Hidden Hodge symmetries and Hodge correlators

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To Don Zagier for his 60-th birthday

1 Hidden Hodge symmetries

There is a well known parallel between Hodge and étale theories, still incomplete and rather mysterious:

l-adic Étale Theory	**Hodge Theory**
Category of l-adic Galois modules	Abelian category $\mathcal{MH}_{\mathbb{R}}$ of real mixed Hodge strucrures
Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$	Hodge Galois group $G_{\mathrm{Hod}} :=$ Galois group of the category $\mathcal{MH}_{\mathbb{R}}$
$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $H^*_{\mathrm{et}}(\overline{X}, \mathbb{Q}_l)$, where X is a variety over $\mathbb{Q}$	$H^*(X(\mathbb{C}), \mathbb{R})$ has a functorial real mixed Hodge structure
étale site	??
$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the étale site, and thus on categories of étale sheaves on X, e.g. on the category of l-adic perverse sheaves	?? ?? ??
$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant perverse sheaves	Saito's Hodge sheaves

The current absense of the "Hodge site" was emphasized by A.A. Beilinson [B].

The Hodge Galois group. A weight n *pure real Hodge structure* is a real vector space H together with a decreasing filtration $F^\bullet H_{\mathbb{C}}$ on its complexification satisfying

$$H_{\mathbb{C}} = \oplus_{p+q=n} F^p H_{\mathbb{C}} \cap \overline{F^q} H_{\mathbb{C}}.$$

A real Hodge structure is a direct sum of pure ones. The category $\mathcal{PH}_{\mathbb{R}}$ real Hodge structures is equivalent to the category of representations of the real algebraic group $\mathbb{C}^*_{\mathbb{C}/\mathbb{R}}$. The group of complex points of $\mathbb{C}^*_{\mathbb{C}/\mathbb{R}}$ is $\mathbb{C}^* \times \mathbb{C}^*$; the complex conjugation interchanges the factors.

A real *mixed Hodge structure* is given by a real vector space H equipped with the weight filtration $W_\bullet H$ and the Hodge filtration $F^\bullet H_{\mathbb{C}}$ of its complexification, such that the Hodge filtration induces on $\mathrm{gr}^W_n H$ a weight n real Hodge structure. The category

$\mathcal{MH}_{\mathbb{R}}$ of real mixed Hodge structures is an abelian rigid tensor category. There is a fiber functor to the category of real vector spaces

$$\omega_{\rm Hod}: \mathcal{MH}_{\mathbb{R}} \longrightarrow {\rm Vect}_{\mathbb{R}}, \qquad H \longrightarrow \oplus_n {\rm gr}^W_n H.$$

The Hodge Galois group is a real algebraic group given by automorphisms of the fiber functor:

$$G_{\rm H} := {\rm Aut}^{\otimes}\omega_{\rm Hod}.$$

The fiber functor provides a canonical equivalence of categories

$$\omega_{\rm Hod}: \mathcal{MH}_{\mathbb{R}} \stackrel{\sim}{\longrightarrow} G_{\rm Hod} - \text{modules}.$$

The Hodge Galois group is a semidirect product of the unipotent radical $U_{\rm Hod}$ and $\mathbb{C}^*_{\mathbb{C}/\mathbb{R}}$:

$$0 \longrightarrow U_{\rm Hod} \longrightarrow G_{\rm Hod} \longrightarrow {\mathbb{C}^*}_{\mathbb{C}/\mathbb{R}} \longrightarrow 0, \qquad {\mathbb{C}^*}_{\mathbb{C}/\mathbb{R}} \hookrightarrow G_{\rm Hod}. \tag{1}$$

The projection $G_{\rm Hod} \rightarrow {\mathbb{C}^*}_{\mathbb{C}/\mathbb{R}}$ is provided by the inclusion of the category of real Hodge structurs to the category of mixed real Hodge structures. The splitting $s: \mathbb{G}_m \rightarrow G_{\rm Hod}$ is provided by the functor $\omega_{\rm Hod}$.

