(\phi^s(x)) &= \int_s^1 \ell(\phi^t(x)) \, dt + \int_1^{1+s} \ell(\phi^t(x)) \, dt \\ &= \int_s^1 \ell(\phi^t(x)) \, dt + \int_0^s \ell(\phi^{1+t}(x)) \, dt \\ &= \int_s^1 \ell(\phi^t(x)) \, dt + \int_0^s \ell(\phi^1(\phi^t(x)) \, dt \end{aligned}$$

If $x \in \mathcal{R}_C(\phi^t)$, then $\ell(\phi^t(x)) \in \mathcal{R}_C(\phi^1)$, implying that $\ell(\phi^1(\phi^t(x)) = \ell(\phi^t(x))$ and hence that

$$L(\phi^s(x)) = \int_0^1 \ell(\phi^t(x)) = L(x).$$

On the other hand, if $x \in X - \mathcal{R}_C(\phi^t)$, then $\ell(\phi^t(x)) \in X - \mathcal{R}_C(\phi^1)$, implying that $\ell(\phi^1(\phi^t(x)) < \ell(\phi^t(x))$ and hence that

$$L(\phi^s(x)) < \int_0^1 \ell(\phi^t(x)) = L(x).$$

The first requirement is thus immediate from the above results.

For the remaining requirements, we first note that the chain transitive components of $\mathcal{R}_C(\phi^1)$ and of $\mathcal{R}_C(\phi^t)$ coincide. Indeed each chain transitive component of $\mathcal{R}_C(\phi^1)$ is contained in a chain transitive component of $\mathcal{R}_C(\phi^t)$, since the transitivity by ϕ^1 implies the transitivity by ϕ^t. On the other hand, each chain transitive component of $\mathcal{R}_C(\phi^t)$ is contained in a chain transitive component of $\mathcal{R}_C(\phi^1)$, since the chain transitive components of $\mathcal{R}_C(\phi^1)$

are unions of connected components of $\mathcal{R}_C(\phi^1)$, while the chain transitive components of $\mathcal{R}_C(\phi^t)$ are exactly the connected components of $\mathcal{R}_C(\phi^t)$.

Now consider $c \in \ell(\mathcal{R}_C(\phi^1))$. Since $\ell^{-1}(c)$ is a chain transitive component of $\mathcal{R}_C(\phi^1)$, it is a chain transitive component of $\mathcal{R}_C(\phi^1)$ and thus is invariant by ϕ^t. Hence, if $x \in \ell^{-1}(c)$, we have that $\phi^t(x) \in \ell^{-1}(c)$, which implies that

$$L(x) = \int_0^1 \ell(\phi^t(x))\, dt = c,$$

showing that $\ell^{-1}(c) \subset L^{-1}(c)$. Since $\{\ell^{-1}(c) : c \in \ell(\mathcal{R}_C(\phi^1))\}$ is a partition of $\mathcal{R}_C(\phi^t)$, we have that $\ell(x) \in [0,1] - \ell(\mathcal{R}_C(\phi^1))$, for every $x \in X - \mathcal{R}_C(\phi^t)$. Using that $X - \mathcal{R}_C(\phi^t)$ is invariant by ϕ^t and using the continuity of the map $t \mapsto \ell(\phi^t(x))$, we get that the set $\{\ell(\phi^t(x)) : t \in [0,1]\}$ is a closed interval contained in $[0,1] - \ell(\mathcal{R}_C(\phi^1))$. This implies that $L(x) \in [0,1] - \ell(\mathcal{R}_C(\phi^1))$, for every $x \in X - \mathcal{R}_C(\phi^t)$, showing that $L^{-1}(c)$ is contained in $\mathcal{R}_C(\phi^t)$, for every $c \in \ell(\mathcal{R}_C(\phi^1))$. Since $\ell^{-1}(c) \subset L^{-1}(c)$ and since $\{\ell^{-1}(c) : c \in \ell(\mathcal{R}_C(\phi^1))\}$ is a partition of $\mathcal{R}_C(\phi^t)$, we get that $\ell^{-1}(c) = L^{-1}(c)$, for every $c \in \ell(\mathcal{R}_C(\phi^1))$. This implies that $L(\mathcal{R}_C(\phi^t)) = \ell(\mathcal{R}_C(\phi^1))$, showing that it is nowhere dense in $[0,1]$ and that $L^{-1}(c)$ is a chain transitive component of $\mathcal{R}_C(\phi^t)$, for every $c \in L(\mathcal{R}_C(\phi^t))$.

The last assertion follows by the first part, applying to the time-one map of the semiflow the main result of [6], which assures the existence of a complete Lyapunov function for maps on separable metric spaces.

□

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