# Invariant number triangles, eigentriangles and Somos-4 sequences

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**Abstract**

Using the language of Riordan arrays, we look at two related iterative processes on matrices and determine which matrices are invariant under these processes. In a special case, the invariant sequences that arise are conjectured to have Hankel transforms that obey Somos-4 recurrences. A notion of eigentriangle for a number triangle emerges and examples are given, including a construction of the Takeuchi numbers.

## 1 Introduction

In this note, we shall define transformations on invertible lower-triangular matrices involving the down-shifting of elements and taking an inverse. The invariant matrices for these transformations turn out to be simple Riordan arrays [9], with generating functions easily described by continued fractions [4, 13]. These matrices have close links to the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. In the case of a particular two-parameter transformation, special sequences defined by this process appear to have Hankel transforms that satisfy Somos-4 type recurrences [3]. Again using Riordan arrays we can characterize these sequences.

We recall that the *Riordan group* [9, 11], is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of generating functions $g(x) = 1 + g_1 x + g_2 x^2 + \dots$ and $f(x) = f_1 x + f_2 x^2 + \dots$ where $f_1 \neq 0$ [11]. The associated matrix is the matrix whose $i$-th column is generated by $g(x)f(x)^i$ (the first column being indexed by 0). The matrix corresponding to the pair $f, g$ is denoted by $(g, f)$. The group law is then given by

$$(g, f) \cdot (h, l) = (g(h \circ f), l \circ f).$$

The identity for this law is $I = (1, x)$ and the inverse of $(g, f)$ is $(g, f)^{-1} = (1/(g \circ \bar{f}), \bar{f})$ where $\bar{f}$ is the compositional inverse of $f$. This is also called the (series) reversion of $f$. A Riordan array of the form $(g(x), x)$, where $g(x)$ is the generating function of the sequence $a_n$, is called the *sequence array* of the sequence $a_n$. Its general term is $a_{n-k}$, or more accurately $[k \leq n]a_{n-k}$ (where $[P]$ is the Iverson bracket [6], defined by $[\mathcal{P}] = 1$ if the proposition $\mathcal{P}$ is true, and $[\mathcal{P}] = 0$ if $\mathcal{P}$ is false). Such arrays are also called *Appell* arrays as they form the elements of the so called Appell subgroup.

If $\mathbf{M}$ is the matrix $(g, f)$, and $\mathbf{a} = (a_0, a_1, \ldots)'$ is an integer sequence with ordinary generating function $\mathcal{A}(x)$, then the sequence $\mathbf{Ma}$ has ordinary generating function $g(x)\mathcal{A}(f(x))$. The (infinite) matrix $(g, f)$ can thus be considered to act on the ring of integer sequences $\mathbf{Z}^{\mathbf{N}}$ by multiplication, where a sequence is regarded as a (infinite) column vector. We can extend this action to the ring of power series $\mathbf{Z}[[x]]$ by

$$(g, f) : \mathcal{A}(x) \longrightarrow (g, f) \cdot \mathcal{A}(x) = g(x)\mathcal{A}(f(x)).$$

**Example 1.** The binomial matrix $\mathbf{B}$ is the element $(\frac{1}{1-x}, \frac{x}{1-x})$ of the Riordan group. It has general element $\binom{n}{k}$. More generally, $\mathbf{B}^m$ is the element $(\frac{1}{1-mx}, \frac{x}{1-mx})$ of the Riordan group, with general term $\binom{n}{k}m^{n-k}$. It is easy to show that the inverse $\mathbf{B}^{-m}$ of $\mathbf{B}^m$ is given by $(\frac{1}{1+mx}, \frac{x}{1+mx})$.

In the sequel, we shall assume that all matrices and sequences are integer valued.

