Tilings by $(0.5, n)$-Crosses and Perfect Codes

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Abstract

The existence question for tiling of the n-dimensional Euclidian space by crosses is well known. A few existence and nonexistence results are known in the literature. Of special interest are tilings of the Euclidian space by crosses with arms of length one, known also as Lee spheres with radius one. Such a tiling forms a perfect code. In this paper crosses with arms of length half are considered. These crosses are scaled by two to form a discrete shape. We prove that an integer tiling for such a shape exists if and only if $n = 2^t - 1$ or $n = 3^t - 1$, $t > 0$. A strong connection of these tilings to binary and ternary perfect codes in the Hamming scheme is shown.

Keywords. cross, perfect code, semicross, tiling.

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1 Introduction

Packing and covering are two fundamental concepts in combinatorics. Tiling is a concept which combines both packing and covering and hence it attracts a substantial interest. Tiling of the Euclidian space with specific shapes is one of the main interest in this respect. Two of the shapes in this context are the semicross and the cross. A (k, n)-*semicross* is an n-dimensional shape whose center is an n-dimensional unit cube from which n *arms* consisting of k n-dimensional unit cubes are spanned in the n positive directions. A (k, n)-*cross* is an n-dimensional shape whose center is an n-dimensional unit cube from which $2n$ *arms* consisting of k n-dimensional unit cubes are spanned in the n directions (one for the positive and one for the negative). Examples of a $(2, 3)$-cross and a $(2, 3)$-semicross are given in Figure 1. Packing and tiling with semicrosses and crosses is a well studied topic (see [18, 19] and their references).

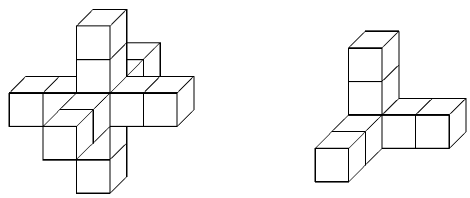

Figure 1: A $(2, 3)$-cross and a $(2, 3)$-semicross.

As mentioned in [19] the origins of the study of the cross and semicross are in several independent sources [6, 9, 15, 23], some of which are pure mathematics and some are connected to coding theory. Semicross and cross are two types of "error spheres" as explained in [5]. Golomb and Welch [6] proved that the $(1, n)$-cross tiles the n-dimensional Euclidian space for all $n \geq 1$. Such a tiling is a perfect code in the Manhattan metric and if the tiling is periodic then it is also a perfect code in the Lee metric. Their work inspired future work (see [3] and its references) on perfect codes in the Lee (and Manhattan) metric.

As said before, packing and tiling with semicrosses and crosses are well studied topics [2, 4, 6, 7, 8, 11, 15, 16, 17, 20, 21, 22]. The results in these research works include bounds on the size of the arms, constructions for such packings and tilings, parameters for which such tilings cannot exist, lattice and non-lattice tilings, etc. Recently, the topic has gained a new interest since the (k, n)-semicross is the error sphere of the *asymmetric error model* associated with flash memories [1, 10], the most advanced type of storage used currently. Recently, Schwartz [13] investigated lattice tilings with generalized crosses and semicrosses in the connection of *unbalanced limited magnitude error model* for multi level flash memories.

Not much is known about tiling of crosses with arms which are not of integer length. Moreover, most tilings considered in the literature are integer lattice tilings. In this paper we study the existence of tiling of the n-dimensional Euclidian space with a $(0.5, n)$-cross. The $(0.5, n)$-cross consists of one complete (non-fractional) unit cube and $2n$ halves unit cubes. Usually, it is more convenient to handle tiling with complete unit cubes. Hence, we scale the $(0.5, n)$-cross by two to obtain a new shape, which will be denoted in the sequel by Υ_n. Υ_n consists of $2^n(n + 1)$ complete unit cubes. For Υ_n we will discuss only integer tiling (also known as $\mathbb{Z}$-tiling) which is a tiling in which the centers of the unit cubes are placed on points of $\mathbb{Z}^n$. We prove that such a tiling exists if and only if $n = 2^t - 1$ or $n = 3^t - 1$,

$t > 0$. The related tiling with a $(0.5, n)$-cross (obtained after scaling by 0.5) will be called a $\frac{1}{2}\mathbb{Z}$-tiling. We present an analysis of the structure obtained from such a tiling. The tiling which is considered for the $(0.5, n)$-cross is usually not an integer tiling. Moreover, we discuss general tilings and not just lattice tilings as done in most literature.

The rest of this paper is organized as follows. In Section 2 we present the basic concepts used throughout this paper. We define what is a tiling, a lattice tiling, an integer tiling, and a periodic tiling. We discuss how to handle a tiling with a $(0.5, n)$-cross. We discuss three distance measures which are used in our discussion, the well known Hamming distance, the Manhattan distance which is used for codes in $\mathbb{Z}^n$, and a new distance measure needed for the $(0.5, n)$-crosses, the cross distance. We also discuss how a tiling with a $(0.5, n)$-cross can be analyzed. In Section 3 we make an analysis of a tiling with a $(0.5, n)$-cross and prove that such a $\frac{1}{2}\mathbb{Z}$-tiling can exist only if $n = 2^t - 1$ or $n = 3^t - 1$, $t > 0$. Some necessary conditions for the existence of such a tiling are given. In Section 4 we show how we can construct a tiling with a $(0.5, n)$-cross from a binary perfect single-error-correcting code of length $n = 2^t - 1$ and vice-versa. We show how to construct a tiling with a $(0.5, n)$-cross from a ternary perfect single-error-correcting code of length $\frac{n}{2} = \frac{3^t-1}{2}$.

2 Basic Concepts

Let $\mathcal{S}$ be an n-dimensional shape in the n-dimensional Euclidian space ($\mathbb{R}^n$). We say that two copies of $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$, are *disjoint* if their intersection is contained in at most an $(n-1)$-dimensional space. A *tiling* $\mathcal{T}$ of the n-dimensional Euclidian space with the shape $\mathcal{S}$ is a set of disjoint copies of $\mathcal{S}$ such that each point $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is contained in at least one copy of $\mathcal{S}$. For a given shape $\mathcal{S}$ we choose a fixed point which will be called the *balanced point* of the shape. In any other copy of $\mathcal{S}$ the balanced point will be chosen in the same relative position. The set of balanced points in the copies of $\mathcal{S}$ contained in the tiling $\mathcal{T}$ defines the tiling. Hence, in the sequel a tiling $\mathcal{T}$ will be defined by a set of points $\mathbb{T}$ in $\mathbb{R}^n$ and a shape $\mathcal{S}$. Henceforth, $\mathbb{T}$ will be called a tiling if the shape $\mathcal{S}$ is known. For example, the set of points $\mathbb{T} = \{(i, i+5j) \ : \ i, j \in \mathbb{Z}\}$ and the $(2,2)$-semicross defines a tiling of $\mathbb{R}^2$ (part of the tiling is presented in Figure 2). A tiling $\mathbb{T}$ with a shape $\mathcal{S}$ is called an *integer tiling* (also a $\mathbb{Z}$-*tiling*) if $\mathbb{T} \subseteq \mathbb{Z}^n$. An n-dimensional *unit cube* centered at a point $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ consists of the points $\{(x_1, x_2, \ldots, x_n) \ : \ |x_i - c_i| \leq 0.5, \ 1 \leq i \leq n\}$. The n-dimensional shape $\mathcal{S}$ is a *discrete shape* if $\mathcal{S}$ is a union of n-dimensional unit cubes, whose centers are in $\mathbb{Z}^n$. Hence, a discrete n-dimensional shape $\mathcal{S}$ can be defined by a set of points from $\mathbb{Z}^n$. Therefore, in an integer tiling with a discrete shape $\mathcal{S}$, each point of $\mathbb{Z}^n$ is contained in exactly one copy of $\mathcal{S}$.

A *lattice* Λ is a discrete, additive subgroup of the real n-space $\mathbb{R}^n$. W.l.o.g. (without loss of generality), we can assume that

$$\Lambda \stackrel{\text{def}}{=} \{u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \ : \ u_1, u_2, \cdots, u_n \in \mathbb{Z}\}$$

where $\{v_1, v_2, \ldots, v_n\}$ is a set of linearly independent vectors in $\mathbb{R}^n$. The set of vectors

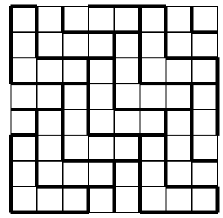

Figure 2: Tiling of $\mathbb{R}^2$ with a (2,2)-semicross.

$\{v_1, v_2, \ldots, v_n\}$ is called *the basis* for Λ, and the $n \times n$ matrix

$$\mathbf{G} \stackrel{\text{def}}{=} \begin{bmatrix} v_{11} & v_{12} & \ldots & v_{1n} \\ v_{21} & v_{22} & \ldots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \ldots & v_{nn} \end{bmatrix}$$

having these vectors as its rows is said to be the *generator matrix* for Λ.

