The 1/3-2/3 conjecture for N-free ordered sets

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Abstract

A balanced pair in a finite ordered set $P = (V, \leq)$ is a pair (x, y) of elements of V such that the proportion of linear extensions of P that put x before y is in the real interval $[1/3, 2/3]$.
We prove that every finite N-free ordered set which is not totally ordered has a balanced pair.
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1 Introduction

Throughout, $P = (V, \leq)$ denotes a *finite ordered set*, that is a finite set V and a binary relation $\leq$ on V which is reflexive, antisymmetric and transitive. A *linear extension* of $P = (V, \leq)$ is a linear ordering $\preceq$ of V which extends $\leq$, i.e. such that $x \preceq y$ whenever $x \leq y$.

For a pair (x, y) of elements of V we denote by $\mathbb{P}(x \prec y)$ the proportion of linear extensions of P that put x before y. Call a pair (x, y) of elements of V a *balanced pair* in $P = (V, \leq)$ if $\mathbb{P}(x \prec y)$ is in the real interval $[1/3, 2/3]$. The 1/3-2/3 Conjecture states that every finite ordered set which is not totally ordered has a balanced pair. If true, then the ordered set consisting of the disjoint sum of a two element chain and a one element chain would show that the result is best possible. The 1/3-2/3 Conjecture first appeared in a paper of Kislitsyn [6]. It was also formulated independently by Fredman in about 1975 and again by Linial [7].

The 1/3-2/3 Conjecture is known to be true for ordered sets with a nontrivial automorphism [5], for ordered sets of width two [7], for semiorders [2], for bipartite ordered sets [10], for 5-thin posets [4], and for 6-thin posets [8]. See [3] for a survey.

The purpose of this paper is to prove that the 1/3-2/3 Conjecture is true for N-free ordered sets.

Let $P = (V, \leq)$ be an ordered set. For $x, y \in V$ we say that x and y are *comparable* if $x \leq y$ or $y \leq x$; otherwise we say that x and y are *incomparable*. A set of pairwise incomparable elements is called an *antichain*. A *chain* is a totally ordered set.
Define $D(x) := \{y \in V : y < x\}$ and $U(x) := \{y \in V : x < y\}$.

A pair (a, b) of elements of V is said to be *chain dominated* if the following holds in P:

$$D(a) \subseteq D(b) \text{ and } \{b\} \cup U(b) \setminus U(a) \text{ is a chain.}$$

A pair (a, b) is said to be *good* if it is chain dominated and $\mathbb{P}(a \prec b) \leq \frac{1}{2}$. Notice that a good pair is necessarily a pair of incomparable elements. Our first result is this.

Theorem 1. *If one of P and its dual has a good pair, then P has a balanced pair.*

The proof of Theorem 1 is similar to the proof of Theorem 2 of [7] stating that the 1/3-2/3 Conjecture is true for finite ordered sets of width two (these being the ordered sets covered by two chains). We will prove that a finite N-free ordered set which is not a chain has a good pair.

For $x, y \in V$ we say that y is an *upper cover* of x or that x is *a lower cover* of y if $x < y$ and there is no element $z \in V$ such that $x < z < y$. A subset $\{a, b, c, d\}$ of V is an N in P if b is an upper cover of a and c, d is an upper cover of c and if these are the only comparabilities between the elements a, b, c, d. The ordered set P is *N-free* if it does not contain an N. Notice that every finite ordered set can be embedded into a *finite* N-free ordered set (see for example [9]). It was proved in [1] that the number of (unlabeled) N-free ordered sets is

$$2^{n \log_2(n) + o(n \log_2(n))}.$$

Our second result is this.

Theorem 2. *Let P be a finite N-free ordered set, then P has a good pair. Hence, P satisfies the $1/3 - 2/3$ Conjecture.*

2 Proof of Theorem 1

We recall that an incomparable pair (x, y) of elements *critical* if $U(y) \subseteq U(x)$ and $D(x) \subseteq D(y)$.

