RANDOM WALK ON A CO-COMPACT FUCHSIAN GROUP: BEHAVIOR OF THE GREEN'S FUNCTION AT THE SPECTRAL RADIUS

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Abstract. It is proved that the Green's function of a symmetric finite range random walk on a co-compact Fuchsian group decays exponentially in distance at the radius of convergence R. It is also shown that Ancona's inequalities extend to R, and therefore that the Martin boundary for $R-$potentials coincides with the natural geometric boundary S^1, and that the Martin kernel is uniformly Hölder continuous. Finally, this implies a local limit theorem for the transition probabilities: in the aperiodic case, $p^n(x, y) \sim C_{x,y} R^{-n} n^{-3/2}$.

1. Introduction

1.1. **Green's function and Martin boundary.** A (right) *random walk* on a countable group Γ is a discrete-time Markov chain whose transition probabilities are $\Gamma-$invariant; equivalently, it is a stochastic process $\{X_n\}_{n\geq 0}$ of the form

$$X_n = x\xi_1\xi_2 \cdots \xi_n$$

where $\xi_1, \xi_2, \ldots$ are independent, identically distributed $\Gamma-$valued random variables. The distribution of ξ_i is the *step distribution* of the random walk. The random walk is said to be *symmetric* if its step distribution is invariant under the mapping $x \mapsto x^{-1}$, and *finite-range* if the step distribution has finite support. The *Green's function* is the generating function of the transition probabilities: for $x, y \in \Gamma$ and $0 \leq r < 1$ it is defined by the absolutely convergent series

$$G_r(x, y) := \sum_{n=0}^{\infty} P^x\{X_n = y\} r^n = G_r(1, x^{-1}y); \tag{1}$$

here P^x is the probability measure on path space governing the random walk with initial point x. If the random walk is irreducible (that is, if the semigroup generated by the support of the step distribution is Γ) then the radius of convergence R of the series (1) is the same for all pairs x, y. Moreover, if the random walk is symmetric, then $1/R$ is the *spectral radius* of the transition operator. By a fundamental theorem of Kesten [21], if the group Γ is finitely generated and nonamenable then $R > 1$. Moreover, in this case the Green's function is finite at its radius of convergence (cf. [35], ch. 2): for all $x, y \in \Gamma$,

$$G_R(x, y) < \infty. \tag{2}$$

The Green's function is of central importance in the study of random walk. Clearly, it encapsulates information about the transition probabilities; in Theorem 9.1, we show

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that the local asymptotic behavior of the transition probabilities can be deduced from the singular behavior of the Green's function at its radius of convergence. The Green's function is also the key to the potential theory associated with the random walk: in particular, it determines the Martin boundary for $r-$potential theory. A prominent theme in the study of random walks on nonabelian groups has been the relationship between the geometry of the group and the nature of the Martin boundary. A landmark result here is a theorem of Ancona [2] describing the Martin boundary for random walks with finitely supported step distributions on *hyperbolic* groups: Ancona proves that for every $r \in (0, R)$ the Martin boundary for $r-$potential theory coincides with the *geometric* (Gromov) boundary, in a sense made precise below. (Series [31] had earlier established this in the special case $r = 1$ when the group is co-compact Fuchsian. See also [3] and [1] for related results concerning Laplace-Beltrami operators on Cartan manifolds.)

It is natural to ask whether Ancona's theorem extends to $r = R$, that is, if the Martin boundary is stable (see [29] for the terminology) through the entire range $(0, R]$. One of the main results of this paper (Theorem 1.3) provides an affirmative answer in the special case of symmetric, finite-range random walk on a co-compact Fuchsian group, i.e., a co-compact, discrete subgroup of $PSL(2, \mathbb{R})$. Any Fuchsian group acts as a discrete group of isometries of the hyperbolic disk, and so its Cayley graph can be embedded quasi-isometrically in the hyperbolic disk; this implies that its Gromov boundary is the circle S^1 at infinity.

Theorem 1.1. *For any symmetric, irreducible, finite-range random walk on a co-compact Fuchsian group Γ, the Martin boundary for $R-$potentials coincides with the geometric boundary $S^1 = \partial\Gamma$.*

This assertion means that (1) for every geodesic ray $y_0, y_1, y_2, \dots$ in the Cayley graph that converges to a point $\zeta \in \partial\Gamma$ and for every $x \in \Gamma$,

$$\lim_{n\to\infty} \frac{G_R(x, y_n)}{G_R(1, y_n)} = K_R(x, \zeta) = K(x, \zeta) \tag{3}$$

exists; (2) for each $\zeta \in \partial\Gamma$ the function $K_\zeta(x) := K(x, \zeta)$ is minimal positive $R-$harmonic in x; (3) for distinct points $\zeta, \zeta' \in \partial\Gamma$ the functions K_ζ and $K_{\zeta'}$ are different; and (4) the topology of pointwise convergence on $\{K_\zeta\}_{\zeta\in\partial\Gamma}$ coincides with the usual topology on $\partial\Gamma = S^1$.

Our results also yield explicit rates for the convergence (3), and imply that the Martin kernel $K_r(x, \zeta)$ is *Hölder* continuous in ζ relative to the usual Euclidean metric (or any visual metric — see [19] for the definition) on $S^1 = \partial\Gamma$.

Theorem 1.2. *For any symmetric, irreducible, finite-range random walk on a co-compact Fuchsian group Γ, there exists $\varrho < 1$ such that for every $1 \le r \le R$ and every geodesic ray $1 = y_0, y_1, y_2, \dots$ converging to a point $\zeta \in \partial\Gamma$,*

$$\left| \frac{G_r(x, y_n)}{G_r(1, y_n)} - K_r(x, \zeta) \right| \le C_x \varrho^n. \tag{4}$$

The constants $C_x < \infty$ depend on $x \in \Gamma$ but not on $r \le R$. Consequently, for each $x \in \Gamma$ and $r \le R$ the function $\zeta \mapsto K_r(x, \zeta)$ is Hölder continuous in ζ relative to the Euclidean metric on $S^1 = \partial\Gamma$, for some exponent not depending on $r \le R$.

The exponential convergence (4) and the Hölder continuity of the Martin kernel for $r = 1$ were established by Series [30] for random walks on Fuchsian groups. Similar results for the Laplace-Beltrami operator on negatively curved Cartan manifolds were proved by Anderson and Schoen [3]. The methods of [3] were adapted by Ledrappier [26] to prove that Series' results extend to all random walks on a free group, and Ledrappier's proof was extended by Izumi, Neshvaev, and Okayasu [18] to prove that for random walk on a non-elementary hyperbolic group the Martin kernel $K_1(x, \zeta)$ is Hölder continuous in ζ. All of these proofs rest on inequalities of the type discussed in section 1.2 below. Theorem 1.3 below asserts (among other things) that similar estimates are valid for all G_r *uniformly for* $r \leq R$. Given these, the proof of [18] applies almost verbatim to establish Theorem 1.2. We will give some additional details in Paragraph 4.2.

1.2. **Ancona's boundary Harnack inequalities.** The crux of Ancona's argument in [2] was a system of inequalities that assert, roughly, that the Green's function $G_r(x, y)$ is nearly submultiplicative in the arguments $x, y \in \Gamma$. Ancona [2] proved that such inequalities always hold for $r < R$: in particular, he proved, for a symmetric random walk with finitely supported step distribution on a hyperbolic group, that for each $r < R$ there is a constant $C_r < \infty$ such that for every geodesic segment $x_0 x_1 \cdots x_m$ in (the Cayley graph of) Γ,

$$G_r(x_0, x_m) \leq C_r G_r(x_0, x_k) G_r(x_k, x_m) \qquad \forall\, 1 \leq k \leq m. \tag{5}$$

His argument depends in an essential way on the hypothesis $r < R$ (cf. his Condition (*)), and it leaves open the possibility that the constants C_r in the inequality (5) might blow up as $r \to R$. For finite-range random walk on a *free* group it can be shown, by direct calculation, that the constants C_r remain bounded as $r \to R$, and that the inequalities (5) remain valid at $r = R$ (cf. [23]). The following result asserts that the same is true for random walks on a co-compact Fuchsian group.

Theorem 1.3. *For any symmetric, finite-range random walk on a co-compact Fuchsian group* Γ,

(A) *the Green's function* $G_R(1, x)$ *decays exponentially in* $|x| := d(1, x)$*; and*
(B) *Ancona's inequalities* (5) *hold for all* $r \leq R$*, with a constant* C *independent of* r.

Note 1.4. Here and throughout the paper $d(x, y)$ denotes the distance between the vertices x and y in the Cayley graph G^Γ, equivalently, distance in the word metric. *Exponential decay* of the Green's function means *uniform* exponential decay in all directions, that is, there are constants $C < \infty$ and $\varrho < 1$ such that for all $x, y \in \Gamma_g$,

$$G_R(x, y) \leq C \varrho^{d(x,y)}. \tag{6}$$

A very simple argument (see Lemma 2.1 below) shows that for a symmetric random walk on any nonamenable group $G_R(1, x) \to 0$ as $|x| \to \infty$. Given this, it is routine to show that exponential decay of the Green's function follows from Ancona's inequalities. However, we will argue in the other direction, first providing an independent proof of exponential decay in subsection 3.3, and then deducing Ancona's inequalities from it in section 4.

Note 1.5. Theorem 1.3 (A) is a discrete analogue of one of the main results (Theorem B) of Hamenstaedt [17] concerning the Green's function of the Laplacian on the universal cover of a compact negatively curved manifold. Unfortunately, Hamenstaedt's proof appears to have a serious error.[1] The approach taken here bears no resemblance to that of [17].

[1] The error is in the proof of Lemma 3.1: The claim is made that a lower bound on a finite measure implies a lower bound for its Hausdorff-Billingsley dimension relative to another measure. This is false – in fact such a lower bound on measure implies an *upper* bound on its Hausdorff-Billingsley dimension.

Theorem 1.3 is proved in sections 3 and 4 below. The argument uses the *planarity* of the Cayley graph in an essential way. It also relies on the simple estimate

$$\lim_{|x|\to\infty} G_R(1,x) = 0,$$

that we derive from the symmetry of the random walk. While this estimate is not true in general without the symmetry assumption, we nevertheless conjecture that Ancona's inequalities and the identification of the Martin boundary at $r = R$ hold in general.

1.3. **Decay at infinity of the Green's function.** Neither Ancona's result nor Theorem 1.3 gives any information about how the uniform exponential decay rate ϱ depends on the step distribution of the random walk. In fact, the Green's function $G_r(1,x)$ decays at different rates in different directions $x \to \partial\Gamma$. To quantify the overall decay, consider the behavior of the Green's function over the entire sphere S_m of radius m centered at 1 in the Cayley graph G^Γ. If Γ is a nonelementary Fuchsian group then the cardinality of the sphere S_m grows exponentially in m (see Corollary 5.5 in section 5), that is, there exist constants $C > 0$ and $\zeta > 1$ such that as $m \to \infty$,

$$|S_m| \sim C\zeta^m.$$

Theorem 1.6. *For any symmetric, irreducible, finite-range random walk on a co-compact Fuchsian group* Γ*,*

$$\lim_{m\to\infty} \sum_{x\in S_m} G_R(1,x)^2 = C > 0 \tag{7}$$

exists and is finite, and

$$\#\{x \in \Gamma \,:\, G_R(1,x) \geq \varepsilon\} \asymp \varepsilon^{-2} \tag{8}$$

as $\varepsilon \to 0$*. (Here* $\asymp$ *means that the ratio of the two sides remains bounded away from* 0 *and* ∞*.)*

The proof is carried out in sections 6–7 below (cf. Propositions 6.2 and 7.1), using the fact that any hyperbolic group has an *automatic structure* [15]. The automatic structure will permit us to use the theory of *Gibbs states* and *thermodynamic formalism* of Bowen [8], ch. 1. Theorem 1.2 is essential for this, as the theory developed in [8] applies only to Hölder continuous functions.

It is likely that $\asymp$ can be replaced by $\sim$ in (8). There is a simple heuristic argument that suggests why the sums $\sum_{x\in S_m} G_R(1,x)^2$ should remain bounded as $m \to \infty$: Since the random walk is R−transient, the contribution to $G_R(1,1) < \infty$ from random walk paths that visit S_m and then return to 1 is bounded (by $G_R(1,1)$). For any $x \in S_m$, the term $G_R(1,x)^2/G_R(1,1)$ is the contribution to $G_R(1,1)$ from paths that visit x before returning to 1. Thus, if $G_R(1,x)$ is not substantially larger than

$$\sum_{n=1}^{\infty} P^1\{X_n = x \text{ and } \tau(m) = n\}R^n,$$

where $\tau(m)$ is the time of the first visit to S_m, then the sum in (7) should be of the same order of magnitude as the total contribution to $G_R(1,1) < \infty$ from random walk paths that visit S_m and then return to 1. Of course, the difficulty in making this heuristic argument rigorous is that *a priori* one does not know that paths that visit x are likely to be making their first visits to S_m; it is Ancona's inequality (5) that ultimately fills the gap.

Note 1.7. A simple argument shows that for $r > 1$ the sum of the Green's function on the sphere S_m, unlike the sum of its square, explodes as $m \to \infty$. Fix $1 < r \le R$ and $m \ge 1$. Let C_0 bound the size of the jumps of the random walk, and let $\tilde{S}_m$ be the set of points with $d(1,x) \in [m, m + C_0)$. Since X_n is transient, it will, with probability one, eventually visit the annulus $\tilde{S}_m$. The minimum number of steps needed to reach $\tilde{S}_m$ is at least m/C_0. Hence,

$$\begin{aligned}\sum_{x\in\tilde{S}_m} G_r(1,x) &= \sum_{n=m/C_0}^{\infty} \sum_{x\in\tilde{S}_m} P^1\{X_n = x\} r^n \\ &\ge r^{m/C_0} \sum_{n=m/C}^{\infty} P^1\{X_n \in \tilde{S}_m\} \\ &\ge r^{m/C_0} P^1\{X_n \in \tilde{S}_m \text{ for some } n\} \\ &= r^{m/C_0}.\end{aligned}$$

Hence, $\sum_{x\in\tilde{S}_m} G_r(1,x)$ diverges. The divergence of $\sum_{x\in S_m} G_r(1,x)$ readily follows if the random walk is irreducible.

Note 1.8. There are some precedents for the result (7). Ledrappier [25] has shown that for Brownian motion on the universal cover of a compact Riemannian manifold of negative curvature, the integral of the Green's function $G_1(x,y) = \int_0^\infty p_t(x,y)\,dt$ over the sphere $S(\varrho, x)$ of radius ϱ centered at a fixed point x converges as $\varrho \to \infty$ to a positive constant C independent of x. Hamenstaedt [17] proves in the same context that the integral of G_R^2 over $S(\varrho, x)$ remains bounded as the radius $\varrho \to \infty$. Our arguments (see Note 6.3 in sec. 6) show that for finite range irreducible random walk on a co-compact Fuchsian group the following is true: for each value of r there exists a power $1 \le \theta = \theta(r) \le 2$ such that

$$\lim_{m\to\infty} \sum_{x\in S_m} G_r(1,x)^\theta = C_r > 0.$$

1.4. **Critical exponent for the Green's function.**

Theorem 1.6 implies that the behavior of the Green's function $G_R(x,y)$ at the radius of convergence as y approaches the geometric boundary is intimately related to the behavior of $G_r(x,y)$ as $r \uparrow R$. The connection between the two is rooted in the following set of differential equations.