The *volume* of a lattice Λ, denoted by $V(\Lambda)$, is inversely proportional to the number of lattice points per a unit volume. More precisely, $V(\Lambda)$ may be defined as the volume of the *fundamental parallelogram* $\Pi(\Lambda)$, which is given by

$$\Pi(\Lambda) \stackrel{\text{def}}{=} \{\xi_1 v_1 + \xi_2 v_2 + \cdots + \xi_n v_n \ : \ 0 \leq \xi_i < 1, \ 1 \leq i \leq n\} \ .$$

There is a simple expression for the volume of Λ, namely, $V(\Lambda) = |\det \mathbf{G}|$.

Λ is a *lattice tiling* with $\mathcal{S}$ if $\mathbb{T} \stackrel{\text{def}}{=} \Lambda$ forms a tiling with $\mathcal{S}$. A lattice tiling Λ is an *integer lattice tiling* with $\mathcal{S}$ if all entries of $\mathbf{G}$ are integers. The following lemma is well known.

Lemma 1. *A necessary condition that Λ defines a lattice tiling with the shape $\mathcal{S}$ is that $V(\Lambda) = |\mathcal{S}|$.*

The vector $(x_1, x_2, \ldots, x_n)$ is called the *r-th unit vector* and will be denoted by e_r if $x_r = 1$ and for all $i \neq r$, $x_i = 0$. For two vectors $X = (x_1, x_2, \ldots, x_n), Y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$, we define the *vector addition* $X + Y \stackrel{\text{def}}{=} (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ and the *scalar multiplication* $\alpha X \stackrel{\text{def}}{=} (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$. For two sets $\mathbb{S}_1 \subseteq \mathbb{R}^n$ and $\mathbb{S}_2 \subseteq \mathbb{R}^n$ we define the *set addition* $\mathbb{S}_1 + \mathbb{S}_2 \stackrel{\text{def}}{=} \{X + Y \ : \ X \in \mathbb{S}_1, \ Y \in \mathbb{S}_2\}$.

For a set $\mathbb{S} \in \mathbb{Z}^n$ and a vector $U \in \mathbb{R}^n$ the *shift* of $\mathbb{S}$ by U is $U + \mathbb{S} \stackrel{\text{def}}{=} \{U + X \ : \ X \in \mathbb{S}\}$. The *multiplication* of $\mathbb{S}$ by a scalar $\alpha \in \mathbb{R}$ is defined by $\alpha \mathbb{S} \stackrel{\text{def}}{=} \{\alpha X \ : \ X \in \mathbb{S}\}$.

Lemma 2. *If $\mathbb{T}$ is a tiling with a shape $\mathcal{S}$ and $U \in \mathbb{R}^n$ then $U + \mathbb{T}$ is also a tiling with $\mathcal{S}$.*

By Lemma 2 we can assume that the origin, denoted by $\mathbf{0}$, is always a point in the tiling. Therefore, given a tiling $\mathbb{T}$ we assume w.l.o.g. that the origin is an element in $\mathbb{T}$. For a set $\mathbb{S} \subseteq \mathbb{R}^n$ and a permutation σ of $\{1, 2, \ldots, n\}$, let $\sigma(\mathbb{S}) \stackrel{\text{def}}{=} \{\sigma(X) \ : \ X \in \mathbb{S}\}$.

Lemma 3. *If $\mathbb{T}$ is a tiling with an n-dimensional shape $\mathcal{S}$ and σ is a permutation of $\{1, 2, \ldots, n\}$ then $\sigma(\mathbb{T})$ is a tiling with the n-dimensional shape $\sigma(\mathcal{S})$.*

A set $\mathbb{S}$ is called *periodic* with period π if $X \in \mathbb{S}$ implies that $X + \alpha\pi e_i \in \mathbb{S}$ for all $\alpha \in \mathbb{Z}$ and $1 \leq i \leq n$. A tiling $\mathbb{T}$ with the shape $\mathcal{S}$ is a periodic tiling if it is a periodic set. The following simple lemma is left for the reader.

Lemma 4. *$\mathbb{T}$ is a periodic tiling with period π if and only if $X \in \mathbb{T}$ implies that $X + \pi e_i \in \mathbb{T}$ for all i, $1 \leq i \leq n$.*

A *code* $\mathcal{C}$ of length n over $\mathbb{Z}_q$ ($\mathbb{Z}$) is a subset of $\mathbb{Z}_q^n$ ($\mathbb{Z}^n$). Let Λ_n be the lattice generated by the basis $\{q \cdot e_i \; : \; 1 \leq i \leq n\}$. A code $\mathcal{C} \subseteq \mathbb{Z}_q^n$ can be viewed also as a subset of $\mathbb{Z}^n$. The code $E(\mathcal{C}) = \mathcal{C} + \Lambda_n$ is the *expanded code* of $\mathcal{C}$. If $E(\mathcal{C})$ is a tiling of $\mathbb{Z}^n$ with the shape $\mathcal{S}$ then we also call $\mathcal{C}$ a tiling of $\mathbb{Z}_q^n$ with the shape $\mathcal{S}$. A tiling $\mathbb{T} \subseteq \mathbb{Z}^n$ with a period π can be viewed as an expanded code $E(\mathcal{C})$ of a code $\mathcal{C}$ of length n over $\mathbb{Z}_\pi$, where $\mathcal{C} = \mathbb{T} \cap \{0, 1, \ldots, \pi - 1\}^n$. In the sequel we denote $\tilde{\mathbb{Z}}_\pi \stackrel{\text{def}}{=} \{0, 1, \ldots, \pi - 1\}$; we will also refer to $\mathbb{T}$ as a code and to its elements as codewords.

To handle a tiling with a $(0.5, n)$-cross we will need to use three distance measures, the well known Hamming distance, the Manhattan distance, and the new defined cross distance.

For any two given words $X, Y \in \mathbb{Z}_q^n$ the *Hamming distance* $d_H(X, Y)$ is the number of positions in which X and Y differ, i.e.

$$d_H(X, Y) \stackrel{\text{def}}{=} |\{i \; : \; x_i \neq y_i, \; 1 \leq i \leq n\}| \; .$$

For any two given points $X, Y \in \mathbb{Z}^n$ the *Manhattan distance* $d_M(X, Y)$ is defined by

$$d_M(X, Y) \stackrel{\text{def}}{=} \sum_{i=1}^{n} |x_i - y_i| \; .$$

For any two given points $X, Y \in \mathbb{Z}^n$ we defined the *cross distance* $d_C(X, Y)$ as follows

$$d_C(X, Y) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \max\{0, |y_i - x_i| - 1\}.$$

Remark 1. *The cross distance can be generalized for two points $X, Y \in \mathbb{R}^n$. We will use this generalization only in this section.*

Remark 2. *The Hamming distance is a scheme, while the Manhattan distance is only a metric distance and not a scheme (see [12] for the definition of a scheme). The cross distance is not a metric, but it will be most important in the discussion on tilings with a $(0.5, n)$-cross.*

For each distance measure we defined the *weight* of a point (word) X to be the distance between X and the point $\mathbf{0}$. The cross weight of a point X will be denoted by $w_C(X)$.

A unit cube centered at $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ is a union of two disjoint half unit cubes in one of the n directions. For the r-th direction one *half unit cube* is defined by the set of points $\{(x_1, x_2, \ldots, x_n) \; : \; 0 \leq x_r - c_r \leq 0.5, \; |x_i - c_i| \leq 0.5, \; 1 \leq i \leq n, \; i \neq r\}$ and a second *half unit cube* is defined by the set of points $\{(x_1, x_2, \ldots, x_n) \; : \; -0.5 \leq x_r - c_r \leq 0,$ $|x_i - c_i| \leq 0.5, \; 1 \leq i \leq n, \; i \neq r\}$. A $(0.5, n)$-cross is a unit cube to which two half unit cubes are attached in the r-th direction for each $1 \leq r \leq n$, one in its negative direction and one in its positive direction. It is more convenient to handle shapes with complete unit cubes (discrete shapes) and therefore we will scale the $(0.5, n)$-cross by two to obtain a new shape which will be called Υ_n. An example of a $(0.5, 3)$-cross and an Υ_3 is given in Figure 3. The complete unit cube in the $(0.5, n)$-cross is transferred into an n-dimensional cube with sides of length two in Υ_n. This cube in Υ_n will be called the *core* of Υ_n; the core consists of 2^n unit cubes. In the sequel we will be interested only in integer tilings with Υ_n. In such an integer tiling Υ_n can be represented by $2^n(n+1)$ points of $\mathbb{Z}^n$ which are the centers of its $2^n(n+1)$ unit cubes. If $\mathbb{C} \subset \mathbb{Z}^n$ is the core of Υ_n then $\Upsilon_n = \{U \ : \ d_M(X, U) = 1, \ X \in \mathbb{C}\}$. Given a point $(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ the set $\{(c_1, c_2, \ldots, c_n) \ : \ c_i \in \{a_i - 1, a_i\}, \ 1 \leq i \leq n\}$ is an example for a core of Υ_n; the set $\{(c_1, c_2, \ldots, c_n) \ : \ c_i \in \{a_i + 1, a_i\}, \ 1 \leq i \leq n\}$ is another example for a core of Υ_n. If $\mathbb{T}$ is a tiling with Υ_n then $0.5\mathbb{T}$ is a tiling with a $(0.5, n)$-cross. Clearly, if for each $(x_1, x_2, \ldots, x_n) \in \mathbb{T}$, x_i is even for all $1 \leq i \leq n$, then also $0.5\mathbb{T}$ is an integer tiling. However, if there exists a point $(x_1, x_2, \ldots, x_n) \in \mathbb{T}$ such that for at least one j we have that x_j is odd then $0.5\mathbb{T}$ is not an integer tiling. To this end we define a $\frac{1}{2}\mathbb{Z}$-tiling. A tiling $\mathbb{T}$ is a $\frac{1}{2}\mathbb{Z}$-*tiling* if $\mathbb{T} \subseteq 0.5\mathbb{Z}^n$.