Lemma 1. *Suppose (x, y) is a critical pair in P and consider any linear extension of P in which $y < x$. Then the linear order obtained by swapping the positions of y and x is also a linear extension of P. Hence, $\mathbb{P}(x \prec y) \geq \frac{1}{2}$.*

Proof. Let L be a linear extension that puts y before x and let z such that $y \prec z \prec x$ in L. Then z is incomparable with both x and y since (x, y) is a critical pair of P. Therefore, the linear order L' obtained by swapping x and y, that is L' puts x before y, is a linear extension of P. Then map $L \mapsto L'$ from the set of linear extensions that put y before x into the set of linear extensions that put x before y is clearly one-to-one. Hence, $\mathbb{P}(y \prec x) \leq \mathbb{P}(x \prec y)$ and therefore $\mathbb{P}(x \prec y) \geq \frac{1}{2}$. □

We now turn to the proof of Theorem 1.

Proof. Let (a, b) be a good pair in P. If $U(b) \setminus U(a) = \emptyset$, then $U(b) \subseteq U(a)$ and hence (a, b) is a critical pair. Therefore $\mathbb{P}(a \prec b) \geq \frac{1}{2}$ (see Lemma 1). Since $\mathbb{P}(a \prec b) \leq \frac{1}{2}$ we infer that $\mathbb{P}(a \prec b) = \frac{1}{2}$ and we are done. So we may assume without loss of generality that $U(b) \setminus U(a) \neq \emptyset$. Hence, $U(b) \setminus U(a)$ is a chain, say $\{b\} \cup U(b) \setminus U(a)$ is the chain $b = b_1 < \cdots < b_n$. We prove the theorem by contradiction. Then

$$\mathbb{P}(a \prec b_1) < \frac{1}{3}.$$

Define now the following quantities

$$\begin{aligned} q_1 &= \mathbb{P}(a \prec b_1), \\ q_j &= \mathbb{P}(b_{j-1} \prec a \prec b_j)(2 \leq j \leq n), \\ q_{n+1} &= \mathbb{P}(b_n \prec a). \end{aligned}$$

The following lemma appeared in [7]. We now adapt its proof to our situation.

Lemma. The real numbers q_j $(1 \leq j \leq n+1)$ satisfy:

(i) $0 \leq q_{n+1} \leq \cdots \leq q_1 \leq \frac{1}{3}$,

(ii) $\sum_{j=1}^{n+1} q_j = 1$.

Proof. Since $q_1, \cdots, q_{n+1}$ is a probability distribution, all we have to show is that $q_{n+1} \leq \cdots \leq q_1$. To show this we exhibit a one-to-one mapping from the event whose probability is q_{j+1} into the event with probability q_j $(1 \leq j \leq n)$. Notice that in a linear extension for which $b_j \prec a \prec b_{j+1}$ every element z between b_j and a is incomparable to both b_j and a. Indeed, such an element z cannot be comparable to b_j because otherwise $b_j < z$ in P but the only element above b_j is b_{j+1} which is above a in the linear extension. Now z cannot be comparable to a as well because otherwise $z < a$ in P and hence $z < b = b_1 < b_j$ (by assumption we have that $D(a) \subseteq D(b)$). The mapping from those linear extensions in which $b_j \prec a \prec b_{j+1}$ to those in which $b_{j-1} \prec a \prec b_j$ is obtained by swapping the positions of a and b_j. This mapping clearly is well defined and one-to-one.

Theorem 1 can be proved now: let r be defined by

$$\sum_{j=1}^{r-1} q_j \leq \frac{1}{2} < \sum_{j=1}^{r} q_j$$

Since $\sum_{j=1}^{r-1} q_j = \mathbb{P}(a \prec b_{r-1}) \leq \frac{1}{2}$, it follows that $\sum_{j=1}^{r-1} q_j < \frac{1}{3}$. Similarly $\sum_{j=1}^{r} q_j = \mathbb{P}(a \prec b_r)$ must be $> \frac{2}{3}$. Therefore $q_r > \frac{1}{3}$, but this contradicts $\frac{1}{3} > q_1 \geq q_r$. □

3 Proof of Theorem 2

Let $P = (V, \leq)$ be a finite ordered set which is not a chain. If P has a minimum element p_0, then p_0 will be the minimum element in every linear extension of the poset. Therefore, nothing will change if p_0 is deleted from the ordered set. So we may assume without loss of generality that P has at least two distinct minimal elements a and b.

Next we suppose that P is N-free. We start by stating some useful properties of N-free ordered sets.

