

# Dimension of the moduli space and Hamiltonian analysis of BF field theories

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## Abstract

By using the Atiyah-Singer theorem through some similarities with the instanton and the anti-instanton moduli spaces, the dimension of the moduli space for two and four-dimensional BF theories valued in different background manifolds and gauge groups scenarios is determined. Additionally, we develop Dirac's canonical analysis for a four-dimensional modified BF theory, which reproduces the topological YM theory. This framework will allow us to understand the local symmetries, the constraints, the extended Hamiltonian and the extended action of the theory.

KEYWORDS: *Index theorem, moduli space, Hamiltonian dynamics*

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## I. Introduction and motivations

The BF formalism [1, 2, 3] in arbitrary space or space-time dimensions has shown profound relationships between topological field theory and quantum field theory, from a simplified version of general relativity [4, 5], to a new formulation of Yang-Mills (YM) theory [6, 7, 8, 9, 10, 11]. In the so called first-order formulation, YM theory can be viewed as a perturbative expansion in the coupling constant  $g$  around the pure topological BF theory; additionally the BF first-order formulation is *on shell* equivalent to the usual (second-order) YM theory. In this context, both formulations of the theory possess the same perturbative quantum properties [12, 13]. Furthermore, the Feynman rules, the structure of one loop divergent diagrams, and renormalization have been studied, and the equivalence of the  $uv$ -behavior of both formulations has been verified [14]. However, in spite of these developments, there exist certain basic aspects poorly understood in the specific case of four-dimensional BF theories, which is ironic as already mentioned by J. Baez [4, 5], since four-dimensional gauge theories are the main motivation, and the obvious subject of research; this is in part due to certain technical complications that the four-dimensional case has in relation to low-dimensional scenarios. It is surprising that, for example, the dimension of the corresponding

moduli spaces of four-dimensional BF theory have not been determined for a general base manifold. Specifically, the natural question to be asked is whether there exists any relationship between four-dimensional YM instantons moduli space and the corresponding moduli space of four-dimensional BF theory. In fact, such a relationship there exists in the case of BF fields on a Riemann surface, and the two-dimensional YM instantons on it ([1],[15], and references therein). In the present work we attempt to explore the BF moduli space using some similarities between the BF complex and YM instanton complex in four dimensions, employing as the main tool the Atiyah-Singer theorem. On the other hand, the instanton moduli space can be considered as a starting point for quantizing a field theory around a non-discrete space of classical minima, where the functional integration over the moduli space is treated non-perturbatively, whereas the integration over the quantum fluctuations “perpendicular” to the moduli space can be treated perturbatively [16, 17]. This scheme has been called the “topological embedding”, where the essential idea is that the moduli space around which the field theory is studied perturbatively possesses an enhanced gauge symmetry, the topological invariance; it is here where the topological BF theory fits naturally within the *topological embedding* setting since it constitutes the topological sector around which QCD or general relativity can be expanded. Therefore, it is crucial to obtain geometrical, topological and physical information about the moduli space of the BF theory, particularly in the case of four dimensions.

In order to obtain relevant information on the moduli space of the BF theory, it is possible to draw on the closely relation between four-dimensional BF moduli space and four-dimensional YM instantons and anti-instantons moduli spaces; this relation is based on the flatness condition, which is one of the equations of motion of the BF theory. It follows that connections meeting self-duality and anti-self-duality conditions simultaneously, namely, the connections belonging to the intersection of the instanton and the anti-instanton moduli spaces, are hence solutions of the flatness condition on the curvature. The space formed by these solutions up to gauge transformations is known as the moduli space of BF; as in the case of YM instantons corresponds generally to a finite-dimensional smooth manifold. This manifold is usually non-compact, partly due to conformal invariance of the equations of motion, leading to technical difficulties in the applications. It turns out that in order to define a well behaved gauge field theory, one needs to regularize the model to avoid problems with reducible connections. Reducible connections are source of great difficulty in making sense to the quantization of gauge field theories in general; the problem is that at reducible connections the path integrals related to partition functions diverge. This regularization amounts to considering a modified four-dimensional BF theory, reproducing in the limit the usual BF theory and the four-dimensional topological YM theory [18]. The main reason to use a modified version of the BF theory as we shall see within the Hamiltonian analysis, is that unlike the BF-YM theory, it shares the same gauge symmetries with the usual BF theory, referring with particular emphasis on diffeomorphisms. However, both cases can be treated perturbatively around the moduli space defined for flat connections within the *topological embedding* mentioned above.

On the other hand, all the information we need to calculate the dimension of the moduli space using the Atiyah-Singer theorem, is given by the equations of motion and the local symmetries of the theory. In order to know these local properties the Hamiltonian analysis is performed, identifying the relevant symmetries of the theory such as the extended action, the extended Hamiltonian and the gauge transformations, with particular emphasis to the latest since they allow us to build the elliptic complex which will provide all information about the global degrees of freedom of the theory under study.

In the next sections we outline the basic aspects of a BF theory and the index theorem calculus given by the historic works by Atiyah *et al* [19], but following the detailed calculations given in [20]; this will allow us to extend certain aspects and to modify other ones, for adapting to the special features of the BF moduli space. In Section IV, as a simple example the dimension of the moduli spaces for (non-Abelian) two-dimensional BF theory on a Riemann surface are determined in terms of the topological and geometrical invariants of the base manifold and the gauge bundle. In Section V following the example of the previous section, we characterize the dimension of moduli spaces for a four-dimensional BF theory and the general expression founded is used in particular base manifolds and gauge groups scenarios. In Section VI we present the Hamiltonian analysis for another BF-topological YM theory. As important results we shall find the extended action, the extended Hamiltonian and the gauge symmetries for the theory. In particular we prove that the theory under study is invariant under diffeomorphisms. We finish in Section VII with some concluding remarks and prospects.

## II. BF theory in four dimensions

Let  $M$  be an oriented smooth four-dimensional manifold,  $B$  a differential two-form, and  $F_A$  the curvature induced by a connection  $A$  on a principal bundle over  $M$  with structure group  $G$ ; the BF action is given by [1, 2, 3]

$$S_{BF} = \int_M Tr B \wedge F_A; \quad (1)$$

this non-Abelian action has the symmetry

$$A \rightarrow A + d_A \Lambda \quad (\text{and then } F_A \rightarrow F_A + [F_A, \Lambda]), \quad B \rightarrow B + [B, \Lambda], \quad (2)$$

and additionally,

$$A \rightarrow A, \quad B \rightarrow B + d_A \chi, \quad (3)$$

where  $\Lambda$  corresponds to an arbitrary 0-form on  $M$ , and  $\chi$  to an arbitrary 1-form; the symmetry (3) requires the Bianchi identities  $d_A F = 0$ , and is satisfied modulo a total derivative.

The equations of motion obtained from (1) read,

$$F_A = 0, \quad d_A B = 0. \quad (4)$$

The action (1) can be obtained in the limit of vanishing coupling ( $g \rightarrow 0$ ) of the first order formulation of YM theory given by the action [6, 7, 8, 9]

$$S_{\text{BF-YM}} = \int_M \text{Tr}(iB \wedge F + \frac{g^2}{4} B \wedge *B), \quad (5)$$

where  $*$  stands for the Hodge-duality operation, and with the gauge symmetry given in (2); it is only in the limit  $g \rightarrow 0$  that the second gauge symmetry (3) is present. Furthermore, the equations of motion of the action (5) read

$$F = i\frac{g^2}{2} *B, \quad d_A B = 0; \quad (6)$$

thus, the substitution of equations (6) into the action (5) leads to the standard YM action

$$S_{\text{YM}} = \frac{1}{g^2} \int \text{Tr} F \wedge *F; \quad (7)$$

therefore the BF-YM theory is *on-shell* equivalent to YM theory. Similarly we can find an (on-shell) equivalence between topological BF-YM theory and topological YM theory considering Eqs. (5) (with the symmetry (2)), (6), and (7) with the Hodge-duality operations removed; thus, the dependence on a metric structure of  $M$  is removed, (see section V). In this case the action (1) is also the vanishing coupling limit of the topological BF-YM theory. The action functional (5) and its topological version allow us to understand YM theory and topological YM theory as *perturbative expansions* in the coupling  $g$  around the topological pure BF theory (1) [10, 11], which defines an authentic topological quantum field theory[1, 2, 3].