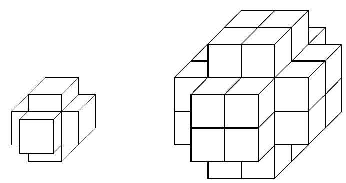

Figure 3: A $(0.5, 3)$-cross and an Υ_3.

Lemma 5. *$\mathbb{T}$ is an integer tiling with Υ_n if and only if $0.5\mathbb{T}$ is a $\frac{1}{2}\mathbb{Z}$-tiling with a $(0.5, n)$-cross.*

Given a set $\mathbb{T} \subset \mathbb{Z}^n$, we would like to know whether $\mathbb{T}$ is a tiling with Υ_n. To show that $\mathbb{T}$ is a tiling we have to prove

($\mathcal{P}$.1) For each point $Y \in \mathbb{Z}^n$ there exists a copy $\mathcal{S}_1$ of Υ_n in the tiling such that $\mathcal{S}_1$ contains Y.

($\mathcal{P}$.2) A point $Y \in \mathbb{Z}^n$ is not contained in more than one copy of Υ_n in the tiling, i.e. for each two copies $\mathcal{S}_1$ and $\mathcal{S}_2$ of Υ_n in the tiling we have $\mathcal{S}_1 \cap \mathcal{S}_2 = \varnothing$.

A set $\mathbb{T} \subset \mathbb{Z}^n$ is a *covering* with Υ_n if it satisfies property **($\mathcal{P}$.1)** and it is a *packing* with Υ_n if it satisfies property **($\mathcal{P}$.2)**. A tiling is clearly both a covering and a packing.

The following two lemmas are immediate results from the definition of Υ_n.

Lemma 6. *If $\mathcal{S}$ is a copy of Υ_n and $X \in \mathcal{S}$ is not a core point of $\mathcal{S}$ then there exists a core point $Y \in \mathcal{S}$ such that $d_M(X, Y) = 1$.*

Lemma 7. *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two copies of Υ_n for which $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \varnothing$ then there exists a point $X \in \mathcal{S}_1 \cap \mathcal{S}_2$ which is not in the core of $\mathcal{S}_1$.*

Corollary 1. *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two copies of Υ_n for which $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \varnothing$ then there exist two core points $X_1 \in \mathcal{S}_1$ and $X_2 \in \mathcal{S}_2$ such that $d_M(X_1, X_2) \leq 2$.*

Lemma 8. *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two copies of Υ_n for which there exist two core points $X_1 \in \mathcal{S}_1$ and $X_2 \in \mathcal{S}_2$ such that $d_M(X_1, X_2) \leq 2$, then $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \varnothing$.*

Proof. If $d_M(X_1, X_2) \leq 2$ then there exists a point $Y \in \mathbb{Z}^n$ such that $d_M(X_1, Y) \leq 1$ and $d_M(X_2, Y) \leq 1$. By definition $Y \in \mathcal{S}_1 \cap \mathcal{S}_2$. $\square$

Corollary 2. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two copies of Υ_n. $\mathcal{S}_1 \cap \mathcal{S}_2 = \varnothing$ if and only if for any two core points $X_1 \in \mathcal{S}_1$ and $X_2 \in \mathcal{S}_2$ we have $d_M(X_1, X_2) \geq 3$.*

Theorem 1. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two copies of Υ_n with balanced points $X, Y \in \mathbb{Z}^n$. Then $\mathcal{S}_1 \cap \mathcal{S}_2 = \varnothing$ if and only if $d_C(X, Y) \geq 3$.*

Proof. Let $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ and $\tilde{Y} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_n)$ be the centers of mass of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Clearly, $\tilde{X}$ and $\tilde{Y}$ are in $(0.5, 0.5, \ldots, 0.5) + \mathbb{Z}^n$. The core points of $\mathcal{S}_1$ are $\{(c_1, c_2, \ldots, c_n) \ : \ c_i \in \{\tilde{x}_i - 0.5, \tilde{x}_i + 0.5\}\}$ and the core points of $\mathcal{S}_2$ are $\{(c_1, c_2, \ldots, c_n) \ : \ c_i \in \{\tilde{y}_i - 0.5, \tilde{y}_i + 0.5\}\}$. Let $X' = (x'_1, x'_2, \ldots, x'_n)$ and $Y' = (y'_1, y'_2, \ldots, y'_n)$ be the two core points of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, defined as follows. If $\tilde{x}_i = \tilde{y}_i$ then $x'_i \stackrel{\text{def}}{=} \tilde{x}_i + 0.5$ and $y'_i \stackrel{\text{def}}{=} \tilde{y}_i + 0.5$. If $\tilde{x}_i < \tilde{y}_i$ then $x'_i \stackrel{\text{def}}{=} \tilde{x}_i + 0.5$ and $y'_i \stackrel{\text{def}}{=} \tilde{y}_i - 0.5$. If $\tilde{x}_i > \tilde{y}_i$ then $x'_i \stackrel{\text{def}}{=} \tilde{x}_i - 0.5$ and $y'_i \stackrel{\text{def}}{=} \tilde{y}_i + 0.5$. Clearly, $d_C(X, Y) = d_C(\tilde{X}, \tilde{Y}) = d_M(X', Y')$ and for any two core points $\hat{X} \in \mathcal{S}_1$ and $\hat{Y} \in \mathcal{S}_2$ we have that $d_M(\hat{X}, \hat{Y}) \geq d_M(X', Y')$. Now, by Corollary 2 we have that $\mathcal{S}_1 \cap \mathcal{S}_2 = \varnothing$ if and only if $d_C(X, Y) \geq 3$. $\square$

Corollary 3. *The set $\mathbb{T}$ induces a packing of the n-dimensional Euclidian space with Υ_n if and only if for any two elements $X, Y \in \mathbb{T}$, we have $d_C(X, Y) \geq 3$.*

To prove that a set is a tiling with Υ_n we will have to show that it satisfies properties **($\boldsymbol{\mathcal{P}}$.1)** and **($\boldsymbol{\mathcal{P}}$.2)**. For this purpose we will have to show that each point of $\mathbb{Z}^n$ is contained (covered) in exactly one copy $\mathcal{S}$ of Υ_n in the tiling. A point $A \in \mathbb{Z}^n$ is *covered* by a codeword X in a tiling $\mathbb{T}$ if A is contained in the copy of Υ_n in the tiling whose balanced point is X. In this case we say that X *covers* A.

Given a tiling $\mathbb{T}$ with Υ_n it has to satisfy properties **($\boldsymbol{\mathcal{P}}$.1)** and **($\boldsymbol{\mathcal{P}}$.2)**. By considering how each point $A \in \mathbb{Z}^n$ is covered by a codeword $X \in \mathbb{T}$ (property **($\boldsymbol{\mathcal{P}}$.1)**) we will discover the structure of $\mathbb{T}$. To this end we will also use property **($\boldsymbol{\mathcal{P}}$.2)**, i.e. for each two codewords $X, Y \in \mathbb{T}$ we have that $d_C(X, Y) \geq 3$ (by Theorem 1 and Corollary 3).

3 The Nonexistence of other Integer Tilings

In this section we will prove that an integer tiling $\mathbb{T}$ with Υ_n exists only if $n = 2^t - 1$ or $n = 3^t - 1$, $t > 0$. In subsection 3.1 we prove this claim for odd n and for lattice tiling. In subsection 3.2 we complete the proof for even n. We will obtain this goal by proving that given a tiling $\mathbb{T}$ with Υ_n, certain elements of $\mathbb{Z}^n$ must be contained in $\mathbb{T}$. It will be proved by considering how elements with small cross weight are covered. For the rest of this section let $\mathbb{T}$ be a tiling with Υ_n. We remind that w.l.o.g. we assumed that $\mathbf{0} \in \mathbb{T}$ and hence by Corollary 3, if $X, Y \in \mathbb{T} \setminus \{\mathbf{0}\}$, $X \neq Y$, then $w_C(X) \geq 3$, $w_C(Y) \geq 3$, and $d_C(X, Y) \geq 3$.