The complexified Lie algebra of $U_{\rm Hod}$ has *canonical* generators $G_{p,q}$, $p, q \geq 1$, satisfying the only relation $\overline{G}_{p,q} = -G_{q,p}$, defined in [G1]. For the subcategory of Hodge-Tate structures they were defined in [L]. Unlike similar but different Deligne's generators [D], they behave nicely in families. So to define an action of the group $G_{\rm Hod}$ one needs to have an action of the subgroup $\mathbb{C}^*_{\mathbb{C}/\mathbb{R}}$ and, in addition to this, an action of a single operator

$$G := \sum_{p,q \geq 1} G_{p,q}.$$

The twistor Galois group. Denote by $\mathbb{C}^*$ the real algebraic group with the group of complex points $\mathbb{C}^*$. The extension induced from (1) by the diagonal embedding $\mathbb{C}^* \subset \mathbb{C}^*_{\mathbb{C}/\mathbb{R}}$ is the *twistor Galois group.* It is a semidirect product of the groups $U_{\rm Hod}$ and $\mathbb{C}^*$.

$$0 \longrightarrow U_{\rm Hod} \longrightarrow G_{\rm T} \stackrel{\longleftarrow}{\longrightarrow} \mathbb{C}^* \longrightarrow 0. \tag{2}$$

It is not difficult to prove

Lemma 1.1 *The category of representations of $G_{\rm T}$ is equivalent to the category of mixed twistor structures defined by Simpson [Si2].*

We suggest the following fills the ??-marks in the dictionary related the Hodge and Galois. Below X is a smooth projective complex algebraic variety.

Conjecture 1.2 *There exists a functorial homotopy action of the twistor Galois group $G_{\rm T}$ by A_∞-equivalences of an A_∞-enhancement of the derived category of perverse sheaves on X such that the category of equivariant objects is equivalent to Saito's category real mixed Hodge sheaves.*[1]

[1] We want to have a natural construction of the action first, and get Saito's category real mixed Hodge sheaves as a consequence, not the other way around.

Denote by $D^b_{\rm sm}(X)$ the category of smooth complexes of sheaves on X, i.e. complexes of sheaves on X whose cohomology are local systems.

Theorem 1.3 *There exists a functorial for pull-backs homotopy action of the twistor Galois group $G_{\rm T}$ by A_∞-equivalences of an A_∞-enhancement of the category $D^b_{\rm sm}(X)$.*

The action of the subgroup $\mathbb{C}^*$ is not algebraic. It arises from Simpson's action of $\mathbb{C}^*$ on semisimple local systems [Si1]. The action of the Lie algebra of the unipotent radical $U_{\rm Hod}$ is determined by a collection of numbers, which we call the *Hodge correlators for semisimple local systems*. Our construction uses the theory of harmonic bundles [Si1]. The Hodge correlators can be interpreted as correlators for a certain Feynman integral. This Feynman integral is probably responsible for the "Hodge site".

For the trivial local system the construction was carried out in [G2]. A more general construction for curves, involving the constant sheaves and delta-functions, was carried out in [G1].

In the case when X is the universal modular curve, the Hodge correlators contain the special values $L(f,n)$ of weight $k \geq 2$ modular forms for $GL_2(\mathbb{Q})$ outside of the critical strip – it turns out that the simplest Hodge correlators in this case coincide with the Rankin-Selberg integrals for the non-critical special values $L(f,k+n)$, $n \geq 0$ – the case $k=2, n=0$ is discussed in detail in [G1].

2 Hodge correlators for local systems

2.1 An action of G_T on the " minimal model"of $\mathcal{D}_{\rm sm}(X)$.

Tensor products of irreducible local systems are semisimple local systems. The *category of harmonic bundles* ${\rm Har}_X$ is the graded category whose objects are semi-simple local systems on X and their shifts, and morphisms are given by graded vector spaces

$$
{\rm Hom}^\bullet_{{\rm Har}_X}(V_1,V_2) := H^\bullet(X, V_1^\vee \otimes V_2). \qquad (3)
$$

Here is our main result.