Therefore, our fundamental topological sector is given by the pure BF action (1), whose moduli spaces are defined as the spaces of solutions of the corresponding equations of motion (4) modulo the gauge symmetries (2) and (4). More specifically we can define the  $A$ -moduli space as the space of (flat) connections satisfying the first of equations (4) modulo the gauge symmetry (2); additionally we define the  $B$ -moduli space as the space of two-forms satisfying the second of equations (4) modulo the gauge symmetry (3).

### III. Atiyah-Singer index theorem

Let  $M$  be a  $n$ -dimensional compact smooth manifold without boundary,  $\Gamma(E_p)$  sections of the (complex) vector bundles  $E_p$  on  $M$ ,  $D_p$  differential operators mapping between sections as indicated in the following finite sequence

$$\begin{aligned} \cdots &\longrightarrow \Gamma(E_{p-1}) \xrightarrow{D_{p-1}} \Gamma(E_p) \xrightarrow{D_p} \Gamma(E_{p+1}) \longrightarrow \cdots \\ \cdots &\longleftarrow \Gamma(E_{p-1}) \xleftarrow{D_{p-1}^\dagger} \Gamma(E_p) \xleftarrow{D_p^\dagger} \Gamma(E_{p+1}) \longleftarrow \cdots \end{aligned} \quad (8)$$

where  $D_p^\dagger$  corresponds to the dual operator of  $D_p$ ; if the Laplacian of the sequence  $\Delta_p = D_p^\dagger D_p + D_{p-1} D_{p-1}^\dagger$  is an *elliptic* differential operator [19, 21], then the sequence (8) defines an *elliptic complex* with an *index* expressed as

$$\text{Index}(E, D) = (-1)^{\frac{n}{2}(n+1)} \int_M \frac{\sum_p (-1)^p \text{ch}(E_p)}{e(T(M))} \text{td}(T(M) \otimes C), \quad (9)$$

where  $ch(E)$  correspond to the Chern characters of the vector bundles  $E$ ,  $td(T(M) \otimes C)$  to the Todd class of the complexified tangent bundle  $T(M) \times C$  of the manifold  $M$ , and  $e(T(M))$  is the Euler class of the tangent bundle  $T(M)$ .

It is important to mention that in general the procedure to calculate a moduli space dimension through the Atiyah-Singer index theorem is actually a way to estimate such a dimension, since the dimension may have unexpected and inadmissible values; a reason is the presence of a nontrivial second cohomology group in the corresponding elliptic complex. In this sense the index calculated corresponds to a *virtual* dimension, which will require additional considerations in order to obtain a real dimension. For example it is common the appearance of negative values of the virtual dimension, which will be associated with a empty moduli space; this will be a basic criterion in the present work, as usual in the instanton calculus scenario.

#### IV. BF moduli space on a Riemann surface

In their own right, gauge theories in two dimensions have for a long time served as useful laboratories for testing ideas and gaining insight into the properties of field theories in general, specifically quantum gauge theories on arbitrary Riemann surfaces. Our basic concern in this section is with the space of flat connections (gauge fields) on a compact Riemann surface of genus  $g$ ,  $M = \Sigma_g$ , and a compact gauge group  $G$ . A connection  $A$  on a  $G$  bundle over  $M$ , or a gauge field on  $M$ , is said to be flat when its curvature tensor  $F_A$  vanishes,

$$F_A = d\Lambda + \frac{1}{2}[\Lambda, \Lambda] = 0. \quad (10)$$

Flatness is preserved under gauge transformations  $A \rightarrow A^U$  where

$$A^U = U^{-1}AU + U^{-1}dU, \quad (11)$$

as  $F_A$  transforms to  $U^{-1}F_AU$ . The moduli space of flat connections  $\mathcal{M}_F(M, G)$  is the space of gauge inequivalent solutions to (10). This means that solutions to (10) which are not related by a gauge transformation are taken to be different points of  $\mathcal{M}_F(M, G)$ . On the other hand, if the solutions are related by a gauge transformation they are taken to be the same point in  $\mathcal{M}_F(M, G)$ , that is  $\{A^U\} = \{A\}$ .

The usual description of the moduli space in terms of representation of the fundamental group  $\pi_1(M)$  of the manifold  $M$  [21],

$$\mathcal{M}_F(M, G) = Hom(\pi_1, G)/G, \quad (12)$$

that is of equivalent classes of homomorphisms

$$\varphi : \pi_1(M) \rightarrow G, \quad (13)$$

up to homotopic conjugation.  $\pi_1(M)$  is made up of loops on the manifold  $M$  with two loops are identified if they can be smoothly deformed into each other; for example all contractible loops can

be identified. Using homotopy [22] there is a standard presentation of  $\pi_1$  in terms of  $2g$  generators which are not independent, since they satisfy relations on the Riemann surface. To give an index approach we use the Atiyah-Singer theorem allowing us naturally to make an extension, particularly in the case of four dimensions where the calculations through homotopy classes could be rather involve. We now concentrate on the dimension of  $\mathcal{M}_F(\Sigma_g, G)$ , which in this case it is known that is smooth except at singular points which arise at reducible connections [19]. To achieve this, the natural elliptic complex to use for our index calculation is

$$0 \xrightarrow{i} \Omega^0(\Sigma_g, \text{ad } P) \xrightarrow{d_A} \Omega^1(\Sigma_g, \text{ad } P) \xrightarrow{d_A} \Omega^2(\Sigma_g, \text{ad } P) \xrightarrow{d_A} 0, \quad (14)$$

being  $\text{ad } P$  the bundle where the gauge group  $G$  is defined, ensuring the correct behavior of flat connections under gauge transformations. This complex is the corresponding finite sequence of differential operators defined in (8). On a two dimensional Riemannian manifold, the Atiyah-Singer theorem reads

$$\chi(\Sigma_g) = \sum_{r=0}^2 (-1)^r h_r(\Sigma_g), \quad (15)$$

where  $\chi(\Sigma_g)$  is the *Euler characteristic* and  $h_r(\Sigma_g)$  corresponds to the dimension of the Hodge groups  $H^r(\Sigma_g \otimes \text{ad } P)$ [19]. It turns out that  $h_0 = 0$ , since  $h_0$  comes from a cohomology group of dimension zero, more precisely it is the dimension of the space of sections of  $\text{ad } P$  which are covariantly constant[21]; the case of  $h_2$  is more subtle, and it is possible to pick the bundle  $\Sigma_g \otimes \text{ad } P$  carefully ensuring that there are no reducible connections [23]; using this assumption  $h_2 = 0$  turning this surface into a *bona fide* manifold. Since  $\chi(\Sigma_g) = 2 - 2g$  is the *Euler characteristic* of a Riemann surface of genus  $g$ , the dimension of the moduli space for  $g > 0$  and  $G$  simple is given by

$$\dim \mathcal{M}_F(\Sigma_g, G) = b_1(\Sigma_g) = (2g - 2) \dim G, \quad (16)$$

where it has been used differential operators instead homotopy groups. When the manifold is the two sphere,  $g = 0$ , and  $\mathcal{M}_F(\Sigma_g, G)$  is one point, this means that up to gauge equivalence the only flat connection is the trivial connection. For the torus the situation changes somewhat, as in the homotopy approach the generators commute; this property is reflected on the characteristic classes [21] resulting  $\dim \mathcal{M}_F(\Sigma_g, G) = 2 \text{rank } G$ . In the case of  $U(1)$  as everything must commute then we have  $\dim \mathcal{M}_F(\Sigma_g, G) = 2g$ . Eventhough this approach could not seem natural, it is possible to avoid some difficulties to build loops on higher dimensional spaces with complex topologies. We will use this approach in the next section for the case of four dimensional *BF* theories.

## V. BF moduli spaces in four dimensions

Since one of the equations of motion of BF theories is the vanishing of the curvature (4), it is possible to use some properties of instanton and anti-instanton complexes as the condition  $F_A = 0$  fullfills the usual self-dual and anti-self-dual condition simultaneously, *i.e.* the connections that

generate zero curvature are at the same time self-dual and anti-self-dual connections. If we think the BF equations as a non-linear generalization of Hodge theory such as in the case of Yang-Mills equations, they are not elliptic as they stand. The reason for this is that they possess a symmetry group; from a physical point of view, this symmetry is the invariance under gauge transformations; hence to obtain an elliptic problem we have to choose a gauge. As we shall see in the next section, the gauge transformations corresponding to a BF theory are given by diffeomorphisms, in order to fix the gauge it is possible to make use of the action (5) which is not diffeomorphism invariant and then take the limit when  $g$  goes to zero.