Lemma 2. *Let $P = (V, \leq)$ be an N-free ordered set. If $x, y \in V$ have a common upper cover, then x and y have the same upper covers. Dually, if $x, y \in V$ have a common lower cover, then x and y have the same lower covers.*

Proof. Trivial. □

Let $P = (V, \leq)$ be an ordered set. An element $m \in V$ is called *minimal* if for all $x \in V$ comparable to m we have $x \geq m$. We denote by $Min(P)$ the set of all minimal elements of P. We recall that the decomposition of P into *levels* is the sequence $P_0, \cdots, P_l, \cdots$ defined by induction by the formula

$$P_l := Min(P - \cup\{P_{l'} : l' < l\}).$$

In particular, $P_0 = Min(P)$.

Lemma 3. *Let $P = (V, \leq)$ be an N-free ordered set and let $P_0, \cdots, P_h$ be the sequence of its levels. Then for every $x \in V$, there exists $i \leq h$ such that all upper covers of x are in P_i.*

Proof. If x has at most one upper cover, then the conclusion of the lemma holds. So we may assume that x has at least two distinct upper covers x_1 and x_2 belonging to two distinct levels. Let $j < k$ such that $x_1 \in P_j$ and $x_2 \in P_k$. Then x_2 has a lower cover $x_3 \in P_{k-1}$. We claim that $\{x, x_1, x_2, x_3\}$ is an N in P contradicting our assumption that P is N-free. Indeed, since x_1 and x_2 are upper covers of x we infer that they must be incomparable. Moreover, x_1 and x_3 are incomparable because otherwise $x_1 < x_3 < x_2$ (notice that $x_3 < x_1$ is not possible since $j \leq k - 1$) which contradicts our assumption that x_2 is an upper cover of x. Similarly we have that x and x_3 are incomparable proving our claim. The proof of the lemma is now complete. □

Corollary 1. *Let P be an N-free ordered set and let $P_0, \cdots, P_h$ be the sequence of its levels. Let $0 \leq i \leq h$ such that i is the largest with the property that P_i contains two distinct elements with the same set of lower covers. Then for every $x \in P_i$ we have that $U(x) \cup \{x\}$ is a chain. Hence, P has a good pair.*

Proof. Let $x \in P_i$ such that $U(x) \neq \emptyset$ and suppose that $U(x)$ is not a chain. There is then an element $y \in U(x) \cup \{x\}$ having at least two distinct upper covers, say y_1, y_2. From Lemma 3 we deduce that y_1 and y_2 are in the same level P_j with $i < j$. Because P is N-free it follows

from Lemma 2 that y_1 and y_2 have the same set of lower covers. This contradicts our choice of i.
Pick any two distinct elements $a, b \in P_i$. If $U(a)$ and $U(b)$ are chains, then both (a, b) and (b, a) are good in P. Otherwise, one of $U(a)$ and $U(b)$ is empty, say $U(a) = \emptyset$, in which case (a, b) is good in P. $\square$

References

[1] Bayoumi I. Bayoumi, M. El-Zahar and Soheir M. Khamis, *Asymptotic enumeration of N-free partial orders*, Order **6** (1989), 219-225.

[2] G. Brightwell, *Semiorders and the* $1/3 - 2/3$ *conjecture,* Order **5** (1989), 369-380.

[3] G. Brightwell, *Balanced pairs in Partial orders,* Discrte Mathematics **201** (1999), 25-52.

[4] G. Brightwell and C. D. Wright, *The* $1/3 - 2/3$ *conjecture for 5-thin posets,* SIAM. J. Discrte Mathematics **5** (1992), 467-474.

[5] B. Ganter, G. Hafner and W. Poguntke, *On linear extensions of ordered sets with a symmetry,* Special issue: ordered sets (Oberwolfach, 1985). Discrete Math. **63** (1987), 153-156.

[6] S. S. Kislitsyn, *Finite partially ordered sets and their associated set of permutations.* Matematicheskiye Zametki. **4** (1968), 511-518.

[7] N. Linial, *The information theoretic bound is good for merging.* SIAM J. Comput. **13** (1984), 795-801.

[8] Peczarski, Marcin, *The gold partition conjecture for 6-thin posets.* Order **25** (2008), 91-103

[9] M. Pouzet and N. Zaguia, *N-free extensions of posets. Note on a theorem of P. A. Grillet,* Contrib. Discrete Math. **1** (2006), 80-87.(electronic).

[10] W. T. Trotter, W. G. Gehrlein, P. C. Fishburn, *Balance theorems for height-2 posets.* Order **9** (1992), 43–53.