3.1 Tiling for odd n and lattice tiling

Throughout this subsection we assume that if $X = (x_1, x_2, \ldots, x_n) \in \mathbb{T}$ is a balanced point then the set $\{(c_1, c_2, \ldots, c_n) \ : \ c_i \in \{x_i - 1, x_i\}, \ 1 \leq i \leq n\}$ is the related core of Υ_n. The first lemma is an immediate result from this choice of the core and the definition of Υ_n.

Lemma 9. *Let $X \in \mathbb{T}$ and $A = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$. A is covered by X if and only if $x_i \in \{a_i - 1, a_i, a_i + 1, a_i + 2\}$, $1 \leq i \leq n$, and for at most one i we have $x_i \in \{a_i - 1, a_i + 2\}$.*

Let $\mathcal{D}_1$ be the set of points from $\{0, 1, 2, 3\}^n$ in which 2 and 3 appear exactly once.

Lemma 10. *If $X \in \mathcal{D}_1 \cap \mathbb{T}$ then $X = 3e_r + 2e_s$ for some $r \neq s$.*

Proof. Assume w.l.o.g. that $X = (3, 2, 1, x_4, \ldots, x_n)$, $x_i \in \{0, 1\}$, $4 \leq i \leq n$. The point $A = (1, 1, -1, 0, ..., 0)$ is covered by a codeword $Y \in \mathbb{T}$. By Lemma 9 we have that $Y \notin \{X, \mathbf{0}\}$ and we can distinguish between three cases:
Case 1: If $y_i \in \{a_i, a_i + 1\}$ for all i then $w_C(Y) \leq 2$, a contradiction.
Case 2: There exists a j such that $y_j = a_j - 1$ and $y_i \in \{a_i, a_i + 1\}$ for all $i \neq j$. Since $w_C(Y) \geq 3$ it follows that $j = 3$ and hence $Y = (2, 2, -2, y_4, \ldots, y_n)$, where $y_i \in \{0, 1\}$, $4 \leq i \leq n$. This implies that $d_C(X, Y) = 2$, a contradiction.
Case 3: There exists a j such that $y_j = a_j + 2$ and $y_i \in \{a_i, a_i + 1\}$ for all $i \neq j$. Since $w_C(Y) \geq 3$ it follows that $j \neq 3$. W.l.o.g. it implies that $Y = (3, 2, y_3, y_4, \ldots, y_n)$, $Y = (2, 3, y_3, y_4, \ldots, y_n)$ ($y_4 \in \{0, 1\}$), or $Y = (2, 2, y_3, 2, y_5, \ldots, y_n)$, where $y_3 \in \{-1, 0\}$, $y_i \in \{0, 1\}$, $5 \leq i \leq n$. Hence, $d_C(X, Y) \leq 2$, a contradiction.

Therefore, there is no codeword $Y \in \mathbb{T}$ which covers A, a contradiction. Thus, if $X \in \mathcal{D}_1 \cap \mathbb{T}$ then $X = 3e_r + 2e_s$ for some $r \neq s$. □

Let $\mathcal{D}_2$ be the set of points from $\{0, 1, 4\}^n$ in which 4 appears exactly once.

Lemma 11. *If $X \in \mathcal{D}_2 \cap \mathbb{T}$ then $X = 4e_r$ for some $1 \leq r \leq n$.*

Proof. Assume w.l.o.g. that $X = (4, 1, x_3, \ldots, x_n)$, $x_i \in \{0, 1\}$, $3 \leq i \leq n$. The point $A = (1, 1, 0, ..., 0)$ is covered by a codeword $Y \in \mathbb{T}$. By Lemma 9 we have that $Y \notin \{X, \mathbf{0}\}$ and we can distinguish between two cases:
Case 1: If $y_i \in \{a_i, a_i + 1\}$ for all i, with a possible exception for at most one j, for which $y_j = a_j - 1$, then $w_C(Y) \leq 2$, a contradiction.
Case 2: There exists a j such that $y_j = a_j + 2$ and $y_i \in \{a_i, a_i + 1\}$ for all $i \neq j$. W.l.o.g. it implies that $Y = (3, 2, y_3, \ldots, y_n)$, $Y = (2, 3, y_3, \ldots, y_n)$ ($y_3 \in \{0, 1\}$), or $Y = (2, 2, 2, y_4, \ldots, y_n)$, where $y_i \in \{0, 1\}$, $4 \leq i \leq n$. Hence, $d_C(X, Y) \leq 2$, a contradiction.

Therefore, there is no codeword $Y \in \mathbb{T}$ which covers A, a contradiction. Thus, if $X \in \mathcal{D}_2 \cap \mathbb{T}$ then $X = 4e_r$ for some $1 \leq r \leq n$. □

Corollary 4. *For each r, $1 \leq r \leq n$, the point $2e_r$ is covered by a codeword $X \in \mathbb{T}$, where either $X = 4e_r$ or $X = 3e_r + 2e_s$ for some $s \neq r$.*

Proof. Follows immediately from Lemmas 9, 10, and 11. □

Let $\mathcal{D}_3$ be the set of points from $\{0, 1, 2\}^n$ in which 2 appears exactly three times.

Lemma 12. *If $X = 3e_r + 2e_s \in \mathbb{T}$ then for every $k \notin \{r, s\}$ there exists a unique $j \notin \{r, s, k\}$ and a codeword $Y \in \mathcal{D}_3 \cap \mathbb{T}$ such that $y_r = 1, y_s = y_k = y_j = 2$.*

Proof. Let $k \notin \{r, s\}$ and consider the point $A = e_r + e_s + e_k$. A is covered by a codeword $Y \in \mathbb{T}$. W.l.o.g. we assume that $r = 1$, $s = 2$, and $k = 3$, i.e. $X = (3, 2, 0, \ldots, 0)$ and $A = (1, 1, 1, 0 \ldots, 0)$. By Lemma 9 we have that $Y \notin \{X, \mathbf{0}\}$ and we can distinguish between three cases:
Case 1: If $y_i \in \{a_i, a_i+1\}$ for all i then since $w_C(Y) \geq 3$ it follows that $Y = (2, 2, 2, y_4, \ldots, y_n)$, where $y_i \in \{0, 1\}$, $4 \leq i \leq n$. Hence, $d_C(X, Y) = 1$, a contradiction.
Case 2: There exists a j such that $y_j = a_j - 1$ and $y_i \in \{a_i, a_i + 1\}$ for all $i \neq j$. If $j \leq 3$ then $w_C(Y) \leq 2$, a contradiction. If $j > 3$ then since $w_C(Y) \geq 3$ it follows that $Y = (2, 2, 2, y_4, \ldots, y_n)$, where $-1 \leq y_i \leq 1$, $4 \leq i \leq n$, and hence $d_C(X, Y) = 1$, a contradiction.
Case 3: There exists a j such that $y_j = a_j+2$ and $y_i \in \{a_i, a_i+1\}$ for all $i \neq j$. If $j \leq 3$ then since $w_C(Y) \geq 3$ and $d_C(X, Y) \geq 3$ it follows that $Y = (1, 2, 3, y_4, \ldots, y_n)$, where $0 \leq y_i \leq 1$, $4 \leq i \leq n$, a contradiction to Lemma 10.

Therefore, there exists a $j > 3$ such that $y_j = a_j + 2$ and $y_i \in \{a_i, a_i + 1\}$ for all $i \neq j$. W.l.o.g. we assume that $j = 4$. Since $w_C(Y) \geq 3$ and $d_C(X, Y) \geq 3$ it follows that $Y = (1, 2, 2, 2, y_5, \ldots, y_n)$, where $0 \leq y_i \leq 1$, $5 \leq i \leq n$. The uniqueness of j follows from the fact that if there exists another j and a related codeword Y' then $d_C(Y, Y') \leq 2$. □

Corollary 5. *If $3e_r + 2e_s \in \mathbb{T}$ then n is even.*

From Corollaries 4 and 5 we infer that

Corollary 6. *If n is odd then for all $X \in \mathbb{T}$ and $1 \leq r \leq n$ we have $X + 4e_r \in \mathbb{T}$, i.e. $\mathbb{T}$ is a periodic tiling with period 4.*

Theorem 2. *If $\mathbb{T}$ is an integer tiling with Υ_n, where n is an odd integer, then $n = 2^t - 1$ for some $t > 0$.*

Proof. By Corollary 6 we have that $\mathbb{T}$ is a periodic tiling with period 4. Therefore, the size of Υ_n divides 4^n. $|\Upsilon_n| = 2^n(n + 1)$ and hence $n = 2^t - 1$ for some $t > 0$. □