Let  $A$  a flat connection; when gauge equivalence is taken properly into account, the space of such  $A$  forms a finite dimensional space  $\mathcal{M}$  which we call the flat connection moduli space. This space should be viewed as a finite dimensional subspace of the infinite dimensional configuration space  $\mathcal{A}/\mathcal{G}$ , being  $\mathcal{A}$  the space of all connections and  $\mathcal{G}$  the group of gauge transformations. To obtain a *good* moduli space we have to cut down, like the instantons case, both the configuration space and the group defined on the bundle, and to restrict  $\mathcal{A}/\mathcal{G}$  to the subspace of irreducible connections[21]. The reason for this is that one needs to regularize the model to avoid problems with reducible connections. Then there are some gauge transformations that act trivially on the connections. This mean that  $\mathcal{A}/\mathcal{G}$  is not in general a manifold as the quotienting out by the gauge group. Generally the connections are irreducible, and there will be isolated reducible connections; therefore  $\mathcal{A}/\mathcal{G}$  is then at least an orbifold. Reducible connections are a source of great difficulty in making sense of topological field theories in general. The problem is that at reducible connections, path integrals related to partitions functions diverge. Hence our new configuration space is therefore the quotient

$$\mathcal{M} = \mathcal{A}^{irred}/\mathcal{G}. \quad (17)$$

Our next task is to find the dimension of  $\mathcal{M}$ . We employ a similar idea to that used on a Riemann surface in section IV. The main idea is to work infinitesimally, by which we mean to work with the tangent space to  $\mathcal{M}$ . The advantage of doing this is that the dimension of the tangent space can be calculated using the Atiyah-Singer theorem. Let  $A + ta$  be a one parameter family of smooth connections. Hence, by construction,  $a$  is tangent to this family, so that

$$a \in T_A\mathcal{A}. \quad (18)$$

We wish to obtain from  $a$ , the tangent space of the moduli space. Using  $[A]$  to denote the point of the moduli space to which  $A$  belongs, then we obtain an element of  $T_{[A]}\mathcal{M}$ . Firstly we must request this family to be flat, and secondly we must project out those  $a$ 's which correspond to gauge directions, *i.e.* those  $a$  which belong to the tangent space in the orbit-gauge directions  $T_{\mathcal{G}\cdot A}\mathcal{A}$ . To achieve our first condition from the equation of motion  $F = 0$  we see that

$$F(A + ta) = F(A) + td_Aa + t^2a \wedge a, \quad (19)$$

then, working infinitesimally  $a$  satisfies  $d_Aa = 0$ . To achieve our second goal we must identify those  $a$  which differ by an element of  $T_{\mathcal{G}\cdot A}\mathcal{A}$ . But, since  $T_{\mathcal{G}\cdot A}\mathcal{A} \simeq \text{Imag } d_A$ , it means taking those  $A$ 's

gauge equivalent; the two requirements are satisfied if

$$a \in \frac{\ker d_A}{\text{Imag } d_A}, \quad (20)$$

this has a cohomological interpretation which we now exploit using the index theorem. The Lie algebra valued 1-forms  $a$  are sections of the bundle  $\text{ad } P \otimes \Lambda^1 T^* M$ , where the first factor ensures that  $a$  has the correct behavior under gauge transformations, namely  $a \mapsto g^{-1} a g$ ; the second factor is simply because it is a 1-form. Let  $\Omega^i(M, \text{ad } P)$  the spaces of sections where  $\Omega^i(M, \text{ad } P) = \Gamma(M, \text{ad } P \otimes \Lambda^i T^* M)$ ; the natural elliptic complex to use for our index calculation is

$$0 \xrightarrow{i} \Omega^0(M, \text{ad } P) \xrightarrow{d_A} \Omega^1(M, \text{ad } P) \xrightarrow{d_A} \Omega^2(M, \text{ad } P) \xrightarrow{\pi_+ \pi_-} 0. \quad (21)$$

Where  $\pi_+$ ,  $\pi_-$  are the operators which project a two-form onto its self-dual part and onto its anti-self-dual part respectively. This sequence of differential operators is defined as in (8) in order to calculate the moduli space using the Atiyah-Singer theorem. It is a complex in the sense that  $d_A \circ d_A = 0$ . The cohomology data for this complex are

$$\begin{aligned} H^0(E) &= \ker d_A^{(0)}, & \dim H^0(E) &= h_0, \\ H^1(E) &= \frac{\ker d_A}{\text{Imag } d_A^{(0)}}, & \dim H^1(E) &= h_1, \\ H^2(E) &= \frac{\ker \pi_+ \pi_-}{\text{Imag } d_A}, & \dim H^2(E) &= h_2, \end{aligned} \quad (22)$$

where  $d_A^{(0)}$  denotes the exterior covariant derivative acting on  $\text{ad } P \otimes \Lambda^0 T^* M$ . Only one of these dimensions corresponds to our moduli space calculation, this being  $h_1$ , in other words, we wish to compute

$$h_1 = \dim T_{[A]} \mathcal{M} = \dim \mathcal{M}. \quad (23)$$

However, the index of the complex is the alternating sum

$$h_0 - h_1 + h_2. \quad (24)$$

Nevertheless, it turns out that  $h_0 = 0$  since  $h_0$  comes from a cohomology group of dimension zero, i.e. the dimension of space of sections which are covariantly constant. Similarly  $h_2 = 0$ , which requires the use of a vanishing theorem. This is done by a Bochner-Weitzenbech technique [20] used in the case of self-dual and anti-self-dual connections. This means that because of a two-form can be expressed as a combination of its self-dual and its anti-self-dual part respectively, both terms are mapped to zero assuming that each term corresponds to the instanton and the anti-instanton term in the Yang-Mills complex, in fact flat connections satisfy both conditions; as we shall see this assumption will restrict our manifold. The associated Laplacian

$$\Delta_A = d_A (d_A)^\dagger + (\pi_+ \pi_-) (\pi_+ \pi_-)^\dagger, \quad (25)$$



is positive definite and hence has no kernel; the second term on the right hand side is zero due to flatness condition on the curvature; computing the remaining term in local coordinates shows that

$$\Delta_A = \frac{1}{2}d_A(d_A)^\dagger + \frac{R}{6} - W_- - W_+, \quad (26)$$

where  $R$  is the scalar curvature of  $M$  and  $W_+, W_-$  the self-dual part and the anti-self-dual part of its Weyl tensor. Positivity will result if we assume  $W_+$  and  $W_-$  are zero; that is,  $M$  is known as a conformally flat manifold. The case when  $h_2 \neq 0$  can be obtained considering corrections to the associated Laplacian. Since  $h_0 = 0$  y  $h_2 = 0$  we have

$$\text{Index} = -h_1 = -\dim \mathcal{M}. \quad (27)$$

After complexification we can use our index formula (9)

$$\text{Index} = (-1)^{n(\frac{n}{2})} \int_M \frac{\text{ch}(\sum_p (-1)^p [E^p])}{e(M)} \cdot \text{td}(T(M)_C)[M], \quad (28)$$

in the present case  $n = 4$ , and the  $E^p$  are given by

$$E^0 = \text{ad}_C P \otimes \Lambda^0 T^* M_C, \quad E^1 = \text{ad}_C P \otimes \Lambda^1 T^* M_C, \quad E^2 = \text{ad}_C P \otimes \Lambda^2 T^* M_C,$$

with  $\text{ad}_C P$  the complexification of the adjoint bundle  $\text{ad} P$ . Now using the decomposition theorem of fiber bundles [24, 25] and the multiplicative property of Todd classes, we have

$$T(M)_C = L_1 \oplus \bar{L}_1 \oplus L_2 \oplus \bar{L}_2, \quad (29)$$

then

$$\text{td}(T(M)_C) = \left(\frac{x_1}{1 - \exp[-x_1]}\right) \left(\frac{-x_1}{1 - \exp[x_1]}\right) \left(\frac{x_2}{1 - \exp[-x_2]}\right) \left(\frac{-x_2}{1 - \exp[x_2]}\right), \quad (30)$$

where  $x_1$  and  $x_2$  are two forms proportional to independent eigenvalues of the curvature 2-form. In the same way using the properties of the Euler class we have  $e(T(M)) = x_1 x_2$ . To deal with the rest of the formula we need to know  $\text{ch}(E^0 - E^1 + E^2)$ , then using the properties of the Chern character we have that  $\text{ch}(E^0 - E^1 + E^2) = \text{ch}(\text{ad}_C P) \text{ch}(\Lambda^0 T^*(M)_C - \Lambda^1 T^*(M)_C + \Lambda^2 T^*(M)_C)$ . From the splitting principle [25] we obtain that

$$\begin{aligned} \text{ch}(\Lambda^0 T^*(M)_C) &= 1, \\ \text{ch}(\Lambda^1 T^*(M)_C) &= e^{x_1} + e^{-x_1} + e^{x_2} + e^{-x_2}, \\ \text{ch}(\Lambda^2 T^*(M)_C) &= 2 + e^{x_1+x_2} + e^{x_1-x_2} + e^{-x_1+x_2} + e^{-x_1-x_2}. \end{aligned} \quad (31)$$