Theorem 2.1 *There is a homotopy action of the twistor Galois group G_T by A_∞-equivalences of the graded category ${\rm Har}_X$, such that the action of the subgroup $\mathbb{C}^*$ is given by Simpson's action of $\mathbb{C}^*$ on semi-simple local systems.*

This immediately implies Theorem 1.3. Indeed, given a small A_∞-category $\mathcal{A}$, there is a functorial constraction of the triangulated envelope ${\rm Tr}(\mathcal{A})$ of $\mathcal{A}$, the smallest triangulated category containing $\mathcal{A}$. Since $\mathcal{D}^b_{\rm sm}(X)$ is generated as a triangulated category by semi-simple local systems, the category ${\rm Tr}({\rm Har}_X)$ is equivalent to $\mathcal{D}^b_{\rm sm}(X)$ as a triangulated category, and thus is an A_∞-enhancement of the latter. On the other hand, the action of the group G_T from Theorem 2.1 extends by functoriality to the action on ${\rm Tr}({\rm Har}_X)$.

Below we recall what are A_∞-equivalences of DG categories and then define the corresponding data in our case.

2.2 A_∞-equivalences of DG categories

The Hochshild cohomology of a small dg-category $\mathcal{A}$. Let $\mathcal{A}$ be a small dg category. Consider a bicomplex whose n-th column is

$$\prod_{[X_i]} \mathrm{Hom}\Big(\mathcal{A}(X_0, X_1)[1] \otimes \mathcal{A}(X_1, X_2)[1] \otimes \ldots \otimes \mathcal{A}(X_{n-1}, X_n)[1], \mathcal{A}(X_0, X_n)[1]\Big), \qquad (4)$$

where the product is over isomorphism classes $[X_i]$ of objects of the category $\mathcal{A}$. The vertical differential d_1 in the bicomplex is given by the differential on the tensor product of complexes. The horisontal one d_2 is the degree 1 map provided by the composition

$$\mathcal{A}(X_i, X_{i+1}) \otimes \mathcal{A}(X_{i+1}, X_{i+2}) \longrightarrow \mathcal{A}(X_i, X_{i+2}).$$

Let $\mathrm{HC}^*(\mathcal{A})$ be the total complex of this bicomplex. Its cohomology are the Hochshild cohomology $\mathrm{HH}^*(\mathcal{A})$ of $\mathcal{A}$. Let $\mathrm{Fun}_{A_\infty}(\mathcal{A}, \mathcal{A})$ be the space of A_∞-functors from $\mathcal{A}$ to itself. Lemma 2.2 can serve as a definition of A_∞-functors considered modulo homotopy equivalence.

Lemma 2.2 *One has*

$$H^0\mathrm{Fun}_{A_\infty}(\mathcal{A}, \mathcal{A}) = \mathrm{HH}^0(\mathcal{A}). \qquad (5)$$

Indeed, a cocycle in $\mathrm{HC}^0(\mathcal{A})$ is the same thing as an A_∞-functor. Coboundaries corresponds to the homotopic to zero functors.

The cyclic homology of a small rigid dg-category $\mathcal{A}$. Let $(\alpha_0 \otimes ... \otimes \alpha_m)_{\mathcal{C}}$ be the projection of $\alpha_0 \otimes ... \otimes \alpha_m$ to the coinvariants of the cyclic shift. So, if $\overline{\alpha} := \mathrm{deg}\alpha$,

$$(\alpha_0 \otimes ... \otimes \alpha_m)_{\mathcal{C}} = (-1)^{\overline{\alpha}_m(\overline{\alpha}_0+...+\overline{\alpha}_{m-1})}(\alpha_1 \otimes ... \otimes \alpha_m \otimes \alpha_0)_{\mathcal{C}}.$$

We assign to $\mathcal{A}$ a bicomplex whose n-th column is

$$\prod_{[X_i]} \Big(\mathcal{A}(X_0, X_1)[1] \otimes \ldots \otimes \mathcal{A}(X_{n-1}, X_n)[1] \otimes \mathcal{A}(X_n, X_0)[1]\Big)_{\mathcal{C}}.$$

The differentials are induced by the differentials and the composition maps on Hom's. The cyclic homology complex $\mathrm{CC}_*(\mathcal{A})$ of $\mathcal{A}$ is the total complex of this bicomplex. Its homology are the cyclic homology of $\mathcal{A}$.

