BIRATIONAL AUTOMORPHISM GROUPS AND THE MOVABLE CONE THEOREM FOR CALABI-YAU MANIFOLDS OF WEHLER TYPE VIA UNIVERSAL COXETER GROUPS

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Abstract. We prove that the birational automorphism group of any Calabi-Yau manifold given by a generic hypersurface of multi-degree two in $(\mathbf{P}^1)^{n+1}$ is isomorphic to the universal Coxeter group of rank $n+1$ and satisfies the Morrison-Kawamata movable cone conjecture. Schröer and I found a new series of Calabi-Yau manifolds of even dimension, namely, the universal covers of punctual Hilbert schemes of Enriques surfaces. We also prove that they admit a biregular action of the universal Coxeter group of rank 3 with positive entropy for generic Enriques surfaces.

1. Introduction

Throughout this note, we work over the complex number field $\mathbf{C}$.

Coxeter groups (see eg. [Hum], [Vi]) are fundamental in the group theory. They also appear in the theory of algebraic surfaces mostly as hyperbolic reflection groups on the Néron-Severi groups (see for instance [Ni], [Bor], [Do], [Mc2], [To]). Among all Coxeter groups generated by N involutions, the group called the universal Coxeter group of rank N

$$\mathrm{UC}(N) := \underbrace{\mathbf{Z}_2 * \mathbf{Z}_2 * \cdots * \mathbf{Z}_2}_{N} ,$$

where $\mathbf{Z}_2$ is a cyclic group of order 2, is the most fundamental one in the sense that the generators have no non-trivial relation. UC(2) is almost abelian in the sense that it has an abelian subgroup $\mathbf{Z}$ as index 2 subgroup. But, UC(N) with $N \geq 3$ are essentially non-commutative in the sense that they contain non-commutative free group $\mathbf{Z} * \mathbf{Z}$.

In a more explicit, algebro-geometric context, Wehler ([We], the last three lines) pointed out the following pretty result without proof:

Theorem 1.1. *Let S be a generic surface of multi-degree $(2,2,2)$ in $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. Then, $\mathrm{Aut}\,(S)$ is isomorphic to the universal Coxeter group of rank 3, i.e., $\mathrm{Aut}\,(S) \simeq \mathrm{UC}(3)$.*

See section 6 for proof of a slightly more general result. Wehler's K3 surfaces and their variants sometimes appear in the study of complex dynamics and arithmetic dynamics as handy, concrete examples ([Mc1], [Sil]).

The aim of this short note is to generalize Theorem (1.1) for higher dimensional Calabi-Yau manifolds in both birational and biregular ways. Main results are Theorems (1.2), (1.4).

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Theorem 1.2. *Let $n \geq 3$ and let X be a generic hypersurface of multi-degree $(2, \ldots, 2)$ in $(\mathbf{P}^1)^{n+1}$. Then:*

(1) the biregular automorphism group of X is trivial, i.e., $\mathrm{Aut}\,(X) = \{1\}$ *but the birational automorphism group of X is isomorphic to the universal Coxeter group of rank $n+1$, i.e.,* $\mathrm{Bir}\,(X) \simeq \mathrm{UC}(n+1)$.

(2) The abstract version of the Morrison-Kawamata movable cone conjecture ([Ka2], *Conjecture (1.12)) is true for X, i.e., the natural action of* $\mathrm{Bir}\,(X)$ *on the movable effective cone $\mathcal{M}^e(X)$ (*[Ka1], *Definition (1,1)) has a finite rational polyhedral cone as its fundamental domain.*

Note that X is a Calabi-Yau manifold of dimension n. Here and hereafter, by a *Calabi-Yau manifold*, we mean a simply-connected projective manifold M such that $H^0(M, \Omega_M^k) = 0$ for all k such that $1 < k < d$ and $H^0(M, \Omega_M^d) = \mathbf{C}\omega_M$. Here $\dim M = d$ and ω_M is a nowhere vanishing holomorphic d-form. Any projective K3 surface is a Calabi-Yau manifold of dimension 2. For the precise formulation of the Morrison-Kawamata movable cone conjecture, see [Ka2] Conjecture (1.12). See also [Mo] for the biregular version of the conjecture and relation with mirror symmetry. This conjecture is true for log K3 surfaces ([To]), abelian varieties ([Ka2], [PS]). The relative version of the conjecture is true for fibered Calabi-Yau threefolds ([Ka2]). This conjecture is also checked for some interesting examples of Calabi-Yau threefolds, for instance [GM] in a biregular situation, [Fr] in a birational situation. A weaker version of the conjecture is also discussed for compact hyperkähler manifolds ([Ma]). To my best knowledge, our theorem is, however, the first non-trivial result in which the movable cone conjecture is checked for non-trivial examples of Calabi-Yau manifolds of any dimension ≥ 4. Our proof is a combination of a part of recent important progress in the minimal model theory in higher dimension due to Birkar-Cascini-Hacon-McKernan and Kawamata ([BCHM], [Ka3]) and the geometric representation of the universal Coxeter groups (Theorems (2.1), (2.3)). We shall prove slightly more explicit versions of Theorem (1.2) (1), (2) in sections 3, 4.

As already indicated by Theorem (1.2), in higher dimensional algebraic geometry, birational automorphisms are more natural, and in general easier to find, than biregular automorphisms. Nevertheless, it is also of fundamental interest to find non-trivial *biregular* automorphisms of higher dimensional algebraic varieties.

In our previous work, Schröer and I ([OS], Theorem (3.1)) found the following new series of Calabi-Yau manifolds of any even dimension (as failure in some sense):

Theorem 1.3. *Let S be an Enriques surface and $\mathrm{Hilb}^n(S)$ be the Hilbert scheme of n points on S, where $n \geq 2$. Let $\pi : \widetilde{\mathrm{Hilb}^n(S)} \to \mathrm{Hilb}^n(S)$ be the universal cover of $\mathrm{Hilb}^n(S)$. Then π is of degree 2 and $\widetilde{\mathrm{Hilb}^n(S)}$ is a Calabi-Yau manifold of dimension $2n$.*

Our second main result is the following:

Theorem 1.4. *Let S be a generic Enriques surface. Then, for each $n \geq 2$, the biregular automorphism group of $\widetilde{\mathrm{Hilb}^n(S)}$ contains the universal Coxeter group of rank 3 as its subgroup, i.e.,* $\mathrm{UC}(3) \subset \mathrm{Aut}\,(\widetilde{\mathrm{Hilb}^n(S)})$, *such that* $\mathrm{UC}(3)$ *contains an element of positive entropy.*

We shall prove Theorem (1.4) in section 5.

Calabi-Yau manifolds in Theorems (1.2), (1.4) are fairly concrete. I hope that these two examples will also provide non-trivial handy examples for complex dynamics, birational complex dynamics ([DS], [Zh]) and arithmetic dynamics ([Sil], [Kg]) in higher dimension.

2. Universal Coxeter groups and their geometric reprsentations

The goal of this section is Theorem (2.3).

First, we shall review some general theory of Coxeter groups that are crucial in the sequel. Our main source is [Hum].

Let W be a group with a finite set of generators $S = \{s_j\}_{j=1}^N$. We call the pair (W, S) a Coxeter system if

$$W = \langle s_j \in S \,|\, (s_i s_j)^{m_{ij}} = 1 \rangle \ ,$$

where $m_{jj} = 1$, i.e., $s_j^2 = 1$ for all j and $2 \le m_{ij} = m_{ji} \le \infty$ when $i \ne j$. Here $m_{ij} = \infty$ means that $s_i s_j$ is of infinite order. A group W is called a *Coxeter group* if W has a finite subset S such that (W, S) forms a Coxeter system. Let

$$\mathrm{UC}(N) := \underbrace{\mathbf{Z}_2 * \mathbf{Z}_2 * \cdots * \mathbf{Z}_2}_{N}$$

be the free product of N cyclic groups $\mathbf{Z}_2$ of order 2. We denote by t_j the generator of the j-th factor of $\mathrm{UC}(N)$. If (W, S) is a Coxeter system with $|S| = N$, then we have a surjective homomorphism $\mathrm{UC}(N) \to W$, $t_j \mapsto s_j$ whose kernel is the minimal normal subgroup containing $\{(t_i t_j)^{m_{ij}}\}$. In this sense, the group $\mathrm{UC}(N)$ is the universal one among the Coxeter groups with N generators. We call $\mathrm{UC}(N)$ (resp. $(\mathrm{UC}(N), \{t_j\}_{j=1}^N)$) the *universal Coxeter group* (resp. the universal Coxeter system) *of rank* N.