Finally replacing on the index formula (9) we have

$$\begin{aligned} \text{Index}(E) &= \int_M [1 - (e^{x_1} + e^{-x_1} + e^{x_2} + e^{-x_2}) + (2 + e^{x_1+x_2} + e^{x_1-x_2} + e^{-x_1+x_2} + e^{-x_1-x_2})] \\ &\quad \frac{1}{x_1 x_2} \cdot \left(\frac{x_1}{1 - \exp[-x_1]}\right) \left(\frac{-x_1}{1 - \exp[x_1]}\right) \left(\frac{x_2}{1 - \exp[-x_2]}\right) \left(\frac{-x_2}{1 - \exp[x_2]}\right) \text{ch}(\text{ad}_C P). \end{aligned}$$

Since the manifold is four-dimensional, no terms higher than 4-form will appear in the characteristic classes, then developing the polynomials we have

$$\text{Index}(E) = \int_M \text{ch}(\text{ad}_C P) \left( \frac{3}{x_1 x_2} + \frac{x_1}{x_2} + \frac{x_2}{x_1} + x_1 x_2 \right) \left[ 1 - \frac{1}{12} (x_1^2 + x_2^2) \right], \quad (32)$$

In four dimensions the Chern character takes the form

$$\text{ch}(\text{ad}_C P) = \text{rank}(\text{ad}_C P) + c_1(\text{ad}_C P) + \frac{1}{2} (c_1^2(\text{ad}_C P) - 2c_2(\text{ad}_C P)), \quad (33)$$

however,  $\text{ad}_C P$  is the complexification of a real bundle, then it is self-conjugated and has only even dimensional Chern classes; also it is clear that  $\text{rank}(\text{ad}_C P) = \dim G$ . Finally we can employ the properties of the Pontrjagin classes to write  $p_1(\text{ad}_C P) = -2c_2(\text{ad}_C P)$ . This gives the result that

$$\text{ch}(\text{ad}_C P) = \dim G + \frac{1}{2} p_1(\text{ad}_C P). \quad (34)$$

Replacing (34) into (32) we have

$$\text{Index}(E) = \int_M \left( \dim G + \frac{1}{2} p_1(\text{ad}_C P) \right) \left( \frac{3}{x_1 x_2} + \frac{x_1}{x_2} + \frac{x_2}{x_1} + x_1 x_2 \right) \left[ 1 - \frac{1}{12} (x_1^2 + x_2^2) \right]; \quad (35)$$

however, the singular terms  $\frac{3}{x_1 x_2} + \frac{x_1}{x_2} + \frac{x_2}{x_1}$  require evidently a *regularization* in order to get regular polynomials. This is achieved considering that

$$\frac{3}{x_1 x_2} + \frac{x_1}{x_2} + \frac{x_2}{x_1} = \lim_{\xi \rightarrow 0} \left[ \frac{3}{(x_1 + \xi)(x_2 + \xi)} + \frac{x_1 + \xi}{x_2 + \xi} + \frac{x_2 + \xi}{x_1 + \xi} \right];$$

and making the expansion about the zero of the right-hand-side expression keeping only polynomials of order four we have an expression depending on the Euler class  $x_1 x_2 = e(M)$ , and the Pontrjagin class  $x_1^2 + x_2^2 = p_1(M)$ ,

$$\frac{3}{x_1 x_2} + \frac{x_1}{x_2} + \frac{x_2}{x_1} = \lim_{\xi \rightarrow 0} \frac{1}{\xi^4} [(3 + \xi^2)(x_1^2 + x_2^2) + (3 - 2\xi^2)x_1 x_2 + \xi^2(3 + 2\xi^2)];$$

*regularization* requires then the integration of the singular terms through  $\lim_{\xi \rightarrow 0} \int \lambda \xi^4$  (singular terms), where  $\lambda$  is a global factor to be determined:

$$\text{Index}(E) = \int_M \dim G [(3\lambda + 1)x_1 x_2 + 3\lambda(x_1^2 + x_2^2)],$$

where we have considered that  $p_1(\text{ad}_C P)$  is proportional to a 4-form. Let for example  $E$  be a  $SU(2)$ -bundle, then  $E$  carries the fundamental two dimensional representation of  $SU(2)$  and, if we make the tensor product of  $E$  with itself, there is a natural decomposition of this tensor product bundle into three-dimensional and one-dimensional representations. However, because  $E \otimes E$  is quadratic in  $E$ , it is clear that the elements of the fundamental representation are both mapped onto the same element in the tensor product, thus the bundle  $\text{ad}_C P$  is the three dimensional part of the tensor product  $E \otimes E$  [25]. Decomposing  $E \otimes E$  into the sum of a symmetric and an anti-symmetric part

$$E \otimes E = S^2 E \oplus \Lambda^2 E, \quad (36)$$

then applying the properties of the Chern character to (36) gives

$$\text{ch}(E) \text{ch}(E) = \text{ch}(\text{ad}_C P) + \text{ch}(\Lambda^2 E), \quad (37)$$

if we expand both sides using the properties of Chern character we get

$$(2 + c_1(E) + \frac{1}{2}(c_1^2(E)))^2 = 3 + \frac{1}{2}p_1(\text{ad}_C P) + 1 + c_1(E). \quad (38)$$

But on  $M$  we need only keep polynomials of dimension four, so that

$$\begin{aligned} 4 - 4c_2(E) &= 4 + \frac{1}{2}p_1(\text{ad}_C P) \\ \Rightarrow p_1(\text{ad}_C P) &= -8c_2(E), \end{aligned} \quad (39)$$

which corresponds essentially a 4-form. As we expected our Index will be written in terms of topological invariants that describe global properties of the  $SU(N)$ -bundle and of the background manifold  $M$ . To see this, according to the Gauss-Bonnet theorem [24] and to the Hirzebruch signature theorem [26] the *virtual* dimension of the moduli space for a BF theory on a  $SU(N)$ -bundle is given finally by

$$h_1 = -\text{Index}(E) = -\dim G[(3\lambda + 1)\chi + 9\lambda|\tau|]; \quad (40)$$

where  $SU(N)$  is the structure group of the bundle that in the fundamental representation has  $\dim G = N^2 - 1$ . This index represents on the one hand the dimension of flat connections, and on the other hand the dimension of the intersection of the spaces of connections that generate 4-instantons and 4-anti-instantons simultaneously.