Lemma 13. *If there exist two distinct codewords $X = 3e_i + 2e_j$ and $X' = 3e_r + 2e_s$ in $\mathbb{T}$ then $\{i, j\} \cap \{r, s\} = \varnothing$.*

Proof. W.l.o.g. we assume that $i = 1$ and $j = 2$. Since $d_C(X, X') \geq 3$ it follows that $r \neq 1$ and $X' \neq 3e_2 + 2e_1$. If $r = 2$ or $s = 2$ then w.l.o.g. we assume that $X' = (0, 3, 2, 0, \ldots, 0)$ or $X' = (0, 2, 3, 0, \ldots, 0)$. By Lemma 12 we have a codeword $Y = (1, 2, 2, y_4, \ldots, y_n) \in \mathcal{D}_3 \cap \mathbb{T}$. It implies that $d_C(X', Y) = 1$, a contradiction. The case where $s = 1$ and $r > 2$ is symmetric to the case where $r = 2$ and $s > 2$. □

From Corollary 4 and Lemma 13 we have that

Corollary 7. *If $3e_r + 2e_s \in \mathbb{T}$ then $4e_s \in \mathbb{T}$.*

Theorem 3. *If $\mathbb{T}$ is an integer lattice tiling with Υ_n then either $n = 2^t - 1$ or $n = 3^t - 1$ for some $t > 0$.*

Proof. Assume that there are exactly k codewords of the form $3e_i+2e_j$ in $\mathbb{T}$. From Corollaries 4 and 7 and by Lemma 13 the lattice $\mathbb{T}$ contains a sublattice defined by these k codewords and $n-k$ codewords of the form $4e_s$. Therefore, the volume of the lattice is of the form $2^m 3^\ell = 2^n(n+1)$. By Theorem 2 we have that if n is odd then $n = 2^t - 1$ for some $t > 0$. If n is even then we must have $n = m$ and thus $n = 3^\ell - 1$ for some $\ell > 0$. □

3.2 Tiling for even n

A *packing triple system* of order n is a pair $(Q, \mathcal{B})$, where Q is an n-set and B is a collection of 3-subsets of Q, called *blocks* such that each 2-subset of Q is contained in at most one block of $\mathcal{B}$. Spencer [14] proved that if $n \not\equiv 5 \pmod 6$ then

$$|\mathcal{B}| \leq \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor . \tag{1}$$

Lemma 14. *For each $1 \leq i < j \leq n$, the point $e_i + e_j$ is covered by a codeword $X \in \mathbb{T}$, where $X = 3e_i + 2e_j$ or $X = 3e_j + 2e_i$ or $X \in \mathcal{D}_3$, where $x_i = x_j = 2$.*

Proof. Follows from Lemmas 9 and 10. □

Let

$$\mathcal{F}_1 \stackrel{\text{def}}{=} \{\{i,j\} \ : \ 3e_i + 2e_j \in \mathbb{T}\}$$

and

$$\mathcal{F}_2 \stackrel{\text{def}}{=} \{\{i,j,k\} \ : \ 2e_i + 2e_j + 2e_k + \sum_{m \notin \{i,j,k\}} \alpha_m e_m \in \mathbb{T}, \ \alpha_m \in \{0,1\}\} .$$

Since $\mathbb{T}$ is a tiling it follows that each point $e_i + e_j$, $i \neq j$, is covered by exactly one codeword of $\mathbb{T}$. As a consequence of Lemmas 9, 10, and since $w_C(X) \geq 3$ for each $X \in \mathbb{T}$, we have that each pair $\{r, s\}$ is a subset of exactly one element from $\mathcal{F}_1 \cup \mathcal{F}_2$. Therefore, $\mathcal{F}_2$ is a packing triple system of order n.

Theorem 4. *If $\mathbb{T}$ is an integer tiling with Υ_n then $n \not\equiv 4 \pmod 6$.*

Proof. Assume $\mathbb{T}$ is an integer tiling with Υ_n, $n \equiv 4 \pmod 6$. By (1) we have that

$$|\mathcal{F}_2| \leq \frac{n^2 - 2n - 2}{6} .$$

Since each pair $\{i,j\} \subset \{1, 2, \ldots, n\}$ is contained in either $\mathcal{F}_1$ or $\mathcal{F}_2$ it follows that

$$|\mathcal{F}_1| + 3|\mathcal{F}_2| = \binom{n}{2} .$$

Hence, $|\mathcal{F}_1| \geq \frac{n}{2} + 1$, contradicting Lemma 13. □

By using the same arguments as in the proof of Theorem 4 we have that if $n \equiv 0$ or $2 \pmod 6$ then $|\mathcal{F}_1| \geq \frac{n}{2}$. Hence, by Lemma 13 we infer

Lemma 15. *If n is even and $\mathbb{T}$ is an integer tiling with Υ_n, then there are exactly $\frac{n}{2}$ codewords of the form $3e_r + 2e_s$.*

Combing Lemmas 13 and 15 we infer

Corollary 8. *If n is even and $\mathbb{T}$ is an integer tiling with Υ_n, then there are $\frac{n}{2}$ codewords of the form $3e_r+2e_s$ and the set $\{i\ :\ 3e_i+2e_j \in \mathbb{T}$ or $3e_j+2e_i \in \mathbb{T}\}$ contains all the integers between 1 and n.*

Let $\mathbb{T}'$ be the tiling of $\mathbb{Z}^n$ with Υ_n defined by $\mathbb{T}' \stackrel{\text{def}}{=} \{X\ :\ -X \in \mathbb{T}\}$. Since $\mathbb{T}'$ is a tiling of $\mathbb{Z}^n$ with Υ_n, it follows that the 5 lemmas and the 4 corollaries of subsection 3.1 hold also for $\mathbb{T}'$. They imply new 5 lemmas and 4 corollaries for $\mathbb{T}$. For example we have

Corollary 9. *For each r, $1 \leq r \leq n$, the point $-2e_r$ is covered by a codeword $X \in \mathbb{T}$, where either $X=-4e_r$ or $X=-3e_r-2e_s$ for some $s \neq r$.*

In a similar way we can define 2^n different tiling of $\mathbb{Z}^n$ with Υ_n. For $A=(a_1,a_2,\ldots,a_n)$, where $a_i \in \{-1,1\}$, let $\mathbb{T}_A$ be the tiling of $\mathbb{Z}^n$ with Υ_n defined by $\mathbb{T}_A \stackrel{\text{def}}{=} \{(x_1,x_2,\ldots,x_n)\ :\ (a_1x_1,a_2x_2,\ldots,a_nx_n) \in \mathbb{T}\}$. As for $\mathbb{T}_{(-1,-1,\ldots,-1)}$, each lemma and each corollary of subsection 3.1 holds, and each one induces 2^n new claims on $\mathbb{T}$. W.l.o.g. we assume (based on Lemma 3, Corollaries 7 and 8) that $3e_{2i-1}+2e_{2i} \in \mathbb{T}$ and $4e_{2i} \in \mathbb{T}$, for all $1 \leq i \leq \frac{n}{2}$.

Lemma 16. *If $X=3e_r+2e_s \in \mathbb{T}$ then $-4e_s \in \mathbb{T}$.*

Proof. W.l.o.g. we will prove the claim for $r=1$ and $s=2$; let $A=(1,-1,1,\ldots,1)$. Since $3e_{2i-1}+2e_{2i} \in \mathbb{T}$, for all $2 \leq i \leq \frac{n}{2}$, it follows that $3e_{2i-1}+2e_{2i} \in \mathbb{T}_A$, for all $2 \leq i \leq \frac{n}{2}$, and by Corollary 8 we have that either $3e_1+2e_2 \in \mathbb{T}_A$ or $2e_1+3e_2 \in \mathbb{T}_A$. If $2e_1+3e_2 \in \mathbb{T}_A$ then Corollary 7 implies that $Y=4e_1 \in \mathbb{T}_A$. Therefore, $Y=4e_1 \in \mathbb{T}$, and since $d_C(X,Y)=1$ we have a contradiction. Hence, $3e_1+2e_2 \in \mathbb{T}_A$, and therefore, by Corollary 7 we have that $4e_2 \in \mathbb{T}_A$, i.e. $-4e_2 \in \mathbb{T}$. $\square$

Corollary 10. *$4e_s \in \mathbb{T}$ if and only if $-4e_s \in \mathbb{T}$, $1 \leq s \leq n$.*

