Illumination by Tangent Lines

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7/27/11

Abstract

Let f be a differentiable function on the real line, and let $P \in G_f^C =$ all points not on the graph of f. We say that the *illumination index* of P, denoted by $I_f(P)$, is k if there are k distinct tangents to the graph of f which pass through P. In section 2 we prove results about the illumination index of f with $f''(x) \geq 0$ on $\Re$. In particular, suppose that $y = L_1(x)$ and $y = L_2(x)$ are distinct oblique asymptotes of f and let $P = (s,t) \in G_f^C$. If $\max(L_1(s), L_2(s)) < t < f(s)$, then $I_f(P) = 2$. If $L_1(s) \neq L_2(s)$ and $\min(L_1(s), L_2(s)) < t \leq \max(L_1(s), L_2(s))$, then $I_f(P) = 1$.

Finally, if $t \leq \min(L_1(s), L_2(s))$, then $I_f(P) = 0$. We also show that any point below the graph of a convex rational function or exponential polynomial must have illumination index equal to 2. In section 3 we also prove results about the illumination index of polynomials.

2000 Mathematics Subject Classification: 26A06

Key Words: tangent line, oblique asymptote, illumination index

1 Introduction

Let f be a differentiable function on the real line, $\Re$, and let P be any point not on the graph of f. We say that the *illumination index* of P, denoted by $I_f(P)$, equals the non–negative integer, k, if there are k distinct tangents to the graph of f which pass through P. We allow the possibility that $k = \infty$. In [1] we proved some results about $I_f(P)$ and also about illumination by odd order Taylor Polynomials in general. In this paper we focus just on illumination by tangent lines. In particular, in section 1 we prove several theorems about $I_f(P)$ for functions with a non–negative second derivative on $\Re$. An example was given in [1] where $f''(x) \geq 0$ on $\Re$, but where there are points below the graph of f whose illumination index equals 0. In this paper we strengthen these results for functions, f, with $f''(x) \geq 0$ on $\Re$. First, we prove(Theorem 2.1) that if f has oblique asymptotes $L_1$and L_2, then the illumination index equals 2 for any point below the graph of f, but above both L_1 and L_2. For points lying between L_1 and L_2, the illumination index equals 1, while for any point below both L_1 and L_2, the illumination index equals 0. Similar results(Theorem 2.2) are proven when f has one oblique asymptote. In ([1]) we proved that if the second derivative of f is bounded below by a positive number on the entire real line, and if P is any point below the graph of f, then the illumination index of P equals 2. We strengthen this result in Theorem 2.3 by proving that if $\lim_{|x|\to\infty}(xf''(x)) \neq 0$, then the illumination index of any point below the graph of f equals 2. We also show(Propositions 2.1 and 2.2) that any point below the graph of a convex rational function or exponential polynomial must have illumination index equal to 2. Finally in section 3 we prove several results about the illumination index for polynomials.

Notation 1 $\Re$ = *real numbers. Given any function,* $y = f(x)$ *defined on* $\Re$*, we let*

$$G_f = (x, f(x)) : x \in \Re,$$

the graph of f*, and*

$$G_f^C = (x, y) : x \in \Re, y \neq f(x),$$

all points in the xy plane ***not*** *on the graph of* f*.*

$$y = T_c(x) = f(c) + f'(c)(x - c)$$

denotes the tangent line to f *at* $(c, f(c))$*.*

For $s \in \Re$*, we let*

$$I_1 = (-\infty, s), I_2 = (s, \infty).$$

Definition 1.1 *Let* f *be a differentiable function on the real line, and let* $P \in$

G_f^C. *We say that the illumination index of* P*, denoted by* $I_f(P)$*, equals the non–negative integer* k*, if there are* k *distinct tangents to the graph of* f *which pass through* P*.*

Remark 1.1 *In the definition above, one could allow for points,* $P \in G_f$*. However, we prefer to just define* $I_f(P)$ *for* $P \in G_f^C$*. Also, if there are* k *distinct tangents to the graph of* f *which pass through* P*, and if at least one of the tangent lines is tangent to the graph of* f *at more than one point, we still count the illumination index as* k*. One could, of course, define* $I_f(P)$ *so as to count the number of points at which* T *is tangent to the graph of* f *.*

Before stating and proving our main results, it is useful to define the following function: If f is a differentiable function on $\Re$ and $s \in \Re$, let

$$g_s(c) = f(c) + (s - c)f'(c) = T_c(s).$$

Then

$$T_{c_0}(s) = t \iff g_s(c_0) = t. \tag{1.1}$$

That is, the tangent line at $(c_0, f(c_0))$ passes thru $P = (s, t)$ if and only if $g_s(c_0) = t$.

Remark 1.2 *We find it convenient to use the notation* $g_s(c)$ *rather than* $T_c(s)$ *since we want to keep* s *fixed while allowing* c *to vary.*

2 Functions with Non–negative Second Derivative

In this section we prove some results about the illumination index for functions, f, with $f''(x) \geq 0$ on $\Re$. We do not assume continuity of f'', but the existence of f'' on $\Re$ implies that g_s is differentiable on $\Re$. First we need the following result about *multiple tangent lines*, which is a tangent line which is tangent to the graph of f at more than one point.