Now we need to fix  $\lambda$  in order to obtain values of  $h_1$  physical and geometrically admissible as moduli space dimension. One may try with different values of  $\lambda$ , but the algebraic structure of the above expression and the fact that in general  $\chi \geq |\tau|$  (see the tables below), lead to an expression essentially of the form  $\alpha(\lambda)(m|\tau| - \chi)$ , with  $m$  rational, and  $\alpha$  a constant depending on  $\lambda$ ; a direct comparison leads to  $3\lambda + 1 = \alpha$ , and  $-9\lambda = \alpha m$ , which allows to obtain an expression in terms of  $m$ :

$$h_1 = \dim G \frac{3}{3+m} (m|\tau| - \chi); \quad (41)$$

reducing the problem of fixing  $\lambda$  to choose an appropriate rational number  $m$ . If  $m < 0$ , positivity of  $h_1$  will require  $m < -3$ ; however, in the case of  $S^4$  as base manifold this condition will lead to a moduli space dimension of flat connections bigger than the corresponding to instanton or anti-instantons, which is inadmissible since flat connections can be viewed as the intersection of the space of those field configurations; hence we can consider as first restriction  $m > -3$ . More specifically if we consider that the dimension of the moduli space for  $SU(2)$ -instantons on  $S^4$  with instantonic number  $k = 0$  is 5, then the restriction is  $\lambda > -11/18$ , and considering that  $\lambda = \frac{m}{9+3m}$  we obtain consistently the restriction  $m > -3$ , at least for the case  $S^4$ . However, this restriction on  $m$  leads to  $h_1 < 0$ , and the moduli space will be considered empty. Using the equation (16) we can observe the case of a two-dimensional sphere  $S^2$ , and that it is true also for a four-dimensional sphere  $S^4$ , both have as dimensional moduli spaces just one point, this means that up to gauge transformations the only flat connection is the trivial connection, property observed in homotopy theory and in other calculus with different complexes [27, 28, 29].

For most of base manifolds, the cases with integers  $m = -2, -1, 0, 1$  are ruled out due to yield non-positive  $h_1$  and the corresponding moduli space is considered empty; additionally  $m = 2$  is also ruled out due to yields positive but non-integer  $h_1$ ; but this last value may make sense only for  $SU(4)$  since  $\dim[SU(4)] = 15$ , which is divisible by 5. However, there will exist fractional values of  $m$  leading to integer  $h_1$  for arbitrary gauge symmetry group as we shall see below in the figures.

Continuing with integer values of  $m$  that yield admissible values for  $h_1$ , let us see now the case  $m = 3$ , and hence  $h_1 = \frac{1}{2} \dim G(3|\tau| - \chi)$ ; in the following table we display explicitly the values of  $h_1$  for different background four-manifolds. The symbol  $\emptyset$  denotes a negative virtual dimension, and will be understood as a empty moduli space; it is different of course from a zero dimensional moduli space.

Table 1: Characteristic numbers with $m=3$			
	$\chi$	$\tau$	$h_1 = \frac{1}{2} \dim G(3 \tau  - \chi)$
$S^4$	2	0	$\emptyset$
$CP_2$	3	1	0
$S^2 \times \Sigma_g$	$4(1-g)$	0	$\emptyset$ for $g = 0$ ; $2 \dim G(g - 1)$ for $g \geq 1$ ;
$K3$	24	-16	$12 \dim G$
$K3_{Z_2}$	12	-8	$6 \dim G$
$K3_{Z_2 \otimes Z_2}$	6	-4	$3 \dim G$
$E(n)$	$12n$	$-8n$	$6n \dim G$
$S_d$	$d(6 - d + d^2)$	$\frac{1}{3}(4 - d^2)d$	0 for $d = 1$ ; $\emptyset$ for $d = 2$ ; $\dim G d(2d - 5)$ for $d > 2$

$S^2 \times \Sigma_g$  represent product manifolds of  $S^2$  with Riemann surfaces of genus  $g$ . Note that for  $g = 0$ , we have  $S^2 \times S^2$ , and  $h_1$  is negative and we shall consider it empty; for  $g \geq 1$ ,  $h_1$  is a non-negative number.  $S_d$  represent hypersurfaces of degree  $d$  in  $CP(3)$  associated with the homogeneous polynomials  $\sum_{i=1}^4 z_i^d = 0$ ; for example,  $S_4$  represents the  $K3$  surface.  $E(n)$  represent the so called elliptic surfaces, which can be viewed also as elliptic fibrations where the fibers correspond to elliptic curves; these simply connected four-dimensional manifolds are labeled by a non-negative integer  $n$ . For example  $E(2)$  reduces in particular to  $K3$ .

The cases with  $m = 4, 5$  are ruled out by similarity with the case  $m = 2$ . The cases  $m = 6, 15$  are meaningful with  $h_1 = \frac{1}{3} \dim G(6|\tau| - \chi)$ , and  $h_1 = \frac{1}{6} \dim G(15|\tau| - \chi)$  respectively, and in tables 2 and 3 we display the corresponding characteristic numbers. In the row corresponding to  $S_d$ , a hat  $\hat{\phantom{x}}$  means that the number must be omitted from the sequence.

Table 2: Characteristic numbers with $m=6$ and $h_1 = \frac{1}{3} \dim G(6 \tau  - \chi)$		
	$6 \tau  - \chi$	$h_1$
$S^4$	-2	$\emptyset$
$CP_2$	3	$\dim G$
$S^2 \times \Sigma_g$	$4(g-1)$	$\frac{4}{3}(g-1) \dim G$ for $SU(N)$ and $g = 3l+1, l = 0, 1, 2, 3, \dots$ ; $4(g-1)$ for $SU(2)$ , and $g = 1, 2, 3, 4, \dots$ ; $20(g-1)$ for $SU(4)$ , and $g = 1, 2, 3, 4, \dots$ ; $\emptyset$ for $g = 0$ ;
$K3$	$3(24)$	$24 \dim G$
$K3_{Z_2}$	$3(12)$	$12 \dim G$
$K3_{Z_2 \otimes Z_2}$	12	$4 \dim G$
$E(n)$	$3(12n)$	$12n \dim G$
$S_d$	$3$ for $d = 1$ ; $-4$ for $d = 2$ ; $d[d^2 + 4(d-3)]$ for $d > 2$ ;	$\dim G$ ; $\emptyset$ ; $\frac{1}{3} \dim G d[d^2 + 4(d-3)]$ for $SU(N)$ and $d = 3, \widehat{4}, 5, 6, \widehat{7}, 8, 9, \widehat{10}, 11, \dots$ ; $d[d^2 + 4(d-3)]$ for $SU(2)$ ;

Table 3: Characteristic numbers with $m=15$ and $h_1 = \frac{1}{6} \dim G(15 \tau  - \chi)$		
	$15 \tau  - \chi$	$h_1$
$S^4$	-2	$\emptyset$
$CP_2$	12	$2 \dim G$
$S^2 \times \Sigma_g$	$4(g-1)$	$\frac{2}{3} \dim G(g-1)$ for $SU(N)$ and $g = 3l+1, l = 0, 1, 2, 3, \dots$ ; $2(g-1)$ for $SU(2)$ , and $g = 1, 2, 3, \dots$ ; $\emptyset$ for $g = 0$ ;
$K3$	$9(24)$	$36 \dim G$
$K3_{Z_2}$	$9(12)$	$18 \dim G$
$K3_{Z_2 \otimes Z_2}$	$6(9)$	$9 \dim G$
$E(n)$	$9(12)n$	$18n \dim G$
$S_d$	$12$ for $d = 1$ ; $-4$ for $d = 2$ ; $2d[2d(d+1) - 13]$ for $d > 2$ ;	$2 \dim G$ ; $\emptyset$ ; $d[2d(d+1) - 13]$ for $SU(2)$ ; $\frac{1}{3} \dim G d[2d(d+1) - 13]$ for $SU(N)$ and $d = 3, 4, \widehat{5}, 6, 7, \widehat{8}, 9, 10, \widehat{11}, \dots$ ;

Besides the four-dimensional sphere, there exist two cases with empty moduli space,  $S^2 \times \Sigma_0 = S^2 \times S^2$  and  $S_2$ ; however this fact is not fortuitous, since these last four-dimensional manifolds are

diffeomorphic to each other  $S^2 \times S^2 \simeq S_2$ ; consistently it is well known that there not exist instantons (nor anti-instantons) on  $S^2 \times S^2$ , which is compatible with our interpretation of flat connections lying in the intersection, in this case, of empty spaces.