## 2 The $(a, b)$-Process

We start by defining an operation on lower-triangular matrices which have 1's on the diagonal. Thus let $M$ be of the form

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ m_{2,1} & 1 & 0 & 0 & 0 & 0 & \ldots \\ m_{3,1} & m_{3,2} & 1 & 0 & 0 & 0 & \ldots \\ m_{4,1} & m_{4,2} & m_{4,3} & 1 & 0 & 0 & \ldots \\ m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & 1 & 0 & \ldots \\ m_{6,1} & m_{6,2} & m_{6,3} & m_{6,4} & m_{6,5} & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1}$$

Now form the matrix

$$\tilde{M}(a, b) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ -a & 1 & 0 & 0 & 0 & 0 & \ldots \\ -b & -a & 1 & 0 & 0 & 0 & \ldots \\ -m_{2,1} & -b & -a & 1 & 0 & 0 & \ldots \\ -m_{3,1} & -m_{3,2} & -b & -a & 1 & 0 & \ldots \\ -m_{4,1} & -m_{4,2} & -m_{4,3} & -b & -a & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2}$$

Then we take the inverse $\tilde{M}(a, b)^{-1}$ of this matrix. Let us call this process the $(a, b)$-process. We have the following proposition.

**Proposition 2.** *Let $f(x)$ be the power series defined by*

$$f(x) = \frac{1}{1 - ax - (b-1)x^2 - x^2 f(x)}. \tag{3}$$

*Then the Riordan array*

$$(f(x), x)$$

*is invariant under the* $(a, b)$*-operation.*

*Proof.* By equation (3), we see that $f(x) = \sum_{i=0}^{\infty} a_i x^i$ where $a_0 = 1$. Then

$$x^2 f(x) = x^2 a_0 + x^3 \sum_{i=0} a_{i+1} x^i = x^2 + x^3 \sum_{i=0} a_{i+1} x^i.$$

We obtain

$$1-ax-(b-1)x^2-x^2 f(x) = 1-ax-bx^2+x^2-x^2-x^3 \sum_{i=0} a_{i+1}x^i = 1-ax-bx^2-x^3 \sum_{i=0} a_{i+1}x^i.$$

Thus we wish to prove that

$$(f(x), x) = (1 - ax - (b-1)x^2 - x^2 f(x), x)^{-1},$$

or equivalently that

$$(f(x), x)^{-1} = (1 - ax - (b-1)x^2 - x^2 f(x), x).$$

Now

$$(f(x), x)^{-1} = \left(\frac{1}{f(x)}, x\right)$$

and hence we wish to establish that

$$\frac{1}{f(x)} = 1 - ax - (b-1)x^2 - x^2 f(x).$$

But this follows immediately from the definition of $f$. □

Let $a_n$ denote the $n$-th element of the first column of $(f(x), x)$. Then the $(n, k)$-th element of $(f(x), x)$ is given by

$$[k \le n] a_{n-k}.$$

Thus we need only a knowledge of $a_n$ to describe all elements of the matrix.

**Proposition 3.** *Let*

$$g(x) = \frac{1}{1 - ax - bx^2 - x^2 g(x)}.$$

*Then*

$$[x^n]g(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} b^k \sum_{j=0}^{n-2k} \binom{n-2k}{j} a^{n-2k-j} C_{\frac{j}{2}} \frac{1+(-1)^j}{2},$$

*where* $C_n = \frac{1}{n+1}\binom{2n}{n}$ *is the* $n$*-th Catalan number A000108.*

*Proof.* Solving the equation

$$g(x) = \frac{1}{1 - ax - bx^2 - x^2 g(x)}$$

gives us

$$g(x) = g_{a,b}(x) = \frac{1 - ax - bx^2 - \sqrt{1 - 2ax + (a^2 - 2b - 4)x^2 + 2abx^3 + b^2x^4}}{2x^2}.$$