Lemma 2.1 *If $f''(x) \geq 0$ on $\Re$ and f is not linear on any subinterval of $\Re$, then f has no multiple tangent lines.*

Proof. Suppose that f has a multiple tangent line, T, which is tangent at $(a, f(a))$ and at $(b, f(b))$ for some $a \neq b$. Then $\dfrac{p(b)-p(a)}{b-a} = p'(a) = p'(b)$. Since $f''(x) \geq 0$ on $\Re$, $p'(a) \leq p'(x) \leq p'(b)$ for any $x \in [a, b]$. Thus $p'(x)$ is constant on $[a, b]$, which implies that f is linear on $[a, b]$. ■

For functions, f, with $f''(x) \geq 0$ on $\Re$, part (i) of the following lemma shows that g_s has one local extremum, a local maximum when $c = s$. Part (ii) shows that if $c_i < c_j$ are any two roots of $g_s - t$, then there are two possibilities: Either the tangents to f at $(c_i, f(c_i))$ and at $(c_j, f(c_j))$ are distinct, or f is linear on the closed interval $[c_i, c_j]$.

Lemma 2.2 *(i) If $f''(x) \geq 0$ on $\Re$, then for any given $s \in \Re, g_s(c)$ is non–decreasing on I_1 and non–increasing on I_2.*

(ii) For given $t \in \Re$, there are two possibilities for the number of solutions

of the equation $g_s(c) = t$ *in* $I_j, j = 1, 2$.

(A) $g_s(c) = t$ *has at most one solution in* I_j, *or*

(B) $g_s(c) = t$ *for all* c *in some interval,* I, *contained in* I_j. *In that case* $f(x) = mx + b$ *for all* $x \in I$, *which implies that* $T_c(x) = f(x)$ *for*

$c, x \in I$. *In addition,* $g_s(c) = f(s)$ *if* $s \in I$.

Proof. (i) Since $g_s'(c) = (s - c)f''(c)$, $g_s'(c)$ is $\begin{Bmatrix} \geq 0, c < s \\ \leq 0, c > s \end{Bmatrix}$

(ii) By part (i), $g_s(c) = t$ has at most one solution in I_j, or $g_s(c) = t$ for all c in some interval, I, contained in I_j. If $g_s(c)$ is constant on I, then $g_s'(c) = 0, c \in I$, which implies that $(s - c)f''(c) = 0, c \in I$. Thus $f''(x) = 0$ for $x \in I, x \neq s$, which implies that f is linear on I. That implies that $T_c(x) = f(x)$ for $c, x \in I$. If $s \in I$, then $g_s(c) = T_c(s) = f(s)$. ■

The following lemma was proved in [1] with the assumption that f'' is continuous, non–negative, and has *finitely many* zeros in $\Re$. We have need for a somewhat stronger version here.

Lemma 2.3 *Suppose that* $f''(x) \geq 0$ *on* $\Re$. *Then at most two distinct tangent lines to* f *can pass through any given point* P *in the plane.*

Proof. The details follow exactly as in the proof of ([1], Lemma 2) using the following facts: Suppose that T_1 and T_2 are distinct tangent lines which are tangent to f at $(c_1, f(c_1))$ and $(c_2, f(c_2))$, respectively. Then T_1 and T_2 are not parallel and if $(u, v) =$ intersection point of T_1 and T_2, then $c_1 < u < c_2$. We leave the rest of the details to the reader. ■

Definition 2.1 *A line with equation $y = L(x)$ is said to be an oblique asymptote of f if* $\lim_{x\to-\infty}(f(x) - L(x)) = 0$ *and/or* $\lim_{x\to\infty}(f(x) - L(x)) = 0$.

Lemma 2.4 *Suppose that $f''(x) \geq 0$ on $\Re$, and let $s \in \Re$.*

(i) If $\lim_{x\to\infty}(f(x) - L(x)) = 0$ for some linear function, L, then $\lim_{c\to\infty} g_s(c) = L(s)$

(ii) If $\lim_{x\to-\infty}(f(x) - L(x)) = 0$ for some linear function, L, then $\lim_{c\to-\infty} g_s(c) = L(s)$

Proof. We prove (i). Assume first that $L = 0$–that is, $\lim_{x\to\infty} f(x) = 0$. Choose any $h > 0$ and partition $[s, \infty)$ into infinitely many subintervals, $[x_{k-1}, x_k]$, of constant width h. Since f is convex,

$$\frac{f(x_k) - f(x_{k-1})}{h} \leq f'(x_k) \leq \frac{f(x_{k+1}) - f(x_k)}{h}.$$

Since $f(x_k) - f(x_{k-1}) \to 0$ and $f(x_{k+1}) - f(x_k) \to 0$ as $k \to \infty$, $f'(x_k) \to 0$ as $k \to \infty$ by the Squeeze Theorem. Since h is arbitrary, that proves that

$$\lim_{x\to\infty} f'(x) = 0. \tag{2.1}$$

Now $\frac{d}{dc}(f(c) - cf'(c)) = -cf''(c) \leq 0$ for $c > 0$, which implies that $f(c) - cf'(c)$ is non–increasing for $c > 0$. Since f is convex and $\lim_{x\to\infty} f(x) = 0$, f must be eventually positive and non–increasing, which implies that $f'(c) \leq 0$ for large c. Thus $f(c) - cf'(c)$ is eventually positive and non–increasing, which implies that $f(c) - cf'(c)$ is bounded and monotonic on $[0, \infty)$. Using the integral form of Taylor's Remainder formula, we have $f(s)-T_c(s)=\int\limits_s^c (t-s)f''(t)dt$, which implies that