Assume that there are functorial pairings

$$\mathcal{A}(X, Y)[1] \otimes \mathcal{A}(Y, X)[1] \longrightarrow \mathcal{H}^*.$$

Then there is a morphism of complexes

$$\mathrm{HC}^*(\mathcal{A})^* \longrightarrow \mathrm{CC}_*(\mathcal{A}) \otimes \mathcal{H}. \qquad (6)$$

For the category of harmonic bundles Har_X there is such a pairing with

$$\mathcal{H} := H_{2n}(X)[-2].$$

It provides a map

$$\varphi : \mathrm{Hom}\Big(H_0(\mathrm{CC}_*(\mathrm{Har}_X) \otimes \mathcal{H}, \mathbb{C}\Big) \longrightarrow \mathrm{HH}^0(\mathrm{Har}_X) \stackrel{(5)}{=} H^0\mathrm{Fun}_{A_\infty}(\mathrm{Har}_X, \mathrm{Har}_X). \qquad (7)$$

2.3 The Hodge correlators

Theorem 2.3 *a) There is a linear map, the Hodge correlator map*

$$\mathrm{Cor}_{\mathrm{Har_X}} : H_0(\mathrm{CC}_*(\mathrm{Har}_X) \otimes \mathcal{H}) \longrightarrow \mathbb{C}. \tag{8}$$

Combining it with (7), we get a cohomology class

$$\mathbf{H}_{\mathrm{Har_X}} := \varphi(\mathrm{Cor}_{\mathrm{Har_X}}) \in H^0\mathrm{Fun}_{A_\infty}(\mathrm{Har}_X, \mathrm{Har}_X). \tag{9}$$

b) There is a homotopy action of the twistor Galois group G_{T} *by* A_∞*-autoequivalences of the category* Har_X *such that*

- *Its restriction to the subgroup* $\mathbb{C}^*$ *is the Simpson action [Si1] on the category* Har_X*.*
- *Its restriction to the Lie algebra* $\mathrm{LieU_{Hod}}$ *is given by a Lie algebra map*

$$\mathbb{H}_{\mathrm{Har_X}} : \mathrm{L_{Hod}} \longrightarrow \mathrm{H^0Fun}_{A_\infty}(\mathrm{Har}_X, \mathrm{Har}_X), \tag{10}$$

uniquely determined by the condition that $\mathbb{H}_{\mathrm{Har_X}}(G) = \mathbf{H}_{\mathrm{Har_X}}$*.*

c) The action of the group G_{T} *is functorial with respect to the pull backs.*

2.4 Construction.

To define the Hodge correlator map (8), we define a collection of degree zero maps

$$\mathrm{Cor}_{\mathrm{Hod_X}} : \Big(H^\bullet(X, V_0^* \otimes V_1)[1] \otimes \ldots \otimes H^\bullet(X, V_m^* \otimes V_0)[1]\Big)_{\mathcal{C}} \otimes \mathcal{H} \longrightarrow \mathbb{C}. \tag{11}$$

The definition depends on some choices, like harmonic representatatives of cohomology classes. We prove that it is well defined on HC^0, i.e. its resctriction to cycles is independent of the choices, and coboundaries are mapped to zero.

We picture an element in the sourse of the map (11) by a polygon P, see Fig 1, whose vertices are the objects V_i, and the oriented sides V_iV_{i+1} are graded vector space $\mathrm{Ext}^*(V_i, V_{i+1})(1)$.