Lemma 17. *If $X=3e_r+2e_s \in \mathbb{T}$ then $-3e_r-2e_s \in \mathbb{T}$.*

Proof. W.l.o.g. we will prove the claim for $r=1$ and $s=2$; let $A=(-1,-1,1,\ldots,1)$. Since $3e_{2i-1}+2e_{2i} \in \mathbb{T}$, for all $2 \leq i \leq \frac{n}{2}$, it follows that $3e_{2i-1}+2e_{2i} \in \mathbb{T}_A$, for all $2 \leq i \leq \frac{n}{2}$, and by Corollary 8 we have that either $3e_1+2e_2 \in \mathbb{T}_A$ or $2e_1+3e_2 \in \mathbb{T}_A$. If $2e_1+3e_2 \in \mathbb{T}_A$ then Lemma 16 implies that $-4e_1 \in \mathbb{T}_A$. Therefore, $Y=4e_1 \in \mathbb{T}$, and since $d_C(X,Y)=1$ we have a contradiction. Hence, $3e_1+2e_2 \in \mathbb{T}_A$, and therefore we have that $-3e_1-2e_2 \in \mathbb{T}$. $\square$

Corollary 11. *$3e_r+2e_s \in \mathbb{T}$ if and only if $-3e_r-2e_s \in \mathbb{T}$.*

Lemma 18. *If $3e_r+2e_s \in \mathbb{T}$ then $12e_r, 12e_s \in \mathbb{T}$.*

Proof. By Corollary 7 we have that $4e_s \in \mathbb{T}$. $\mathbb{T}_1=-4e_s+\mathbb{T}$ is a tiling with Υ_n for which $\mathbf{0}, -4e_s \in \mathbb{T}_1$. It follows by Corollary 10 that $4e_s \in \mathbb{T}_1$ and hence $8e_s \in \mathbb{T}$. Similarly, $12e_s \in \mathbb{T}$.

Similarly, by Corollary 11 we have that $\mathbf{0}, 3e_r+2e_s \in \mathbb{T}$ implies that $6e_r+4e_s, 9e_r+6e_s, 12e_r+8e_s \in \mathbb{T}$. $\mathbb{T}_1=-12e_r-8e_s+\mathbb{T}$ is a tiling with Υ_n for which $\mathbf{0}, -3e_r-2e_s \in \mathbb{T}_1$. By Corollary 11 and Lemma 16 we have that $-4e_s \in \mathbb{T}_1$, and hence $12e_r+4e_s \in \mathbb{T}$. Similarly, by Corollary 10 we have $12e_r+4e_s, 12e_r+8e_s \in \mathbb{T}$ implies that $12e_r \in \mathbb{T}$. $\square$

Corollary 12. *If n is even and $\mathbb{T}$ is an integer tiling with Υ_n, then $\mathbb{T}$ is a periodic tiling with period 12.*

Theorem 5. *If $\mathbb{T}$ is an integer tiling with Υ_n, where n is an even integer, then $n = 3^t - 1$ for some $t > 0$.*

Proof. By Corollary 12 we have that $\mathbb{T}$ is a periodic tiling with period 12. Therefore, the size of Υ_n divides 12^n. $|\Upsilon_n| = 2^n(n+1)$ and hence $n+1$ divides $2^n 3^n$. Since n is even it follows that $n+1$ is odd and thus $n = 3^t - 1$ for some $t > 0$. $\square$

Theorems 2 and 5 are combined to obtain

Corollary 13. *If $\mathbb{T}$ is an integer tiling with Υ_n, then either $n = 2^t - 1$ or $n = 3^t - 1$, $t > 0$.*

Corollary 14. *If $\mathbb{T}$ is a $\frac{1}{2}\mathbb{Z}$-tiling with a $(0.5, n)$-cross, then either $n = 2^t - 1$ or $n = 3^t - 1$, $t > 0$.*

4 Tilings based on Perfect Codes

In Section 3 we proved that a $\frac{1}{2}\mathbb{Z}$-tiling with $(0.5, n)$-cross exists only if $n = 2^t - 1$ or $n = 3^t - 1$, $t > 0$. In this section we will prove that this necessary condition is also sufficient. Surprisingly, two constructions which produce the related tilings are based on perfect codes in the Hamming scheme. If $n = 2^t - 1$ then the code is binary of length n and the construction of the tiling is very simple. If $n = 3^t - 1$ then the perfect code is ternary of length $\frac{n}{2}$.

We will refer only to perfect codes with minimum Hamming distance three. A code $\mathcal{C}$ has *minimum Hamming distance* d if for any two distinct codewords $X, Y \in \mathcal{C}$ we have $d_H(X, Y) \geq d$. The minimum Hamming distance of $\mathcal{C}$ will be denoted by $d_H(\mathcal{C})$. Similarly, we define the *minimum cross distance* of a code. A code $\mathcal{C}$ of length n over $\mathbb{Z}_q$, with minimum Hamming distance 3, is called *perfect* if for each word $A \in \mathbb{Z}_q^n$ there exists a codeword $X \in \mathcal{C}$ such that $d_H(A, X) \leq 1$. Such a code is also called a single-error-correcting perfect code since it is capable to correct a single transmission error [12]. The *sphere* of radius ρ centered at $A = (a_1, a_2, \ldots, a_n)$ is the set $\{B \in \mathbb{Z}_q^n \ : \ d_H(A, B) \leq \rho\}$. $\mathcal{C}$ is a single-error-correcting perfect code if and only if $\mathcal{C}$ is a tiling of $\mathbb{Z}_q^n$ with a sphere of radius one. Binary $(q = 2)$ perfect codes exists if and only if $n = 2^t - 1$, $t > 0$. Ternary $(q = 3)$ perfect codes exists if and only if $n = \frac{3^t - 1}{2}$, $t > 0$. These are the only perfect codes which are of interest in this section. Finally, we note that a perfect code is identified by its size, its minimum distance, and the fact that each element of $\mathbb{Z}_q^n$ is covered by at least one codeword. One can easily verify that given any two of these parameters one can determine whether the code is perfect or not perfect. This fact will be used throughout this section.

Remark 3. *A perfect code $\mathcal{C}$ of length n over $\mathbb{Z}_q$ is known to exist if q is a power of a prime and $n = \frac{q^t - 1}{q - 1}$, $t > 0$. The related sphere of radius one can be viewed as a $(q-1, n)$-semicross or as a $(\frac{q-1}{2}, n)$-cross. Thus, these perfect codes form tilings with the related semicrosses and crosses. Only if q is a prime some of the known tilings are lattice tilings (they are related to linear perfect codes). If q is not a prime then the tiling of $\mathbb{Z}^n$ is done first by using any one-to-one mapping between GF(q) (on which the codes are defined) and $\mathbb{Z}_q$. Tilings of this type have applications in flash memories [13]. If $q = 2$ then $\mathcal{C}$ is a tiling of $\mathbb{Z}_2^n$ with $(0.5, n)$-cross and $E(\mathcal{C})$ forms a tiling of $\mathbb{Z}^n$ with $(0.5, n)$-cross.*

4.1 Binary Perfect Codes

Since the size of of a sphere with radius one in $\mathbb{Z}_2^n$ is $n+1$, it follows that a binary perfect code of length $n = 2^t - 1$ has 2^{n-t} codewords.

Theorem 6. *There exists an one-to-one correspondence between the set of binary perfect codes of length $n = 2^t - 1$ and the set of integer tilings with Υ_n in which each codeword has only even entries.*

Proof. Note first, that by Corollary 6 a tiling $\mathbb{T}$ of $\mathbb{Z}^n$ with Υ_n is periodic with period 4 and hence it can be reduced to a tiling of $\mathbb{Z}_4^n$ with Υ_n.

The size of an $(1,n)$-semicross is equal the size of a $(0.5,n)$-cross. It implies that the number of codewords in a binary single-error-correcting perfect code $\mathcal{C}$ of length $n = 2^t - 1$ is equal the number of codewords in a tiling $\mathbb{T}$ of $\mathbb{Z}_4^n$ with Υ_n. If $X, Y \in \{0,2\}^n$ then $0.5X$ and $0.5Y$ are binary words and it is easy to verify that $d_C(X,Y) = d_H(0.5X, 0.5Y)$.

Therefore, if $\mathcal{C}$ is a binary perfect code of length $n = 2^t - 1$ then $2E(\mathcal{C})$ is a tiling of $\mathbb{Z}^n$ with Υ_n in which each codeword has only even entries. Similarly, if $\mathbb{T}$ is a tiling of $\mathbb{Z}^n$ with Υ_n, in which each codeword has only even entries, then $0.5\mathbb{T} \cap \{0,1\}^n$ is a binary perfect code. □

Corollary 15. *There exists an one-to-one correspondence between the set of binary perfect codes of length $n = 2^t - 1$ and the set of integer tilings with $(0.5,n)$-cross.*

Do there exists any integer tilings with Υ_n, $n = 2^t - 1$, except for those implied by Theorem 6? The answer is that there exist many such tilings. Let $\mathcal{C}$ be a binary code of length n. Its *punctured* code $\mathcal{C}'$ of length $n-1$ is defined by $\mathcal{C}' \overset{\text{def}}{=} \{c \ : \ (c,x) \in \mathcal{C}, \ x \in \{0,1\}\}$.