Let $(W, \{s_j\}_{j=1}^N)$ be a Coxeter system with $(s_i s_j)^{m_{ij}} = 1$. For $(W, \{s_j\}_{j=1}^N)$, we associate the N-dimensional real vector space $V = \oplus_{j=1}^N \mathbf{R}\alpha_j$ and the bilinear form $b(*, **)$ on V given by

$$b(\alpha_i, \alpha_j) = -\cos\frac{\pi}{m_{ij}} \ .$$

Then, as explained in [Hum], Page 110, Proposition, we have a well-defined linear representation $\rho : W \to \mathrm{GL}(V)$ defined by

$$\rho(s_j)(\lambda) = \lambda - 2b(\alpha_j, \lambda)\alpha_j \ (\forall \lambda \in V) \ .$$

We make the identification $\mathrm{GL}(V) = \mathrm{GL}(N, \mathbf{R})$ under the basis $\langle \alpha_j \rangle_{j=1}^N$ of V. We call the representation ρ the *geometric representation* of the Coxeter system (W, S). The following theorem (see eg. [Hum], Page 113, Corollary) is one of the most fundamental theorems on Coxeter groups:

Theorem 2.1. *The geometric representation ρ of a Coxeter system (W, S) is faithful. In particular, Coxeter groups are linear.*

Definition 2.2. Let $M_{N,j}$ $(1 \le j \le N)$ be the $N \times N$ matrices with integer coefficients, defined by:

$$M_{N,j} = \begin{pmatrix} 1 & 0 & \dots & 0 & 2 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 & 2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 2 & 0 & \dots & 1 \end{pmatrix} ,$$

where -1 is the (j, j)-entry. For instance,

$$M_{3,1} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} , \ M_{3,2} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{pmatrix} , \ M_{3,3} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} .$$

Theorem 2.3. *The geometric representation ρ of the universal Coxeter system* $(\mathrm{UC}(N), \{t_j\}_{j=1}^N)$ *of rank N is given by $\rho(t_j) = M_{N,j}^t$, where $M_{N,j}^t$ is the transpose of $M_{N,j}$. In particular,*

$$\langle M_{N,j} \,|\, 1 \le j \le N \rangle = \langle M_{N,1} \rangle * \langle M_{N,2} \rangle * \cdots * \langle M_{N,N} \rangle \simeq \mathrm{UC}(N)$$

in $\mathrm{GL}(N, \mathbf{R})$.

Proof. By definition, $m_{jj} = 1$ and $m_{ij} = \infty$ $(i \ne j)$ for the universal Coxeter system. Hence

$$\rho(t_j)(\alpha_i) = \alpha_i + 2\alpha_j \ (i \ne j) \ , \ \rho(t_j)(\alpha_j) = -\alpha_j \ ,$$

i.e., the matrix representation of $\rho(t_j)$ under the basis $\langle \alpha_j \rangle_{j=1}^N$ is $M_{N,j}^t$. □

Remark 2.4. It is in general very hard to see if given matrices have no relation or not. Surprizingly, the fact that $\{M_{N,j}\}_{j=1}^N$ has no relation can be also seen directly. The following elegant argument is due to Professor Mathias Schuëtt.

Professor Mathias Schuëtt's argument. Denote $M_{N,j}^t = m(j)$. We may assume that $N \ge 2$. It is easy to see that $m(j)^2 = I_N$. So, it suffices to show that

$$m(k_\ell)m(k_{\ell-1}) \cdots m(k_1) \ne I_N \tag{2.1}$$

if $k_\ell \ne k_{\ell-1} \ne \cdots \ne k_1$. Let $1 \le i \le k_\ell$ and let

$$\mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} , \ \mathbf{x}_i = \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{iN} \end{pmatrix} ,$$

where $\mathbf{x}_i := m(k_i)m(k_{i-1}) \cdots m(k_1)\mathbf{x}_0$. By induction on i, we will have

$$x_{ij} > 0 \ , \ x_{ik_i} > x_{ij} \ (j \ne k_i) \ . \tag{2.2}$$

In fact, $x_{1j} = 1$ for $j \neq k_1$ and $x_{1k_1} = -1 + 2N > 1$. Assume that (2.2) is true for $i - 1$. Since $\mathbf{x}_i = m(k_i)\mathbf{x}_{i-1}$, it follows that $x_{ij} = x_{i-1,j}$ for $j \neq k_i$ and

$$x_{i,k_i} = 2x_{i-1,k_{i-1}} - x_{i-1,k_i} + \sum_{j \neq k_{i-1}, k_i} 2x_{i-1,j} \ .$$

Hence, by $x_{i-1,k_{i-1}} > x_{i-1,j}$ $(j \neq k_{i-1})$, we obtain that $x_{i,k_i} > x_{i-1,k_{i-1}} = x_{i,k_{i-1}}$ and $x_{i,k_i} > x_{i-1,j} = x_{ij}$ for $j \neq k_{i-1}, k_i$. This proves (2.2) for i.

By (2.2) for $i = \ell$, it follows that $\mathbf{x}_\ell \neq \mathbf{x}_0$. Hence (2.1) follows.

3. Proof of Theorem (1.2) (1)

Theorem 3.1. *Let $n \geq 3$ be an integer and V be an $(n+1)$-dimensional Fano manifold, i.e,, a projective manifold whose anti-canonical divisor $-K_V$ is ample. Assume that M is a smooth member of $|-K_V|$. Let $\tau : M \to V$ be the natural inclusion. Then:*

(1) M is a Calabi-Yau manifold of dimension $n \geq 3$.

(2) $\tau^* : \mathrm{Pic}\,(V) \simeq \mathrm{Pic}\,(M)$ is an isomorphism. Moreover, this isomorphism induces an isomorphism of the ample cones $\tau^* : \mathrm{Amp}\,(V) \simeq \mathrm{Amp}\,(M)$.

(3) $\mathrm{Aut}\,(M)$ is a finite group.

Remark 3.2. For a Calabi-Yau manifold (or for a Fano manifold) M, the natural cycle map $\mathrm{Pic}\,(M) \to \mathrm{NS}\,(M)$ given by $L \mapsto c_1(L)$ is an isomorphism by $h^1(\mathcal{O}_M) = 0$. So, in what follows, we identify the Picard group $\mathrm{Pic}\,(M)$ and the Néron-Severi group $\mathrm{NS}\,(M)$, and discuss several cones in the real vector space $\mathrm{NS}\,(M)_{\mathbf{R}} = \mathrm{NS}\,(M) \otimes_{\mathbf{Z}} \mathbf{R}$. For instance, the ample cone of M is the open convex cone of $\mathrm{NS}\,(M)_{\mathbf{R}}$ generated by the ample classes.

Proof. By the adjunction formula, it follows that $\mathcal{O}_M(K_M) \simeq \mathcal{O}_M$. By the Lefschetz hyperplane section theorem, $\pi_1(M) \simeq \pi_1(V) = \{1\}$. By the long exact sequence of the exact sequence of sheaves

$$0 \to \mathcal{O}_V(-K_V) \to \mathcal{O}_V \to \mathcal{O}_M \to 0$$

together with the Kodaira vanishing theorem, we have $h^k(\mathcal{O}_M) = 0$ for $1 \leq k \leq n-1$. Hence $h^0(\Omega_M^k) = 0$ for $1 \leq k \leq n-1$ by the Hodge symmetrty. This implies (1).

The first assertion of (2) follows from the Lefschetz hyperplane section theorem. Here we need $n \geq 3$. By a result of Kollár ([Bo], Appendix), the natural map $\tau_* : \overline{\mathrm{NE}}(V) \to \overline{\mathrm{NE}}(M)$ is an isomorphism. Taking the dual, we have the second assertion.