$$f(s)-g_s(c)=\int\limits_s^c (t-s)f''(t)dt. \tag{2.2}$$

Let $s=0$ and use integration by parts with $u=t$ and $dv=f''(t)dt$ to obtain $\int\limits_0^c tf''(t)dt=[tf'(t)]_0^c-\int\limits_0^c f'(t)dt=cf'(c)-f(c)+f(0)$. Since $f(c)-cf'(c)+f(0)$ is bounded and monotonic on $[0,\infty)$, the improper integral $\int\limits_0^\infty tf''(t)dt$ converges. Let $G(u)=L\left\{tf''(t)\right\}(u)=\int\limits_0^\infty e^{-ut}tf''(t)dt$, where L denotes the Laplace Transform. Then $G(0)=\int\limits_0^\infty tf''(t)dt$. Using well known formulas for the Laplace Transform, with $F(s)=L(f)$, $L\left(tf''(t)\right)=-\frac{d}{du}L\left(f''(t)\right)=-\frac{d}{du}\left(u^2L(f)-uf(0)-f'(0)\right)=-u^2F'(s)-2uF(s)-f(0)$, which implies that $G(0)=f(0)$. Hence

$$\int\limits_0^\infty tf''(t)dt=f(0). \tag{2.3}$$

Since $\int\limits_0^c tf''(t)dt=f(0)-g_0(c)$ by (2.2), $\lim\limits_{c\to\infty}\left(f(0)-g_0(c)\right)=f(0)$ by (2.3), which implies that $\lim\limits_{c\to\infty}g_0(c)=0$. Now

$$\lim_{c\to\infty}g_s(c)=s\lim_{c\to\infty}f'(c)+\lim_{c\to\infty}\left(f(c)-cf'(c)\right)=s(0)+\lim_{c\to\infty}g_0(c)=0.$$

That proves (i) when $L(x)=0$. Now assume that $\lim\limits_{x\to\infty}\left(f(x)-L(x)\right)=0$ and let $w(x)=f(x)-L(x)$. Then $\lim\limits_{x\to\infty}w(x)=0$ and $\bar{T}_c=T_c-L=$ tangent line to w at $(c,w(c))$. By what we just proved, $\lim\limits_{c\to\infty}\bar{T}_c(s)=0$, which implies that

$\lim_{c\to\infty}(T_c(s) - L(s)) = 0.$ ■

We now prove some theorems about the illumination index of functions convex on the real line. For any convex function, f, it is trivial that if $P = (s,t)$ lies above the graph of f, then $I_f(P) = 0$. Thus we do not bother stating that case in any of the theorems below.

Theorem 2.1 *Suppose that $f''(x) \geq 0$ on $\Re$ and that $y = L_1(x)$ and $y = L_2(x)$ are distinct oblique asymptotes of f. Let $P = (s,t) \in G_f^C$ be given.*

(i) If $\max(L_1(s), L_2(s)) < t < f(s)$, then $I_f(P) = 2$.

(ii) If $L_1(s) \neq L_2(s)$ and $\min(L_1(s), L_2(s)) < t \leq \max(L_1(s), L_2(s))$, then $I_f(P) = 1$.

(iii) If $t \leq \min(L_1(s), L_2(s))$, then $I_f(P) = 0$.

Proof. Without loss of generality we can assume that $\lim_{x\to -\infty}(f(x) - L_1(x)) = 0$ and $\lim_{x\to\infty}(f(x) - L_2(x)) = 0$. Then by Lemma 2.4, $\lim_{c\to -\infty} g_s(c) = L_1(s)$ and $\lim_{c\to\infty} g_s(c) = L_2(s)$. We prove the theorem for the case when $L_1(s) \leq L_2(s)$, the proof when $L_2(s) \leq L_1(s)$ being similar. Thus we have

$$\begin{aligned}
\lim_{c\to -\infty} g_s(c) &= \min(L_1(s), L_2(s)) = L_1(s), \\
\lim_{c\to\infty} g_s(c) &= \max(L_1(s), L_2(s)) = L_2(s).
\end{aligned}$$

To prove (i): $\lim_{c\to -\infty} g_s(c) = L_1(s) < t, g_s(s) = f(s) > t$, and $\lim_{c\to\infty} g_s(c) = L_2(s) < t$. That implies that $g_s - t$ has at least two real roots, $c_1 \in I_1$ and $c_2 \in I_2$, by the Intermediate Value Theorem. Note that $g_s(s) = f(s) \neq t$, so that $c = s$ is not a root of $g_s - t$. Either c_1 is the only root of $g_s - t$ in I_1, or

$g_s(c) = t$ for all c in some interval, I, contained in I_1 by Lemma 2.2(ii). In the latter case, $T_c(x) = f(x)$ for all $c, x \in I$, so that there is only one tangent line for all $c \in I$. In either case, that yields one tangent line from I_1 which passes thru P. The same holds for I_2 by Lemma 2.2(ii).Thus there are precisely two distinct tangent lines to f which pass thru P, which implies that $I_f(P) = 2$.