With this value, we then have the Riordan array factorization

$$\begin{aligned}(g_{a,b}(x), x) &= \left(\frac{1}{1 - ax - bx^2}, \frac{x}{1 - ax - bx^2}\right) \cdot \left(c(x^2), \frac{g_{a,b}(x)}{x}\right) \\ &= \left(\frac{1}{1 - bx^2}, \frac{x}{1 - bx^2}\right) \cdot \left(\frac{1}{1 - ax}, \frac{x}{1 - ax}\right) \cdot \left(c(x^2), \frac{g_{a,b}(x)}{x}\right),\end{aligned}$$

where

$$c(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

is the g.f. of the Catalan numbers, and $c(x^2)$ is the g.f. of the aerated Catalan numbers $1, 0, 1, 0, 2, 0, 5, 0, \ldots$. Thus

$$[x^n]g(x) = [x^n] \left(\frac{1}{1 - bx^2}, \frac{x}{1 - bx^2}\right) \cdot \left(\frac{1}{1 - ax}, \frac{x}{1 - ax}\right) \cdot c(x^2).$$

The result follows from this.

□

**Corollary 4.**

$$a_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (b-1)^k \sum_{j=0}^{n-2k} \binom{n-2k}{j} a^{n-2k-j} C_{\frac{j}{2}} \frac{1 + (-1)^j}{2}.$$

We note that if we start with any matrix of the form (1), and iterate the $(a, b)$-process on it, then the limit matrix is $(f(x), x)$. Thus the element of the Appell subgroup of the Riordan group $(f(x), x)$ where

$$f(x) = \cfrac{1}{1 - ax - (b-1)x^2 - \cfrac{x^2}{1 - ax - (b-1)x^2 - \cfrac{x^2}{1 - \cdots}}},$$

is a "universal element" for the $(a, b)$-process.

## 3 A Somos-4 conjecture

We have the following Somos-4 conjecture.

**Conjecture 5.** *The Hankel transform of the sequence $a_n$ is a $(a^2, b^2-a^2)$ Somos-4 sequence.*

By this we mean that the sequence $h_n$ of Hankel determinants

$$h_n = |a_{i+j}|_{0\le i,j\le n}$$

satisfies an $(\alpha, \beta)$ Somos-4 relation

$$h_n = \frac{\alpha h_{n-1}h_{n-3} + \beta h_{n-2}^2}{h_{n-4}}, \quad n > 3,$$

where $\alpha = a^2$ and $\beta = b^2 - a^2$.

Equivalently the Hankel transform of the sequence with general term

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} b^k \sum_{j=0}^{n-2k} \binom{n-2k}{j} a^{n-2k-j} C_{\frac{j}{2}} \frac{1+(-1)^j}{2}$$

is (conjectured to be) a $(a^2, (b+1)^2 - a^2)$ Somos-4 sequence.

**Example 6.** We let $a = b = 1$. Then $a_n$ is the sequence A128720

$$1, 1, 3, 6, 16, 40, 109, 297, 836, 2377, 6869 \ldots$$

which counts the number of skew Dyck paths of semi-length $n$ with no $UUU$'s. The Hankel transform of this sequence is the $(1, 3)$ Somos-4 sequence A174168 which begins

$$1, 2, 5, 17, 109, 706, 9529, 149057, 3464585, 141172802, 5987285341, \ldots.$$

**Example 7.** We take $a = 1$, $b = 2$ to get the sequence A174171 which begins

$$1, 1, 4, 8, 25, 65, 197, 571, 1753, 5351, 16746 \ldots,$$

with $(1, 8)$ Somos-4 Hankel transform

$$1, 3, 11, 83, 1217, 22833, 1249441, 68570323, 11548470571, 2279343327171, \ldots.$$

This is A097495, or the even-indexed terms of the Somos-5 sequence.