Proposition 1.9. *For any random walk on any discrete group, the Green's functions satisfy*

$$\frac{d}{dr} G_r(x,y) = r^{-1} \sum_{z\in\Gamma} G_r(x,z) G_r(z,y) - r^{-1} G_r(x,y) \quad \forall\, 0 \le r < R. \tag{9}$$

Although the proof is elementary (cf. section 2.1 below) these differential equations have not (to our knowledge) been observed before. Theorem 1.6 implies that the sum in equation (9) blows up as $r \to R-$; this is what causes the singularity of $r \mapsto G_r(1,1)$ at $r = R$. The rate at which the sum blows up determines the *critical exponent* for the Green's function, that is, the exponent α for which $G_R(1,1) - G_r(1,1) \sim C(R-r)^\alpha$. The following theorem asserts that the critical exponent is 1/2.

Theorem 1.10. *For any symmetric, irreducible, finite-range random walk on a co-compact Fuchsian group* Γ*, there exist constants* $C_{x,y} > 0$ *such that as* $r \to R-$*,*

$$G_R(x,y) - G_r(x,y) \sim C_{x,y}\sqrt{R-r} \quad \text{and} \tag{10}$$

$$dG_r(x,y)/dr \sim \frac{1}{2} C_{x,y}/\sqrt{R-r}. \tag{11}$$

The proof of Theorem 1.10 is given in section 8. Like the proof of Theorem 1.6, it uses the existence of an automatic structure and the attendant thermodynamic formalism. It also relies critically on the conclusion of Theorem 1.6, which determines the value of the key thermodynamic variable.

The behavior of the generating function $G_r(1,1)$ in the neighborhood of the singularity $r = R$ is of interest because it reflects the asymptotic behavior of the coefficients $P^1\{X_n = 1\}$ as $n \to \infty$. In section 9 we will show that Theorem 1.10, in conjunction with Karamata's Tauberian Theorem, implies the following *local limit theorem*.

Theorem 1.11. *For any finite-range, aperiodic, symmetric, irreducible random walk on a co-compact Fuchsian group with spectral radius* R^{-1}*, there exist constants* $C_{x,y} > 0$ *such that for all* $x, y \in \Gamma$*,*

$$p^n(x,y) \sim C_{x,y} R^{-n} n^{-3/2}. \tag{12}$$

If the random walk is not aperiodic, these asymptotics hold for even (resp. odd) n *if the distance from* x *to* y *is even (resp. odd).*

According to a theorem of Bougerol [7], the transition probability densities of a random walk on a semi-simple Lie group follow a similar asymptotic law, provided the step distribution is rapidly decaying and has an absolutely continuous component with respect to the Haar measure. The exponential decay rate depends on the step distribution, but the critical exponent (the power of n in the asymptotic formula, in our case $3/2$) depends only on the rank and the number of positive, indivisible roots of the group. Theorem 1.11 shows that – at least for $SL(2,\mathbb{R})$ – the critical exponent is inherited by a large class of co-compact discrete subgroups. For random walks on free groups [14], [23], certain free products [34], and virtually free groups that are not virtually cyclic (including $SL_2(\mathbb{Z})$), [24], [27], [33] local limit theorems of the form (12) have been known for some time. In all of these cases the Green's functions are algebraic functions of r. We expect (but cannot prove) that for symmetric, finite-range random walks on co-compact Fuchsian groups the Green's functions are *not* algebraic. However, we will prove in section 9 (Theorem 9.3) that the Green's function admits an analytic continuation to a doubly slit plane $\mathbb{C} \setminus ([R,\infty) \cup (-\infty, -R(1+\varepsilon)])$, and Theorem 1.10 implies that if the singularity at $r = R$ is a branch point then it must be of order 2. If it could be shown that the singularity is indeed a branch point then it would follow that the transition probabilities have complete asymptotic expansions in powers of $n^{-1/2}$.

2. Green's function: preliminaries

Throughout this section, X_n is a symmetric, finite-range irreducible random walk on a finitely generated, nonamenable group Γ with (symmetric) generating set A. Let S denote the support of the step distribution of the random walk. We assume throughout that S is finite, and hence contained in a ball $B(1, C_0)$ for some $C_0 \geq 1$.

2.1. **Green's function as a sum over paths.** The Green's function $G_r(x,y)$ defined by (1) has an obvious interpretation as a sum over paths from x to y. (Note: Here and in the sequel a *path* in Γ is just a sequence x_n of vertices in the Cayley graph G^Γ with $x_n^{-1}x_{n+1}\in S$ for all n). Denote by $\mathcal{R}(x,y)$ the set of all paths γ from x to y, and for any such path $\gamma=(x_0,x_1,\dots,x_m)$ define the *weight*

$$w_r(\gamma) := r^m\prod_{i=0}^{m-1}p(x_i,x_{i+1}).$$

Then

$$G_r(x,y)=\sum_{\gamma\in\mathcal{R}(x,y)}w_r(\gamma). \tag{13}$$

Since the step distribution $p(x)=p(x^{-1})$ is symmetric with respect to inversion, so is the weight function $\gamma\mapsto w_r(\gamma)$: if γ^R is the reversal of the path γ, then $w_r(\gamma^R)=w_r(\gamma)$. Consequently, the Green's function is symmetric in its arguments:

$$G_r(x,y)=G_r(y,x). \tag{14}$$

Also, the weight function is multiplicative with respect to concatenation of paths, that is, $w_r(\gamma\gamma')=w_r(\gamma)w_r(\gamma')$. Since the random walk is irreducible, every generator of the group can be reached by the random walk in finite time with positive probability. It follows that the Green's function satisfies a system of *Harnack inequalities*: There exists a constant $C<\infty$ such that for each $0<r\le R$ and all group elements x,y,z,

$$G_r(x,z)\le C^{d(y,z)}G_r(x,y). \tag{15}$$

Proof of Proposition 1.9. This is a routine calculation based on the representation (13) of the Green's function as a sum over paths. Since all terms in the power series representation of the Green's function have nonnegative coefficients, interchange of d/dr and $\sum_\gamma$ is permissible, so

$$\frac{d}{dr}G_r(x,y)=\sum_{\gamma\in\mathcal{R}(x,y)}\frac{d}{dr}w_r(\gamma).$$

If γ is a path from 1 to x of length m, then the derivative with respect to r of the weight $w_r(\gamma)$ is $mw_r(\gamma)/r$, so $dw_r(\gamma)/dr$ contributes one term of size $w_r(\gamma)/r$ for each vertex visited by γ after its first step. This, together with the multiplicativity of w_r, yields the identity (9). □

2.2. **First-passage generating functions.** Other useful generating functions can be obtained by summing path weights over different sets of paths. Two classes of such generating functions that will be used below are the *restricted Green's functions* and the *first-passage* generating functions (called the *balayage* by Ancona [2]) defined as follows. Fix a set of vertices $\Omega\subset\Gamma$, and for any $x,y\in\Gamma$ let $\mathcal{P}(x,y;\Omega)$ be the set of all paths from x to y that remain in the region Ω at all except the initial and final points. Define

$$G_r(x,y;\Omega)=\sum_{\mathcal{P}(x,y;\Omega)}w_r(\gamma),\quad\text{and}$$

$$F_r(x,y)=G_r(x,y;\Gamma\setminus\{y\}).$$

Thus, $F_r(x,y)$, the *first-passage generating function*, is the sum over all paths from x to y that first visit y on the last step. This generating function has the alternative representation

$$F_r(x,y)=E^x r^{\tau(y)}$$

where $\tau(y)$ is the time of the first visit to y by the random walk X_n, and the expectation extends only over those sample paths such that $\tau(y) < \infty$. Finally, since any visit to y by a path started at x must follow a *first* visit to y,

$$G_r(x, y) = F_r(x, y)G_r(1, 1). \tag{16}$$

Therefore, since G_r is symmetric in its arguments, so is F_r.

Lemma 2.1.

$$\lim_{n\to\infty} \max_{\{x\in\Gamma\,:\,|x|=n\}} G_R(1, x) = 0.$$

Proof. If γ is a path from 1 to x, and γ' a path from x to 1, then the concatenation $\gamma\gamma'$ is a path from 1 back to 1. Furthermore, since any path from 1 to x or back must make at least $|x|/C_0$ steps, the length of $\gamma\gamma'$ is at least $2|x|/C_0$. Consequently, by symmetry,

$$F_R(1, x)^2 G_R(1, 1) \le \sum_{n=2|x|/C_0}^{\infty} P^1\{X_n = 1\}R^n. \tag{17}$$

Since $G_R(1, 1) < \infty$, by nonamenability of the group Γ, the tail-sum on the right side of inequality (17) converges to 0 as $|x| \to \infty$, and so $F_R(1, x) \to 0$ as $|x| \to \infty$. Consequently, by (16), so does $G_R(1, x)$. □

Several variations on this argument will be used later.

2.3. **Subadditivity and the random walk metric.** A path from x to z visiting y can be uniquely split into a path from x to y with first visit to y at the last point, and a path from y to z. Consequently, by the Markov property (or alternatively the path representation (13) and the multiplicativity of the weight function w_r), $G_r(x, z) \ge F_r(x, y)G_r(y, z)$. Since $G_r(x, z) = F_r(x, z)G_r(1, 1)$ and $G_r(y, z) = F_r(y, z)G_r(1, 1)$, we deduce that the function $-\log F_r(x, y)$ is *subadditive*:

Lemma 2.2. *For each $r \le R$ the first-passage generating functions $F_r(x, y)$ are super–multiplicative, that is, for any group elements x, y, z,*

$$F_r(x, z) \ge F_r(x, y)F_r(y, z).$$

Together with Kingman's subadditive ergodic theorem, this implies that the Green's function $G_r(1, x)$ must decay at a fixed exponential rate along suitably chosen trajectories. For instance, if

$$Y_n = \xi_1\xi_2 \cdots \xi_n$$

where ξ_n is an ergodic Markov chain on the alphabet A, or on the set A^K of words of length K, then Kingman's theorem implies that

$$\lim n^{-1} \log G_r(1, Y_n) = \alpha \quad \text{a.s.} \tag{18}$$

where α is a constant depending only on the transition probabilities of the underlying Markov chain. More generally, if ξ_n is a suitable ergodic stationary process, then (18) will hold. Super-multiplicativity of the Green's function also implies the following.

Corollary 2.3. *The function $d_G(x, y) := \log F_R(x, y)$ is a metric on Γ.*

Proof. The triangle inequality is immediate from Lemma 2.2, and symmetry $d_G(x, y) = d_G(y, x)$ follows from the corresponding symmetry property (14) of the Green's function. Thus, to show that d_G is a metric (and not merely a pseudo-metric) it suffices to show that if $x \neq y$ then $F_R(x, y) < 1$. But this follows from the fact (2) that the Green's function is finite at the spectral radius, because the path representation implies that

$$G_R(x, x) \geq 1 + F_R(x, y)^2 + F_R(x, y)^4 + \cdots . \qquad \square$$

Call d_G the *Green metric*. The Harnack inequalities imply that d_G is dominated by a constant multiple of the word metric d. In general, there is no domination in the other direction. However:

Proposition 2.4. *If the Green's function decays exponentially in $d(x, y)$ (that is, if inequality (6) holds for all $x, y \in \Gamma$), then the Green metric d_G and the word metric d on Γ are quasi-isometric, that is, there are constants $0 < C_1 < C_2 < \infty$ such that for all $x, y \in \Gamma$,*

$$C_1 d(x, y) \leq d_G(x, y) \leq C_2 d(x, y). \tag{19}$$

Proof. If inequality (6) holds for all $x, y \in \Gamma$, then the first inequality in (19) will hold with $C_1 = -\log \varrho$. $\square$

2.4. **Green's function and branching random walks.** There is a simple interpretation of the Green's function $G_r(x, y)$ in terms of the occupation statistics of *branching random walks*. A branching random walk is built using a probability distribution $\mathcal{Q} = \{q_k\}_{k\geq 0}$ on the nonnegative integers, called the *offspring distribution*, together with the step distribution $\mathcal{P} := \{p(x, y) = p(x^{-1}y)\}_{x,y\in\Gamma}$ of the underlying random walk, according to the following rules: At each time $n \geq 0$, each particle fissions and then dies, creating a random number of offspring with distribution $\mathcal{Q}$; the offspring counts for different particles are mutually independent. Each offspring particle then moves from the location of its parent by making a random jump according to the step distribution $p(x, y)$; the jumps are once again mutually independent. Consider the initial condition which places a single particle at site $x \in \Gamma$, and denote the corresponding probability measure on population evolutions by Q^x.

Proposition 2.5. *Under Q^x, the total number of particles in generation n evolves as a Galton-Watson process with offspring distribution $\mathcal{Q}$. If the offspring distribution has mean $r \leq R$, then under Q^x the expected number of particles at location y at time n is $r^n P^x\{X_n = y\}$, where under P^x the process X_n is an ordinary random walk with step distribution $\mathcal{P}$. Therefore, $G_r(x, y)$ is the mean total number of particle visits to location y.*

Proof. The first assertion follows easily from the definition of a Galton-Watson process – see [4] for the definition and basic theory. The second is easily proved by induction on n. The third then follows from the formula (1) for the Green's function. $\square$

There are similar interpretations of the restricted Green's function $G_r(x, y; \Omega)$ and the first-passage generating function $F_r(x, y)$. In particular, if particles of the branching random walk are allowed to reproduce only in the region Ω, then $G_r(x, y; \Omega)$ is the mean number of particle visits to y in this modified branching random walk.

3. Exponential decay of the Green's function

3.1. **Hyperbolic geometry.** In this section, we prove that the Green's function of a symmetric random walk on a co-compact Fuchsian group decays exponentially fast. The proof is most conveniently formulated using some basic ingredients of hyperbolic geometry (see, e.g., [20], chs. 3–4).