Although in the tables the values of  $h_1$  are displayed for fixed values of  $m$ , it is necessary to have a general outlook of the behavior of  $h_1$  without restrictions on  $m$  for different base manifolds; hence the tables correspond only to points in the figure 1 with both  $h_1$  and  $m$  positive integers; these tables can be considered as a display zoom of points on the different curves. Such points are in the region restricted for admissible values of  $h_1$  leading to real moduli space dimensions. However, from this global view it is impossible to look the asymptotic behavior of  $h_1$  as  $m \rightarrow -3$ , where there exist admissible integer values as moduli space dimension. A display zoom of this asymptotic region is given in the figure 2 for  $CP_2$ , showing the generic behavior of the dimension for all base manifolds considered in figure 1; in all cases there will be an infinite (but countable) number of values admissible as real dimensions. Additionally it is possible to define invariants of differentiable structures counting the number of points in zero dimensional moduli spaces like in the case of monopole moduli spaces [30]; in the present case all curves have a cross point with the axis  $h_1 = 0$ , condition satisfied by the rational  $m = \chi / |\tau|$ , except of course the case of manifolds with  $\tau = 0$ , such as  $S^2 \times \Sigma_g$ .

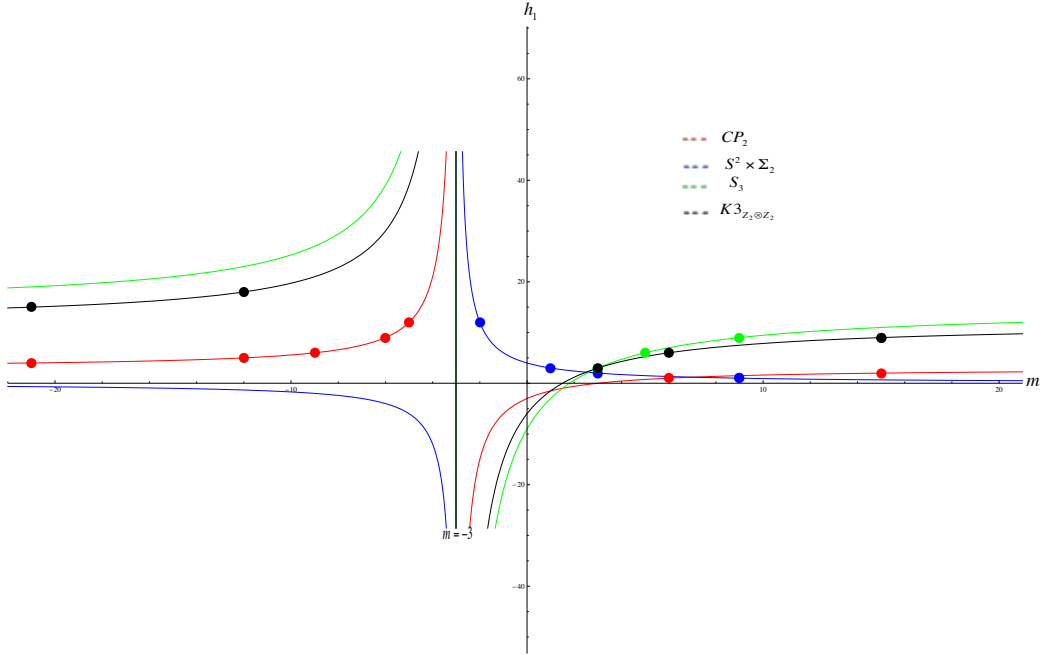


Figure 1: The *virtual* dimension  $h_1$  is represented for different base manifolds using  $[dimG]$  as unit. The continuous lines represent  $h_1$  for  $m \in \mathbb{R}$ , and the points represent non-negative integers for  $m$  rational. The vertical asymptote corresponds to  $\lim_{(m \rightarrow -3)} h_1 = \pm\infty$ ; there exists a horizontal asymptote for each manifold, and corresponds to  $\lim_{(m \rightarrow \pm\infty)} h_1 = 3 \mid \tau \mid$ , separating two disconnected parts of the curves. On this figure the first criterion for obtaining admissible values is  $h_1 \geq 0$ ; although for the virtual dimension  $m \in \mathbb{R}$ , the integer values of  $h_1$  are always contained in a finite range  $m \in \{a, b\}$ , with  $a$  and  $b$  integers, with an infinite and countable number of values of  $h_1$  (see figure 2). For example, the range for  $CP_2$  is  $\{-21, 15\}$ . The curves for  $S^2 \times \Sigma_2$ ,  $K3_{Z_2 \otimes Z_2}$ , and  $S_2$  represent the global behavior of the families  $S^2 \times \Sigma_g$ ,  $K3$ , and  $S_d$  respectively.

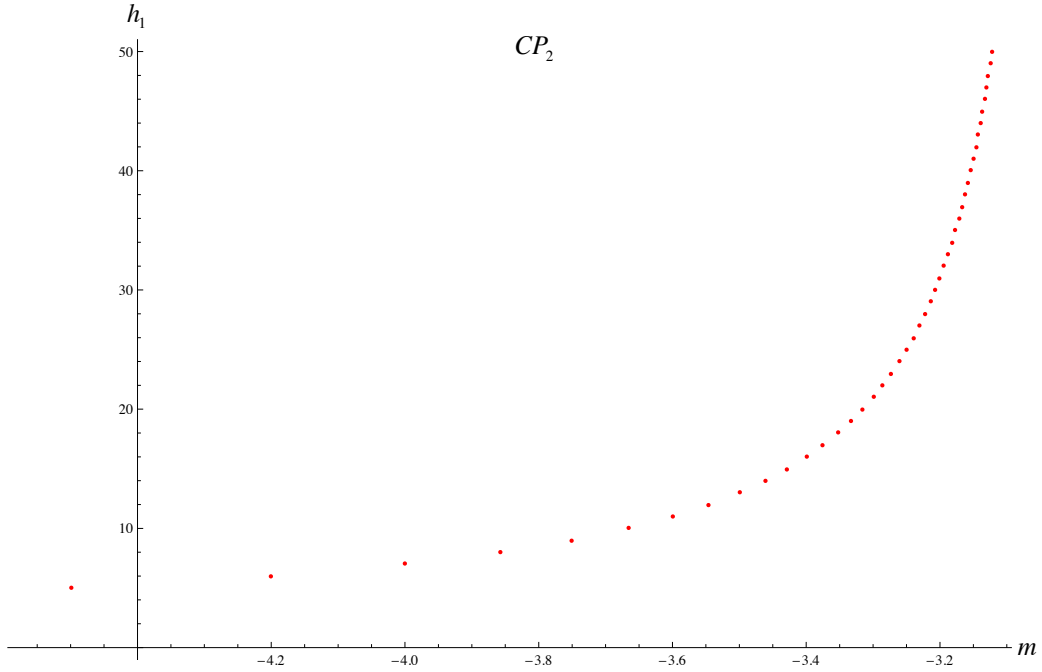


Figure 2: The limiting behavior of  $h_1$  as  $m \rightarrow -3$ ;  $m = -3$  is an accumulation point for non-negative integers of  $h_1$  which are represented by the points in the figure; therefore, the physically admissible values correspond to a set infinite and countable. This figure has been obtained by mean of the below *fifty points*-sequence of the form  $\{m, h_1\}$  with  $m$  rational and  $h_1$  integer, generated from the inverse function  $m = m(h_1)$  of the Eq. (41) for  $CP_2$ , and evaluated for the integers  $h_1 = 1, 2, \dots, 50$ , in  $[\frac{1}{3}dimG]$  units; one can obtain other *n-points* sequences for arbitrary  $n$ .