Let us show (3) following [Wi] Page 389. By (1), $T_M \simeq \Omega_M^{n-1}$. Hence $h^0(T_M) = 0$. Thus $\dim \mathrm{Aut}\,(M) = 0$. Recall that $\mathrm{Amp}\,(V)$, which is the dual of $\overline{\mathrm{NE}}(V)$, is a finite rational polyhedral cone for a Fano manifold V. Hence, by (2), $\mathrm{Amp}\,(M)$ is also a finite rational polyhedral cone. Thus, there are finitely many rationally defined 1-dimensional faces, say $\mathbf{R}_{\geq 0} h_i$ $(1 \leq i \leq \ell)$, of the boundary of $\mathrm{Amp}\,(M)$. Here h_i $(1 \leq i \leq \ell)$ are integral primitive vectors. Let $h = \sum_{i=1}^{\ell} h_i$. Then h is ample and $\mathrm{Aut}\,(M)(h) = h$. Hence, by considering the embedding $M \to \mathbf{P}^N$ by very large multiple of h, we find that $\mathrm{Aut}\,(M)$ is a closed algebraic subgroup of $\mathrm{PGL}(N)$. Since $\mathrm{PGL}(N)$ is noetherian and $\dim \mathrm{Aut}\,(M) = 0$, the assertion (3) now follows. $\square$

In this section, we shall prove Theorem (1.2) (1) in a slightly more explicit form (Theorem (3.3)). From now on until the end of section 4, we denote

$$P(n+1) := (\mathbf{P}^1)^{n+1} = \mathbf{P}^1_1 \times \mathbf{P}^1_2 \times \cdots \times \mathbf{P}^1_{n+1}$$

where $n \ge 3$ and

$$p^j : P(n+1) \to \mathbf{P}^1_j \simeq \mathbf{P}^1$$

$$p_j : P(n+1) \to P(n+1)_j := \mathbf{P}^1_1 \times \cdots \mathbf{P}^1_{j-1} \times \mathbf{P}^1_{j+1} \cdots \times \mathbf{P}^1_{n+1} \simeq P(n)$$

be the natural projections. Let $H_j := (p^j)^*(\mathcal{O}_{\mathbf{P}^1}(1))$. Then $P(n+1)$ is a Fano manifold of dimension $n+1$ such that

$$\mathrm{NS}\,(P(n+1)) = \oplus_{j=1}^{n+1}\mathbf{Z}H_j \ , \ -K_{P(n+1)} = \oplus_{j=1}^{n+1} 2H_j \ ,$$

$$\mathrm{Amp}\,(P(n+1)) = \oplus_{j=1}^{n+1}\mathbf{R}_{>0}H_j \ .$$

Let $X \in |-K_{P(n+1)}|$ be a generic element, i.e., X is a generic hypersurface of multi-degree $(2, 2, \ldots, 2)$ on $P(n+1)$. More explicitly, X is defined by the following equation in $P(n+1)$,

$$F_{j,1}x_{j,0}^2 + F_{j,2}x_{j,0}x_{j,1} + F_{j,3}x_{j,1}^2 = 0 \tag{3.1}$$

where $[x_{j,0} : x_{j,1}]$ is the homogenous coordinate of $\mathbf{P}^1_j$ and $F_{j,k}$ $(1 \le k \le 3)$ are generic homogeneous polynomials of multi-degree $(2, 2, \ldots, 2)$ on $P(n+1)_j$.

Let $\tau : X \to V$ be the natural inclusion and $h_j := \tau^* H_j$. Then X is a Calabi-Yau manifold of dimension n such that

$$\mathrm{NS}\,(X) = \oplus_{j=1}^{n+1}\mathbf{Z}h_j \ , \ \mathrm{Amp}\,(X) = \oplus_{j=1}^{n+1}\mathbf{R}_{>0}h_j \tag{3.2}$$

by the Bertini theorem and Theorem 3.1 (1). Let

$$\pi_j := \tau \circ p_j : X \to P(n+1)_j \simeq P(n) \ .$$

π_j is a surjective morphism of degree 2 and finite over *outside* the codimension ≥ 3 locus

$$B_j := (F_{j,1} = F_{j,2} = F_{j,3} = 0) \ .$$

For $x \in B_j$, we have $\pi_j^{-1}(x) \simeq \mathbf{P}^1$. It follows that π_j contracts no divisor. For $x \notin B_j$, the set $\pi_j^{-1}(x)$ consists of 2 points, say $\{y_1, y_2\}$, and therefore the correspondence $y_1 \leftrightarrow y_2$ defines a birational automorphism ι_j of X of order 2 over $P(n+1)_j$. We also note that $\mathrm{Bir}\,(X)$ naturally acts on $\mathrm{NS}\,(X)$ as a group automorphism. This is because K_X is trivial so that each element of $\mathrm{Bir}\,(X)$ is an isomorphism in codimension 1 (see eg. [Ka3], Page 420).

Theorem 3.3. *(1)* $\iota_j^* = M_{n+1,j}$ *under the basis* $\langle h_k\rangle_{k=1}^{n+1}$ *of* $\mathrm{NS}\,(X)$. *In particular, in* $\mathrm{Bir}\,(X)$,

$$\langle \iota_1 \, , \, \iota_2 \, , \, \cdots \iota_{n+1} \rangle = \langle \iota_1 \rangle * \langle \iota_2 \rangle * \cdots * \langle \iota_{n+1} \rangle \simeq \mathrm{UC}(n+1) \, .$$

(2) $\mathrm{Aut}\,(X) = \{1\}$.

(3) $\mathrm{Bir}\,(X) = \langle \iota_1 \, , \, \iota_2 \, , \, \cdots \iota_{n+1} \rangle \simeq \mathrm{UC}(n+1)$.

Remark 3.4. As our proof shows, the assertion (1) is true whenever X is smooth.

Proof. By definition of ι_j, we have $\iota_j^*(h_k) = h_k$ for $k \neq j$. Write $P(n+1) = P(n+1)_j \times \mathbf{P}^1_j$, where $\mathbf{P}^1_j$ is the j-th factor of $P(n+1)$. Let $(a, [b_0 : b_1])$ be a point of $X \setminus \pi_j^{-1}(B_j)$. Then one can write $\iota_j(a, [b_0 : b_1]) = (a, [c_0 : c_1])$. Here, by the relation of the roots and coefficients of the quadratic equation (3.1), we have:

$$\frac{c_0}{c_1} \cdot \frac{b_0}{b_1} = \frac{F_{j,3}(a)}{F_{j,1}(a)} \ , \ \frac{c_0}{c_1} + \frac{b_0}{b_1} = -\frac{F_{j,2}(a)}{F_{j,1}(a)} \ .$$

Here the polynomial $F_{j,3}$ is not zero and $\mathrm{div}\,(F_{j,k}|X)$ ($k = 1, 2, 3$) have no common divisor. This is because X is smooth. Thus, in $\mathrm{Pic}\,(X) \simeq \mathrm{NS}\,(X)$, we obtain

$$\iota_j^*(h_j) + h_j = \sum_{k \neq j} 2h_k \ ,$$

from the formula above. This proves the first assertion of (1). Since ι_j are of order 2, the second assertion of (1) now follows from Theorem (2.3).

Let us show (2). Let x_j be the standard inhomogeneous coordinate of $\mathbf{P}^1_j$. Then $X \in |-K_{P(n+1)}|$ is represented by an inhomogeneous polynomial $f_X(x_j)$ of degree ≤ 2 with respect to each x_j.

By Theorem (3.1)(3), $\mathrm{Aut}\,(X)$ is a finite group. Since $\mathrm{Aut}\,(X)$ preserves $\mathrm{Amp}\,(X)$, whence the set $\{h_j \,|\, 1 \leq j \leq n+1\}$, it follows from $H^0(\mathcal{O}_{P(n+1)}(H_j)) \simeq H^0(\mathcal{O}_X(h_j))$ that

$$\mathrm{Aut}\,(X) \subset \mathrm{Aut}\,(P(n+1)) = \mathrm{PGL}(1)^{n+1} \cdot S_{n+1} \ .$$

Here S_{n+1} is the group of permutations of $(n+1)$-factors of $P(n+1)$.

Consider the natural action of $\mathrm{PGL}(1)^{n+1} \cdot S_{n+1}$ on $|-K_V|$. $\mathrm{Aut}\,(X)$ is nothing but the stabilizer of the corresponding point $[X]$ of $|-K_V|$. The image of the homomorphism

$$\mathrm{Aut}\,(X) \to \mathrm{PGL}(1)^{n+1} \cdot S_{n+1} \to S_{n+1}$$

is trivial for generic X. In fact, otherwise, there would be $1 \neq g \in S_{n+1}$ such that *every* $X \in |-K_V|$ would admit an automorphism $\tilde{g}_X$ which is a lift of g, because $|-K_{P(n+1)}|$ is irreducible and S_{n+1} is finite. However, it turns out that this is impossible, by considering the actions on special inhomogeneous quadratic polynomials x_j^2 $(1 \leq j \leq n+1)$.