For any co-compact, Fuchsian group Γ there is a *fundamental polygon* $\mathcal{F}$ for the action of Γ. The closure of $\mathcal{F}$ is a compact, finite-sided polygon whose sides are arcs of hyperbolic geodesics. The hyperbolic disk $\mathbb{D}$ is tiled by the images $x\mathcal{F}$ of $\mathcal{F}$, where $x \in \Gamma$; distinct elements $x, y \in \Gamma$ correspond to tiles $x\mathcal{F}$ and $y\mathcal{F}$ which intersect, if at all, in a single side of each tile. Thus, the elements of Γ are in bijective correspondence with the tiles of the tessellation, or alternatively with the points xO in the $\Gamma-$orbit of a distinguished point $O \in \mathcal{F}^\circ$. The Cayley graph of Γ (relative to the standard generators) is gotten by putting *edges* between those *vertices* $x, y \in \Gamma$ such that the tiles $x\mathcal{F}$ and $y\mathcal{F}$ share a side. The *word metric* $d = d_\Gamma$ on Γ is the (Cayley) graph distance. The hyperbolic metric induces another distance $d_{\mathbb{H}}(x, y)$ on Γ, defined to be the hyperbolic distance between the points xO and yO. The metrics d and $d_{\mathbb{H}}$ are both left-invariant.

The following fact is well known (see, e.g., [9], Theorem 1).

Lemma 3.1. *The hyperbolic metric $d_{\mathbb{H}}$ and the word metric d on Γ are quasi-isometric.*

3.2. **Constructing barriers.** To prove that the Green's function decays exponentially fast, we will put barriers along the hyperbolic geodesic between two points xO and yO, such that the weight w_R of paths crossing any of those barriers is small. Let us fix once and for all a constant $C_0 > 0$ such that the jumps of the random walk are bounded by C_0.

Definition 3.2. Let $\xi \neq \eta$ be two distinct points of $S^1 = \partial\mathbb{D}$. A *barrier* between ξ and η is a subset B of Γ (identified with ΓO) such that:

(1) There exist neighborhoods V and W of ξ and η such that every path $x_0, x_1, \dots, x_n$ connecting points $x_0 \in V$ and $x_n \in W$ in Γ with $d(x_i, x_{i+1}) \leq C_0$ for all i must go through B, i.e., $x_i \in B$ for some i.

(2) For all $x \in V$,
$$\sum_{b \in B} G_R(x, b) \leq 1/2.$$

(3) The set $\partial B \subset S^1$ of limit points of B satisfies $\partial B \cap \{\xi, \eta\} = \emptyset$.

The main result of this subsection is the following.

Theorem 3.3. *For any two points $\xi \neq \eta \in S^1$, there exists a barrier between ξ and η.*

The rest of this subsection is devoted to the proof of this theorem. The main idea is to construct the barrier using typical trajectories of the random walk.

Lemma 3.4. *There exist $C > 0$ and $\varrho < 1$ such that $P^1(G_R(1, X_n) \geq \varrho^n) \leq C\varrho^n$.*

Proof. Any path γ of length n from 1 to a point x can be concatenated with any path γ' from x to 1, yielding a path from 1 to itself. Hence, summing the weights of all such paths gives a lower bound for $G_R(1, 1)$. This implies
$$G_R(1, 1) \geq \sum_x R^n P^1(X_n = x) G_R(x, 1) = R^n E^1(G_R(X_n, 1)).$$

Therefore,
$$P^1(G_R(X_n, 1) \geq \varrho^n) \leq \frac{1}{\varrho^n} E^1(G_R(X_n, 1)) \leq CR^{-n}\varrho^{-n}.$$

Taking $\varrho = R^{-1/2}$, we get the conclusion of the lemma. □

Lemma 3.5. *For almost all independent trajectories $X_0 = 1, X_1, \dots$ and $Y_0 = 1, Y_1, \dots$ of the random walk,*

$$\sum_{m,n\in\mathbb{N}} G_R(X_m, Y_n) < \infty.$$

Proof. For fixed m and n, $Y_n^{-1}X_m$ is distributed as X_{n+m}, by symmetry of μ. Therefore,

$$P(G_R(X_m, Y_n) \geq \varrho^{m+n}) = P(G_R(Y_n^{-1}X_m, 1) \geq \varrho^{m+n}) = P(G_R(X_{n+m}, 1) \geq \varrho^{m+n}) \leq C\varrho^{n+m},$$

by the previous lemma. Since this quantity is summable in m and n, Borel-Cantelli ensures that, almost surely, $G_R(X_m, Y_n) \leq \varrho^{m+n}$ for all but finitely many pairs (m, n). □

Instead of barriers, it will be easier to construct *pre-barriers* as defined below.

Definition 3.6. Let $\xi \neq \eta$ be two distinct points in $S^1 = \partial\mathbb{D}$. A pre-barrier between ξ and η is a pair (A, B) of subsets of Γ such that for some neighborhoods V of ξ and W of ζ:

1. Every path $x_0, x_1, \dots, x_n$ connecting points $x_0 \in V$ and $x_n \in W$ in Γ such that $d(x_i, x_{i+1}) \leq C_0$ for all i must go first through A and then through B, i.e., there exist $i < j$ such that $x_i \in A$ and $x_j \in B$.
2. In the same way, every path from V to B with jumps bounded by C_0 must first go through A.
3. We have
$$\sum_{a\in A, b\in B} G_R(a, b) < \infty.$$
4. We have $\partial(A \cup B) \cap \{\xi, \eta\} = \emptyset$.

Lemma 3.7. *For all $\xi \neq \eta$, there exists a pre-barrier between ξ and η.*

Proof. Choose two disjoint compact subintervals J and K in one of the connected components of $S^1 - \{\xi, \eta\}$ such that J is closer to ξ than K, and similarly choose J', K' in the other connected component of $S^1 - \{\xi, \eta\}$.

Let Ω be a probability space that supports 4 independent copies of the random walk starting from 1. Denote their trajectories by X_n, X'_n, Y_n, Y'_n. Let Ω' be the event that X_n converges to J, X'_n converges to J', Y_n converges to K and Y'_n converges to K'. The Poisson boundary of the random walk is identified with the topological boundary S^1, and the limiting measure has full support there since the group action is minimal on the boundary. Therefore, Ω' has positive probability. Moreover, for a typical point in Ω, Lemma 3.5 shows that $\sum G_R(\tilde{X}_m, \tilde{Y}_n) < \infty$ for $\tilde{X} = X$ or X' and $\tilde{Y} = Y$ or Y'. Let us fix such typical sequences in Ω'.

Write $X_{-n} = X'_n$ and $Y_{-n} = Y'_n$ for $n > 0$. By construction, $\sum_{m,n\in\mathbb{Z}} G_R(X_m, Y_n) < \infty$. Moreover, the sequence X_m makes jumps of at most C_0, and converges in positive time (resp. negative time) to a point in J (resp. J'). Similarly, the sequence Y_n makes jumps of at most C_0, and converges to K and K' in positive and negative time. Since $J \cup J'$ is disjoint from $K \cup K'$, these trajectories are disjoint for large $|m|, |n|$. Modifying the trajectories at finitely many places (which does not change the validity of $\sum_{m,n} G_R(X_m, Y_n) < \infty$) and then thickening them, we obtain the required pre-barrier. □

Theorem 3.3 follows from the previous lemma and the next lemma.

Lemma 3.8. *If (A, B) is a pre-barrier between ξ and η, then B is a barrier between those same points.*

Proof. The only nontrivial point is the existence of a neighborhood V of ξ such that, for any $x \in V$, we have $\sum_{b\in B} G_R(x,b) \leq 1/2$.

Let V be a small neighborhood of ξ. Every path from V to B must go first through A. Decomposing such paths according to the first visited point in A, we get for any $b \in B$,

$$G_R(x,b) \leq \sum_{a\in A} G_R(x,a)G_R(a,b).$$

As a consequence,

$$\sum_{b\in B} G_R(x,b) \leq \sup_{a\in A} G_R(x,a) \cdot \sum_{a\in A, b\in B} G_R(a,b) \leq C \sup_{a\in A} G_R(x,a).$$

By Lemma 2.1, $G_R(1,x)$ tends uniformly to 0 when $|x| \to \infty$. Therefore, if V is small enough then $\sup_{a\in A} G_R(x,a)$ is smaller than $1/(2C)$. This concludes the proof. □

3.3. **Exponential decay of the Green's function.**

Theorem 3.9. *The Green's function $G_R(x,y)$ of a symmetric, irreducible, finite-range random walk on a co-compact Fuchsian group Γ, evaluated at its radius of convergence R, decays exponentially in the distance $d(x,y)$: there exist constants $C < \infty$ and $\varrho < 1$ such that for every $x \in \Gamma$,*

$$G_R(1,x) \leq C\varrho^{|x|}.$$

To prove this theorem, we will show that disjoint barriers can be placed consecutively along the hyperbolic geodesic from 1 to x (identified with O and xO) in $\mathbb{D}$.

Lemma 3.10. *There exists $L > 0$ with the following property. If $x \in \Gamma$ satisfies $d(1,x) \geq nL$ for some integer $n \geq 2$, then there exist barriers $B_1, \dots, B_{n-1}$ such that*

- *every path from 1 to x with jumps bounded by C_0 goes successively through $B_1, B_2, \dots, B_{n-1}$.*
- *We have $\sum_{b_1\in B_1} G_R(1,b_1) \leq 1/2$.*
- *For any $i < n-1$ and any $b_i \in B_i$,*

$$\sum_{b_{i+1}\in B_{i+1}} G_R(b_i, b_{i+1}) \leq 1/2. \tag{20}$$

Before proving this, let us explain why it implies Theorem 3.9.

Proof of Theorem 3.9. Let $x \in \Gamma$ be a vertex such that $d(1,x) \in [nL, (n+1)L)$. Decomposing paths from 1 to x according to their first points of entry into B_1, then B_2, and so on, we get

$$\begin{aligned} G_R(1,x) &\leq \sum_{b_1\in B_1,\dots,b_{n-1}\in B_{n-1}} G_R(1,b_1)G_R(b_1,b_2)\dots G_R(b_{n-2},b_{n-1})G_R(b_{n-1},x) \\ &\leq 2^{-n+1} \sup_{b_{n-1}\in B_{n-1}} G_R(b_{n-1},x). \end{aligned}$$

Since G_R tends to 0 at infinity, it is uniformly bounded, by (say) C. Hence, $G_R(1,x) \leq C2^{-n+1}$, which is the desired exponential bound. □

Proof of Lemma 3.10. Because the group Γ is co-compact, there exists $C_1 < \infty$ such that every point in $\mathbb{D}$ is within distance C_1 of a translate xO of O, for some $x \in \Gamma$. Denote by $\mathcal{G}$ the set of pairs of distinct points (ξ,η) in S^1 such that the point $O \in \mathbb{D}$ is at distance at most C_1 from the hyperbolic geodesic from ξ to η. This is a compact subset of $S^1 \times S^1$. By Theorem 3.3, for every $(\xi,\eta) \in \mathcal{G}$, there exist two neighborhoods V, W of ξ and η and a barrier B

between V and W. We can assume that V and W are both halfplanes bounded by small (in the Euclidean metric on $\mathbb{D}$) hyperbolic geodesics around ξ and η. Let $\tilde{V}$ and $\tilde{W}$ be strictly smaller neighborhoods of ξ and η, also chosen to be halfplanes. By compactness, $\mathcal{G}$ is covered by a finite number of sets $\tilde{V} \times \tilde{W}$, say $\tilde{V}^k \times \tilde{W}^k$ for $k \in \{1, \dots, K\}$. Let B^k denote the corresponding barrier, and V^k, W^k the corresponding larger neighborhoods.

We turn now to the assertion of the lemma. Let L be large, and let $x \in \Gamma$ be such that $d(1, x) \geq nL$. Since the Cayley graph is quasi-isometric to the hyperbolic disk, by Lemma 3.1, the hyperbolic distance $d_{\mathbb{H}}(O, xO)$ is at least nL/C, for some constant C determined by the quasi-isometry. Consider the hyperbolic geodesic D between O and xO, which we extend infinitely in both directions, with endpoints ξ_0 and η_0. For every $i \in \{1, \dots, n-1\}$, the point at distance iL/C of O on D is at distance at most C_1 of a translate $x_i O$ of O, by definition of C_1. Therefore, O is at a distance at most C_1 of the geodesic $D_i = x_i^{-1}(D)$, with endpoints $\xi_i = x_i^{-1}(\xi_0)$ and $\eta_i = x_i^{-1}(\eta)$. In particular, $(\xi_i, \eta_i) \in \mathcal{G}$. Therefore, there exists $k_i \in \{1, \dots, K\}$ such that $\xi_i \in \tilde{V}^{k_i}$ and $\eta_i \in \tilde{W}^{k_i}$. Let $B_i = x_i B^{k_i}$, where B^{k_i} is the barrier for the pair V^{k_i}, W^{k_i}. We will show that $B_1, B_2, \dots, B_{n-1}$ is a sequence of barriers satisfying the requirements of the lemma, provided L is large enough.

Let us write $x_{i+1} = x_i y_{i+1}$. Since $x_{i+1}O$ is at distance at most C_1 of D and x_i^{-1} is a hyperbolic isometry, $y_{i+1}O$ is at distance at most C_1 of D_i. Moreover, it is at distance at least $L/C - 2C_1$ of O. Since D_i tends towards the point η_i in $\tilde{W}^{k_i}$, we deduce that $y_{i+1}O$ is in the larger neighborhood W^{k_i} of η_i if L is large. Increasing L, we may even ensure that it is arbitrarily deep in this set (i.e., arbitrarily far from its boundary). Let us apply the isometry y_{i+1}^{-1}. The geodesic ∂W^{k_i} is sent to a geodesic which is arbitrarily far away from $y_{i+1}^{-1}(y_{i+1}O) = O$. Therefore, this geodesic is arbitrarily small for the euclidean distance in $\overline{\mathbb{D}}$. The complement of W^{k_i} is also contracted on a set of arbitrarily small euclidean diameter ε. This set contains $y_{i+1}^{-1}\xi_i = \xi_{i+1}$, as well as $y_{i+1}^{-1}B^{k_i} = x_{i+1}^{-1}B_i$. Since $\xi_{i+1} \in \tilde{V}^{k_{i+1}}$, we deduce that $x_{i+1}^{-1}B_i$ is contained in the slightly larger set $V^{k_{i+1}}$ if ε is small enough. Moreover, $x_{i+1}^{-1}B_{i+1}$ is a barrier between $V^{k_{i+1}}$ and $W^{k_{i+1}}$ (which contains the preimage $x_{i+1}^{-1}(xO)$ of xO if L is large enough). It follows that every trajectory going through B_i and ending at xO has to go through B_{i+1}. We also obtain the estimate (20) from the definition of barriers.

The estimate $\sum_{b_1 \in B_1} G_R(1, b_1) \leq 1/2$ is similar. We have proved that $B_1, \dots, B_{n-1}$ satisfy all the requirements of the lemma. □

4. Ancona's inequalities

In this section, we study Ancona's inequalities (5) and their consequences. First, we will derive such inequalities from the exponential decay of Green's functions that was proved in the previous section.