Construction 1. *Let $\mathcal{C}$ be a binary perfect code of length n and $\mathcal{C}'$ its punctured code. Let $\mathcal{C}'_e$ and $\mathcal{C}'_o$ be the set of codewords from $\mathcal{C}'$ with even weight and odd weight, respectively. We define a code $\mathbb{C}^* \overset{\text{def}}{=} \mathbb{C}_1^* \cup \mathbb{C}_2^*$ over $\mathbb{Z}_4^n$, where*

$$\mathbb{C}_1^* \overset{\text{def}}{=} \{(2c, 2x) \ : \ c \in \mathcal{C}'_e, \ (c,x) \in \mathcal{C}\} \text{ and } \mathbb{C}_2^* \overset{\text{def}}{=} \{(2c, 2x+1) \ : \ c \in \mathcal{C}'_o, \ (c,x) \in \mathcal{C}\} \ .$$

Theorem 7. *$E(\mathbb{C}^*)$ defines a tiling of $\mathbb{Z}^n$ with Υ_n, in which not all entries are even.*

Proof. Since $d_H(\mathcal{C}) = 3$ it follows that $d_H(\mathcal{C}') = d_H(\mathcal{C}'_e) = d_H(\mathcal{C}'_o) = 2$ and $d_C(\mathbb{C}_1^*) = d_C(\mathbb{C}_2^*) = 3$. If $\tilde{c}_1 \in \mathcal{C}'_e$ and $\tilde{c}_2 \in \mathcal{C}'_o$ then $d_H(\tilde{c}_1, \tilde{c}_2)$ is an odd integer. Hence, since $d_H(\mathcal{C}') = 2$, it follows that $d_H(\tilde{c}_1, \tilde{c}_2) \geq 3$. Therefore, if $\tilde{c}_1^* \in \mathbb{C}_1$ and $\tilde{c}_2^* \in \mathbb{C}_2$ then $d_C(\tilde{c}_1^*, \tilde{c}_2^*) \geq 3$ and thus $d_C(\mathbb{C}^*) \geq 3$. The minimum distance of the code $\mathbb{C}^*$ and its number of codewords implies that $\mathbb{C}^*$ is a tiling of $\mathbb{Z}_4^n$ with Υ_n. It is easy to verify that $\mathcal{C}'_o$ has at least one codeword (in fact it can be proved that it contains exactly half of the codewords) and hence the last entry in at least one of the codewords of $\mathbb{C}^*$ is 1 or 3. □

Example 1. *The following code forms a tiling of $\mathbb{Z}_4^7$ with Υ_7:*

0000000	0000222	2222000	2222222
2200201	2200023	0022201	0022023
2020021	2020203	0202021	0202203
2002002	2002220	0220002	0220220

Remark 4. *Let ξ be a mapping from $\mathbb{Z}_4$ to $\mathbb{Z}_2$ defined by $\xi(0) = \xi(1) = 0$, $\xi(2) = \xi(3) = 1$. If $\mathbb{C}$ forms a tiling of $\mathbb{Z}_4^n$ with Υ_n then the code $\mathcal{C} = \{\xi(X) \ : \ X \in \mathbb{C}\}$, where $\xi(x_1, x_2, \ldots, x_n) = (\xi(x_1), \xi(x_2), \ldots, \xi(x_n))$ is a binary perfect code of length n.*

Remark 5. *By Corollary 6 an integer tiling with Υ_n, n odd, has period 4. Hence, the related $\frac{1}{2}\mathbb{Z}$-tiling $\mathbb{T}$ with $(0.5, n)$-cross has period 2. It implies that this tiling is also a tiling with the $(1, n)$-semicross (even if $\mathbb{T}$ is not a $\mathbb{Z}$-tiling).*

4.2 Ternary Perfect Codes

Let $n = 3^t - 1$, $t > 0$, and $\nu = \frac{n}{2}$. Since the size of of a sphere with radius one in $\mathbb{Z}_3^\nu$ is $2\nu + 1$, it follows that a ternary perfect code of length ν has $3^{\nu - t}$ codewords. Let Λ_n be the lattice generated by the basis $\{3e_{2i-1} + 2e_{2i} \ : \ 1 \leq i \leq \nu\} \cup \{4e_{2i} \ : \ 1 \leq i \leq \nu\}$. Let G_n be the quotient group $\mathbb{Z}^n / \Lambda_n$. The following lemma can be readily verified.

Lemma 19. *$|G_2| = 12$ and the 12 representatives of elements from G_2 (the cosets of Λ_2 in $\mathbb{Z}^2$) can be taken as $\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4$.*

Let $(\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4)^m \stackrel{\text{def}}{=} \underbrace{(\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4) \times (\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4) \times \ \cdots \ \times (\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4)}_{m \text{ times}}$.

Corollary 16. *$|G_n| = 12^\nu$ and the 12^ν representatives of elements from G_n (the cosets of Λ_n in $\mathbb{Z}^n$) can be taken as the elements of $(\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4)^\nu$.*

Consider the mapping $\Phi : \mathbb{Z}_3^\nu \to G_n$ defined by

$$\Phi(x_1, x_2, ..., x_\nu) = (\phi(x_1), \phi(x_2), ..., \phi(x_\nu)) \ ,$$

where $\phi : \mathbb{Z}_3 \to G_2$ is a mapping defined by

$$\phi(x) = \begin{cases} (0,0) & \text{if } x = 0 \\ (1,2) & \text{if } x = 1 \\ (2,0) & \text{if } x = 2 \end{cases} .$$

It is easy to verify that both ϕ and Φ are group homomorphisms.

Let $\mathcal{C}$ be a ternary perfect code of length ν with $3^{\nu - t}$ codewords, and let $\Phi(\mathcal{C}) \stackrel{\text{def}}{=} \{\Phi(\tilde{c}) : \tilde{c} \in \mathcal{C}\}$. Since the elements of $\Phi(\mathcal{C})$ are representatives of elements of G_n (see Corollary 16) it follows that the elements of $\Phi(\mathcal{C})$ can be considered as elements in $\mathbb{Z}^n$. Let $\mathbb{T}_n \stackrel{\text{def}}{=} \Phi(\mathcal{C}) + \Lambda_n$.

Theorem 8. *$\mathbb{T}_n$ is a tiling of $\mathbb{Z}^n$ with Υ_n.*

Proof. Clearly, Λ_n is a lattice with period 12 and hence $\mathbb{T}_n$ is a periodic code of $\mathbb{Z}^n$ with period 12. Therefore, w.l.o.g. we can restrict our discussion to $\tilde{\mathbb{Z}}_{12}^n$. i.e. codewords of $\mathbb{T}_n \cap \tilde{\mathbb{Z}}_{12}^n$. Since $|\Upsilon_n| = 2^{2\nu} 3^t$ it follows that the size of the tiling $\mathbb{T}_n$ in $\tilde{\mathbb{Z}}_{12}^n$, $|\mathbb{T}_n \cap \tilde{\mathbb{Z}}_{12}^n|$, should be $2^{2\nu} 3^{2\nu - t}$. To prove that $\mathbb{T}_n$ is a tiling of $\mathbb{Z}^n$ with Υ_n we will show that the size of $\mathbb{T}_n \cap \tilde{\mathbb{Z}}_{12}^n$ is $2^{2\nu} 3^{2\nu - t}$ and we will prove that each point of $\mathbb{Z}^n$ is covered by an element of $\mathbb{T}_n$.

Claim: For any two codewords $\tilde{c}_1, \tilde{c}_2 \in \mathcal{C}$, and two lattice points $Y_1, Y_2 \in \Lambda_n$, we have $\Phi(\tilde{c}_1) + Y_1 \neq \Phi(\tilde{c}_2) + Y_2$, unless $\tilde{c}_1 = \tilde{c}_2$ and $Y_1 = Y_2$.
Proof: Assume that $\Phi(\tilde{c}_1) + Y_1 = \Phi(\tilde{c}_2) + Y_2$, i.e. $\Phi(\tilde{c}_1) - \Phi(\tilde{c}_2) = Y_2 - Y_1$, $\tilde{c}_1, \tilde{c}_2 \in \mathcal{C}$ and $Y_1, Y_2 \in \Lambda_n$. $Y_2 - Y_1 = (\alpha_1, \ldots, \alpha_n)$ is a lattice point and unless $Y_1 = Y_2$ we have that for at least one i, $|\alpha_i| > 2$. $\Phi(\tilde{c}_1) - \Phi(\tilde{c}_2) = (\beta_1, \ldots, \beta_n)$ and by definition of Φ, for each i, $1 \leq i \leq n$, we have $|\beta_i| \leq 2$. Therefore, $Y_1 = Y_2$ and $\Phi(\tilde{c}_1) = \Phi(\tilde{c}_2)$ and since Φ is an injective mapping it implies that $\tilde{c}_1 = \tilde{c}_2$ and the claim is proved.