Thus $\mathrm{Aut}\,(X) \subset \mathrm{PGL}(1)^{n+1}$ for generic X. Let $1 \neq g \in \mathrm{PGL}\,(1)^{n+1}$ be an element of finite order. Then *up to the conjugate action* of $\mathrm{PGL}(1)^{n+1}$, the co-action of g is written as $g^*(x_j) = c_j x_j$ $(1 \leq j \leq n+1)$, where c_j are all root of 1 and at least one c_j, say c_1, is not 1. Suppose that $f_X(x_j)$ is g^*-semi-invariant. Then, in terms of the 3^{n+1} monimials basis of $H^0(-K_{P(n+1)})$;

$$x_1^{k_1} x_2^{k_2} \cdots x^{k_{n+1}} \ , \ k_j = 0, 1, 2 \ ,$$

$f_X(x_j)$ has to be a linear combination of those monimials that satisfy the relation

$$c_1^{k_1} c_2^{k_2} \cdots c_{n+1}^{k_{n+1}} = c \ ,$$

for some constant c. Since $c_1 \neq 1$, for each fixed choice of $(k_2, k_3, \dots, k_{n+1})$, at most two of the three monomials

$$x_1^2 x_2^{k_2} \cdots x^{k_{n+1}} \ , \ x_1 x_2^{k_2} \cdots x^{k_{n+1}} \ , \ x_2^{k_2} \cdots x^{k_{n+1}}$$

satisfy the relation above. Hence the number of monimials in $f_X(x_j)$ is at most $2 \cdot 3^n$ for each c. Note that the candidates of such c are only finitely many for each g. Thus

$X \in |-K_{P(n+1)}|$ with automorphism $g \neq 1$ belongs to a finitely many hyperplanes of dimension at most $2 \cdot 3^n - 1$ in $|-K_{P(n+1)}|$. On the other hand,

$$(3^{n+1} - 1) - (2 \cdot 3^n - 1 + \dim \mathrm{PGL}\,(1)^{n+1}) = 3^n - 3(n+1) \geq 3$$

for $n \geq 1$. Thus, $X \in |-K_{P(n+1)}|$ with non-trivial automorphism from $\mathrm{PGL}\,(1)^{n+1}$ is in a countable union of codimension ≥ 3 subsets of $|-K_{P(n+1)}|$. This proves (2).

Let us show (3). Let

$$X \stackrel{\overline{\pi}_j}{\to} \overline{X}_j \stackrel{q_j}{\to} P(n+1)_j$$

be the Stein factorization of π_j. Then, $\overline{\pi}_j$ is the small contraction corresponding to the codimension 1 face $F_j := \sum_{k \neq j} \mathbf{R}_{\geq 0} h_k$ of the nef cone $\overline{\mathrm{Amp}}\,(X)$. Thus, $\rho(X/\overline{X}_j) = 1$ with a $\overline{\pi}_j$-ample generator h_j. Hence $\overline{\pi}_j$ is a flopping contraction of X. Let us describe the flop of $\overline{\pi}_j$.

By definition of the Stein factorization, ι_j induces a *biregular* automorphism $\overline{\iota}_j$ of $\overline{X}_j$ that satisfies $\overline{\iota}_j \circ \overline{\pi}_j = \overline{\pi}_j \circ \iota_j$. We set

$$\overline{\pi}_j^+ := (\overline{\iota}_j)^{-1} \circ \overline{\pi}_j : X \to \overline{X}_j\ .$$

Then, $\iota \circ \overline{\pi}_j^+ = \overline{\pi}_j$ and $\iota_j^*(h_j) = -h_j + \sum_{k \neq j} 2h_k$ by (1). In particular, $\iota_j^*(h_j)$ is $\overline{\pi}_j^+$-anti-ample. Hence $\overline{\pi}_j^+ : X \to \overline{X}_j$, or by abuse of language, the associated birational transformation ι_j, is the flop of $\overline{\pi}_j : X \to \overline{X}_j$.

Recall that any flopping contraction of a Calabi-Yau manifold is given by a codimension one face of $\overline{\mathrm{Amp}}\,(X)$ up to automorphisms of X ([Ka2], Theorem (5.7)). Since there is no codimension one face of $\overline{\mathrm{Amp}}\,(X)$ other than F_j $(1 \leq j \leq n+1)$, it follows that there is no flop other than ι_j $(1 \leq j \leq n+1)$ up to $\mathrm{Aut}\,(X)$. On the other hand, by a fundamental result of Kawamata ([Ka3], Theorem 1), any birational map between minimal models is decomposed into finitely many flops up to automorphisms of the source variety. Thus any $\varphi \in \mathrm{Bir}\,(X)$ is decomposed into a finite sequence of flops modulo automorphisms of X.

Hence $\mathrm{Bir}\,(X)$ is generated by $\mathrm{Aut}\,(X)$ and ι_j $(1 \leq j \leq n+1)$. Since $\mathrm{Aut}\,(X) = \{1\}$ by (2), the first equality of (3) now follows. The last isomorphism in (3) is proved in (1). □

Remark 3.5. In general, minimal models of a given variety are not unique up to isomorphisms. See for instance [LO] for an impressive example for Calabi-Yau threefold case. However, by the proof of (3), our X has no minimal model other than X.

4. Proof of Theorem (1.2) (2)

We shall prove Theorem (1.2) (2) in a slightly more explicit form (Theorem (4.2)).

Before entering the proof, we recall definition of various cones in $\mathrm{NS}\,(X)_{\mathbf{R}}$ relevant to us, following [Ka1], [Ka2].

Let M be a Calabi-Yau manifold. An integral divisor D on M is called *movable* (resp. *effective*, $\mathbf{Q}$-*effective*) if the complete linear system $|D|$ has no fixed components (resp. $|D| \neq \emptyset$, $|mD| \neq \emptyset$ for some positive integer m). $g^*(D)$ is again movable (resp. effective, $\mathbf{Q}$-*effective*) for $g \in \mathrm{Bir}\,(M)$, because g is an isomorphism in codimension 1.

The *big cone* $\mathcal{B}\,(M)$ is the convex cone generated by the classes of big divisors. The *movable cone* $\overline{\mathcal{M}}\,(M)$ is the closure of the convex cone generated by the classes of movable divisors. The *effective cone* $\mathcal{B}^e(M)$ is the convex cone generated by the classes of effective divisors. The *movable effective cone* $\mathcal{M}^e(M)$ is defined to be $\overline{\mathcal{M}}(M) \cap \mathcal{B}^e(M)$.

$\mathrm{Bir}(M)$ naturally acts on these four cones $\overline{\mathcal{M}}(M)$, $\mathcal{B}^e(M)$ and $\mathcal{M}^e(M)$.

Note that $\mathcal{B}(M) \subset \mathcal{B}^e(M)$ and the *pseudo-effective cone* $\overline{\mathcal{B}}(M)$ is the closure of both $\mathcal{B}(M)$ and $\mathcal{B}^e(M)$. Both nef cone $\overline{\mathrm{Amp}}(M)$ and the movable cone $\overline{\mathcal{M}}(M)$ involve the closure. So, *in apriori*, $\overline{\mathcal{M}}(M)$ is not necessarily a subset of $\mathcal{B}^e(M)$. Similarly, $\overline{\mathrm{Amp}}(M)$ is not necessarily a subset of $\mathcal{B}^e(M)$ nor $\mathcal{M}^e(M)$, while it is always true that $\overline{\mathrm{Amp}}(M) \subset \overline{\mathcal{M}}(M)$. We also recall that the ample cone $\mathrm{Amp}(M)$ is the set of interior points of the nef cone $\overline{\mathrm{Amp}}(M)$.

The abstract version of the Morrison-Kawamata movable cone conjecture is the following:

Conjecture 4.1. The action of $\mathrm{Bir}(M)$ on the movable effective cone $\mathcal{M}^e(M)$ has a finite rational polyhedral cone Δ as a fundamental domain. Here Δ is called a fundamental domain if

$$g^*(\Delta^\circ) \cap \Delta^\circ = \emptyset \, , \, \forall g^* \neq 1 \ ,$$

where Δ° is the set of interior points of Δ, and

$$\mathrm{Bir}(M) \cdot \Delta = \mathcal{M}^e(M) \, .$$

Let us return back to our special Calabi-Yau manifold. From now until the end of this section, we denote by X a generic hypersurface of multi-degree $(2, 2, \dots, 2)$ in $P(n+1)$ ($n \geq 3$). We also use the same notation as in section 3. Recall that $\mathrm{NS}(X)_{\mathbf{R}} = \oplus_{j=1}^{n+1} \mathbf{R} h_j$, and the nef cone is $\overline{\mathrm{Amp}}(X) = \oplus_{j=1}^{n+1} \mathbf{R}_{\geq 0} h_j$. In particular, $\overline{\mathrm{Amp}}(X)$ is a finite rational polyhedral cone for our X. In general, $\overline{\mathrm{Amp}}(M)$ is not necessarily a finite rational polyhedral cone for a Calabi-Yau manifold.