4.1. **Relative Ancona's inequalities.** For technical reasons, we will need Ancona's inequalities for relative Green's functions (as defined in subsection 2.2). Such inequalities guarantee that, for any point z on a geodesic segment between two points x and y, and for a suitable class of domains Ω, one has

$$G_r(x, y; \Omega) \leq C G_r(x, z; \Omega) G_r(z, y; \Omega).$$

The constant C should be independent of x, y, z, of Ω, and of $r \in [1, R]$. The usual Ancona's inequalities (5) correspond to the case $\Omega = \Gamma$. Of course, if x and y belong to Ω (and there is a positive probability to go from x to y in Ω) but z is not in the neighborhood of size C_0 of Ω, such an inequality can not hold. Therefore, one should put restrictions upon the set Ω.

Relative's Ancona inequalities play an important role in [1] (see th. 1′ there), [26, 18]. In those two last papers, relative Ancona's inequalities are proved whenever the domain Ω contains a neighborhood of fixed size C of the geodesic segment $[xy]$. We are only able to deal with a smaller class of domains, that will nevertheless be sufficient for the applications.

Theorem 4.1. *Consider a symmetric, irreducible, finite-range random walk on a co-compact Fuchsian group Γ. There exist positive constants C_1 and C_2 with the following property. Consider two points x, y, and a point z on a geodesic segment $[xy]$ between those two points. Let Ω be a subset of Γ such that, for any $w \in [xy]$, the ball $B(w, C_1 + d(w, z)/2)$ is included in Ω. Then, for all $r \in [1, R]$,*

$$G_r(x, y; \Omega) \leq C_2 G_r(x, z; \Omega) G_r(z, y; \Omega).$$

Let us note that the converse inequality $G_r(x, y; \Omega) \geq C G_r(x, z; \Omega) G_r(z, y; \Omega)$ is trivial (see Lemma 2.2), hence this theorem really says that $G_r(x, y; \Omega) \asymp G_r(x, z; \Omega) G_r(z, y; \Omega)$ (meaning that the ratio between the two sides of this equation remains bounded away from 0 and ∞).

We start with the following lemma, which follows easily from the exponential decay of the Green's function and the geometry of the hyperbolic disk.

Lemma 4.2. *There exist $C > 0$ and $\alpha > 0$ such that, for any geodesic segment $[xy]$, for any $z \in [xy]$ and for any $k \geq 0$,*

$$G_R(x, y; B(z, k)^c) \leq C e^{-e^{\alpha k}}.$$

Here, $B(z, k)^c$ is the complement of the ball $B(z, k)$. Since the distance from x to y in the complement of $B(z, k)$ is exponential in k, and the Green's function itself decays exponentially, it is not surprising to get two composed exponentials in the conclusion of the lemma.

Proof. It suffices to prove the lemma for large enough k. We work again in the hyperbolic disk $\mathbb{D}$. Composing everything by an element of Γ, we can assume without loss of generality that $z = 1$. Consider the hyperbolic circle $\mathcal{C}$ of radius k/C around O, where C is large enough so that this circle is contained in the ball $B(1, k)$ for the Cayley graph distance. Since the hyperbolic length of $\mathcal{C}$ is exponential in terms of k, one can set along this circle a number $e^{\alpha k}$ of straight lines D_i starting from O and going to infinity, such that the distance d from D_i to D_{i+1} along $\mathcal{C}$ is arbitrarily large. Let $\tilde{D}_i$ be a thickening of D_i so that a path making jumps of at most C_0 can not jump across $\tilde{D}_i$. If d is large enough, one has $\sum_{a_i \in \tilde{D}_i, a_{i+1} \in \tilde{D}_{i+1}} G_R(a_i, a_{i+1}) \leq 1/2$ (this sum is at most a geometric series thanks to the exponential decay of the Green's function, and it has arbitrarily small first term if d is large). A path from x to y avoiding $B(1, k)$ has to go around the circle in one direction or the other, and should therefore cross at least half the domains $\tilde{D}_i$. As in the conclusion of the proof of Theorem 3.9, this yields $G_R(x, y; B(1, k)^c) \leq C 2^{-e^{\alpha k}/2}$. □

Proof of Theorem 4.1. Let $[xy]$ be a geodesic segment of some length m (for the group distance), let z be a point on $[xy]$, and let Ω be a domain as in the statement of the theorem.

We construct by induction points x_n, y_n on the geodesic segment $[xy]$ such that $z \in [x_n y_n]$ for all n, as follows. First, let $x_0 = x$ and $y_0 = y$. At step n, if z is in the left half of $[x_n y_n]$, we let $x_{n+1} = x_n$, and y_{n+1} is the point in $[x_n y_n]$ at distance $d(x_n, y_n)/4$ of y_n. Otherwise, the construction is symmetric: $y_{n+1} = y_n$ and $d(x_n, x_{n+1}) = d(x_n, y_n)/4$. We stop this construction at the first step N such that $d(x_N, y_N) \leq A$, for some large enough fixed $A > 0$. By construction, $d(x_n, y_n) = (3/4)^n m$ for all $n \leq N$, and $z \in [x_n y_n]$.

Let c_{n+1} be the midpoint of the discarded interval at step n (i.e., $[y_{n+1} y_n]$ in the first case, $[x_n x_{n+1}]$ in the second case). In both cases, $d(c_{n+1}, z) \geq d(x_n, y_n)/4$. Therefore, the ball $B_{n+1} = B(c_{n+1}, d(x_n, y_n)/100)$ is included in Ω. Moreover, those balls are disjoint.

We will decompose $G_r(x, y; \Omega)$ according to the visits of the trajectories to B_i or not. Let us write $u_0 = x$ and $v_0 = y$. Assume for instance that at the first step we are in the first case (i.e., z is in the first half of $[u_0 v_0]$). In this case, considering the last visit of a path to B_1, we get

$$G_r(u_0, v_0; \Omega) = G_r(u_0, v_0; \Omega \cap B_1^c) + \sum_{v_1 \in B_1} G_r(u_0, v_1; \Omega) G_r(v_1, v_0; \Omega \cap B_1^c).$$

If z is in the second half of $[u_0 v_0]$, considering rather the first visit to B_1, we get

$$G_r(u_0, v_0; \Omega) = G_r(u_0, v_0; \Omega \cap B_1^c) + \sum_{u_1 \in B_1} G_r(u_0, u_1; \Omega \cap B_1^c) G_r(u_1, v_0; \Omega).$$

To get manageable formulas, we will introduce a more symmetric notation. Let $H(u, v; B) = G_r(u, v; \Omega \cap B)$ if $u \neq v$, and 1 if $u = v$. Write moreover $u_1 = u_0$ in the first case, and $v_1 = v_0$ in the second case. The formulas become

$$G_r(u_0, v_0; \Omega) = H(u_0, v_0; B_1^c) + \sum_{u_1, v_1} H(u_0, u_1; B_1^c) G_r(u_1, v_1; \Omega) H(v_1, v_0; B_1^c).$$

In the second sum, we can again decompose $G_r(u_1, v_1; \Omega)$ according to the visits to B_2. Inductively, we obtain for all $k \leq N$

$$\begin{aligned} G_r(u_0, v_0; \Omega) = & \\ & \sum_{j=0}^{k-1} \sum_{\substack{u_1, \dots, u_j \\ v_1, \dots, v_j}} H(u_0, u_1; B_1^c) \cdots H(u_{j-1}, u_j; B_j^c) H(u_j, v_j; B_{j+1}^c) \times \\ & \qquad \times H(v_j, v_{j-1}; B_j^c) \cdots H(v_1, v_0; B_1^c) \\ & + \sum_{\substack{u_1, \dots, u_k \\ v_1, \dots, v_k}} H(u_0, u_1; B_1^c) \cdots H(u_{k-1}, u_k; B_k^c) G_r(u_k, v_k; \Omega) H(v_k, v_{k-1}; B_k^c) \cdots H(v_1, v_0; B_1^c). \end{aligned} \tag{21}$$

The quantity $H(u_j, v_j; B_{j+1}^c)$ measures the r-weight of paths from u_j to v_j that avoid the ball B_{j+1}. By hyperbolicity, the center of B_{j+1} is at a bounded distance from the geodesic segment from u_j to v_j. Since the radius of B_{j+1} is $m(3/4)^j/100$, Lemma 4.2 shows that $H(u_j, v_j; B_{j+1}^c) \leq C e^{-e^{\alpha' m (3/4)^j}}$. As $d(u_j, v_j) = m(3/4)^j$ and Ω contains a fixed size neighborhood of the geodesic segment from u_j to v_j thanks to our geometric assumptions, we have $G_r(u_j, v_j; \Omega) \geq C \varrho^{m(3/4)^j}$ for some $\varrho < 1$. Therefore, $H(u_j, v_j; B_{j+1}^c)$ is much smaller than $G_r(u_j, v_j; \Omega)$, say

$$H(u_j, v_j; B_{j+1}^c) \leq \lambda_j G_r(u_j, v_j; \Omega),$$

where λ_j is superexponentially small in terms of $m(3/4)^j$. The term corresponding to j in (21) is therefore bounded by

$$\lambda_j \sum_{\substack{u_0, \dots, u_j \\ v_0, \dots, v_j}} H(u_0, u_1; B_1^c) \cdots H(u_{j-1}, u_j; B_j^c) G_r(u_j, v_j; \Omega) H(v_j, v_{j-1}; B_j^c) \cdots H(v_1, v_0; B_1^c).$$

This is the last line of (21) (with j instead of k). Since everything in (21) is nonnegative, this is bounded by $\lambda_j G_r(u_0, v_0; \Omega)$.

Let us now take $k = N$ (i.e., the last step of the construction), and let us consider the last line of (21). Since the points u_N and v_N are within distance A of z, we have $G_r(u_N, v_N; \Omega) \asymp 1 \asymp G_r(u_N, z; \Omega) G_r(z, v_N; \Omega)$. This makes it possible to separate the sum over the u_is and the v_is. Doing the same argument but in the other direction (i.e., gluing trajectories instead of splitting them), one shows that those two factors are bounded respectively by $G_r(u_0, z; \Omega)$ and $G_r(z, v_0; \Omega)$. We have obtained

$$G_r(u_0, v_0; \Omega) \leq \left(\sum_{j=0}^{N-1} \lambda_j \right) G_r(u_0, v_0; \Omega) + C(A) G_r(u_0, z; \Omega) G_r(z, v_0; \Omega).$$

If A is large enough, the sum $\sum \lambda_j$ is small, say $\leq 1/2$. We get

$$G_r(u_0, v_0; \Omega) \leq 2C(A) G_r(u_0, z; \Omega) G_r(z, v_0; \Omega). \qquad \square$$

4.2. **Hölder continuity of the Green's function.** In this paragraph, we explain how the controls on the Martin boundary given by Theorem 1.2 follow from the relative Ancona's inequalities of the previous paragraph. The method is the same as in [18], but we give some details since our Ancona's inequalities are proved for a more restricted class of domains.

Fix some $C > 0$ (its precise value is irrelevant for what follows, it should just be large enough). We say that a family $\mathcal{G}$ of geodesic segments is fellow traveling during time k if there are two points at distance k that are both at distance at most C of every geodesic segment in $\mathcal{G}$ (as a consequence, the geodesic segment of length k between those two points is also C' close to every geodesic segment in $\mathcal{G}$, by hyperbolicity). This condition can also be formulated in terms of approximating trees (see [15]): up to a constant, there is an arc of length k in the tree, and the endpoints of any geodesic segment in $\mathcal{G}$ are on the opposite sides of this arc.

Theorem 4.3. *There exist constants $C > 0$ and $\varrho < 1$ with the following property. Consider point x, x', y, y' such that the four geodesic segments from x or x' to y or y' are fellow traveling during time k. Then*

$$\left| \frac{G_r(x, y)/G_r(x', y)}{G_r(x, y')/G_r(x', y')} - 1 \right| \leq C \varrho^k,$$

for all $r \in [1, R]$.

Denoting by $[x_0 y_0]$ the geodesic segment that is close to all geodesics from x or x' to y or y', Ancona's inequalities give $G_r(x, y) \asymp G_r(x, x_0) G_r(x_0, y_0) G_r(y_0, y)$, and similarly for the other geodesics. Therefore,

$$\frac{G_r(x, y)}{G_r(x', y)} \asymp \frac{G_r(x, x_0)}{G_r(x', x_0)} \asymp \frac{G_r(x, y')}{G_r(x', y')}. \tag{22}$$

The theorem is a quantitative strengthening of this estimate, showing that the ratio between those quantities not only stays bounded, but tends exponentially fast to 1 when k tends to infinity.

In particular, let us take $x' = 1$, and y, y' points y_n, y_m on a geodesic ray converging towards a point $\zeta \in \partial\Gamma$. The geodesic segments from x, x' to y_n, y_m are fellow traveling for a time at least $\min(m, n) - C_x$. Therefore,

$$\left| \frac{G_r(x, y_n)/G_r(1, y_n)}{G_r(x, y_m)/G_r(1, y_m)} - 1 \right| \leq C_x \varrho^{\min(m,n)}.$$

This shows that the sequence $G_r(x,y_n)/G_r(1,y_n)$ is Cauchy, therefore convergent to a limit $K_r(x,\zeta)$. Moreover, letting m tend to infinity, one get $|G_r(x,y_n)/G_r(1,y_n) - K_r(x,\zeta)| \leq C_x\varrho^n K_r(x,\zeta)$, which proves Theorem 1.2.

We will not give all the details of the proof (regarding especially the precise values of the constants or the hyperbolic geometry computations), since the interested reader may find them in [18]. Constructions are most conveniently formulated using the Gromov product $(x|y)_z = (d(x,z)+d(y,z)-d(x,y))/2$, which measures the time during which two geodesic segments from z to x and from z to y are fellow traveling.

Proof of Theorem 4.3. Let L be a large enough constant. Under the assumptions of the theorem, consider a geodesic segment $[x_0y_0]$ of length k which is close to the four geodesics from x or x' to y or y', such that x_0 is closer to x,x' and y_0 is closer to y,y'. Let $\ell = k/(3L)$. We define a sequence of domains Ω_i (for $1\leq i\leq \ell$) as the set of points z such that $(x_0|z)_{y_0} \geq 3Li$. In particular, $x,x' \in \Omega_\ell$ and $y,y' \notin \Omega_1$. Let us start with the function $u(z) = G_r(z,y)/G_r(x_0,y)$, which is r-harmonic on Ω_1 and normalized by $u(x_0) = 1$. Let $u_1 = u$, we will construct a sequence of functions u_i on Ω_i, with $u_{i-1} = u_i + \varphi_i$ such that φ_i does not depend on the initial normalized harmonic function u, is r-harmonic on Ω_i, and $u_{i-1} \geq \varphi_i \geq \varepsilon u_{i-1}$ on Ω_i, for some $\varepsilon > 0$. Let us first show how this gives the conclusion of the theorem.