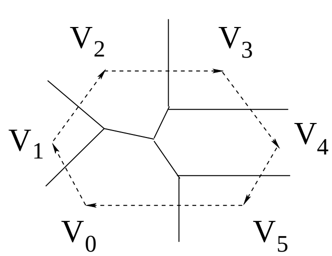

Figure 1: A decorated plane trivalent tree; V_i are harmonic bundles.

Green currents for harmonic bundles. Let V be a harmonic bundle on X. Then there is a Doulbeaut bicomplex $(\mathcal{A}^\bullet(X,V); D', D'')$ where the differentials D', D'' are provided by the complex structure on X and the harmonic metric on V. It satisfies the D', D''-lemma.

Choose a splitting of the corresponding de Rham complex $\mathcal{A}^\bullet(X,V)$ into an arbitrary subspace $\mathcal{H}ar^\bullet(X,V)$ isomorphically projecting onto the cohomology $H^\bullet(X,V)$ ("harmonic forms") and its orthogonal complement. If $V = \mathbb{C}_X$, we choose $a \in X$ and take the δ-function δ_a at the point $a \in X$ as a representative of the fundamental class.

Let δ_Δ be the Schwarz kernel of the identity map $V \to V$ given by the δ-function of the diagonal, and $P_{\rm Har}$ the Schwarz kernel of the projector onto the space $\mathcal{H}ar^\bullet(X,V)$, realized by an (n,n)-form on $X \times X$. Choose a basis $\{\alpha_i\}$ in $\mathcal{H}ar^\bullet(X,V)$. Denote by $\{\alpha_i^\vee\}$ the dual basis. Then we have

$$P_{\rm Har} = \sum \alpha_i^\vee \otimes \alpha_i, \qquad \int_X \alpha_i \wedge \alpha_j^\vee = \delta_{ij}.$$

Let $p_i : X \times X \to X$ be the projections onto the factors.

Definition 2.4 *A Green current $G(V;x,y)$ is a $p_1^*V^* \otimes p_2^*V$-valued current on $X \times X$,*

$$G(V;x,y) \in \mathcal{D}^{2n-2}(X \times X, p_1^*V^* \otimes p_2^*V), \quad n = {\rm dim}_{\mathbb{C}}X,$$

which satisfies the differential equation

$$(2\pi i)^{-1} D''D'G(V;x,y) = \delta_\Delta - P_{\rm Har}. \tag{12}$$

The two currents on the right hand side of (12) represent the same cohomology class, so the equation has a solution by the $D''D'$-lemma.

Remark. The Green current depends on the choice of the "harmonic forms". So if $V = \mathbb{C}$, it depends on the choise of the base point a. Solutions of equation (12) are well defined modulo ${\rm Im}D'' + {\rm Im}D' + \mathcal{H}ar^\bullet(X,V)$.

Construction of the Hodge correlators. *Trees.* Take a plane trivalent tree T dual to a triangulation of the polygon P, see Fig 1. The complement to T in the polygon P is a union of connected domains parametrized by the vertices of P, and thus decorated by the harmonic bundles V_i. Each edge E of the tree T is shared by two domains. The corresponding harmonic bundles are denoted V_{E-} and V_{E+}. If E is an external edge, we assume that V_{E-} is before V_{E+} for the clockwise orientation.

Given an internal vertex v of the tree T, there are three domains sharing the vertex. We denote the corresponding harmonic bundles by V_i, V_j, V_k, where the cyclic order of the bundles agrees with the clockwise orientation. There is a natural trace map

$${\rm Tr}_v : V_i^* \otimes V_j \otimes V_j^* \otimes V_k \otimes V_k^* \otimes V_i \longrightarrow \mathbb{C}. \tag{13}$$

It is invariant under the cyclic shift.