Main result of this section is the following:

Theorem 4.2. *The nef cone $\overline{\mathrm{Amp}}(X)$, which is a finite rational polyhedral cone, is the fundamental domain of the action of $\mathrm{Bir}(X)$ on the movable effective cone $\mathcal{M}^e(X)$.*

Remark 4.3. Since $\mathrm{Bir}(X)$ is much bigger than $\mathrm{Aut}(X)$, our theorem (4.2) says that the movable effective cone $\mathcal{M}^e(X)$, whence the psuedo effective cone $\overline{\mathcal{B}}(X)$, is much bigger than the nef cone $\overline{\mathrm{Amp}}(X)$. On the other hand, for the ambient space $P(n+1)$, we have that $\overline{\mathcal{B}}(P(n+1)) = \overline{\mathrm{Amp}}(P(n+1))$. This is a direct consequence of the fact that the intersection numbers

$$(v.H_1 \cdots H_{k-1} \cdot H_{k+1} \cdots H_{n+1})_{P(n+1)} \ , \ 1 \leq k \leq n+1$$

are non-negative if $v \in \overline{\mathcal{B}}(P(n+1))$. So, under the isomorphism $\mathrm{NS}(P(n+1)) \simeq \mathrm{NS}(X)$, we have $\overline{\mathrm{Amp}}(P(n+1)) \simeq \overline{\mathrm{Amp}}(X)$ (Theorem (3.1)) but $\overline{\mathcal{B}}(P(n+1)) \not\simeq \overline{\mathcal{B}}(X)$ even if X is generic. This gives an explicit negative answer to a question of Mister Yoshinori Gongyo (in any dimension ≥ 3) for me by e-mail.

In the rest of this section, we prove Theorem (4.2).

Lemma 4.4. *(1)* $\overline{\mathrm{Amp}}(X) \subset \mathcal{M}^e(X)$.

(2) $g^*(\mathrm{Amp}(X)) \cap \mathrm{Amp}(X) = \emptyset$ *for* $g \neq 1$.

Proof. Since h_j $(1 \le j \le n+1)$ are free (hence movable), (1) follows. If $g \in \mathrm{Aut}\,(X)$ satisfies $g^*(\mathrm{Amp}\,(X)) \cap \mathrm{Amp}\,(X) \neq \emptyset$, then $g \in \mathrm{Aut}\,(X)$ by [Ka2], Lemma (1.5). Since $\mathrm{Aut}\,(X) = \{1\}$, the assertion (2) now follows. □

Lemma 4.5. *(1) For any effective* $\mathbf{Q}$*-divisor (class) D, there is $g \in \mathrm{Bir}\,(X)$ such that $g^*(D) \in \overline{\mathrm{Amp}}\,(X)$.*

(2) In particular, any effective divisor on X is movable and $\mathcal{M}^e(X) = \mathcal{B}^e(X)$.

Proof. By linearlity, we may assume that D is integral. Put $D_1 := D$. In $\mathrm{NS}\,(X)$, we can write

$$D_1 = \sum_{j=1}^{n+1} a_j(D_1)h_j \ ,$$

where $a_j(D_1)$ are integers. Put

$$s(D_1) := \sum_{j=1}^{n+1} a_j(D_1) \ .$$

By definition, $s(D_1)$ is an integer. Since D_1 is an effective divisor class and h_i are nef, it follows that

$$a_n(D_1) + a_{n+1}(D_1) = (D.h_1.h_2.\dots.h_{n-1})_X \ge 0\,.$$

For the same reason, $a_i(D) + a_j(D_1) \ge 0$ for any $1 \le i \neq j \le n+1$. Hence there is *at most* one i such that $a_i(D_1) < 0$. In fact, if otherwise, there would be $1 \le i \neq j \le n+1$ such that $a_i(D_1) < 0$ and $a_j(D_1) < 0$. However, then $a_i(D_1) + a_j(D_1) < 0$, a contradiction to $a_i(D_1) + a_j(D_1) \ge 0$.

Moreover $s(D_1) \ge 0$. In fact, if all $a_i(D_1) \ge 0$, then this is true by definition of $s(D_1)$. If there is i such that $a_i(D_1) < 0$, say, $a_1(D_1) < 0$, then by $a_1(D_1) + a_2(D_1) \ge 0$ and $a_k(D_1) \ge 0$ $(k \ge 2)$ as observed above, we have

$$s(D_1) = (a_1(D_1) + a_2(D_1)) + a_3(D_1) + \cdots + a_{n+1}(D_1) \ge 0 \ .$$

If $D_1 \in \overline{\mathrm{Amp}}\,(X)$, then we can take $g = 1$. So, we may assume that $D_1 \notin \overline{\mathrm{Amp}}\,(X)$. There is then i such that $a_i(D_1) < 0$. Then consider the new divisor class $D_2 := \iota_i^*(D_1)$. D_2 is an effective divisor class. By definition of $a_j(*)$ and Theorem (3.3)(1), we have

$$\sum_{j=1}^{n+1} a_j(D_2)h_j = D_2 = -a_i(D_1)h_i + \sum_{j \neq i}(a_j(D_1) + 2a_i(D_1))h_j \ .$$

Hence, by definition of $s(*)$ and $a_i(D_1) < 0$, it follows that

$$s(D_2) = \sum_{j=1}^{n+1} a_j(D_2) = -a_i(D_1) + \sum_{j \neq i}(a_j(D_1) + 2a_i(D_1)) = s(D_1) + (2n-1)a_i(D_1) < s(D_1) \ .$$

If $a_j(D_2) \ge 0$ for all j, then $D_2 \in \overline{\mathrm{Amp}}\,(X)$, and we are done. Otherwise, there is j such that $a_j(D_2) < 0$. Consider then the divisor class $D_3 := \iota_j^*(D_2)$. For the same reason as above, D_3 is an effective divisor class such that $s(D_3) < s(D_2)$. We repeat this process until all the coefficients $a_j(D_\ell)$ of D_ℓ become non-negative, i.e., D_ℓ becomes a nef divisor class. This is possible. In fact, if this process would *not* terminate, then there would be an *infinite* sequence of effective divisor classes D_k $(k \ge 1)$ such that

$$s(D_1) > s(D_2) > s(D_3) > \cdots > s(D_k) > \cdots \ge 0 \ .$$

However, this is impossible, because $s(D_k) \in \mathbf{Z}$. This proves (1).

By (1), for any effective divisor class D, there is $g \in \mathrm{Bir}\,(X)$ such that $g^*(D) \in \overline{\mathrm{Amp}}\,(X)$, thus $g^*(D) = \sum_{j=1}^{n+1} c_j h_j$ with $c_j \ge 0$. Since h_j $(1 \le j \le n+1)$ are represented by free divisors, it follows that $g^*(D) \in \overline{\mathcal{M}}\,(X)$. Thus $D = (g^{-1})^*(g^*(D)) \in \overline{\mathcal{M}}\,(X)$. Hence by definition of $\mathcal{B}^e(X)$, we have $\mathcal{B}^e(X) \subset \overline{\mathcal{M}}\,(X)$, whence

$$\mathcal{M}^e(X) = \overline{\mathcal{M}}\,(X) \cap \mathcal{B}^e(X) = \mathcal{B}^e(X) \ .$$

This proves (2). □

Lemma 4.6. *If $v \in \mathcal{B}\,(X)$, then there is $g \in \mathrm{Bir}\,(X)$ such that $g^*(v) \in \overline{\mathrm{Amp}}\,(X)$.*

Proof. Our proof here is communicated by Doctor Arthur Prendergast-Smith. We may assume that v itself is an effective $\mathbf{R}$-divisor. Then, for sufficiently small $\epsilon > 0$, the pair $(X, \epsilon v)$ is klt. Since $K_X = 0$ and ϵv is big, one can run the minimal model program with scaling for $(X, \epsilon v)$ by a fundamental result of Birkar-Cascini-Hacon-McKernan ([BCHM], Corollary (1.4.2)). The resulting pair is a pair (X', v') such that v' is nef (and big) on X'. Note that any log-extremal contraction of X is given by a divisor in the interior of a codimension 1 face of $\overline{\mathrm{Amp}}\,(X)$. Thus, by the explicit description of $\overline{\mathrm{Amp}}\,(X)$ and by the observations in section 3, any log-extremal contraction of X is a small contraction and the log-flip corresponding to the contraction is one of the flops ι_j $(1 \le j \le n+1)$. Hence, in this process, X is unchanged. Thus $X' = X$ and $v' = g^*(\epsilon v)$, where g is a finite composition of ι_j $(1 \le j \le n+1)$. Since v' is nef, this proves Lemma (4.6). □