To prove (ii): It follows easily, as in the proof of Lemma 2.4, that $L_2(s) \leq f(s)$, which implies that $t < f(s)$ since $(s,t) \in G_f^C$. Thus $g_s(s) \neq t$, so again $c = s$ is not a root of $g_s - t$. Note also that $g_s(c) \neq t$ for any $c \in I_2$ since g_s is non–increasing on (s, ∞), $t \leq L_2(s)$, and $\lim\limits_{c \to \infty} g_s(c) = L_2(s)$. Since $\lim\limits_{c \to -\infty} g_s(c) = L_1(s) < t$ and $g_s(s) = f(s) > t$, $g_s - t$ has at least one real root, $c_0 \in I_1$. Either c_0 is the only root of $g_s - t$ in I_1, or $g_s(c) = t$ for all c in some interval, I, contained in I_1 by Lemma 2.2(ii). In the latter case, $T_c(x) = f(x)$ for all $c, x \in I$, so that there is only one tangent line for all $c \in I$. In either case, that yields one tangent line from I_1 which passes thru P, which implies that $I_f(P) = 1$.

To prove (iii): If $t < L_1(s)$, then it follows easily that $g_s(c) = t$ has no solution. If $t = L_1(s)$ and $g_s(c) = t$, then $g_s(c) = t$ for all $c \in I = (-\infty, k)$ for some $k < s$. Arguing as in the proof of Lemma 2.2(ii), it follows easily that $f(x) = L_1(x)$ for all $x \in I$, which implies that $g_s(c) = f(s)$ and thus $f(s) = t$, which contradicts the assumption that $(s,t) \in G_f^C$. Hence $I_f(P) = 0$. ■

Example 1 *Let* $f(x) = x \tan^{-1} x$. *Then* $f''(x) = \dfrac{2}{(1+x^2)^2} > 0$ *on* $\Re$, *and* $y = \pm \dfrac{\pi}{2} x - 1$*are distinct oblique asymptotes of* f. *Thus Theorem 2.1 applies* *with* $L_1(x) = -\frac{\pi}{2}x - 1$ *and* $L_2(x) = \frac{\pi}{2}x - 1$.

$$\begin{aligned} \min\left(L_1(s), L_2(s)\right) &= \begin{cases} \frac{\pi}{2}s - 1 & \text{if } s < 0 \\ -\frac{\pi}{2}s - 1 & \text{if } s \geq 0 \end{cases} \\ \max\left(L_1(s), L_2(s)\right) &= \begin{cases} -\frac{\pi}{2}s - 1 & \text{if } s < 0 \\ \frac{\pi}{2}s - 1 & \text{if } s \geq 0 \end{cases} \end{aligned}$$

Let $P = (s, t)$. *If* $s < 0$ *and* $-\frac{\pi}{2}s - 1 < t < s\tan^{-1} s$, *or* $s \geq 0$ *and* $\frac{\pi}{2}s - 1 < t < s\tan^{-1} s$, *then* $I_f(P) = 2$.

If $s < 0$ *and* $\frac{\pi}{2}s - 1 < t \leq -\frac{\pi}{2}s - 1$, *or* $s > 0$ *and* $-\frac{\pi}{2}s - 1 < t \leq \frac{\pi}{2}s - 1$, *then* $I_f(P) = 1$.

Finally, if $s < 0$ *and* $t \leq \frac{\pi}{2}s - 1$, *or* $s \geq 0$ *and* $t \leq -\frac{\pi}{2}s - 1$, *then* $I_f(P) = 0$.

Before proving our next result, we need the following lemma.

Lemma 2.5 *Suppose that* $f''(x) \geq 0$ *for* $|x| > b$, *where* b *is a positive real number. Let* $g_s(c) = f(c) + (s - c)f'(c)$ *for given* $s \in \Re$.

(i) If $\lim_{x \to -\infty} (xf''(x)) = A$, *where* $-\infty \leq A < 0$, *then* $\lim_{c \to -\infty} g_s(c) = -\infty$

(ii) If $\lim_{x \to \infty} (xf''(x)) = A$, *where* $0 < A \leq \infty$, *then* $\lim_{c \to \infty} g_s(c) = -\infty$

Remark 2.1 *A weaker version of this lemma was given in ([1], Lemma 1) where it was assumed that* $f''(x) \geq m > 0$ *for* $|x| > b$, *where* m *and* b *are positive real numbers.*

Proof. We prove (ii), the proof of (i) being similar. Let $\{x_k\} \subset (b, \infty)$ be any sequence with $x_k \to \infty$. Suppose that $\lim_{k \to \infty} f''(x_k) = m > 0$. Then

$\lim_{k\to\infty} [(s-x_k)f''(x_k)] = -\infty \neq 0$. Second, suppose that $\lim_{k\to\infty} f''(x_k) = 0$. Then $\lim_{k\to\infty} [(s-x_k)f''(x_k)] = -\lim_{k\to\infty} [x_k f''(x_k)] \neq 0$ by (ii). Hence $\lim_{x\to\infty} [(s-x)f''(x)] \neq 0$, which implies that $\lim_{c\to\infty} g_s'(c) = \lim_{c\to\infty} [(s-c)f''(c)] \neq 0$. Since $f''(x) \geq 0$ for $x > b$, $g_s(c)$ is eventually decreasing. Since $\lim_{c\to\infty} g_s'(c) \neq 0$, it follows that $\lim_{c\to\infty} g_s(c) = -\infty$. ■

The following theorem is similar to Theorem 2.1 for the case when f has only one oblique asymptote.