As $\varphi_i \geq \varepsilon u_{i-1}$, we have $u_i \leq (1-\varepsilon)u_{i-1}$, hence $u_\ell \leq (1-\varepsilon)^{\ell-1}u_1$. Applying the same construction to $v(z) = G_r(z,y')/G_r(x_0,y')$, we obtain $v = \sum\varphi_i + v_\ell$ (for the same functions φ_i), which gives on Ω_ℓ the estimate $|u-v| = |u_\ell - v_\ell| \leq C(1-\varepsilon)^\ell(u+v)$. Since $u \asymp v$ by (22), we get $|u/v - 1| \leq C(1-\varepsilon)^\ell$ on Ω_ℓ, which is the desired inequality.

Let us now describe the construction of φ_i. Assume that $u_1,\dots,u_i$ are constructed. By harmonicity, for any $z\in\Omega_{i+1}$,

$$u_i(z) = \sum_{w\in\Omega_i^c} G_r(z,w;\Omega_i)u_i(w).$$

Let $\Lambda_i = \{z \ :\ (x_0|z)_{y_0} \in [3Li+L-C_0, 3Li+L+C_0]\}$ (where C_0 is the maximal size of the jumps of the random walk). This set is contained in Ω_i, but away from Ω_{i+1}, and any trajectory from Ω_{i+1} to the complement of Ω_i has to cross Λ_i. Splitting a trajectory from $z\in\Omega_{i+1}$ to $w\in\Omega_i^c$ according to its last visit to Λ_i, we get $G_r(z,w;\Omega_i) = \sum_{w'\in\Lambda_i} G_r(z,w';\Omega_i)G_r(w',w;\Omega_i\cap\Lambda_i^c)$.

We are now in a position to estimate $G_r(z,w';\Omega_i)$ using Ancona's inequalities. Indeed, the geodesic segment from z to w' passes close to the point z_i at distance $3Li+2L$ from y_0 on $[x_0y_0]$, by hyperbolicity. Moreover, the domain Ω_i satisfies the assumptions of Theorem 4.1 (those two statements readily follow from a tree approximation). Hence, $G_r(z,w';\Omega_i) \leq CG_r(z,z_i;\Omega_i)G_r(z_i,w';\Omega_i)$. We obtain

$$\begin{aligned}
u_i(z) &= \sum_{w\in\Omega_i^c}\sum_{w'\in\Lambda_i} G_r(z,w';\Omega_i)G_r(w',w;\Omega_i\cap\Lambda_i^c)u_i(w)\\
&\leq CG_r(z,z_i;\Omega_i)\sum_{w\in\Omega_i^c}\sum_{w'\in\Lambda_i} G_r(z_i,w';\Omega_i)G_r(w',w;\Omega_i\cap\Lambda_i^c)u_i(w)\\
&= CG_r(z,z_i;\Omega_i)u_i(z_i).
\end{aligned}$$

Replacing Ancona's inequality by the trivial bound $G_r(z,w';\Omega_i) \geq C'G_r(z,z_i;\Omega_i)G_r(z_i,w';\Omega_i)$, we also get a lower bound in the last equation. Hence, $u_i(z) \asymp G_r(z,z_i;\Omega_i)u_i(z_i)$ on Ω_{i+1}. Using this estimate for $z = x_0$, we obtain $u_i(z)/u_i(x_0) \asymp G_r(z,z_i;\Omega_i)/G_r(x_0,z_i;\Omega_i)$. In particular, if c is small enough, the function $\varphi_{i+1}(z) = cu_i(x_0)G_r(z,z_i;\Omega_i)/G_r(x_0,z_i;\Omega_i)$ satisfies $\varepsilon u_i \leq \varphi_{i+1} \leq u_i$ on Ω_{i+1}. Moreover, this function only depends on u_i through the value of

$u_i(x_0)$. By induction, it only depends on $u(x_0)$. Since we have normalized u so that $u(x_0) = 1$, this shows that φ_{i+1} is independent of the initial function u. □

5. Automatic structure

5.1. **Strongly Markov groups and hyperbolicity.** A finitely generated group Γ is said to be *strongly Markov* (fortement Markov – see [15]) if for each finite, symmetric generating set A there exists a finite directed graph $\mathcal{A} = (V, E, s_*)$ with distinguished vertex s_* ("start") and a labeling $\alpha : E \to A$ of edges by generators that meets the following specifications. A *path* in the graph is a sequence of edges $e_0, \dots, e_{m-1}$ such that the endpoint of e_i is the starting point of e_{i+1}. Let

$$\mathcal{P} := \{\text{finite paths in } \mathcal{A} \text{ starting at } s_*\},$$

and for each path $\gamma = e_0 e_1 \cdots e_{m-1}$, denote by

$$\alpha(\gamma) = \text{path in } G^\Gamma \text{ through } 1, \alpha(e_0), \alpha(e_0)\alpha(e_1), \dots, \quad \text{and}$$
$$\alpha_*(\gamma) = \text{right endpoint of } \ \alpha(\gamma) = \alpha(e_0)\alpha(e_1)\cdots\alpha(e_{m-1}).$$

Definition 5.1. The labeled automaton $(\mathcal{A}, \alpha)$ is a strongly Markov automatic structure for Γ if:

(A) No edge $e \in E$ ends at s_*.
(B) Every vertex $v \in V$ is accessible from the start state s_*.
(C) For every path γ, the path $\alpha(\gamma)$ is a geodesic path in G^Γ.
(D) The endpoint mapping $\alpha_* : \mathcal{P} \to \Gamma$ induced by α is a bijection of $\mathcal{P}$ onto Γ.

Theorem 5.2. *Every word hyperbolic group is strongly Markov.*

See [15], Ch. 9, Th. 13. The result is essentially due to Cannon (at least in a more restricted form) — see [11], [10] — and in important special cases (co-compact Fuchsian groups) to Series [30]. Henceforth, we will call the directed graph $\mathcal{A} = (V, E, s_*)$ the *Cannon automaton* (despite the fact that it is not quite the same automaton as constructed in [11]).

Properties (C)-(D) of Definition 5.1 imply that for each $x \in \Gamma$ there is a *unique* geodesic segment in the Cayley graph from the group identity 1 to x that is the image of a path in the automaton. We shall denote this distinguished geodesic segment by $L(1, x)$.

5.2. **Automatic structures for the surface groups.** The existence of an automatic structure will be used to connect the behavior of the Green's function at infinity to the theory of Gibbs states and Ruelle operators (see [8], ch. 1). For these arguments, the detailed structure of the automaton will not be important (except for those aspects discussed in sec. 5.3 below). Nevertheless, we note here that an automatic structure $\mathcal{A}$ for the *surface group* Γ_g is easily constructed. Let $A = A_g = \{a_i^\pm, b_i^\pm\}$ be the standard generating set, with the generators satisfying the basic relation

$$\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1. \tag{23}$$

Define the set V of vertices for the automaton to be the set of all reduced words in the generators of length $\leq 2g$, with s_* = the empty word. Directed edges are set according to the following rules:

(A) If a (reduced) word w' is obtained by adding a single letter x to the end of word w, then draw an edge $e(w, w')$ from w to w', and label it with the letter x.

(B) If a word w' of maximal length $2g$ is obtained from another word w of length $2g$ by deleting the first letter and adding a new letter x to the end, then draw an edge $e(w, w')$ from w to w' with label x *unless* the word wx constitutes the first $2g+1$ letters of a cyclic permutation of the basic relation (23).

That properties (C)–(D) of Definition 5.1 are satisfied follows from Dehn's algorithm. The words of maximal length $2g$ are the *recurrent vertices* of this automaton, while the words of length $< 2g$ are the *transient vertices* (see sec. 5.3 below for the definitions). It is easily verified that for any vertex w and any *recurrent* vertex w', there is a path in the automaton from w to w'.

5.3. **Recurrent and transient vertices.** Let $\mathcal{A}$ be a Cannon automaton for the group Γ with vertex set V and (directed) edge set E. Call an edge $e \in E$ *recurrent* if there is a path in $\mathcal{A}$ of length ≥ 2 that begins and ends with e; otherwise, call it *transient*. Denote by $\mathcal{A}_R$ the restriction of the digraph $\mathcal{A}$ to the set $\mathcal{R}$ of recurrent edges. For certain hyperbolic groups — among them the co-compact Fuchsian groups — the automatic structure can be chosen so that the digraph $\mathcal{A}_R$ is *connected* (see [30]). Henceforth we restrict attention to word-hyperbolic groups with this property:

Assumption 5.3. *The automatic structure can be chosen so that the digraph $\mathcal{A}_R$ is connected..*

Assumption 5.4. *The incidence matrix of the digraph $\mathcal{A}_R$ is aperiodic.*

Both assumptions hold for any co-compact Fuchsian group. Assumption 5.4 is for ease of exposition only — the results and arguments below can be modified to account for any periodicities that might arise if the assumption were to fail. Assumption 5.3, however, is essentially important.

5.4. **Symbolic dynamics.** We shall assume for the remainder of the paper that the automaton $\mathcal{A}$ has been chosen satisfying Assumptions 5.3 and 5.4.

Set

$$\begin{aligned}
\Sigma &= \{\text{semi-infinite paths in } \mathcal{A}\}, \\
\Sigma^n &= \{\text{paths of length } n \text{ in } \mathcal{A}\}, \\
\Sigma^* &= \cup_{n=0}^{\infty} \Sigma^n, \\
\overline{\Sigma} &= \Sigma \cup \Sigma^*.
\end{aligned}$$

We will also need bilateral versions of those sets, that we will denote with a superscript $\mathbb{Z}$. For instance, $\overline{\Sigma}^{\mathbb{Z}}$ is the set of (finite or infinite) bilateral paths in $\mathcal{A}$. Equivalently, it is the set of sequences $(\omega_n)_{n\in\mathbb{Z}}$ where ω_n is an edge of $\mathcal{A}$ for n in some interval of $\mathbb{Z}$ (with admissible transition from ω_n to ω_{n+1}), and ω_n is empty for n outside of this interval. Let σ be the forward shift operator on $\overline{\Sigma}$ and $\overline{\Sigma}^{\mathbb{Z}}$. The spaces $\overline{\Sigma}$ and $\overline{\Sigma}^{\mathbb{Z}}$ are given metrics in the usual way, that is,

$$d(\omega, \omega') = 2^{-n(\omega,\omega')}$$

where $n(\omega, \omega')$ is the maximum integer n such that $\omega_j = \omega'_j$ for all $|j| < n$. With the topology induced by d the space Σ is a Cantor set, Σ is the set of accumulation points of Σ^*, and $\overline{\Sigma}$ and $\overline{\Sigma}^{\mathbb{Z}}$ are compact. Observe that, relative to the metrics d, Hölder-continuous real-valued functions on Σ^* extend by continuity to Hölder-continuous functions on $\overline{\Sigma}$, and then pull back to Hölder-continuous functions on $\overline{\Sigma}^{\mathbb{Z}}$.

Each $\omega \in \Sigma$ projects via the edge-labeling map α to a geodesic ray in G^Γ starting at the vertex 1 (more precisely, the sequence of finite prefixes of ω project to the vertices along a geodesic ray). Each geodesic ray in G^Γ must converge in the Gromov topology to a point of $\partial\Gamma$, so α induces on Σ a mapping α_* to $\partial\Gamma$. By construction, this mapping is Hölder continuous relative to any visual metric on $\partial\Gamma$. Each point $\zeta \in \partial\Gamma$ is the limit of a geodesic ray corresponding to a semi-infinite path ω in $\mathcal{A}$ that begins at s_*.

In a somewhat different way, the edge-labeling map α determines a map from the space $\overline{\Sigma}^{\mathbb{Z}}$ to the set of two-sided (finite or infinite) geodesics in G^Γ that pass through the vertex 1. This map is defined as follows: if $\omega \in \overline{\Sigma}^{\mathbb{Z}}$ then the image of ω is the two-sided geodesic that passes through

$$\dots, \alpha(\omega_{-1}^{-1})\alpha(\omega_{-2}^{-1}), \alpha(\omega_{-1}^{-1}), 1, \alpha(\omega_0), \alpha(\omega_0)\alpha(\omega_1), \dots,$$

equivalently, it is the concatenation of the geodesic rays starting at 1 that are obtained by reading successive steps from the sequences

$$\omega_0\omega_1\omega_2\cdots \quad \text{and} \quad \omega_{-1}^{-1}\omega_{-2}^{-1}\omega_{-3}^{-1}\cdots,$$

respectively. When ω is biinfinite, each of these geodesic rays converges to a point of $\partial\Gamma$, so α induces a mapping from $\Sigma^{\mathbb{Z}}$ into $\partial\Gamma \times \partial\Gamma$. This mapping is neither injective nor surjective, but it is Hölder-continuous.

Let E_* be the set of edges originating from s_*, and let $\Sigma^m(E_*)$ be the set of sequences of length m in Σ^m with $\omega_0 \in E_*$. By definition of the Cannon automaton, the mapping α_* induces a bijection between $\Sigma^m(E_*)$ and the sphere S_m of radius m in G^Γ.

Corollary 5.5. *Let ζ be the spectral radius of the incidence matrix of the digraph $\mathcal{A}$. If Assumptions 5.3 and 5.4 hold, then $\zeta > 1$, and there exists $C > 0$ such that*

$$|S_m| \sim C\zeta^m \quad \text{as} \quad m \to \infty.$$

Proof. This follows directly from the Perron-Frobenius theorem, with the exception of the assertion that the spectral radius $\zeta > 1$. That $\zeta > 1$ follows from the fact that the group Γ is nonelementary. Since Γ is nonelementary, it is nonamenable, and so its Cayley graph has positive Cheeger constant; this implies that $|S_m|$ grows exponentially with m. □

Corollary 5.6. *The shift (Σ, σ) has positive topological entropy.*

Proof. This follows from the exponential growth of the group, cf. Corollary 5.5. □

6. Thermodynamic formalism

Assume throughout this section and sections 7–8 that the group Γ is co-compact Fuchsian, and that the random walk is symmetric and finite-range.

6.1. **The potential functions φ_r.** The machinery of thermodynamic formalism and Gibbs states developed in [8] applies to Hölder continuous functions on Σ (or on $\overline{\Sigma}$). To make use of this machinery, we will lift the Green function and the Martin kernel from $\overline{\Gamma}$ to the sequence space $\overline{\Sigma}$. For this the results of Theorem 1.2 and Theorem 4.3 are crucial, as they ensure that those lifts are Hölder-continuous. The lift is defined as follows. Fix $\omega \in \Sigma^*$, and let

$$\varphi_r(\omega) := \log \frac{G_r(1, \alpha_*(\omega))}{G_r(1, \alpha_*(\sigma\omega))}.$$

If ω is not the empty word, one can also write

$$\varphi_r(\omega) = \log \frac{G_r(1, \alpha_*(\omega))}{G_r(\alpha_*(\omega_0), \alpha_*(\omega))}.$$

Therefore, Theorem 4.3 shows that, if two paths ω and ω' coincide up to time n, then $|\varphi_r(\omega) - \varphi_r(\omega')| \leq C\varrho^n$, for some $\varrho < 1$. By definition of the distance on Σ^*, this means that φ_r is Hölder-continuous. In particular, it extends to a Hölder-continuous function (that we still denote by φ_r) on $\overline{\Sigma}$. On Σ, it is given by

$$\varphi_r(\omega) = \log \frac{K_r(1, \alpha_*(\omega))}{K_r(\alpha_*(\omega_0), \alpha_*(\omega))} = -\log K_r(\alpha_*(\omega_0), \alpha_*(\omega)). \tag{24}$$

The mapping $r \mapsto \varphi_r$ is clearly continuous at every point of Σ^*, and all the functions φ_r are uniformly Hölder-continuous for some fixed exponent. Therefore, $r \mapsto \varphi_r$ is also continuous for the sup norm, and it follows that it is continuous for the Hölder topology respective to any Hölder exponent strictly less than the initial one.