Now the following Lemma will complete the proof:

Lemma 4.7. *If $v \in \mathcal{M}^e(X) \setminus \mathcal{B}\,(X)$, then there is $g \in \mathrm{Bir}\,(X)$ such that $g^*(v) \in \overline{\mathrm{Amp}}\,(X)$.*

Proof. We may assume that $v \neq 0$. Then by definition, we can write v as a finite positive linear combination

$$v = \sum_{k=1}^{N} \alpha_k D_k \ ,$$

where α_k are positive real numbers and D_k are effective, hence movable by Lemma (4.5), divisor classes. We may assume that D_k themselves are movable effective divisors. Since v is not in $\mathcal{B}\,(X)$, *none of D_k is big.*

By Lemma (4.5), for each D_k, there is $g_k \in \mathrm{Bir}\,(X)$ such that $g_k^*(D_k) \in \overline{\mathrm{Amp}}\,(X)$, i.e., $g_k^*(D_k) = \sum_{j=1}^{n+1} a_{k,j} h_j$ for some non-negative integers $a_{k,j} \ge 0$ $(1 \le j \le n+1)$. Since D_k, whence $g_k^*(D_k)$, is not big, *at least two* of $a_{j,k}$ $(1 \le j \le n+1)$ are 0 for each k, by the description of $\overline{\mathrm{Amp}}\,(X)$. Let

$$m(D_k, g_k) := |\{j \,|a_{k,j} = 0\}| \ge 2 \ , \ m := m(v) := \min\{m(D_k, g_k)\} \ ,$$

where the minimum is taken under *all possible* expressions of v as positive linear combinations of movable divisor classes and *all possible* g_k with $g_k^*(D_k) \in \overline{\mathrm{Amp}}\,(X)$ as above. Since

$m(D_k, g_k)$ are integers greater than or equal to two, the minimum $m = m(v)$ exists and it is an integer greater than or equal to 2. Let

$$v = \sum_{k=1}^{N} \alpha_k D_k$$

be one of the expressions which attain the minimum $m = m(v)$. Then (after renumbering D_k's if necessary), we may assume that $m = m(D_1, g_1)$ in this expression. Then, after renumbering h_j's if necessary, we may further assume that

$$a_{1,1} = a_{1,2} = \cdots a_{1,m} = 0 \ , \ a_{1,m+1} > 0 \ \ldots , a_{1,n+1} > 0 \ ,$$

in the expression $g_1^* D_1 = \sum_{j=1}^{n+1} a_{1,j} h_j$, i.e.,

$$g_1^* D_1 = \sum_{j=m+1}^{n+1} a_{1,j} h_j \ ,$$

where $a_{1,j} > 0$ for all j $(m + 1 \le j \le n + 1)$. Set $g_1^* D_2 = \sum_{j=1}^{n+1} b_{2,j} h_j$. Since $g_1^* D_2$ is movable, $b_{2,j} + b_{2,k} \ge 0$. Assume that $b_{2,j} < 0$ for some $j \ge m + 1$. Then $b_{2,k} > 0$ for all $1 \le k \le m$ by the inequality above. Then, any very small positive rational number q, we would have that $g_1^*(D_1 + qD_2) = \sum_{j=1}^{n+1} c_j h_j$ with $c_j > 0$ for all $1 \le j \le n + 1$. However, then $v = \alpha_1(D_1 + qD_2) + (\alpha_2 - \alpha_1 q)D_2 + \sum_{k\ge 3} \alpha_k D_k$ would be big, a contradiction to the choice of v. Hence $a_{2,j} \ge 0$ for all $j \ge m + 1$ and therefore $k \le m$ if $a_{2,k} < 0$. Assume that $a_{2,k} < 0$ for some $k \le m$. Then, performing ι_k^*, we have that

$$\sum_{j=1}^{n+1} a_{2,j}^{(2)} h_j := \iota_k^* g_1^* D_2 = (a_{2,1} + 2a_{2,k})h_1 + \cdots + -a_{2,k} h_k + \cdots + (a_{2,n+1} + 2a_{2,k})h_{n+1}$$

and, by $k \le m + 1$, we also have that $\iota_k^* g_1^* D_1 = g_1^* D_1$. For the same reason as before, $a_{2,j}^{(2)} \ge 0$ for all $j \ge m+1$. If some of $a_{2,k'}^{(2)}$ $(k' \le m)$ is still negative, we will repeat the same process as above, until all the coefficients become non-negative. This is possible as follows. Recalling the function $s(*)$ in the proof of Lemma (4.5), we have

$$0 \le s(\iota_k^* g_1^* D_2) < s(g_1^* D_2) \ .$$

Thus, this process certainly terminates and we finally obtain an element $g_2 \in \mathrm{Bir}\,(X)$ such that

$$g_2^* D_1 = \sum_{j=m+1}^{n+1} a_{1,j} h_j \ , \ g_2^* D_2 = \sum_{j=1}^{n+1} a_{2,j} h_j$$

where $a_{1,j} \ge 0$ and $a_{2,j} \ge 0$ for all j. Then, again by the minimality of m applied for the expression

$$v = \alpha_1(D_1 + qD_2) + (\alpha_2 - \alpha_1 q)D_2 + \sum_{k\ge 3} \alpha_k D_k$$

with very small positive number q, we find that $a_{2,j} = 0$ for all $j \le m$. Hence

$$g_2^* D_1 = \sum_{j=m+1}^{n+1} a_{1,j} h_j \ , \ g_2^* D_2 = \sum_{j=m+1}^{n+1} a_{2,j} h_j \ , \ a_{1,j} \ge 0 \ , \ a_{2,j} \ge 0 \ .$$

We can repeat the same process for the movable divisor class $g_2^*D_3$ and we obtain $g_3 \in \mathrm{Bir}\,(X)$, which is by the process above the compsition of g_2 and ι_k with $k \le m+1$, such that

$$g_3^*D_r = \sum_{j=m+1}^{n+1} a_{r,j}h_j \ ,\ a_{r,j} \ge 0$$

for all $r = 1$, 2, 3 and for all $j \ge m+1$. Here we note that $\iota_k^*D_1 = D_1$ and $\iota_k^*D_2 = D_2$ for $k \le m$ by the descriptions of $g_2^*D_1$ and $g_2^*D_2$ above and the description of ι_k^* in Definition (2.2). Now we can repeat this process inductively until D_N and finally obtain a birational automorphism $g_N \in \mathrm{Bir}\,(X)$ such that

$$g_N^*D_r = \sum_{j=m+1}^{n+1} a_{r,j}h_j \ ,\ a_{r,j} \ge 0$$

for all $1 \le r \le N$ and for all $j \ge m+1$. We may then take $g = g_N$. □

It may be natural to ask the following question also in the view of Theorem (3.1):

Question 4.8. Let V be a Fano manifold of dimension ≥ 4 such that $|-K_V|$ is free (or very ample if you like). How extent can one generalize Theorem (1.2) to Calabi-Yau manifolds being generic in $|-K_V|$?

5. Proof of Theorem (1.4)

In this section, we shall prove Theorem (1.4).

Let S be an Enriques surface, that is, a compact complex surface whose universal cover is a K3 surface $\tilde{S}$. It is well-known that S is projective, $\pi_1(S) \simeq \mathbf{Z}/2$, and the Enriques surfaces form 10-dimensional family (See eg. [BHPV], Chapter VIII for basic facts on Enriques surfaces). The free part of the Néron-Severi group $\mathrm{NS}_f\,(S)$ is isomorphic to the lattice $U \oplus E_8(-1)$. Here U is the unique even unimodular lattice of signature $(1,1)$ and $E_8(-1)$ is the unique even unimodular *negative* definite lattice of rank 8 (see eg. [BHPV], Chapter I, section 2 for details). From now, we identify the lattices $\mathrm{NS}_f\,(S)$ and $U \oplus E_8(-1)$. We denote the group of isometries of $\mathrm{NS}_f\,(S) = U \oplus E_8(-1)$ preserving the positive cone by $\mathrm{O}^+(U \oplus E_8(-1))$. Here the positive cone $P(S)$ is the connected component of $\{\,x \in \mathrm{NS}\,(S)_{\mathbf{R}} \,|\, (x^2) > 0\}$, containing the ample cone. We define

$$\mathrm{O}^+(U\oplus E_8(-1))[2] := \{\varphi \in \mathrm{O}^+(U\oplus E_8(-1)) \,|\, \varphi(x) - x \in 2(U\oplus E_8(-1))\, \forall x \in U\oplus E_8(-1)\,\}\ .$$

Here $2(U \oplus E_8(-1)) = \{2y \,|\, y \in U \oplus E_8(-1)\}$. By an important result of Barth and Peters ([BP], Theorem (3.4), Proposition (2.8)), among all Enriques surfaces, generic ones satisfy the following:

Theorem 5.1. *Let S be a generic Enriques surface. Then S has no $\mathbf{P}^1$ and*

$$\mathrm{Aut}\,(S) \simeq \mathrm{O}^+(U \oplus E_8(-1))[2]$$

under the natural map $g \mapsto g^*$.