Theorem 2.2 *Suppose that $f''(x) \geq 0$ on $\Re$ and that one of the following two conditions holds, where L is a linear function.*

$\lim_{x\to-\infty} (xf''(x)) = A$, where $-\infty \leq A < 0$ and $\lim_{x\to\infty} (f(x) - L(x)) = 0$, or $\lim_{x\to-\infty} (f(x) - L(x)) = 0$ and $\lim_{x\to\infty} (xf''(x)) = A$, where $0 < A \leq \infty$. Let $P = (s,t) \in G_f^C$ be given.

(i) If $L(s) < t < f(s)$, then $I_f(P) = 2$

(ii) If $t \leq L(s)$, then $I_f(P) = 1$

Proof. We prove the case when $\lim_{x\to-\infty} (xf''(x)) = A$, $-\infty \leq A < 0$ and $\lim_{x\to\infty} (f(x)-L(x)) = 0$, the proof of the other case being similar. By Lemma 2.5, $\lim_{c\to-\infty} g_s(c) = -\infty$, and by Lemma 2.4, $\lim_{c\to\infty} g_s(c) = L(s)$. If $L(s) < t < f(s)$, then $\lim_{c\to-\infty} g_s(c) < t$, $g_s(s) = f(s) > t$, and $\lim_{c\to\infty} g_s(c) < t$. That implies that $g_s - t$ has at least two real roots, $c_1 \in I_1$ and $c_2 \in I_2$, by the Intermediate Value Theorem. Arguing exactly as in the proof of Theorem 2.1, part (i), it follows that exactly two tangent lines pass thru P, which implies that $I_f(P) = 2$. That proves (i). It follows easily, as in the proof of Lemma 2.4, that $L(s) \leq f(s)$. If

$t \leq L(s)$, then $\lim_{c \to -\infty} g_s(c) < t$ and $g_s(s) = f(s) > t$ implies that $g_s - t$ has at least one real root, $c_0 \in I_1$. Arguing exactly as in the proof of Theorem 2.1, part (ii), it follows that $I_f(P) = 1$. ■

Remark 2.2 *It is possible to prove Theorem 2.2 with slightly weaker hypotheses. However, we believe, but have not been able to prove, that the conclusion of Theorem 2.2 holds with only the assumption that f has one oblique asymptote.*

Example 2 *Let $f(x) = e^x$. Then $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} (xf''(x)) \neq 0$, so that Theorem 2.2 applies with $L(x) = 0$. Hence,if $P = (s, t)$ with $0 < t < e^s$, then $I_f(P) = 2$. If $P = (s, t)$ with $t \leq 0$, then $I_f(P) = 1$.*

Theorem 2.3 *Suppose that $f''(x) \geq 0$ on $\Re$ and that $\lim_{x \to -\infty} (xf''(x)) = A$, where $-\infty \leq A < 0$ and $\lim_{x \to \infty} (xf''(x)) = A$, where $0 < A \leq \infty$. Then $I_f(P) = 2$ for any point $P = (s, t)$ below the graph of f.*

Proof. The proof follows from Lemma 2.5 as in the proof of Theorem 2.2 and we omit the details. ■

We now apply Theorem 2.3 to show that any point below the graph of a convex rational function or exponential polynomial must have illumination index equal to 2.

Proposition 2.1 *Let R be a rational function defined on $\Re$ with $R'' \geq 0$ on $\Re$. Then $I_R(P) = 2$ for any point P below the graph of R.*

Proof. Write $R(x) = \dfrac{p(x)}{q(x)}$, where p and q are polynomials of degree m and n respectively. If $n \geq m$, then R has a horizontal asymptote and thus R'' cannot be non–negative on $\Re$. Thus we have $n < m$. If $m = n + 1$, then R has one oblique asymptote, L, which implies that $\lim_{x\to\pm\infty}(R(x)-L(x)) = 0$ and again R'' would not be non–negative on $\Re$. Thus $m - n - 1 \neq 0$. Also, since R is defined on $\Re$, n must be even. Let $p(x) = \sum_{k=0}^{m} a_k x^k$, $q(x) = \sum_{k=0}^{n} b_k x^k, a_m \neq 0 \neq b_n$. Then $R' = \frac{qp' - pq'}{q^2} \Rightarrow$

$R'' = \dfrac{q^2 p'' - pqq'' - 2p'q'q + 2p\,(q')^2}{q^3}$, which implies that

$$\left(\sum_{k=0}^{n} b_k x^k\right)^3 R''(x) =$$
$$\left(\sum_{k=0}^{n} b_k x^k\right)^2 \left(\sum_{k=2}^{m} k(k-1)a_k x^{k-2}\right) -$$
$$\left(\sum_{k=0}^{m} a_k x^k\right)\left(\sum_{k=0}^{n} b_k x^k\right)\left(\sum_{k=2}^{n} k(k-1)b_k x^{k-2}\right)$$
$$-2\left(\sum_{k=1}^{m} k a_k x^{k-1}\right)\left(\sum_{k=1}^{n} k b_k x^{k-1}\right)\left(\sum_{k=0}^{n} b_k x^k\right)$$
$$+2\left(\sum_{k=0}^{m} a_k x^k\right)\left(\sum_{k=1}^{n} k b_k x^{k-1}\right)^2 =$$
$$(m-n)\,(m-n-1)\,a_m b_n^2 x^{2n+m-2} + \cdots .$$