$$\begin{aligned}
& \{\infty, 1\} \quad , \quad \{-9, 2\}, \quad \{-6, 3\}, \quad \{-5, 4\}, \quad \{-\frac{9}{2}, 5\}, \quad \{-\frac{21}{5}, 6\}, \quad \{-4, 7\}, \quad \{-\frac{27}{7}, 8\}, \quad \{-\frac{15}{4}, 9\}, \\
& \{-\frac{11}{3}, 10\} \quad , \quad \{-\frac{18}{5}, 11\}, \quad \{-\frac{39}{11}, 12\}, \quad \{-\frac{7}{2}, 13\}, \quad \{-\frac{45}{3}, 14\}, \quad \{-\frac{24}{7}, 15\}, \quad \{-\frac{17}{5}, 16\}, \\
& \{-\frac{27}{8}, 17\} \quad , \quad \{-\frac{57}{17}, 18\}, \quad \{-\frac{10}{3}, 19\}, \quad \{-\frac{63}{19}, 20\}, \quad \{-\frac{33}{10}, 21\}, \quad \{-\frac{23}{7}, 22\}, \quad \{-\frac{36}{11}, 23\}, \\
& \{-\frac{75}{23}, 24\} \quad , \quad \{-\frac{13}{4}, 25\}, \quad \{-\frac{81}{25}, 26\}, \quad \{-\frac{42}{13}, 27\}, \quad \{-\frac{29}{9}, 28\}, \quad \{-\frac{45}{14}, 29\}, \quad \{-\frac{93}{29}, 30\}, \\
& \{-\frac{16}{5}, 31\} \quad , \quad \{-\frac{99}{31}, 32\}, \quad \{-\frac{51}{16}, 33\}, \quad \{-\frac{35}{11}, 34\}, \quad \{-\frac{54}{17}, 35\}, \quad \{-\frac{111}{35}, 36\}, \quad \{-\frac{19}{6}, 37\}, \\
& \{-\frac{117}{37}, 38\} \quad , \quad \{-\frac{60}{19}, 39\}, \quad \{-\frac{41}{13}, 40\}, \quad \{-\frac{63}{20}, 41\}, \quad \{-\frac{129}{41}, 42\}, \quad \{-\frac{22}{27}, 43\}, \quad \{-\frac{135}{43}, 44\}, \\
& \{-\frac{69}{22}, 45\} \quad , \quad \{-\frac{47}{15}, 46\}, \quad \{-\frac{72}{23}, 47\}, \quad \{-\frac{147}{47}, 48\}, \quad \{-\frac{25}{8}, 49\}, \quad \{-\frac{153}{49}, 50\}.
\end{aligned} \tag{42}$$

As discussed previously, the expression for the *virtual* dimension makes sense for  $S^4$  only under the restriction  $m > -3$ ; if this restriction is imposed as universal criterion on the other base manifolds, then only the region defined by  $m > -3$  and  $h_1 \geq 0$  will contain the admissible values as real dimensions. Under these conditions the only case with a infinite and countable number of values will be  $S^2 \times \Sigma_g$ , being the other cases a finite number of isolated points in the permitted region (the branches of the curves to the left of the asymptote  $m = -3$  will be not admissible). However, it is not the unique criterion, since that the expression for the *virtual* dimension has been obtained under the only assumption of a 4-dimensional background compact manifold, and it is reasonable consider that will admit independent restrictions for obtaining real dimensions; hence under this new criterion the restriction valid for the four-dimensional sphere will not affect to the curves corresponding to other manifolds, being the restriction  $h_1 \geq 0$  the only universal criterion (and the branches of the curves eliminated under the first criterion will be now recovered). Anyway, the results of the tables 1,2, and 3 are valid under both criteria (they were constructed deliberately taken into the account these considerations).

It is mandatory to try find possible physical or geometrical explanations on the appearance of a rational  $m$  characterizing the moduli space dimensions for flat connections; a possibility is as follows, and it is connected with the question formulated in the introduction on a possible relationship between 4-dimensional YM instanton moduli space and the corresponding one to 4-dimensional BF field theory, interpreted here as 4-dimensional flat connections lying in the intersection of the instantons and anti-instantons spaces. The later are characterized as well known by the instantonic number  $k$  (infinite and countable), and defined in terms of the squared norm of the self-dual or anti-self-dual curvature respectively. Instead a rational  $m$  (infinite and countable) appears in the case at hand; therefore, the relationship mentioned may be through a possible correlation between the numbers  $k$ , and  $m$ : for different  $m$ 's, we shall have in general different intersections of the instantons and anti-instantons spaces, and such intersection spaces will be labeled with a  $m$  through the correlation  $m = m(k)$  (finally both  $k$  and  $m$  are infinite and countable). The seeking for this correlation (if any) will be the subject of forthcoming communications.

## VI. Dirac's canonical analysis

In this section, we shall develop Dirac's canonical analysis for a four-dimensional modified BF theory, reproducing on shell topological YM theory. We will see later that the theory studied in this section shares the same moduli space with both BF theory and the BF-YM theory in the limit when the gauge coupling goes to zero, and additionally preserves the same gauge symmetries of BF theory at Hamiltonian level. The Hamiltonian framework developed in this section, will allow us to understand the principal symmetries of the theory as well as its constraints, the extended Hamiltonian and the gauge transformations. The theory under study will depend on a connection valued in the Lie algebra of  $SU(N)$  [31, 32].

Our starting point is the following action

$$S[A, B] = \int_M Tr \left( iB \wedge F(A) + \frac{g^2}{4} B \wedge B \right), \quad (43)$$

where  $F_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I + f^{IJK} A_\mu^J A_\nu^K$  is the curvature of the connection 1-form  $A_\mu^I dx^\mu$ ; being  $f^{IJK}$  the structure constants of the Lie algebra  $SU(N)$  and  $B_{\alpha\beta}^I$  is a set of  $6(N^2 - 1)$   $SU(N)$  components of valued 2-forms. Here,  $\mu, \nu = 0, 1, \dots, 3$  are spacetime indices,  $x^\mu$  are the coordinates that label the points of the 4-dimensional manifold  $M$  and  $I, J, K = 0, 1, \dots, N^2 - 1$ , are the internal indices that can be raised and lowered by the Cartan-Killing metric given by the Lie algebra.

The action (43) yields the next equations of motion

$$F^I(A) = i\frac{g^2}{2} B^I, \quad DB^I = 0, \quad (44)$$

where  $F^I$  satisfies Bianchi's identities  $DF^I = 0$ . By substituting the equations of motion (44) into (43) we obtain the topological YM theory. To perform the Hamiltonian analysis, we shall consider that the manifold  $M$  has a topology  $\Sigma \times R$ , where  $\Sigma$  corresponds to a Cauchy surface and  $R$  represents an evolution parameter. In this manner, by making the 3 + 1 decomposition the action (43) takes the form

$$S[A, B] = \frac{1}{2} \int \int_\Sigma d^3x dt \epsilon^{0ijk} \{ i(\dot{A}_k^I - D_k A_0^I) B_{ij}^I + B_{0i}^I (iF_{jk}^I + \frac{g^2}{2} B_{jk}^I) \}, \quad (45)$$

where we are able to identify the corresponding Lagrangian density

$$\mathcal{L} = \frac{1}{2} \eta^{ijk} \{ i(\dot{A}_k^I - D_k A_0^I) B_{ij}^I + B_{0i}^I (iF_{jk}^I + \frac{g^2}{2} B_{jk}^I) \}, \quad (46)$$

here  $\epsilon^{0ijk} \equiv \eta^{ijk}$ ,  $i, j, k = 1, 2, 3$  and  $\eta^{123} = 1$ . To carry out the Hamiltonian analysis, we will consider as dynamical variables those with time derivatives occurring in the action; an alternative procedure can also be considered in [33], where a pure Dirac's analysis of other topological theories is performed.

Then, the canonically conjugate momenta  $\Pi^{iI}$  to the  $A_i^I$  are given

$$\Pi^{iI} \equiv \frac{\delta \mathcal{L}}{\delta \dot{A}_i^I} = \frac{i}{2} \eta^{ijk} B_{jk}^I. \quad (47)$$

In this manner, by using the definition of the momenta in the action (45) we obtain

$$S[A_i^I, \Pi^{iI}, B_{0i}^I, A_0^I] = \int_M d^4x \left( \Pi^{iI} \dot{A}_i^I - A_0^I D_k \Pi^{kI} - \frac{i}{2} \eta^{ijk} B_{0i}^I F_{jk}^I + i \frac{g^2}{2} \Pi^{iI} B_{0i}^I \right). \quad (48)$$

From the action (48) we can identify the non-vanishing fundamental Poisson brackets for the theory

$$\{A_i^I(x^0, \vec{x}), \Pi_j^I(x^0, \vec{y})\} = \delta_i^j \delta_j^I \delta^3(x - y), \quad (49)$$

and the corresponding Hamiltonian of this theory given by

$$H_c = \dot{A}_i^I \Pi^{iI} - \mathcal{L} = \int d^3x \left( -A_0^I D_k \Pi^{kI} - i B_{0i}^I \left( \frac{1}{2} \eta^{ijk} F_{jk}^I - \frac{g^2}{2} \Pi^{iI} \right) \right). \quad (50)$$

Calculating the variation of (48) with respect to  $A_i^I$ ,  $\Pi^{iI}$  the equations of motion read

$$\begin{aligned} \delta A_i^I &: \eta^{ijk} D_j B_{0k}^I = 0, \\ \delta \Pi^{iI} &: D_i A_0^I = \dot{A}_i^I - i \frac{g^2}{2} B_{0i}^I, \end{aligned} \quad (51)$$

and the variations respect to  $B_{0i}^I$ , and  $A_0^I$  yield the following  $4(N^2 - 1)$  primary constraints

$$\begin{aligned} \phi^I &: D_k \Pi^{kI} \approx 0, \\ \phi^{iI} &: \frac{g^2}{2} \Pi^{iI} - \frac{1}{2} \eta^{ijk} F_{jk}^I \approx 0. \end{aligned} \quad (52)$$

As we can observe, the Hamiltonian (50) is a linear combination of the constraints (52) and  $A_0^I$ ,  $B_{0i}^I$ , both correspond to Lagrange multipliers.