Form now until the end of this section, S is a generic Enriques surface. Let e_1, e_2 be the standard basis of U, i.e., $U = \mathbf{Z}\langle e_1, e_2\rangle$ and $(e_1^2)_S = (e_2^2)_S = 0$ and $(e_1.e_2)_S = 1$. We choose

e_1 so that it is in the closure of the positive cone. Let $v \in E_8(-1)$ be an element such that $(v^2)_S = -2$. Put $e_3 := e_1 + e_2 + v$. Then, we have

$$(e_3^2)_S = 0 \ , \ (e_3, e_1)_S = (e_3, e_2)_S = 1 \ .$$

So, e_2 and e_3 are also in the closure of the positive cone and the three sublattices

$$U_3 = \mathbf{Z}\langle e_1, e_2\rangle \ , U_2 = \mathbf{Z}\langle e_1, e_3\rangle \ , \ U_1 = \mathbf{Z}\langle e_2, e_3\rangle$$

of $\mathrm{NS}_f(S)$ are isomorphic to U. According to these three sublattices, we have three orthogonal decompositions of NS_f:

$$\mathrm{NS}_f = U_3 \oplus U_3^{\perp} \ , \ \mathrm{NS}_f = U_2 \oplus U_2^{\perp} \ \ \mathrm{NS}_f = U_1 \oplus U_1^{\perp} \ .$$

Consider the three isometries of NS_f:

$$\iota_3^* = id_{U_3} \oplus -id_{U_3^{\perp}} \ , \ \iota_2^* = id_{U_2} \oplus -id_{U_2^{\perp}} \ , \ \iota_1^* = id_{U_1} \oplus -id_{U_1^{\perp}} \ .$$

Then, $\iota_3^*, \iota_2^*, \iota_1^* \in \mathrm{O}^+(U \oplus E_8(-1))[2]$, whence (by abuse of notation) they come from automorphisms $\iota_3, \iota_2, \iota_1 \in \mathrm{Aut}\,(S)$ by Theorem (5.1). Let $\langle \iota_1, \iota_2, \iota_3\rangle$ be the subgroup of $\mathrm{Aut}\,(S)$ generated by ι_1, ι_2 and ι_3. Recall that an element of $g \in \mathrm{Aut}\,(S)$ is *of positive entropy* if the spectral radius of the action of g on the even cohomology ring $\oplus_k H^{2k}(S, \mathbf{Z})$ is strictly bigger than 1.

Theorem 5.2. *In* $\mathrm{Aut}\,(S)$, $\langle \iota_1, \iota_2, \iota_3\rangle = \langle \iota_1\rangle * \langle \iota_2\rangle * \langle \iota_3\rangle \simeq \mathrm{UC}(3)$. *Moreover, the maximal eigenvalue of* $\iota_1^*\iota_2^*\iota_3^*$ *on* $\mathrm{NS}\,(S)$ *is greater than* 1. *In particular,* $\iota_3\iota_2\iota_1$ *is of positive entropy.*

Proof. Since ι_j^* $(j = 1, 2, 3)$ are involutions, so are ι_j by Theorem (5.1). Hence the result follows from the next Lemma:

Lemma 5.3. *Define the sublattice L of* $\mathrm{NS}\,(S)$ *by* $L = \mathbf{Z}\langle e_1, e_2, v\rangle = \mathbf{Z}\langle e_1, e_2, e_3\rangle$. *Then L is* $\langle \iota_1, \iota_2, \iota_3\rangle$*-invariant sublattice. Moreover, under the basis* $\langle e_1, e_2, e_3\rangle$,

$$\iota_3^*|L = M_{3,3} \ , \ \iota_2^*|L = M_{3,2}, \ , \ \iota_1^*|L = M_{3,1} \ ,$$

where $M_{3,3}$, $M_{3,2}$ and $M_{3,1}$ are the matrices in Definition (2.2). Moreover, the maximal eigenvalues of the product matrix $M_{3,1}M_{3,2}M_{3,3}$ is $9 + 4\sqrt{5} > 1$.

Proof. Let us check that L is ι_1-invariant and $\iota_1^*|L = M_{3,1}$, that is, $\iota_1^*(e_1) = -e_1 + 2e_2 + 2e_3$, $\iota_3^*(e_2) = e_2$, $\iota_1^*(e_3) = e_3$. By definition of ι_1^*, it follows that $\iota_1^*(e_2) = e_2$ and $\iota_1^*(e_3) = e_3$. Since $(2e_2 + v, e_2)_S = 0$, $(2e_2 + v, e_3)_S = (2e_2 + v, e_1 + e_2 + v)_S = 0$, it follows that $2e_2 + v \in U_1^{\perp}$. Thus $\iota_1^*(2e_2 + v) = -(2e_2 + v)$. Hence $\iota_1^*(v) = -4e_2 - v$. Therefore

$$\iota_1^*(e_1) = \iota_1^*(e_3 - e_2 - v) = e_3 - e_2 + 4e_2 + v = e_3 + 3e_2 + e_3 - e_2 - e_1 = -e_1 + 2e_2 + 2e_3 \ .$$

One can check that L is invariant under ι_2 and ι_3 and $\iota_2^*|L = M_{3,2}$ and $\iota_3^*|L = M_{3,3}$ in a similar manner. This proves the first part of Lemma (5.3). An explicit computation shows that the eigenvalues of $M_{3,1}M_{3,2}M_{3,3}$ are 1, $9 \pm 4\sqrt{5}$. This implies the second part of Lemma (5.3). □

This completes the proof of Theorem (5.2). □

Remark 5.4. Professor Shigeru Mukai kindly told me the following a more geometric description of the group action above.

The line bundle $h = e_1 + e_2 + e_3$ is an ample line bundle of degree 6 and the projective model of S associated to $|h|$ is a sextic surface in $\mathbf{P}^3$ singular along the 6-lines joining 2 of the 4 vertices $[0:0:0:1]$, $[0:0:1:0]$, $[0:1:0:0]$, $[1:0:0:0]$. Then the universal covering K3 surface $\tilde{S}$ has a projective model of degree 12. It is a quadratic section of the Segre manifold $P(3) = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^7$. So, $\tilde{S}$ is a K3 surface of multi-degree $(2,2,2)$ in $P(3)$, i.e., a K3 surface of Wehler type. One can also see this fact in a different way as follows. Let $\pi : \tilde{S} \to S$ be the universal covering map. $\pi^* e_i$ $(i = 1, 2, 3)$ define three different elliptic fibrations $\varphi_i : \tilde{S} \to \mathbf{P}^1$ with no reducible fiber. Hence $\varphi_1 \times \varphi_2 \times \varphi_3$ gives an embedding of $\tilde{S}$ into $P(3)$, which is necessarily a surface of multi-degree $(2,2,2)$.

In anyway, the action of UC(3) on S is the descend of the natural action UC(3) on $\tilde{S}$ (Theorem (6.1) (2) in section 6).

Let us prove Theorem (1.4). We have a natural biregular action $\mathrm{UC}(3) = \langle \iota_1, \iota_2, \iota_3 \rangle$ on the Hilbert scheme $\mathrm{Hilb}^n(S)$ induced by the action on S. Each $\iota_j \in \mathrm{Aut}\,(\mathrm{Hilb}^n(S))$ then lifts to a biregular action $\tilde{\iota}_j \in \mathrm{Aut}\,(\widetilde{\mathrm{Hilb}^n(S)})$ on the universal cover $\widetilde{\mathrm{Hilb}^n(S)}$ equivariantly.

It is suffices to show that each $\tilde{\iota}_j$ is still an involution. In fact, we will then have a natural surjective group homomorphisms

$$\mathrm{UC}(3) \to \langle \tilde{\iota}_1, \tilde{\iota}_2, \tilde{\iota}_3 \rangle \to \langle \iota_1, \iota_2, \iota_3 \rangle \simeq \mathrm{UC}(3)\ .$$

Hence all the arrows will be isomorphic.