**Example 8.** We let $a = 2$, and $b = -1$. Then $a_n$ is the sequence A187256 which begins

$$1, 2, 4, 10, 28, 82, 248, 770, 2440, 7858, 25644, \ldots.$$

This sequence counts peakless Motzkin paths where the level steps come in two colours (Deutsch). The Hankel transform of this sequence is the Somos-4 variant A162547 that begins

$$1, 0, -4, -16, -64, 0, 4096, 65536, 1048576, 0, -1073741824, \ldots.$$

# 4 The "$(a)$-process" and Narayana numbers

We now look at the simpler "$(a)$-process", whereby we send the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ m_{2,1} & 1 & 0 & 0 & 0 & 0 & \dots \\ m_{3,1} & m_{3,2} & 1 & 0 & 0 & 0 & \dots \\ m_{4,1} & m_{4,2} & m_{4,3} & 1 & 0 & 0 & \dots \\ m_{5,1} & m_{5,2} & m_{5,3} & m_{5,4} & 1 & 0 & \dots \\ m_{6,1} & m_{6,2} & m_{6,3} & m_{6,4} & m_{6,5} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4}$$

to the matrix

$$\tilde{M}_a = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ -a & 1 & 0 & 0 & 0 & 0 & \dots \\ -m_{2,1} & -a & 1 & 0 & 0 & 0 & \dots \\ -m_{3,1} & -m_{3,2} & -a & 1 & 0 & 0 & \dots \\ -m_{4,1} & -m_{4,2} & -m_{4,3} & -a & 1 & 0 & \dots \\ -m_{5,1} & -m_{5,2} & -m_{5,3} & -m_{5,4} & -a & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and then take the inverse to obtain $\tilde{M}_a^{-1}$. We have the following result.

**Proposition 9.** *Let $f(x)$ be the power series defined by*

$$f(x) = \frac{1}{1-(a-1)x - xf(x)}.$$

*Then the Riordan array*

$$(f(x), x)$$

*is invariant under the $(a)$-process.*

*Proof.* We wish to show that

$$(f(x), x) = (1-(a-1)x - xf(x), x)^{-1},$$

or equivalently that

$$(f(x), x)^{-1} = \left(\frac{1}{f(x)}, x\right) = (1-(a-1)x - xf, x).$$

But this follows immediately since by definition

$$f(x) = \frac{1}{1-(a-1)x - xf(x)}.$$

□

We now remark that the continued fraction

$$f(x) = \cfrac{1}{1-(a-1)x-\cfrac{x}{1-(a-1)x-\cfrac{x}{1-\cdots}}}$$

is the generating function of the Narayana polynomials $\mathcal{N}_n(a) = \sum_{k=0}^n N_{n,k}a^k$ [1, 2, 12] where the matrix $(N_{n,k})$ is the matrix of Narayana numbers A090181

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 3 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 6 & 6 & 1 & 0 & \ldots \\ 0 & 1 & 10 & 20 & 10 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hence the terms of the first column of $(f(x), x)$ are precisely the Narayana polynomials in $a$:

$$a_n = \mathcal{N}_n(a) = \sum_{k=0}^n N_{n,k}a^k.$$

In particular, for $a = 1$, we get

$$a_n = C_n,$$

the Catalan numbers.

As before, we note that if we start from an arbitrary matrix of the form Eq. (4), and iterate the $(a)$-process, then the limit matrix is $(f(x), x)$. In particular, if $a = 1$, the limit matrix is the Catalan numbers sequence array $(C_{n-k})$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 5 & 2 & 1 & 1 & 0 & 0 & \ldots \\ 14 & 5 & 2 & 1 & 1 & 0 & \ldots \\ 42 & 14 & 5 & 2 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

This is the Riordan array $(c(x), x)$.

By solving the equation

$$f(x) = \frac{1}{1-(a-1)x-xf(x)}$$

we see that

$$(f(x), x) = \left( \frac{1-(a-1)x-\sqrt{1-2(a+1)x+(a-1)^2x^2}}{2x}, x \right),$$

which by the above is the matrix with $(n,k)$-th term

$$[k \leq n]\mathcal{N}_{n-k}(a).$$

# 5 Eigentriangles

We also have the following result.