If $2n + m - 2 \leq 3n$, then $m \leq n + 2$, which implies that $m = n + 2$ since $m > n$ and $m \neq n + 1$. If $2n + m - 2 > 3n$, then $2n + m - 2 - 3n = m - n - 2$ must be even since $R'' \geq 0$ as $|x| \to \infty$. In either case, m is also even. Since $m > n$ and m is even, it follows that $m - n - 1 > 0$. Since $R'' \geq 0$ as $|x| \to \infty$, $(m-n)\,(m-n-1)\,\frac{a_m}{b_n} > 0$. Thus $\lim_{x\to-\infty}[xR''(x)] =$
$\lim_{x\to-\infty} \dfrac{(m-n)\,(m-n-1)\,a_m b_n^2 x^{2n+m-1} + \cdots}{b_n^3 x^{3n} + \cdots} =$
$\lim_{x\to-\infty} \dfrac{(m-n)\,(m-n-1)\,a_m}{b_n} x^{m-n-1} = -\infty$ and

$\lim\limits_{x\to\infty}[xR''(x)] = \lim\limits_{x\to\infty}\frac{(m-n)(m-n-1)a_m}{b_n}x^{m-n-1} = \infty$. By Theorem 2.3, $I_R(P) = 2$ for any point P below the graph of R. ∎

Proposition 2.2 *Suppose that p and q are polynomials of degree m and n respectively, and let $f(x) = p(x)e^{q(x)}$. Suppose that $f'' \geq 0$ on $\Re$. Then $I_f(P) = 2$ for any point P below the graph of f.*

Proof. Let $p(x) = \sum\limits_{k=0}^{m} a_k x^k$, $q(x) = \sum\limits_{k=0}^{n} b_k x^k, a_m \neq 0 \neq b_n$. Now we must have $b_n > 0$ since if $b_n < 0$, then $\lim\limits_{|x|\to\infty} f(x) = 0$. That would imply that $f'' \ngeq 0$ on $\Re$. A simple computation gives

$$f'' = (p\,(q')^2 + 2p'q' + pq'' + p'')e^q$$

and

$$p\,(q')^2 + 2p'q' + pq'' + p'' =$$

$$\left(\sum_{k=0}^{m} a_k x^k\right)\left(\sum_{k=1}^{n} kb_k x^{k-1}\right)^2 + 2\left(\sum_{k=1}^{m} ka_k x^{k-1}\right)\left(\sum_{k=1}^{n} kb_k x^{k-1}\right) +$$
$$\left(\sum_{k=0}^{m} a_k x^k\right)\left(\sum_{k=2}^{n} k(k-1)b_k x^{k-2}\right) + \left(\sum_{k=2}^{m} k(k-1)a_k x^{k-2}\right) =$$

$$n^2 a_m b_n^2 x^{2n+m-2} + \cdots + 2mna_m b_n x^{n+m-2} + \cdots +$$

$$n(n-1)a_m b_n x^{n+m-2} + \cdots + m(m-1)a_m x^{m-2} + \cdots$$

$f'' \geq 0$ on $\Re$ implies that $p\,(q')^2 + 2p'q' + pq'' + p'' \geq 0$ on $\Re \Rightarrow 2n + m - 2$ is even and $a_m > 0$. Thus $2n + m - 1$ is odd and $\lim\limits_{x\to -\infty}[xf''(x)] =$

$n^2 a_m b_n^2 \lim_{x\to -\infty} x^{2n+m-1} = -\infty$. Similarly, $\lim_{x\to\infty}[xf''(x)] = \infty$. By Theorem 2.3, $I_f(P) = 2$ for any point P below the graph of f. ■

3 Polynomials

We now prove some results about the illumination index for polynomials with real coefficients. As earlier, for given differentiable f, we let $g_s(c) = f(c) + (s - c)f'(c)$. We also let π_n = polynomials of degree $\leq n$.

Remark 3.1 *If one or more of the tangent lines which pass thru P is a multiple tangent line, then the illumination index of $P = (s,t)$ could be strictly smaller than the number of real roots of $g_s(c) - t$. This will need to be taken into account for some of the proofs below.*

The following lemma holds for more than just the polynomials, but we just consider that case in this section.

Lemma 3.1 *Let f be a polynomial and let $s \in \Re$ be given. Then the local extrema of $g_s(c)$ occur at precisely the following values of c.*

*(i) $c = s$ if $(s, f(s))$ is **not** an inflection point of f*

(ii) $c = d \neq s$ if $(d, f(d))$ is an inflection point of f

Proof. It is easy to show that

$$g_s^{(k)}(c) = (s-c)f^{(k+1)}(c) - (k-1)f^{(k)}(c), k \geq 1.$$

To prove (i): First, suppose that $f''(s) \neq 0$. Then $g_s(s)$ is a local extremum of $g_s(c)$ since then $g_s'(s) = 0$ and $g_s''(s) = -f''(s) \neq 0$. Now suppose that $f^{(k)}(s) = 0$ for $k = 2, ..., m-1$ and $f^{(m)}(s) \neq 0, m \geq 3$. Then m is even since $(s, f(s))$ is not an inflection point of f. $g_s^{(k)}(s) = -(k-1)f^{(k)}(s)$, which implies that $g_s^{(k)}(s) = 0$ for $k = 2, ..., m-1$ and $g_s^{(m)}(s) \neq 0$. Hence $g_s(s)$ is a local extremum of $g_s(c)$ since m is even. That proves (i).