The claim implies that $|\mathbb{T}_n \cap \tilde{\mathbb{Z}}_{12}^n| = |\Phi(\mathcal{C})| \cdot |\Lambda_n \cap \tilde{\mathbb{Z}}_{12}^n|$. Since Φ is an injective mapping we also have that $|\Phi(\mathcal{C})| = |\mathcal{C}|$. Since Λ_n has period 12 and $V(\Lambda_n) = 12^\nu$ it follows that $|\Lambda_n \cap \tilde{\mathbb{Z}}_{12}^n| = 12^\nu$. Therefore,

$$|\mathbb{T}_n \cap \tilde{\mathbb{Z}}_{12}^n| = |\Phi(\mathcal{C})| \cdot |\Lambda_n \cap \tilde{\mathbb{Z}}_{12}^n| = |\mathcal{C}| \cdot |\Lambda_n \cap \tilde{\mathbb{Z}}_{12}^n| = 3^{\nu-t}12^\nu = 2^{2\nu}3^{2\nu-t}$$

as required.

To show that every point of $\mathbb{Z}^n$ is covered by an element of $\mathbb{T}_n$ we first partition the elements of $\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4$ into three classes:

$$\begin{array}{l} [(0,0)] = \{(0,0),(0,3),(2,2),(2,1)\} \\ [(1,2)] = \{(1,2),(1,1),(0,1),(0,2)\} \\ [(2,0)] = \{(2,0),(1,3),(2,3),(1,0)\} \end{array} ,$$

The following two properties are readily verified (as can be verified from the following table):

(P.1) For each element (x_1, x_2) in a class $[(y_1, y_2)]$ there exists an element $(u_1, u_2) \in \Lambda_2$ such that $u_i + y_i \in \{x_i, x_i + 1\}$, $i \in \{1, 2\}$.

(P.2) For each element $(x_1, x_2) \in \tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4$ and each class $[(y_1, y_2)]$ there exists an element $(u_1, u_2) \in \Lambda_2$ such that $u_i + y_i \in \{x_i - 1, x_i, x_i + 1, x_i + 2\}$, $i \in \{1, 2\}$, and for at most one i we have $u_i + y_i \in \{x_i - 1, x_i + 2\}$.

	class $[(0,0)]$	$(0,0)$	$(0,3)$	$(2,2)$	$(2,1)$	$\longleftarrow (x_1,x_2)$
$(\mathbf{P.1})$	$[(y_1,y_2)] = [(0,0)]$	$(0,0)$	$(0,4)$	$(3,2)$	$(3,2)$	$\longleftarrow (u_1,u_2)+(y_1,y_2)$
$(\mathbf{P.2})$	$[(y_1,y_2)] = [(1,2)]$	$(1,2)$	$(1,2)$	$(1,2)$	$(1,2)$	$\longleftarrow (u_1,u_2)+(y_1,y_2)$
$(\mathbf{P.2})$	$[(y_1,y_2)] = [(2,0)]$	$(2,0)$	$(2,4)$	$(2,4)$	$(2,0)$	$\longleftarrow (u_1,u_2)+(y_1,y_2)$
	class $[(1,2)]$	$(1,2)$	$(1,1)$	$(0,1)$	$(0,2)$	$\longleftarrow (x_1,x_2)$
$(\mathbf{P.2})$	$[(y_1,y_2)] = [(0,0)]$	$(3,2)$	$(3,2)$	$(0,0)$	$(0,4)$	$\longleftarrow (u_1,u_2)+(y_1,y_2)$
$(\mathbf{P.1})$	$[(y_1,y_2)] = [(1,2)]$	$(1,2)$	$(1,2)$	$(1,2)$	$(1,2)$	$\longleftarrow (u_1,u_2)+(y_1,y_2)$
$(\mathbf{P.2})$	$[(y_1,y_2)] = [(2,0)]$	$(2,4)$	$(2,0)$	$(-1,2)$	$(-1,2)$	$\longleftarrow (u_1,u_2)+(y_1,y_2)$
	class $[(2,0)]$	$(2,0)$	$(1,3)$	$(2,3)$	$(1,0)$	$\longleftarrow (x_1,x_2)$
$(\mathbf{P.2})$	$[(y_1,y_2)] = [(0,0)]$	$(3,2)$	$(0,4)$	$(3,2)$	$(0,0)$	$\longleftarrow (u_1,u_2)+(y_1,y_2)$
$(\mathbf{P.2})$	$[(y_1,y_2)] = [(1,2)]$	$(4,0)$	$(1,2)$	$(4,4)$	$(1,2)$	$\longleftarrow (u_1,u_2)+(y_1,y_2)$
$(\mathbf{P.1})$	$[(y_1,y_2)] = [(2,0)]$	$(2,0)$	$(2,4)$	$(2,4)$	$(2,0)$	$\longleftarrow (u_1,u_2)+(y_1,y_2)$

Consider the mapping $\Psi : (\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4)^\nu \to \mathbb{Z}_3^\nu$ defined by

$$\Psi(x_1, x_2, ..., x_n) = (\psi(x_1, x_2), \psi(x_3, x_4), ..., \psi(x_{n-1}, x_n)) ,$$

where $\psi : \tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4 \to \mathbb{Z}_3$ is a mapping defined by

$$\psi(x_1, x_2) = \begin{cases} 0 & \text{if } (x_1, x_2) \in [(0,0)] \\ 1 & \text{if } (x_1, x_2) \in [(1,2)] \\ 2 & \text{if } (x_1, x_2) \in [(2,0)] \end{cases} .$$

For a given point $A = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ we will exhibit a point $X \in \mathbb{T}_n$ which covers A. By Corollary 16 we have that there exists an element $Y \in \Lambda_n$ such that $A + Y \in (\tilde{\mathbb{Z}}_3 \times \tilde{\mathbb{Z}}_4)^\nu$. Let $B = A + Y = (b_1, b_2, \ldots, b_n)$ and let $\Psi(B) = (\alpha_1, \alpha_2, \ldots, \alpha_\nu) \in \mathbb{Z}_3^\nu$. Since $\mathcal{C}$ is a perfect code of length ν over $\mathbb{Z}_3$ it follows that there exists a codeword $(c_1, c_2, \ldots, c_\nu) \in \mathcal{C}$ such that $d_H((\alpha_1, \alpha_2, \ldots, \alpha_\nu), (c_1, c_2, \ldots, c_\nu)) \leq 1$. Let $(\gamma_1, \gamma_2, \ldots, \gamma_n) = \Phi(c_1, c_2, \ldots, c_\nu)$. Note that by the definitions of Φ and Ψ it follows that (b_{2i-1}, b_{2i}) and $\phi(\alpha_i)$ are in the same class, for all $1 \leq i \leq \nu$. Now, we distinguish between two cases:

Case 1: If $(\alpha_1, \alpha_2, \ldots, \alpha_\nu) = (c_1, c_2, \ldots, c_\nu)$ then by property **(P.1)** there exists an element $(u_1, u_2, \ldots u_n) \in \Lambda_n$ such that $u_i + \gamma_i \in \{b_i, b_i+1\}$, $1 \leq i \leq n$. Therefore, by Lemma 9 we have that $(u_1, u_2, \ldots, u_n) + (\gamma_1, \gamma_2, \ldots, \gamma_n)$ covers B and hence the required X is $(u_1, u_2, \ldots u_n) + (\gamma_1, \gamma_2, \ldots, \gamma_n) - Y$.

Case 2: If $(\alpha_1, \alpha_2, \ldots, \alpha_\nu) \neq (c_1, c_2, \ldots, c_\nu)$ then $d_H((\alpha_1, \alpha_2, \ldots, \alpha_\nu), (c_1, c_2, \ldots, c_\nu)) = 1$ and hence there exist exactly one coordinate s such that $\alpha_s \neq c_s$. By properties **(P.1)** and **(P.2)** there exist an element $(u_1, u_2, \ldots, u_n) \in \Lambda_n$ such that $u_i + \gamma_i \in \{b_i - 1, b_i, b_i + 1, b_i + 2\}$, $1 \leq i \leq n$, and for at most one i we have $u_i + \beta_i \in \{b_i - 1, b_i + 2\}$. Therefore, by Lemma 9 we have that $(u_1, u_2, \ldots, u_n) + (\gamma_1, \gamma_2, \ldots, \gamma_n)$ covers B and hence the required X is $(u_1, u_2, \ldots, u_n) + (\gamma_1, \gamma_2, \ldots, \gamma_n) - Y$.

Since we proved that the size of $\mathbb{T}_n \cap \tilde{\mathbb{Z}}_{12}^n$ is $2^{2\nu} 3^{2\nu - t}$ and each point of $\mathbb{Z}^n$ is covered by an element of $\mathbb{T}_n$, the theorem is proved. $\square$

Theorem 9. *If $\mathcal{C}$ is a linear code then $\mathbb{T}_n$ is a lattice tiling.*

Proof. Follows immediately from Theorem 8 and the facts that $\mathcal{C}$ is a linear code and Φ is a group homomorphism. $\square$

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