**Proposition 10.** *Let $M$ be a matrix as in Eq. (1). Then $\tilde{M}_1^{-1}$ is an eigentriangle of $M$.*

By this we mean that if

$$\tilde{M}_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ r_{3,1} & 1 & 1 & 0 & 0 & 0 & \dots \\ r_{4,1} & r_{4,2} & 1 & 1 & 0 & 0 & \dots \\ r_{5,1} & r_{5,2} & r_{5,3} & 1 & 1 & 0 & \dots \\ r_{6,1} & r_{6,2} & r_{6,3} & r_{6,4} & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{5}$$

then

$$M\tilde{M}_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ r_{3,1} & 1 & 0 & 0 & 0 & 0 & \dots \\ r_{4,1} & r_{4,2} & 1 & 0 & 0 & 0 & \dots \\ r_{5,1} & r_{5,2} & r_{5,3} & 1 & 0 & 0 & \dots \\ r_{6,1} & r_{6,2} & r_{6,3} & r_{6,4} & 1 & 0 & \dots \\ r_{7,1} & r_{7,2} & r_{7,3} & r_{7,4} & r_{7,5} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that the first column of $\tilde{M}_1^{-1}$ is then an *eigensequence* of $M$.

*Proof.* We have

$$\tilde{M}\tilde{M}_1^{-1} = I$$

and hence

$$-\sum_{j=1}^{k-1} m_{k-1,j} r_{j,l} + r_{k,l} = 0 \quad \text{for} \quad k \neq l.$$

Then for $k \neq l$, we have

$$r_{k,l} = \sum_{j=0}^{k-1} m_{k-1,j} r_{j,l}.$$

Thus the $(k-1,l)$-th element of $M\tilde{M}_1^{-1}$ is $r_{k,l}$. ∎

**Example 11.** The eigentriangle of the binomial matrix $(\binom{n}{k})$ is given by

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 1 & 0 & 0 & 0 & \dots \\ 5 & 3 & 1 & 1 & 0 & 0 & \dots \\ 15 & 9 & 4 & 1 & 1 & 0 & \dots \\ 52 & 31 & 14 & 5 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the first column entries are the Bell numbers. We note in passing that the production matrix [5] of the matrix $E$ is equal to

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 1 & 0 & 0 & \dots \\ 5 & 3 & 1 & 0 & 1 & 0 & \dots \\ 15 & 9 & 4 & 1 & 0 & 1 & \dots \\ 52 & 31 & 14 & 5 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In this case, we have

$$a_n = \sum_{k=0}^{n-1} \binom{n-1}{k} a_k, \quad n > 0, \quad a_0 = 1,$$

or

$$a_n = Bell(n),$$

the Bell numbers A000110.

**Example 12.** The eigentriangle of the skew binomial matrix $(\binom{k}{n-k})$ is given by

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 1 & 0 & 0 & \dots \\ 4 & 4 & 3 & 1 & 1 & 0 & \dots \\ 11 & 11 & 7 & 4 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the first column

$$1, 1, 1, 2, 4, 11, 33, 114, 438, 1845, 8458, \dots$$

or A127782 is thus an eigensequence of $(\binom{k}{n-k})$ (remark by Gary W. Adamson). We have

$$a_n = \sum_{k=0}^{n-1} \binom{k}{n-k-1} a_k, \quad n > 0, \quad a_0 = 1.$$

**Example 13.** The eigentriangle of the sequence array for the Motzkin numbers $M_n$ (i.e., the matrix with $(n,k)$-th term $[k \le n]M_{n-k}$ where $M_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k$) is the sequence array for the sequence A005773 of directed animals $A_n$ of size $n$. Thus

$$A_n = \sum_{k=0}^{n-1} M_{n-k-1} A_k.$$

We can characterize the eigentriangle $E = (E(n,k))$ corresponding to a matrix $A = (A(n,k))$ as follows. We define

$$\tilde{E}(n,j) = \sum_{k=0}^{n-1} A(n-1+j, k+j)\tilde{E}(k,j), \quad \text{with} \quad \tilde{E}(0,j) = 1. \tag{6}$$