To prove (ii): Suppose that $(d, f(d))$ is an inflection point of $f, d \neq s$. Suppose that $f^{(k)}(d) = 0$ for $k = 2, ..., m-1$ and $f^{(m)}(d) \neq 0$, $m \geq 3$. Then m is odd since $(d, f(d))$ is an inflection point of f. Since $g_s^{(k)}(d) = 0$ for $k = 1, ..., m-2$ and $g_s^{(m-1)}(d) = (s-d)f^{(m)}(d) - (k-1)f^{(m-1)}(d) = (s-d)f^{(m)}(d) \neq 0$, $g_s(d)$ is a local extremum of $g_s(c)$ since $m-1$ is even. That proves (ii). ■

Suppose that $f(x) = \sum\limits_{k=0}^{n} a_k x^k$. Then a simple computation yields

$$g_s(c) - t = -(n-1)a_n c^n + \sum_{k=1}^{n-1} \left[s(k+1)a_{k+1} - (k-1)a_k\right] c^k + a_0 + sa_1 - t, n \geq 2. \tag{3.1}$$

Remark 3.2 *By (3.1) it follows immediately that if $n \geq 3$ is odd, then $I_f(P) \geq 1$ for any $P \in G_f^C$.*

Lemma 3.2 *Let $f(x) = \sum\limits_{k=0}^{n} a_k x^k, a_n \neq 0$ and let $s \in \Re$.*

(i) If $n \geq 3$ and odd, then

$\lim\limits_{c \to -\infty} g_s(c) = [sgn(a_n)]\infty$ *and* $\lim\limits_{c \to \infty} g_s(c) = [-sgn(a_n)]\infty$

(ii) If n is even, then

$\lim\limits_{c \to -\infty} g_s(c) = [-sgn(a_n)]\infty$ *and* $\lim\limits_{c \to \infty} g_s(c) = [-sgn(a_n)]\infty$

Proof. The proof follows immediately from (3.1). ■

Our first theorem in this section is about cubic polynomials. We shall prove some more results below for the case when n is odd.

Theorem 3.1 *Let f be a cubic polynomial. Then for any $k = 1, 2, 3$ there exists $P \in G_f^C$ such that $I_f(P) = k$.*

Proof. Suppose, without loss of generality, that $f(x) = \sum_{k=0}^{3} a_k x^k$ with $a_3 > 0$. Then f has exactly one inflection point, $(d, f(d))$. Choose any $s \neq d$. Then $g_s(c)$ has two exactly local extrema, $g_s(c_1)$ and $g_s(c_2)$, by Lemma 3.1, part (ii). Since $\lim_{c \to -\infty} g_s(c) = \infty$ and $\lim_{c \to \infty} g_s(c) = -\infty$ by Lemma 3.2, we may assume that $g_s(c_1)$ equals the local maximum and $g_s(c_2)$ equals the local minimum, with $c_1 < c_2$. If $t < g(c_2)$ or $t > g(c_1)$, then the horizontal line $y = t$ intersects the graph of g_s in one point, which implies that $I_f(P) = 1$. If $g(c_2) < t < g(c_1)$, then the horizontal line $y = t$ intersects the graph of g_s in three points, $(d_i, g_s(d_i))$, $i = 1, 2,$ or 3. Then $I_f(P) = 3$ since a cubic polynomial cannot have any multiple tangent lines. Finally, the horizontal lines $y = g(c_1)$ and $y = g(c_2)$ intersect the graph of g_s in two points, which yields a point, $P = (s, t)$, such that $I_f(P) = 2$. One could also use the fact that $g_d(c)$ has one local extremum(by Lemma 3.1, part (i)) to obtain a point, $P = (s, t)$, such that $I_f(P) = 2$. ■

The following example shows that Theorem 3.1 does not hold in general for n odd, $n \geq 5$.

Example 3 *Let $f(x) = x^n, n \geq 5$ and odd. Then for any $s \in \Re$, $g_s(c) - t = c^n + (s - c)nc^{n-1} - t = -(n-1)c^n + nsc^{n-1} - t$ and $g_s(-c) - t = (n-1)c^n + nsc^{n-1} - t$.*

We consider the following six cases.

Case 1: $s, t > 0$. Then $g_s(c) - t$ has 2 sign changes and $g_s(-c) - t$ has 1 sign change, which implies that $g_s(c) - t$ has at most 3 distinct real roots.

Case 2: $s > 0, t < 0$. Then $g_s(c) - t$ has 1 sign change and $g_s(-c) - t$ has 0 sign changes, which implies that $g_s(c) - t$ has at most 1 real root.

Case 3: $s < 0, t > 0$. Then $g_s(c) - t$ has 0 sign changes and $g_s(-c) - t$ has 1 sign change, which implies that $g_s(c) - t$ has at most 1 real root.

Case 4: $s, t < 0$. Then $g_s(c) - t$ has 1 sign change and $g_s(-c) - t$ has 2 sign changes, which implies that $g_s(c) - t$ has at most 3 distinct real roots.

Case 5: $s = 0$. Then $g_s(c) - t = -(n-1)c^n - t$, which has 1 real root.

Case 6: $t = 0$. Then $g_s(c) - t = -(n-1)c^n + nsc^{n-1} = c^{n-1}\left[-(n-1)c + ns\right]$ has 2 distinct real roots.

Hence for any point $P \in G_f^C$, $I_f(P) \leq 3$

The example above shows that there are odd polynomials of any degree such that $I_f(P) \leq 3$ for all $P \in G_f^C$. Our next result is a positive result about the illumination index of all odd polynomials.