By construction, if ω is of length n,

$$G_r(1, \alpha_*(\omega)) = G_r(1, 1) \exp(S_n \varphi_r(\omega)) \tag{25}$$

where (in Bowen's notation [8])

$$S_n\varphi := \sum_{j=0}^{n-1} \varphi \circ \sigma^j.$$

(Unfortunately, the notation $S_n\varphi$ conflicts with the notation S_m for the sphere of radius m in Γ; however, both notations are standard, and the meaning should be clear in the following by context.)

6.2. **Gibbs states: background.** According to a fundamental theorem of ergodic theory (cf. [8], Th. 1.2 and sec. 1.4), for each Hölder continuous function on a topologically mixing subshift of finite type, there is a unique Gibbs state for this potential. Unfortunately, the subshift of finite type coming from a Cannon automaton is not topologically mixing, since the edges originating from s_* are always transient (and there can also be terminal edges, i.e., edges where a path can not be continued). However, under Assumptions 5.3 and 5.4, the recurrent part of the graph is topologically mixing. Therefore, generalized versions of the above results hold in our setting. We will in particular need the following general version of Ruelle's Perron-Frobenius Theorem: it admits transient parts, and it works on the larger space $\overline{\Sigma}$ instead of Σ, allowing finite paths in the automaton (this will be very convenient below, see for instance the proof of Proposition 6.2).

Consider a finite directed graph whose recurrent part is connected and aperiodic, let $\overline{\Sigma}$ be the set of finite or infinite paths in this graph, and let $\sigma : \overline{\Sigma} \to \overline{\Sigma}$ be the left shift. The assumption that the recurrent part of the graph is connected and aperiodic implies that the restriction of σ to the set of infinite paths in the recurrent set is topologically mixing. Let $\mathcal{H}$ be the space of real-valued Hölder-continuous functions on $\overline{\Sigma}$ (for some fixed Hölder exponent). Let $\mathcal{R}$ be the set of recurrent edges in the graph, $\mathcal{R}^+$ the set of edges that can be reached from a recurrent edge, and $\mathcal{R}^-$ the set of edges from which a recurrent edge can be reached.

Theorem 6.1. *Consider a potential $\varphi \in \mathcal{H}$, and define an operator $\mathcal{L}_\varphi$ acting on continuous functions by*

$$\mathcal{L}_\varphi f(x) = \sum_{\sigma(y)=x} e^{\varphi(y)} f(y),$$

where if x is the empty word the sum is restricted to the preimages y of positive length. There exist a real number $\Pr(\varphi)$ (the pressure *of φ), a number $\varepsilon > 0$, a Hölder-continuous function $h_\varphi : \overline{\Sigma} \to \mathbb{R}^+$ and a probability measure ν_φ on $\overline{\Sigma}$ such that, for any $f \in \mathcal{H}$, the following asymptotics hold in $\mathcal{H}$:*

$$\mathcal{L}_\varphi^n f = e^{n \Pr(\varphi)} \left(\int f \, d\nu_\varphi \right) h_\varphi + O(e^{-\varepsilon n} e^{n \Pr(\varphi)}). \tag{26}$$

The support of the function h_φ is the set of sequences whose elements all belong to $\mathcal{R}^+$ (and h_φ is bounded away from zero there). The support of the measure ν_φ is the set of infinite sequences whose elements all belong to $\mathcal{R}^-$.

The measure $\mu_\varphi = h_\varphi \nu_\varphi$ is the Gibbs measure *associated to the potential φ: it is a probability measure, supported on the recurrent part $\Sigma^{\mathcal{R}}$ of Σ, it is σ-invariant, and satisfies for any $\omega = (\omega_n) \in \Sigma^{\mathcal{R}}$*

$$C_1 \leq \frac{\mu_\varphi[\omega_0, \dots, \omega_{n-1}]}{e^{S_n \varphi(\omega) - nP(\varphi)}} \leq C_2, \tag{27}$$

where $C_1, C_2 > 0$ are two constants.

Finally, all the quantities in the statement of the theorem (i.e., $\Pr(\varphi)$, ε, h_φ, ν_φ, C_1, C_2, μ_φ and the implicit constant in the O–term in (26)*) vary continuously with $\varphi \in \mathcal{H}$.*

When the subshift of finite type is topologically mixing, this theorem is proved in [8]. Since the arguments there are easily adapted to obtain the above version, we will only sketch a proof, insisting on the arguments that differ from the classical case.

Proof. Classical arguments with Lasota-Yorke estimates (see for instance [28] or [5]) show that $\mathcal{L}_\varphi$ has a spectral gap on $\mathcal{H}$: denoting by $e^{\Pr(\varphi)}$ the spectral radius of $\mathcal{L}_\varphi$, this operator has finitely many eigenvalues of modulus $e^{\Pr(\varphi)}$, and the rest of its spectrum is contained in a disk of strictly smaller radius. Using the positivity of e^φ and the fact that the recurrent part of Σ is topologically mixing, one then proves that there is a unique eigenvalue of maximal modulus, and that it is simple. The asymptotics (26) follows. The eigenfunction and eigenmeasure h_φ and ν_φ satisfy respectively $\mathcal{L}_\varphi h_\varphi = e^{\Pr(\varphi)} h_\varphi$ and $\mathcal{L}_\varphi^* \nu_\varphi = \nu_\varphi$.

Let us now show the claim on the support of h_φ. The results in [8] show that h_φ is positive (and bounded from below) on the recurrent part $\Sigma^{\mathcal{R}}$ of Σ. Since h_φ is Hölder continuous, this implies that, if n is large enough, then h_φ is also positive on words of $\overline{\Sigma}$ of length at least n whose first n symbols are in $\mathcal{R}$. Consider now a word ω whose symbols all belong to $\mathcal{R}^+$. There exists a word α beginning by n symbols in $\mathcal{R}$ such that $\alpha\omega$ is a possible path in the automaton. Therefore,

$$h_\varphi(\omega) = e^{-n \Pr(\varphi)} \sum_{\sigma^n(\eta)=\omega} e^{S_n \varphi(\eta)} h_\varphi(\eta) \geq e^{-n \Pr(\varphi)} e^{S_n \varphi(\alpha\omega)} h_\varphi(\alpha\omega) > 0.$$

On the other hand, if ω contains a symbol not belonging to $\mathcal{R}^+$, then ω has no preimage under σ^n if there is no path of length n in the transient part of the automaton. It follows that $\mathcal{L}_\varphi^n h_\varphi(\omega) = 0$, hence $h_\varphi(\omega) = 0$. This shows that the support of h_φ is exactly those sequences with all symbols in $\mathcal{R}^+$. Since this set is compact, h_φ is bounded from below there.

If f is a continuous function, then

$$\nu_\varphi(f) = e^{-n\,\mathrm{Pr}(\varphi)}\nu_\varphi(\mathcal{L}_\varphi^n f).$$

Since $\mathcal{L}_\varphi^n f$ only depends on the values of f on words of length at least n, this shows that ν_φ has no atom on words of finite length. Let us now take $f = 1_C$ the characteristic function of a cylinder C of length n. Since $\mathcal{L}_\varphi^n 1_C(\omega) = e^{S_n\phi(C\omega)}$ if the concatenation $C\omega$ is an admissible word, and 0 otherwise, we deduce that $\nu_\varphi[C] = 0$ if C can not be extended. If a cylinder contains a symbol not in $\mathcal{R}^-$, it is a union of cylinders that can not be extended, and has therefore 0 measure. On the other hand, if C only contains symbols in $\mathcal{R}^-$, it contains a cylinder C' of some length m that can be followed by a symbol ω_0. Since $\mathcal{L}^m 1_{C'}$ is bounded from below on $[\omega_0]$, we get $\nu_\varphi[C] \geq c\nu_\varphi[\omega_0]$, which is nonzero since ν has full support in the recurrent part of Σ, by [8]. This shows that the support of ν_φ is exactly the set of infinite paths whose symbols all belong to $\mathcal{R}^-$.

The claims on the supports of h_φ and ν_φ show that the probability measure $\mu_\varphi = h_\varphi\nu_\varphi$ is supported on the recurrent part of Σ. It coincides there with the Gibbs measure studied in [8]. Hence, (27) follows.

Finally, all the quantities in the statement of the theorem are constructed from the spectral theory of the operator $\mathcal{L}_\varphi$. It then follows by standard arguments in regular perturbation theory that they all vary continuously with φ in the Hölder topology. □

6.3. **Gibbs states and Green's function on spheres.** Denote by $\lambda_{r,m}$ the probability measure on the sphere $S_m \subset \Gamma$ with density proportional to $G_r(1,x)^2$, that is, such that

$$\lambda_{r,m}(x) = \frac{G_r(1,x)^2}{\sum_{y\in S_m} G_r(1,y)^2} \quad \text{for all } \; x \in S_m. \tag{28}$$

Recall that S_m is in one-to-one correspondence with the paths of length m in the automaton $\mathcal{A}$ that begin at s_*; in particular, each $x \in S_m$ corresponds uniquely to a path ω of length m, where ω_0 belongs to the set E_* of edges originating from s_*. Thus, for each $m \geq 1$, the probability measure $\lambda_{r,m}$ on S_m pulls back to a probability measure on $\Sigma^m(E_*) \subset \overline{\Sigma}$, which we also denote by $\lambda_{r,m}$. This measure has density proportional to

$$G_r(1,\alpha_*(\omega))^2 = G_r(1,1)^2 \exp\{2S_m\varphi_r(\omega)\},$$

where $\omega \in \Sigma^m(E_*)$.

Proposition 6.2. *For each $r \in [1,R]$, the measures $\lambda_{r,m}$ on $\overline{\Sigma}$ converge weakly as $m \to \infty$ to a probability measure λ_r on $\overline{\Sigma}$, and moreover this convergence is uniform in r: if $f : \overline{\Sigma} \to \mathbb{R}$ is continuous, then*

$$\lim_{m\to\infty} \int f\,d\lambda_{r,m} = \int f\,d\lambda_r, \tag{29}$$

uniformly for $r \in [1,R]$.

Furthermore, there exist constants $0 < C = C(r;2) < \infty$ (depending continuously on r) such that the normalizing constants in (28) *satisfy*

$$\sum_{x\in S_m} G_r(1,x)^2 \sim C \exp\left\{m\,\mathrm{Pr}(2\varphi_r)\right\} \tag{30}$$

as $m \to \infty$.

Proof. Let us first prove (30). Let $\mathcal{L}_r$ be the Ruelle operator associated to the potential $2\varphi_r$. Denoting by $\emptyset$ the word of length 0 in $\overline{\Sigma}$, and by $1_{E_*} : \overline{\Sigma} \to \mathbb{R}$ the function equal to 1 on paths originating from s_* and 0 otherwise, we have

$$\mathcal{L}_r^m 1_{E_*}(\emptyset) = \sum e^{2S_m\varphi_r(\omega)} = \sum G_r(1, \alpha_*(\omega))^2 / G_r(1,1)^2,$$

where the sum is over all words ω of length m originating from s_*. Since α_* induces a bijection between such words and S_m, we get

$$\sum_{x \in S_m} G_r(1,x)^2 = G_r(1,1)^2 \mathcal{L}_r^m 1_{E_*}(\emptyset).$$

By the Ruelle–Perron–Frobenius Theorem 6.1, this is asymptotic to

$$G_r(1,1)^2 e^{m \operatorname{Pr}(2\varphi_r)} \left(\int 1_{E_*} \, d\nu_r \right) h_r(\emptyset),$$

where ν_r and h_r are the eigenmeasure and eigenfunction of $\mathcal{L}_r$. Since $(\int 1_{E_*} \, d\nu_r) h_r(\emptyset) > 0$ by Theorem 6.1, we obtain (30).

Let us now turn to $\lambda_{r,m}$. It can also be expressed through the transfer operator, by

$$\int f \, d\lambda_{r,m} = \mathcal{L}_r^m(1_{E_*} f)(\emptyset) / \mathcal{L}_r^m(1_{E_*})(\emptyset). \tag{31}$$

It follows again from Theorem 6.1 that, if f is Hölder-continuous, then $\int f \, d\lambda_{r,m}$ converges to $\int 1_{E_*} f \, d\nu_r / \int 1_{E_*} \, d\nu_r$. Moreover, the convergence is uniform for $r \in [1, R]$. If f is merely continuous, it can be uniformly approximated by a Hölder-continuous function, and the same result follows. The limiting measure λ_r is the normalized restriction of ν_r to the paths starting from s_*. □

Note 6.3. Virtually the same argument shows that for any $\theta \in \mathbb{R}$, as $m \to \infty$,

$$\sum_{x \in S_m} G_r(1,x)^\theta \sim C \exp\left\{ m \operatorname{Pr}(\theta\varphi_r) \right\}.$$

The result (30) implies that $\operatorname{Pr}(2\varphi_r) < 0$ for all $r < R$ (see Lemma 7.2 below), and Note 1.7 implies that $\operatorname{Pr}(\varphi_r) > 0$ for all $r \in (1, R]$. Since $\operatorname{Pr}(\theta\varphi_r)$ varies continuously with θ, it follows that for each $r \in (1, R]$ there exists $\theta \in (1, 2]$ such that $\operatorname{Pr}(\theta\varphi_r) = 0$. It can also be shown that the convergence of the sums is uniform in r for $r \in [1, R]$.

Proposition 6.4. *Let $g : \overline{\Sigma} \to \mathbb{R}$ be any Hölder-continuous function. Then*

$$\lim_{m \to \infty} \lambda_{r,m} \left\{ \omega \in \Sigma^m(E_*) \, : \, \left| \frac{1}{m} \sum_{j=0}^{m-1} g \circ \sigma^j(\omega) - \int g \, d\mu_r \right| > \delta \right\} = 0, \tag{32}$$

and for each $\delta > 0$ the convergence is uniform in $r \in [1, R]$.