Then

$$E(n,k) = [k \le n]\tilde{E}(n-k,k).$$

# 6 The Takeuchi numbers

The Takeuchi numbers $t_n$ A000651 are an example of a sequence that can be defined with the aid of the eigentriangle of the Catalan triangle $(c(x), xc(x))$ A033184. We let $T(x)$ be the generating function of the Takeuchi numbers. Our point of departure is (4) in [8]:

$$T(x) = \frac{c(x)-1}{1-x} + \frac{x(2-c(x))}{\sqrt{1-4x}} T(xc(x)).$$

We now note that

$$\frac{(2-c(x))}{\sqrt{1-4x}} = c(x),$$

so that [8](4) becomes

$$T(x) = \frac{c(x)-1}{1-x} + xc(x)T(xc(x)).$$

In terms of Riordan arrays, we may write this as

$$((1,x) - (xc(x), xc(x))).T(x) = \frac{c(x)-1}{1-x}.$$

Now while the matrix

$$(1,x) - (xc(x), xc(x))$$

is not a Riordan array, it is a special type of invertible matrix. The theory of eigentriangles tells us that its inverse is the eigentriangle of the Catalan matrix

$$(c(x), xc(x)).$$

This eigentriangle begins

$$\mathbf{E} = \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 1 & 0 & 0 & 0 & \dots \\ 6 & 3 & 1 & 1 & 0 & 0 & \dots \\ 22 & 11 & 4 & 1 & 1 & 0 & \dots \\ 92 & 46 & 17 & 5 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

We then have

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & 1 & 0 & 0 & 0 & \dots \\ 5 & 5 & 3 & 1 & 0 & 0 & \dots \\ 14 & 14 & 9 & 4 & 1 & 0 & \dots \\ 42 & 42 & 28 & 14 & 5 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right)\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 1 & 0 & 0 & 0 & \dots \\ 6 & 3 & 1 & 1 & 0 & 0 & \dots \\ 22 & 11 & 4 & 1 & 1 & 0 & \dots \\ 92 & 46 & 17 & 5 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right) =$$

$$\left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & 0 & 0 & \dots \\ 6 & 3 & 1 & 0 & 0 & 0 & \dots \\ 22 & 11 & 4 & 1 & 0 & 0 & \dots \\ 92 & 46 & 17 & 5 & 1 & 0 & \dots \\ 426 & 213 & 79 & 24 & 6 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right).$$

The sequence with g.f. $\frac{c(x)-1}{1-x}$ is the sequence A014138 with general term

$$\sum_{k=0}^{n-1} C_{k+1},$$

and thus the Takeuchi numbers are the image of this sequence by $\mathbf{E}$. Now in this case $A$ of Eq. (6) is the matrix $(c(x), xc(x))$ with $(n,k)$-th term

$$A(n,k) = \binom{2n-k}{n-k}\frac{k+1}{n+1}.$$

Thus we get

$$\tilde{E}(n,j) = \sum_{k=0}^{n-1} \binom{2(n-1)+j-k}{n-1-k}\frac{k+j+1}{n+j}\tilde{E}(k,j), \quad \text{with} \quad \tilde{E}(0,j) = 1,$$

and so

$$t_n = \sum_{k=0}^{n} \tilde{E}(n-k,k)\sum_{j=0}^{k-1} C_{j+1}.$$

We note that the first column of $\mathbf{E}$ is essentially A091768.

# 7 Acknowledgements

There are many examples of eigensequences in [10], many of which are contributed by Paul D. Hanna or Gary W. Adamson. One can find a different but related notion of eigentriangle therein (see A144218, for example). An alternative iterative construction of eigensequences is given, for instance, in A168259. The "(1)-process" and the $(1,1)$-process are looked at in The Mobius function Blog of Mats Granvik [7]. Examples of eigentriangles as defined here are A172380, A181644,A181651, A181654, A186020, A186023, A172380.

# References

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