Let $\pi : \widetilde{\mathrm{Hilb}^n(S)} \to \mathrm{Hilb}^n(S)$ be the universal covering map and σ be the covering involution. Then $(\tilde{\iota}_j)^2$ is either id or σ. As in [MN], Lemma (1.2), assuming that $(\tilde{\iota}_j)^2 = \sigma$, we shall derive a contradiction. Under the assumption, $\langle \tilde{\iota}_j \rangle$ would be a cyclic group of order 4 and would act on $\widetilde{\mathrm{Hilb}^n(S)}$ freely, as so is σ. However, then,

$$\mathbf{Z} \ni \chi(\mathcal{O}_{\widetilde{\mathrm{Hilb}^n(S)}/\langle \tilde{\iota}_j \rangle}) = \frac{\chi(\mathcal{O}_{\widetilde{\mathrm{Hilb}^n(S)}})}{4} = \frac{2}{4} \notin \mathbf{Z}\ ,$$

a contradiction. Here for the last equality, we used the fact that $\chi(\mathcal{O}_{\widetilde{\mathrm{Hilb}^n(S)}}) = 2$ ([OS], Theorem (3.1), Lemma (3.2)). Hence $(\tilde{\iota}_j)^2 = id$ and the first assertion follows.

Since $\mathrm{NS}_f\,(S)$ is embedded into $H^2(\widetilde{\mathrm{Hilb}^n(S)}, \mathbf{Z})$ equivariantly, it follows that $\tilde{\iota}_3\tilde{\iota}_2\tilde{\iota}_1$ has $9+4\sqrt{5} > 1$ as one of eigenvalues on $H^2(\widetilde{\mathrm{Hilb}^n(S)}, \mathbf{Z})$. This completes the proof of Theorem (1.4).

Remark 5.5. There are Enriques surfaces S such that $|\mathrm{Aut}\,(S)| < \infty$. (See [Kn], Main Theorem.) On the other hand, Kondo ([Kn], Corollary) shows that $|\mathrm{Aut}\,(\tilde{S})| = \infty$ for *every* K3 surface $\tilde{S}$ being the universal cover of an Enriques surface.

It may be natural to ask the following generalization of Remark (5.5):

Question 5.6. $|\mathrm{Aut}\,(\widetilde{\mathrm{Hilb}^n(S)})| = \infty$ for *every* Enriques surface?

6. Proof of Theorem (1.1).

Let S be a smooth surface of multi-degree $(2, 2, 2)$ in $P(3) := \mathbf{P}^1_1 \times \mathbf{P}^1_2 \times \mathbf{P}^1_3$. Then, S is a projective K3 surface. As in section 3, we denote the covering involution corresponding to the projection $S \to \mathbf{P}^1_i \times \mathbf{P}^1_j$ by ι_k, where $\{i, j, k\} = \{1, 2, 3\}$. Since S is a minimal surface, ι_k are biregular and thus $\langle \iota_1, \iota_2, \iota_3 \rangle \subset \mathrm{Aut}\,(S)$ (See eg. [BHPV], Page 99, Claim). As in sections 3 and 4, we also denote by $\tau : S \to P(3)$ the natural inclusion, by H_j and h_j ($j = 1$, 2, 3) the line bundles on $P(3)$ and S coming from $\mathcal{O}_{\mathbf{P}^1_j}(1)$.

In this section, we shall prove Theorem (1.1) in the following slightly more general form:

Theorem 6.1. *(1) If S is generic, then* $\mathrm{NS}\,(S) = \oplus_{j=1}^3 \mathbf{Z} h_j$.

(2) If S is generic, then $\mathrm{Aut}\,(S) = \langle \iota_1, \iota_2, \iota_3 \rangle = \langle \iota_1 \rangle * \langle \iota_2 \rangle * \langle \iota_3 \rangle \simeq \mathrm{UC}(3)$.

(3) If S is smooth, then $\mathrm{Aut}\,(S) \supset \langle \iota_1, \iota_2, \iota_3 \rangle = \langle \iota_1 \rangle * \langle \iota_2 \rangle * \langle \iota_3 \rangle \ \simeq \mathrm{UC}(3)$.

Proof.

Lemma 6.2. *Let S be generic. Then:*

(1) $\mathrm{NS}\,(S) = \oplus_{i=1}^3 \mathbf{Z} h_i$.

(2) The intersection matrix $((h_i.h_j)_S)$ is

$$((h_i.h_j)_S) = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

(3) $\mathrm{Amp}\,(S) = P(S)$ *in* $\mathrm{NS}\,(S)_{\mathbf{R}}$*. Here $P(S)$ is the positive cone.*

Proof. Since S is generic, it follows from the Noether-Lefschetz theorem ([Vo], Theorem (3.33)) that $\tau^* : \mathrm{NS}\,(P(3)) \to \mathrm{NS}\,(S)$ is an isomorphism. This imples (1). The assertion (2) follows from $(h_i.h_j)_S = (H_i.H_j.2(H_1 + H_2 + H_3))_{P(3)}$. Since there is no $x \in \mathrm{NS}\,(S)$ such that $(x^2) = -2$ by (2), there is no $\mathbf{P}^1$ in S. This implies (3). $\square$

Remark 6.3. (1) By the proof above, if W is a generic element of $|-K_V|$ of a smooth Fano threefold with very ample anti-canonical divisor $-K_V$, then W is a K3 surface and $\mathrm{NS}\,(V) \simeq \mathrm{NS}\,(W)$ under the natural inclusion map.

(2) Even though S is generic and $\tau^* : \mathrm{NS}\,(P(3)) \simeq \mathrm{NS}\,(S)$ is isomorphic, the image of $\tau^* : \mathrm{Amp}\,(P(3)) \to \mathrm{Amp}\,(S)$ is strictly smaller than $\mathrm{Amp}\,(S)$. This is completely different from higher dimensional case (Theorem (3.1) (2)). This also gives an explicit negative answer for a question of Mister Yoshinori Gongyo to me by e-mail.

Lemma 6.4. *Let S be smooth (not necessarily generic) and ι_k^* be the natural action of ι_k on* $\mathrm{NS}\,(S)$ *($k = 1, 2, 3$). Then the subspace $V := \oplus_{j=1}^3 \mathbf{Z} h_j$ is invariant under $\langle \iota_1, \iota_2, \iota_3 \rangle$. Moreover, $\iota_k^* | V = M_{3,k}$ under the basis $\langle h_1, h_2, h_3 \rangle$ of V. Here $M_{3,k}$ is the matrix in Definition (2.2). In particular,* $\langle \iota_1, \iota_2, \iota_3 \rangle = \langle \iota_1 \rangle * \langle \iota_2 \rangle * \langle \iota_3 \rangle \simeq \mathrm{UC}(3)$ *in* $\mathrm{Aut}\,(S)$.

Proof. Same as Theorem (3.3) (1). $\square$

Lemma 6.5. *Let S be generic. Then*

(1) No element other than identity of $\mathrm{Aut}\,(S)$ *is induced by* $\mathrm{Aut}\,(P(3))$.

(2) $\mathrm{Aut}\,(S) = \langle \iota_1, \iota_2, \iota_3 \rangle$.

Proof. Proof of (1) is the same as Theorem (3.3)(2). Note that ι_k^* is the hyperbolic reflection of the positive cone $P(S)$ with respect to the hyperplane $\mathbf{R}\langle h_i, h_j\rangle$, where $\{i,j,k\} = \{1,2,3\}$. Hence the polyhedral cone $V = \sum_{j=1}^{3} \mathbf{R}_{\geq 0} h_j$ is the fundamental domain of the action of the hyperbolic reflection group $\langle \iota_1, \iota_2, \iota_3\rangle$ on the cone generated by the interior points of $P(S)$ and the rational boundaries of $P(S)$. See [Vi] for hyperbolic reflection groups. Hence, for each $g \in \mathrm{Aut}\,(S)$, there is $\iota \in \langle \iota_1, \iota_2, \iota_3\rangle$ such that $\iota^{-1}g$ preserves V. This $\iota^{-1}g$ then preserves the set $\{h_1, h_2, h_3\}$. Thus $\iota^{-1}g$ is induced by $\mathrm{Aut}\,(P(3))$. Hence $\iota^{-1}g = 1$ by (1) and therefore $g = \iota$. This proves the asserion (2). □

Theorem (6.1) follows from Lemmas (6.4) and (6.5). □

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