Proof. Replacing g by $g - \int g \, d\mu_r$ and φ_r by $\varphi_r - \operatorname{Pr}(\varphi_r)$, we may assume that $\int g \, d\mu_r = 0$ and $\operatorname{Pr}(\varphi_r) = 0$. We will prove the proposition by estimating the variance of $\sum_{j=0}^{m-1} g \circ \sigma^j$ with respect to $\lambda_{r,m}$. We have

$$\int \left(\sum_{j=0}^{m-1} g \circ \sigma^j \right)^2 d\lambda_{r,m} = \sum \int (g \circ \sigma^j)^2 \, d\lambda_{r,m} + 2 \sum_{j<k} \int g \circ \sigma^j \cdot g \circ \sigma^k \, d\lambda_{r,m}.$$

In the first sum, each term is bounded by the sup of g, resulting in a bound $O(m)$. In the second term, for each $j < k$, one can write using (31)

$$\int g\circ\sigma^j \cdot g\circ\sigma^k \, d\lambda_{r,m} = \mathcal{L}_r^m(1_{E_*} g\circ\sigma^j \cdot g\circ\sigma^k)(\emptyset)/\mathcal{L}_r^m 1_{E_*}(\emptyset).$$

Since $\mathcal{L}(u\cdot v\circ\sigma) = \mathcal{L}(u)\cdot v$, this is equal to

$$\mathcal{L}_r^{m-k}(g\mathcal{L}_r^{k-j}(g\mathcal{L}_r^j 1_{E_*}))(\emptyset)/\mathcal{L}_r^m 1_{E_*}(\emptyset).$$

The denominator in this expression is uniformly bounded from above and below for large m. Using the spectral asymptotics of $\mathcal{L}_r$ described in Theorem 6.1, we can write $\mathcal{L}_r^j 1_{E_*} = \nu_r[E_*]h_r + O(e^{-\varepsilon j})$. Therefore,

$$\mathcal{L}_r^{k-j}(g\mathcal{L}_r^j 1_{E_*})) = \nu_r[E_*]\mathcal{L}_r^{k-j}(gh_r)) + O(e^{-\varepsilon j}) = \nu_r[E_*]\left(\int gh_r \, d\nu_r\right)h_r + O(e^{-\varepsilon(k-j)}) + O(e^{-\varepsilon j}).$$

Since $\int gh_r\,d\nu_r = \int g\,d\mu_r = 0$, this is $O(e^{-\varepsilon(k-j)}) + O(e^{-\varepsilon j})$. Summing over $j < k < m$, we obtain a bound

$$C\sum_{j<k<m}(e^{-\varepsilon j} + e^{-(k-j)}) \le Cm.$$

We have proved that

$$\int\left(\sum_{j=0}^{m-1} g\circ\sigma^j\right)^2 d\lambda_{r,m} \le Cm.$$

Hence,

$$\lambda_{r,m}\left\{\left|\frac{1}{m}\sum_{j=0}^{m-1} g\circ\sigma^j - \int g\,d\mu_r\right| > \delta\right\} \le \frac{1}{\delta^2}\frac{Cm}{m^2},$$

which tends to 0 when $m\to\infty$.

Finally, all those estimates are uniform in r, since all spectral data coming from Theorem 6.1 are already uniform. □

The ergodic average in (32) is expressed as an average over the orbit of a path in the Cannon automaton, but it readily translates to an equivalent statement for ergodic averages along the geodesic segment $L = L(1,x)$. Observe that each vertex $z\in L$ disconnects L into two geodesic segments $L^+ = L_z^+$ and $L^- = L_z^-$, where L^+ is the segment of L from z to x, and L^- is the segment of L from z to 1. These paths determine finite reduced words $e^+ = e^+(z)$ and $e^- = e^-(z)$ in the group generators A (recall that each oriented edge (u,v) of the Cayley graph is labeled by the generator $u^{-1}v$). The word $e^+(z)$ and the reversal of the word $e^-(z)$ are both elements of Σ^*; thus, the concatenation e^-e^+ can be viewed as an element of the space $\Sigma^{*\mathbb{Z}}$.

There is also a natural reference measure on the bilateral shift $\overline{\Sigma}^{\mathbb{Z}}$: since the Gibbs measure μ_r constructed on Σ in Theorem 6.1 is shift–invariant, it extends to a measure (still denoted μ_r) on $\Sigma^{\mathbb{Z}}$.

Corollary 6.5. *Let $f:\overline{\Sigma}^{\mathbb{Z}}\to\mathbb{R}$ be a Hölder continuous function. Then for each $\delta > 0$,*

$$\lim_{m\to\infty}\lambda_{r,m}\left\{x\in S_m : \left|m^{-1}\sum_{z\in L(1,x)} f(e^-(z),e^+(z)) - \int f\,d\mu_r\right| > \delta\right\} = 0, \tag{33}$$

and the convergence is uniform in $r \in [1, R]$.

Proof. If the function f only depends on the positive coordinates, this statement directly reduces to Corollary 6.4. By a theorem of Livsits ([8], Lemma 1.6, whose proof works verbatim in spaces where one allows finite sequences) any Hölder continuous function f on $\overline{\Sigma}^{\mathbb{Z}}$ is *cohomologous* to a Hölder continuous function g that depends only on the forward coordinates, i.e., there exists a Hölder continuous function u such that $g = f + u - u \circ \sigma$. Since it is equivalent to have (33) for f or $f + u - u \circ \sigma$, the general case follows. □

7. Evaluation of the pressure at $r = R$

Proposition 6.2 implies that the sums $\sum_{y \in S_m} G_R(1, y)^2$ grow or decay sharply exponentially at exponential rate $\Pr(2\varphi_R)$. Consequently, to prove the relation (7) of Theorem 1.6 it suffices to prove that this rate is 0.

Proposition 7.1. $\Pr(2\varphi_R) = 0$.

The second assertion (8) of Theorem 1.6 also follows from Proposition 7.1, by the main result of [22]. (If it could be shown that the cocycle φ_R defined by (24) above is *nonlattice* in the sense of [22], then the result (8) could be strengthened from $\asymp$ to $\sim$.)

The remainder of this section is devoted to the proof of Proposition 7.1. The first step, that $\Pr(2\varphi_R) \leq 0$, is a consequence of the differential equations (9). These imply the following.

Lemma 7.2. *For every* $r < R$,

$$\Pr(2\varphi_r) < 0, \quad \textit{and so}$$

$$\Pr(2\varphi_R) \leq 0. \tag{34}$$

Proof. For $r < R$ the Green's function $G_r(1, 1)$ is analytic in r, so its derivative must be finite. Thus, by Proposition 1.9, the sum $\sum_{x \in \Gamma} G_r(1, x)^2$ is finite. (The last term $r^{-1} G_r(1, 1)$ in equation (9) remains bounded as $r \to R-$ because $G_R(1, 1,) < \infty$.) Proposition 6.2 therefore implies that $\Pr(2\varphi_r)$ must be negative. Since $\Pr(\varphi)$ varies continuously in φ, relative to the Hölder norm, (34) follows. □

Proof of Proposition 7.1. To complete the proof it suffices, by the preceding lemma, to show that $\Pr(2\varphi_R)$ cannot be negative. In view of Proposition 6.2, this is equivalent to showing that $\sum_{x \in S_m} G_R(1, x)^2$ cannot decay exponentially in m. This will be accomplished by proving that exponential decay of $\sum_{x \in S_m} G_R(1, x)^2$ would force

$$G_r(1, 1) < \infty \quad \text{for some } r > R, \tag{35}$$

which is impossible since R is the radius of convergence of the Green's function.

To prove (35), we will use the branching random walk interpretation of the Green's function discussed in sec. 2.4.[2] Recall that a branching random walk on the Cayley graph G^Γ is specified by an offspring distribution Q; assume for definiteness that this is the Poisson distribution with mean $r > 0$. At each step, particles first produce offspring particles according to this distribution, independently, and then each of these particles jumps to a randomly chosen neighboring vertex. If the mean of the offspring distribution is $r > 0$, and if the branching random walk is initiated by a single particle at the root 1, then the mean number of particles located at vertex x at time $n \geq 1$ is $r^n P^1\{X_n = x\}$. Thus,

[2] Logically this is unnecessary — the argument has an equivalent formulation in terms of weighted paths, using (13) — but the branching random walk interpretation seems more natural.

in particular, $G_r(1,1)$ equals the expected total number of particle visits to the root vertex 1. The strategy is to show that if $\sum_{x\in S_m} G_R(1,x)^2$ decays exponentially in m, then for some $r > R$ the branching random walk remains *subcritical*, that is, the expected total number of particle visits to 1 is finite.

Recall that the Poisson distribution with mean $r > R$ is the convolution of Poisson distributions with means R and $\varepsilon := r - R$, that is, the result of adding independent random variables U, V with distributions Poisson-R and Poisson-ε is a random variable $U + V$ with distribution Poisson-r. Thus, each reproduction step in the branching random walk can be done by making independent draws U, V from the Poisson-R and Poisson-ε distributions. Use these independent draws to assign *colors* $k = 0, 1, 2, \ldots$ to the particles according to the following rules:

(a) The ancestral particle at vertex 1 has color $k = 0$.
(b) Any offspring resulting from a $U-$draw has the same color as its parent.
(c) Any offspring resulting from a $V-$draw has color equal to 1+the color of its parent.

Lemma 7.3. *For each $k = 0, 1, 2, \ldots$, the expected number of visits to the vertex y by particles of color k is*

$$v_k(y) = \varepsilon^k \sum_{x_1, x_2, \ldots x_k \in \Gamma} G_R(1, x_1) \left(\prod_{i=1}^{k-1} G_R(x_i, x_{i+1}) \right) G_R(x_k, y). \tag{36}$$

Proof. By induction on k. First, particles of color $k = 0$ reproduce and move according to the rules of a branching random walk with offspring distribution Poisson-R, so the expected number of visits to vertex y by particles of color $k = 0$ is $G_R(1, y)$, by Proposition 2.5. This proves (36) in the case $k = 0$. Second, assume that the assertion is true for color $k \geq 0$, and consider the production of particles of color $k + 1$. Such particles are produced only by particles of color k or color $k + 1$. Call a particle a *pioneer* if its color is different from that of its parent, that is, if it results from a $V-$draw. Each pioneer of color $k + 1$ engenders its own branching random walk of descendants with color $k + 1$; the offspring distribution for this branching random walk is the Poisson-R distribution. Thus, for a pioneer born at site $z \in \Gamma$, the expected number of visits to y by its color–$(k + 1)$ descendants is $G_R(z, y)$. Every particle of color $k + 1$ belongs to the progeny of one and only one pioneer; consequently, the expected number of visits to y by particles of color $k + 1$ is

$$\sum_{z \in \Gamma} u_{k+1}(z) G_R(z, y),$$

where $u_{k+1}(z)$ is the expected number of pioneers of color $k + 1$ born at site z during the evolution of the branching process. But since pioneers of color $k + 1$ must be children of parents of color k, and since for any particle the expected number of children of different color is ε, it follows that

$$u_{k+1}(z) = \varepsilon v_k(z).$$

Hence, formula (36) for $k + 1$ follows by the induction hypothesis. □

Recall that our objective is to show that if $\sum_{x\in S_m} G_R(1,x)^2$ decays exponentially in m then $G_r(1,1) < \infty$ for some $r = R + \varepsilon > R$. The branching random walk construction exhibits $G_r(1,1)$ as the expected total number of particle visits to the root vertex 1, and this is the sum over $k \geq 0$ of the expected number $v_k(1)$ of visits by particles of color k. Thus, to complete the proof of Proposition 7.1 it suffices, by Lemma 7.3, to show that for some $\varepsilon > 0$,

$$\sum_{k=0}^{\infty} \varepsilon^k \sum_{x_1, x_2, \dots x_k \in \Gamma} G_R(1, x_1) \left(\prod_{i=1}^{k-1} G_R(x_i, x_{i+1}) \right) G_R(x_k, 1) < \infty.$$

This follows directly from the next lemma. □

Lemma 7.4. *Assume that Ancona's inequalities* (5) *hold at the spectral radius R with a constant $C_R < \infty$. If the sum $\sum_{x \in S_m} G_R(1, x)^2$ decays exponentially in m, then there exist constants $\delta > 0$ and $C, \varrho < \infty$ such that for every $k \geq 1$,*

$$\sum_{x_1, x_2, \dots x_k \in \Gamma} G_R(1, x_1) \left(\prod_{i=1}^{k-1} G_R(x_i, x_{i+1}) \right) (1+\delta)^{|x_k|} G_R(x_k, 1) \leq C \varrho^k. \tag{37}$$

Here $|y| = d(1, y)$ denotes the distance of y from the root 1 *in the word metric.*

Proof. Denote by $H_k(\delta)$ the left side of (37); the strategy will be to prove by induction on k that for sufficiently small $\delta > 0$ the ratios $H_{k+1}(\delta)/H_k(\delta)$ remain bounded as $k \to \infty$. Consider first the sum $H_1(\delta)$: by the hypothesis that $\sum_{x \in S_m} G_R(1, x)^2$ decays exponentially in m and the symmetry $G_r(x, y) = G_r(y, x)$ of the Green's function, for all sufficiently small $\delta > 0$

$$H_1(\delta) := \sum_{x \in \Gamma} G_R(1, x)^2 (1+\delta)^{|x|} < \infty.$$

Now consider the ratio $H_{k+1}(\delta)/H_k(\delta)$. Fix vertices $x_1, x_2, \dots, x_k$, and for an arbitrary vertex $y = x_{k+1} \in \Gamma$, consider its position *vis a vis* the geodesic segment $L = L(1, x_k)$ from the root vertex 1 to the vertex x_k. Let $z \in L$ be the vertex on L nearest y (if there is more than one, choose arbitrarily). By the triangle inequality,

$$|y| \leq |z| + d(z, y).$$

Because the group Γ is word-hyperbolic, all geodesic triangles — in particular, any triangle whose sides consist of geodesic segments from y to z, from z to x_k, and from x_k to y, or any triangle whose sides consist of geodesic segments from y to z, from z to 1, and from 1 to y— are Δ-thin, for some $\Delta < \infty$ (cf. [16] or [19]). Hence, any geodesic segment from x_k to y must pass within distance 8Δ of the vertex z. Therefore, by the Harnack and Ancona inequalities (15) and (5), for some constant $C_* = C_R C_{\text{Harnack}}^{32\Delta} < \infty$ independent of y, x_k,

$$G_R(y, 1) \leq C_* G_R(y, z) G_R(z, 1) \quad \text{and}$$
$$G_R(y, x_k) \leq C_* G_R(y, z) G_R(z, x_k).$$

On the other hand, by the log-subadditivity of the Green's function,

$$G_R(1, z) G_R(z, x_k) \leq C' G_R(x_k, 1).$$

It now follows that

$$\begin{aligned}(1+\delta)^{|y|} G_R(x_k, y) G_R(y, 1) &\leq C_*^2 C' (1+\delta)^{|z| + d(z,y)} G_R(z, x_k) G_R(z, y) G_R(y, z) G_R(z, 1) \\ &\leq C_*^2 C' (1+\delta)^{|z| + d(z,y)} G_R(x_k, 1) G_R(z, y)^2.\end{aligned}$$