Now, we need to identify whether the theory presents secondary constraints. From the temporal evolution of the constraints (52), we can observe that consistency demands that there are no more constraints because

$$\begin{aligned} \dot{\phi}^I &= \{\phi^I(x), H_c\} = f^{IJK} [A_0^J \phi^K - \phi^{iJ} B_{0i}^K] \approx 0, \\ \dot{\phi}^{iI} &= \{\phi^{iI}(x), H_c\} = f^{IJK} A_0^J \phi^{iK} \approx 0. \end{aligned} \quad (53)$$

With all constraints at hand, we need to identify which ones correspond to first and second class. In order to do this, we need to calculate the Poisson brackets between all the constraints, which are given by

$$\begin{aligned} \{\phi^I(x), \phi^J(y)\} &= f^{IJK} \phi^K \delta^3(x - y), \\ \{\phi^I(x), \phi^{iJ}(y)\} &= f^{IJK} \phi^{iK} \delta^3(x - y), \\ \{\phi^{iI}(x), \phi^{jK}(y)\} &= 0, \end{aligned} \quad (54)$$

thus, we observe that the constraints are of first class. Nevertheless, we can see that the  $4(N^2 - 1)$  first class constraints given in (52) are not all independents. The reason is because of Bianchi's identity  $DF^I = 0$  implies

$$D_i \phi^{iI} = \phi^I. \quad (55)$$

Thus, from the  $3(N^2 - 1)$  first class constraints  $\phi^{iI}(x)$ , we identify that  $[3(N^2 - 1) - (N^2 - 1)] = 2(N^2 - 1)$  are independents. Therefore, we are able to calculate the physical degrees of freedom as

follows; we have  $6(N^2 - 1)$  canonical variables,  $3(N^2 - 1)$  independent first class constraints and there are not second class constraints. With this information, we conclude that the action (43) is devoid of physical degrees of freedom; this scheme can be generalized to other BF theories [33, 34]. As we can observe, the action defined in (5) and (43) share a kind of similarity, but (43) has the presence of the Hodge-duality operation, and this fact allows the theory has  $2(N^2 - 1)$  degrees of freedom. Nevertheless, the action given in (43) has not the duality operator and the theory is devoid of physical degrees of freedom.

The identification of the constraints allows us to construct the extended action which is given by

$$S_E[A_i^I, \Pi^{iI}, A_0^I, B_{0i}^I] = \int_M d^4x \left\{ \dot{A}_i^I \Pi^{iI} + A_0^I D_i \Pi^{iI} + \frac{1}{2} \eta^{ijk} B_{0i}^I F_{jk}^I + \Pi^{iI} B_{0i}^I \right\}. \quad (56)$$

From (56) we can identify the extended Hamiltonian

$$H_E = -A_0^I D_i \Pi^{iI} - \frac{i}{2} B_{0i}^I (\eta^{ijk} F_{ij}^I - g^2 \Pi^{iI}), \quad (57)$$

that is a linear combination of first class constraints as expected. As well know, the equations of motion obtained from the extended Hamiltonian in general are mathematically different with the Euler-Lagrange equations, but the difference is unphysical.

The equations of motion obtained from the extended action are

$$\begin{aligned} \delta A_0^I &: D_i \Pi^{iI} = 0, \\ \delta B_{0i}^I &: \frac{1}{2} \epsilon^{0ijk} F_{jk}^I - \frac{g^2}{2} \Pi^{iI} = 0, \\ \delta A_i^I &: \epsilon^{ijk} D_j B_{0k}^I = 0, \\ \delta \Pi^{iI} &: D_i A_0^I = \dot{A}_i^I - i \frac{g^2}{2} B_{0i}^I. \end{aligned} \quad (58)$$

Now we proceed computing the gauge transformations on the phase space. To this aim, we need to use the first class constraints to define the generator of gauge transformations as

$$G = \int_{\Sigma} \varepsilon^I \phi^I + \varepsilon_i^I \phi^{iI}, \quad (59)$$

thus, we find the following gauge transformations on the phase space

$$\begin{aligned} \delta_0 A_i^I &= -D_i \varepsilon^I + \varepsilon_i^I, \\ \delta_0 \Pi^{iI} &= f^{IJK} \varepsilon^J \Pi^{iK} + \epsilon^{0ijk} D_j \varepsilon_k^I, \\ \delta_0 A_0^I &= 0, \\ \delta_0 B_{0i}^I &= 0. \end{aligned} \quad (60)$$

On the other hand, we know that the  $BF$  theory is diffeomorphisms covariant, and apparently that symmetry is not present in (60). Nevertheless, by introducing in (60) the following gauge parameters

$$\begin{aligned} \varepsilon^I &= -\xi^\mu A_\mu^I, \\ \varepsilon_i^I &= \xi^\mu F_{\mu i}^I, \end{aligned} \quad (61)$$

then, we obtain

$$A_i^I \rightarrow A_i^I + \mathcal{L}_\xi A_i^I. \quad (62)$$

Therefore, diffeomorphisms correspond to an internal symmetry of the theory in the phase space. It is important to remark that this symmetry is devoid in (5). In fact, in (5) the gauge transformations on the phase space correspond to  $A \rightarrow A + D\epsilon$  being different to the diffeomorphism symmetry [34]. We can also see that diffeomorphism symmetry implies that the extended Hamiltonian (57) is linear combination of first class constraints unlike Yang-Mills theory where its Hamiltonian is not. In addition, all the information obtained along this section has been performed with the aim to know the local symmetries of the theory under study, furthermore will be useful in future works to study the moduli space of the action (43). It is important to observe, that the action (43) and BF theory share the same local symmetries, and if the coupling constant goes to zero, the action (43) and BF theory has the same moduli space as can be appreciated in (44). Nevertheless, if the coupling constant is not zero, (43) and BF theory has the same local symmetries but the moduli space will be different; this issue will be studied in future works as well.

## VII. Discussions

In this paper, the dimension of the moduli space for two and four-dimensional BF theories valued in different gauge scenarios have been determined using the Atiyah-Singer theorem . As an important fact, we have used the connections that generate simultaneously four-instantons and four-anti-instantons to characterize the connections of a BF theory. This local information allowed us to built the elliptic complex and then define its corresponding moduli space. In addition we applied the results to particular base manifolds and gauge bundles. On the other hand within Dirac's method we have developed the Hamiltonian analysis of a modified BF theory to obtain some significance results such as the extended action, the extended Hamiltonian, the local degrees of freedom and the gauge symmetries. As important results obtained using the Hamiltonian method, we found that the theory is diffeomorphisms covariant and with the use of the constraints we concluded that it has zero physical local degrees of freedom, showing in the appropriate limit only the global degrees of freedom determined as the dimension of the moduli space. In this sense, in the limit when the coupling constant goes to zero this modified version of BF theory shares unlike the usual BF-YM theory, the same moduli space and the same gauge transformations with the usual BF theory. Nevertheless, the moduli space obtained for usual BF theory and modified BF theory correspond to connections satisfying one of the equations of motion, more precisely flat connections; but these solutions are not invariant under all gauge transformations of the theory as we have observed within Dirac's method. In order to take into account this fact, we have to consider the second equation of motion 4 related to the field  $B$ , and build its corresponding elliptic complex; this problem will have to be faced within

the setting of *coupled* elliptic complexes[35]. In this manner the theory has been characterized both globally and locally providing all necessary elements to make progress in the quantization; these subjects will be reported in forthcoming works.

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