Theorem 3.2 *Suppose that f is a polynomial of degree $n \geq 5, n$ odd. Then there exists $P \in G_f^C$ such that $I_f(P) = 3$.*

Proof. Since n is odd, f'' must have at least one real root where it changes sign. Hence f has at least one inflection point, $(d, f(d))$. Choose any $s \neq d$. Arguing as in the proof of Theorem 3.1 above,$g_s(c)$ has two exactly local extrema, $g_s(c_1)$ and $g_s(c_2)$, by Lemma 3.1, part (ii), and we may assume that $g_s(c_1)$ equals the local maximum and $g_s(c_2)$ equals the local minimum, with $c_1 < c_2$. However, it is possible that f has multiple tangent lines. Suppose that for each $t, g(c_2) < t < g(c_1)$, $y = t$ intersects the graph of g_s in the three distinct points $(d_i, g_s(d_i))$, $i = 1, 2, 3$, and the tangent lines at $(d_i, g_s(d_i))$ and at $(d_j, g_s(d_j))$ are identical for some $i \neq j$. Then f would have infinitely many multiple tangent lines since the $(d_i, g_s(d_i))$ change with t. But a polynomial can only have finitely many multiple tangent lines(that is not difficult to prove), and thus we can choose $t, g(c_2) < t < g(c_1)$, such that $(d_i, g_s(d_i))$ yield three distinct tangent lines to f, which yields $I_f(P) = 3$. ■

Theorem 3.3 *Suppose that f is a polynomial of degree $n \geq 2, n$ even. Then there exist points $P_1, P_2 \in G_f^C$ such that $I_f(P_1) = 0$ and $I_f(P_1) = 2$.*

Proof. Since n is even, for any $s \in \Re$, $\lim_{c \to -\infty} g_s(c) = -\infty$ and $\lim_{c \to \infty} g_s(c) = -\infty$, or $\lim_{c \to -\infty} g_s(c) = \infty$ and $\lim_{c \to \infty} g_s(c) = \infty$ by Lemma 3.2. Thus there must be values of t such that the horizontal line $y = t$ does not intersect the graph of g_s, which implies that $I_f(P) = 0$ for $P = (s, t)$. Now suppose that $f'' \geq 0$ on $\Re$. Then $I_f(P) = 2$ for any point P below the graph of f by Proposition 2.1. If $f'' \ngeq 0$ on $\Re$, then f has at least one inflection point, $(d, f(d))$, which implies that g_s has at least one local extremum. Again, since $\lim_{c \to -\infty} g_s(c) = -\infty$ and $\lim_{c \to \infty} g_s(c) = -\infty$, or $\lim_{c \to -\infty} g_s(c) = \infty$ and $\lim_{c \to \infty} g_s(c) = \infty$, there must be values of t such that the horizontal line $y = t$ intersects the graph of g_s in precisely two points,$(d_i, g_s(d_i))$, $i = 1, 2$. In addition, one can choose t so that the tangent lines to the graph of f at $(d_i, f(d_i))$, $i = 1, 2$ are distinct, which yields $I_f(P) = 2$. ■

4 Theta Illumination Index

Let θ be a given angle with $0 \leq \theta \leq \frac{\pi}{2}$. We call L_θ a **theta line** at P if L_θ makes an angle, θ, with the graph of f at P. Here the angle between two lines in a plane is defined to be 0, if the lines are parallel, or the smaller angle having as sides the half-lines starting from the intersection point of the lines and lying on those two lines, if the lines are not parallel.

Definition 4.1 *Let $f(x)$ be a differentiable function on the real line and let $P \in G_f^C$. Let θ be a given angle with $0 \leq \theta \leq \frac{\pi}{2}$. We say that the θ illumination index of P, denoted by $I_{f,\theta}(P)$, is k if there are k distinct theta lines to the graph of f which pass through P. In particular, we use $I_{f,N}(P)$ to denote the $\frac{\pi}{2}$ illumination index of P. In that case, of course, L_θ is a normal line to the graph of f.*

Unlike the case with illumination by tangent lines, where it is clearly possible that **no** tangent line passes thru a given point $P \in G_f^C$, this cannot happen with normal lines.

Theorem 4.1 *For any differentiable f defined on $\Re$ and any point $P \in G_f^C$, $I_{f,N}(P) \geq 1$*

Proof. Given $P = (s,t)$, let $S =$ set of circles centered at P which also intersect G_f, and let C_0 be the circle in S with the smallest radius. Then C_0 is tangent to G_f at some point $(c_0, f(c_0))$. The line $\overleftrightarrow{(s,t)\,(c_0, f(c_0))}$ is then a normal line passing thru P. ■

Remark 4.1 *Alternatively, one could also look at the distance from* P *to* G_f *– the minimal distance is obtained at* $(c_0, f(c_0))$ *with* $\overleftrightarrow{P\,(c_0, f(c_0))}$ *perpendicular to the tangent at* $(c_0, f(c_0))$.

We believe that Theorem 4.1 holds for theta lines in general, but have not been able to prove it.

Conjecture 1 *For any differentiable* f *defined on* $\Re$, *any given* $0 \le \theta \le \dfrac{\pi}{2}$, *and any point* $P \in G_f^C$, $I_{f,\theta}(P) \ge 1$.

References

[1] "Illumination by Taylor Polynomials", International Journal of Mathematics and Mathematical Sciences 27(2001), 125-130.