Decorations. For every edge E of T, choose a graded splitting of the de Rham complex

$$\mathcal{A}^{\bullet}(X, V_{E-}^{*}\otimes V_{E+}) = \mathcal{H}ar^{\bullet}(X, V_{E-}^{*}\otimes V_{E+}) \bigoplus \mathcal{H}ar^{\bullet}(X, V_{E-}^{*}\otimes V_{E+})^{\perp}.$$

Then a decomposable class in $\Big(\otimes_{i=0}^{m} H^{*}(X, V_{i}^{\bullet}\otimes V_{i+1})[1]\Big)_{\mathcal{C}}$ has a harmonic representative

$$W = \Big(\alpha_{0,1}\otimes\alpha_{1,2}\otimes\ldots\otimes\alpha_{m,0}\Big)_{\mathcal{C}}.$$

We are going to assign to W a top degree current $\kappa(W)$ on

$$X^{\{\text{internal vertices of } T\}}. \qquad (14)$$

Each external edge E of the tree T is decorated by an element

$$\alpha_E \in \mathcal{H}ar^{\bullet}(X, V_{E-}^{*}\otimes V_{E+}).$$

Put the current α_E to the copy of X assigned to the internal vertex of the edge E, and pull it back to (14) using the projection p_{α_E} of the latter to the X. Abusing notation, we denote the pull back by α_E. It is a form on (14) with values in the bundle $p_{\alpha_E}^{*}(V_{E-}^{*}\otimes V_{E+})$

Green currents. We assign to each internal edge E of the tree T a Green current

$$G(V_{E-}^{*}\otimes V_{E+}; x_-, x_+). \qquad (15)$$

The order of (x_-, x_+) agrees with the one of (V_{E-}^{*}, V_{E+}) as on Fig 2: the cyclic order of $(V_{E-}^{*}, x_-, V_{E+}, x_+)$ agrees with the clockwise orientation. The Green current (15) is symmetric:

$$G(V_{E-}^{*}\otimes V_{E+}; x_-, x_+) = G(V_{E+}^{*}\otimes V_{E-}; x_+, x_-). \qquad (16)$$

So it does not depend on the choice of orientation of the edge E.

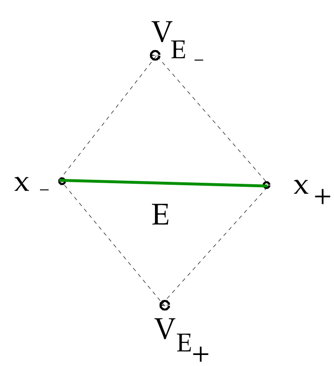

Figure 2: Decorations of the Green current assigned to an edge E.

The map ξ. There is a degree zero map

$$\xi : \mathcal{A}^{\bullet}(X, V_0)[-1]\otimes\ldots\otimes\mathcal{A}^{\bullet}(X, V_m)[-1] \longrightarrow \mathcal{A}^{\bullet}(X, V_0\otimes ...\otimes V_m)[-1]; \qquad (17)$$

$$\varphi_0 \otimes \ldots \otimes \varphi_m \longmapsto \mathrm{Sym}_{\{0,\ldots,m\}}\Big(\varphi_0 \wedge D^{\mathbb{C}}\varphi_1 \wedge \ldots \wedge D^{\mathbb{C}}\varphi_m\Big). \qquad (18)$$

The graded symmetrization in (18) is defined via isomorphisms $V_{\sigma(0)} \otimes ... \otimes V_{\sigma(m)} \to V_0 \otimes ... \otimes V_m$, where σ is a permutation of $\{0, ..., m\}$. It is essential that $\deg D^{\mathbb{C}}\varphi = \deg\varphi + 1$.

An outline of the construction. We apply the operator ξ to the product of the Green currents assigned to the internal edges of T. Then we multiply on (14) the obtained local system valued current with the one provided by the decoration W, with an appropriate sign. Applying the product of the trace maps (13) over the internal vertices of T, we get a top degree scalar current on (14). Integrating it we get a number assigned to T. Taking the sum over all plane trivalent trees T decorated by W, we get a complex number $\mathrm{Cor}_{\mathrm{Har}_X}(W \otimes \mathcal{H})$. Altogether, we get the map (8). One checks that its degree is zero. The signs in this definition are defined the same way as in [G2].

Theorem 2.5 *The maps (11) give rise to a well defined Hodge correlator map (8).*

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