Denote by $\Gamma(z)$ the set of all vertices $y \in \Gamma$ such that z is a closest vertex to y in the geodesic segment L. Then for each $z \in L$,

$$\sum_{y \in \Gamma(z)} (1+\delta)^{d(z,y)} G_R(z, y)^2 \leq \sum_{y \in \Gamma} (1+\delta)^{|y|} G_R(1, y)^2 = H_1(\delta).$$

Finally, because L is a geodesic segment from 1 to x_k there is precisely one vertex $z \in L$ at distance n from x_k for every integer $0 \leq n \leq |x_k|$, so $\sum_{z \in L}(1+\delta)^{|z|} \leq C_\delta(1+\delta)^{|x_k|}$ where $C_\delta = (1+\delta)/(2+\delta)$. Therefore,

$$H_{k+1}(\delta) \leq C_*^2 C' C_\delta H_1(\delta) H_k(\delta).$$

□

8. Critical Exponent of the Green's function at the Spectral Radius

8.1. **Reduction to a simple case.** Consider first the case $x = y = 1$ of Theorem 1.10. The system of differential equations (9) implies that the growth of the derivative $dG_r(1,1)/dr$ as $r \to R-$ is controlled by the growth of the quadratic sums $\sum_{x\in\Gamma} G_r(1,x)^2$. To show that the Green's function has a square root singularity at $r = R$, as asserted in (10), it will suffice to show that the (approximate) derivative behaves as follows as $r \to R-$:

Proposition 8.1. *For some* $0 < C < \infty$,

$$\eta(r) := \sum_{x\in\Gamma} G_r(1,x)^2 \sim C/\sqrt{R-r} \quad \text{as } r \to R-. \tag{38}$$

This will follow from Corollary 8.4 below. The key to the argument is that the growth of $\eta(r)$ as $r \to R-$ is related by Proposition 6.2 to that of $\Pr(2\varphi_r)$: in particular, Proposition 7.1 implies that $\eta(r) \to \infty$ as $r \to R-$, so the dominant contribution to the sum (38) comes from vertices x at large distances from the root vertex 1. Consequently, by equation (30),

$$\eta(r) = \sum_{m=0}^{\infty} \sum_{x\in S_m} G_r(1,x)^2 \sim C(R,2)/(1-\exp\{\Pr(2\varphi_r)\}) \quad \text{as} \quad r \to R-.$$

Before beginning the analysis of $\eta(r)$ near the singularity $r = R$ we will show that the relation (38) implies similar asymptotic behavior for the derivatives of all of the Green's functions $G_r(x,y)$.

Corollary 8.2. *There exist constants* $0 < C_{1,y} < \infty$ *such that*

$$\sum_{x\in\Gamma} G_r(1,x)G_r(x,y) \sim C_{1,y}/\sqrt{R-r} \quad \text{as } r \to R-. \tag{39}$$

Proof. Recall the probability measures $\lambda_{r,m}$ on the spheres S_m with densities proportional to $G_r(1,x)^2$ (see (28)), and recall that these probability transfer to probability measures, also denoted by $\lambda_{r,m}$, on $\Sigma^m(E_*)$, using the correspondence $S_m \leftrightarrow \Sigma^m(E_*)$. By Proposition 6.2, as $m \to \infty$ the measures $\lambda_{r,m}$ converge weakly to a probability measure λ_r, and this convergence is uniform for $r \in [1,R]$, in the sense specified by (29). These measures are related to the sum in (39) as follows:

$$\sum_{x\in S_m} G_r(1,x)G_r(x,y) = \sum_{x\in S_m} G_r(1,x)^2 \frac{G_r(x,y)}{G_r(1,x)} = \left(\sum_{x\in S_m} G_r(1,x)^2\right) \int \frac{G_r(x,y)}{G_r(1,x)}\, d\lambda_{r,m}(x).$$

As $x \to \xi \in \partial\Gamma$ the ratios $G_r(x,y)/G_r(1,x)$ converge to the Martin kernel $K_r(y,\xi)$, and the convergence is uniform, by Theorem 1.2. Hence, the weak convergence $\lambda_{r,m} \Rightarrow \lambda_r$ implies that

$$\lim_{m\to\infty} \int \frac{G_r(x,y)}{G_r(1,x)}\, d\lambda_{r,m}(x) := C_{1,y}(r)$$

exists, and the convergence is uniform for $r \in [1,R]$. Therefore, the corollary follows from Proposition 8.1. □

8.2. **Analysis of the function $\eta(r)$ near the singularity $r = R$.** To analyze the behavior of $\eta(r)$ (or equivalently that of $\Pr(2\varphi_r)$) as $r \to R-$, we use the differential equations (9) to express the derivative of $\eta(r)$ as

$$\frac{d\eta}{dr} = \sum_{x\in\Gamma}\left\{\sum_{y\in\Gamma} 2r^{-1}G_r(1,x)G_r(1,y)G_r(y,x)\right\} - 2r^{-1}G_r(1,x)^2. \tag{40}$$

(Note: The implicit interchange of d/dr with an infinite sum is justified here because the Green's functions $G_r(u,v)$ are defined by power series with nonnegative coefficients.) For $r \approx R$, the sum $\sum_{x\in\Gamma}$ is once again dominated by those vertices x at large distances from the root 1. Because the second term $2r^{-1}G_r(1,x)^2$ in (40) remains bounded as $r \to R-$, it is asymptotically negligible compared to the first term $\sum_x \sum_y$ and so we can ignore it in proving (38).

The strategy for dealing with the inner sum $\sum_{y\in\Gamma}$ in (40) will be similar to that used in the proof of Lemma 7.4 above. For each x, let $L = L(1,x)$ be the unique geodesic segment from the root to x that corresponds to a path in the Cannon automaton, and partition the sum $\sum_{y\in\Gamma}$ according to the nearest vertex $z \in L$:

$$\sum_{y\in\Gamma} = \sum_{z\in L}\sum_{y\in\Gamma(z)} \tag{41}$$

where $\Gamma(z)$ is the set of all vertices $y \in \Gamma$ such that z is a closest vertex to y in the geodesic segment L. (If for some y there are several vertices $z_1, z_2, \dots$ on L all closest to y, put $y \in \Gamma(z_i)$ only for the vertex z_i nearest to the root 1.) By the log-subadditivity of the Green's function and Theorem 1.3 (the Ancona inequalities) there exists a constant $C < \infty$ independent of $1 \le r \le R$ such that for all choices of $x \in \Gamma$, $z \in L(1,x)$, and $y \in \Gamma(z)$,

$$\begin{aligned} G_r(1,x)G_r(1,y)G_r(y,x) &\le CG_r(1,z)^2G_r(z,x)^2G_r(z,y)^2 \\ &\le CG_r(1,x)^2G_r(z,y)^2; \end{aligned} \tag{42}$$

consequently, for each $x \in \Gamma$,

$$\begin{aligned} \sum_{y\in\Gamma} G_r(1,x)G_r(1,y)G_r(y,x) &\le \sum_{z\in L(1,x)}\sum_{y\in\Gamma(z)} CG_r(1,x)^2G_r(z,y)^2 \\ &\le \sum_{z\in L(1,x)}\sum_{y\in\Gamma} CG_r(1,x)^2G_r(z,y)^2 \\ &= CG_r(1,x)^2(|x|+1)\eta(r). \end{aligned} \tag{43}$$

Proposition 8.3 below asserts that for large m and $r \approx R$ this inequality is in fact an approximate equality for "most" $x \in S_m$. This implies that for large m the contribution to the double sum in (40) with $|x| = m$ is dominated by those x that are "generic" for the probability measure $\lambda_{r,m}$ on S_m with density proportional to $G_r(1,x)^2$ (cf. sec. 6.3).

Proposition 8.3. *For each $r \le R$ and each $m = 1, 2, \dots$ let $\lambda_{r,m}$ be the probability measure on the sphere S_m with density proportional to $G_r(1,x)^2$. There is a continuous, positive function $\xi(r)$ of $r \in [1,R]$ such that for each $\varepsilon > 0$, and uniformly for $1 \le r < R$,*

$$\lim_{m\to\infty} \lambda_{r,m}\left\{x \in S_m \,:\left|\frac{1}{m}\sum_{y\in\Gamma} G_r(1,y)G_r(y,x)/G_r(1,x) - \xi(r)\eta(r)\right| > \varepsilon\eta(r)\right\} = 0. \tag{44}$$

This will be deduced from Corollary 6.5 — see section 8.3 below. Given Proposition 8.3, Proposition 8.1 and Theorem 1.10 follow easily, as we now show.

Corollary 8.4. *There exists a positive, finite constant C such that as $r \to R-$,*

$$\frac{d\eta}{dr} \sim C\eta(r)^3. \tag{45}$$

Consequently,

$$\eta(r)^{-2} \sim C(R-r)/2. \tag{46}$$

Proof. We have already observed that as r near R, the dominant contribution to the sum (40) comes from vertices x far from the root. Proposition 8.3 and the uniform upper bound (43) on ergodic averages imply that as $r \to R-$,

$$\begin{aligned}\frac{d\eta}{dr} \sim \sum_{x\in\Gamma}\sum_{y\in\Gamma} 2r^{-1}G_r(1,x)G_r(1,y)G_r(y,x) &\sim 2R^{-1}\xi(R)\eta(r)\sum_{m=1}^{\infty} m \sum_{x\in S_m} G_r(1,x)^2 \\ &\sim C'\eta(r)/(1-\exp\{\Pr(2\varphi_r)\})^2 \\ &\sim C\eta(r)^3\end{aligned}$$

for suitable positive constants C, C'. This proves (45). The relation (46) follows directly from (45). □

8.3. **Proof of Proposition 8.3.** This will be accomplished by showing that the average in (44) can be expressed approximately as an ergodic average of the form (33), to which the result of Corollary 6.5 applies. The starting point is a version of the decomposition (41), but rather using a continuous partition of unity instead of the characteristic functions of the sets $\Gamma(z)$, since we will need a continuous extension to the boundary of the group.

Lemma 8.5. *For K large enough, we can associate to any geodesic segment L in the Cayley graph of length $2K+1$ centered around 1 a function $\gamma_L : \Gamma \to [0,1]$ with the following properties:*

(1) *The function γ_L extends continuously to $\overline{\Gamma} = \Gamma \cup S^1$.*
(2) *Let $\pi_L(y)$ be the set of points on L that are closest to $y \in \Gamma$. Then $\gamma_L(y) = 0$ if $\pi_L(y)$ contains a point at distance $\geq K/4$ of 1.*
(3) *Let L' be any biinfinite geodesic. Adding the functions γ_L along the subsegments of L' of length $2K+1$ of L one gets the function identically equal to 1. More formally, for all $y \in \Gamma$,*

$$\sum_{i\in\mathbb{Z}} \gamma_{L'(i)^{-1}L'[i-K,i+K]}(L'(i)^{-1}y) = 1.$$

If we define functions γ_L by $\gamma_L(y) = 1$ if the closest point to y on L is 1, and 0 otherwise (with ties resolved as explained in Paragraph 8.2), they satisfy the two last properties, but might not extend continuously to the boundary.

Proof. Let K be even. For any geodesic segment L_0 of length $K+1$ centered around 1, consider the set A of points y such that $\pi_{L_0}(y)$ contains a point at distance at most $K/10$ of 1, and the set B of points y such that $\pi_{L_0}(y)$ contains a point at distance at least $K/5$ of 1. By the hyperbolicity of the group Γ, if K is large enough, the two sets A, B are disjoint, and their closures in $\overline{\Gamma}$ are still disjoint. Therefore, there exists a continuous function $0 \leq g_{L_0} \leq 1$ on *overline*Γ equal to 1 on $\overline{A}$ and to 0 on $\overline{B}$.

If L is a geodesic segment of length $2K+1$, let γ_L be equal to $g_{L[-K/2,K/2]}$ divided by the sum of the functions $g_{\tilde L}$ along every subsegment $\tilde L$ of L of length $K+1$. Formally,

$$\gamma_L(y) = g_{L[-K/2,K/2]}(y) / \sum g_{L(i)^{-1}L[-K/2+i,K/2+i]}(L(i)^{-1}y).$$

The sum in the denominator is ≥ 1 on a neighborhood of the support of $g_{L[-K/2,K/2]}$ by construction, so the function γ_L is well defined and continuous. All the properties of the lemma follow easily. □

Let us now define for $r \in [1,R)$ a function f_r on geodesic segments L through 0, as follows. Let a and b be the endpoints of L. If $d(1,a) \leq K$ or $d(1,b) \leq K$, let $f_r(L) = 0$. Otherwise, let

$$f_r(L) = \eta(r)^{-1} \sum_{y \in \Gamma} \gamma_{L[-K,K]}(y) G_r(a,y) G_r(y,b) / G_r(a,b).$$

By construction, for all $x \in \Gamma$,

$$\sum_{y\in\Gamma} G_r(1,y)G_r(y,x)/G_r(1,x) = \eta(r) \sum_{z \in L(1,x)} f_r(e^-(z), e^+(z)) + O(\eta(r)),$$

where $e^-(z)$ and $e^+(z)$ are the segments of $L(1,x)$ between 1 and z and between z and x respectively. This quantity will be estimates thanks to Corollary 6.5 once the following lemma is established.

Lemma 8.6. *The functions f_r are uniformly bounded and Hölder-continuous for $r \in [1,R)$, and they converge in the Hölder topology to a function f_R when $r \to R-$.*

By Hölder continuous, we mean that, if two geodesics L and L' coincide on a ball of size n around 1, then $|f_r(L) - f_r(L')| \leq C\varrho^n$ for some $\varrho < 1$. This readily implies that the canonical lift of f_r to the symbolic space $\Sigma^{*\mathbb{Z}}$ (see Paragraph 5.4) is Hölder continuous for the symbolic distance, making it possible to apply Corollary 6.5.

Proof. By definition of γ_L, any geodesic segment from a point y with $\gamma_L(y) > 0$ to the endpoints of the geodesic segment L (or a geodesic extension of it) passes within bounded distance of 1. Arguing as in (42), one deduces that f_r is uniformly bounded by a constant C.

Consider now two geodesics L_1 and L_2 around 0, and assume that they coincide on a neighborhood of 1 of size $n > K$. In particular, their restriction L to the ball $B(1,K)$ coincide. Let a_1 and b_1 (resp. a_2 and b_2) be the endpoints of L_1 (resp. L_2). For each $y \in \Gamma$ with $\gamma_L(y) > 0$,

$$\frac{G_r(a_1,y)G_r(y,b_1)/G_r(a_1,b_1)}{G_r(a_2,y)G_r(y,b_2)/G_r(a_2,b_2)} = \frac{G_r(a_1,y)/G_r(a_1,b_1)}{G_r(a_2,y)/G_r(a_2,b_1)} \cdot \frac{G_r(y,b_1)/G_r(a_2,b_1)}{G_r(y,b_2)/G_r(a_2,b_2)}.$$

Since the geodesics from a_1 or a_2 to b_1 or y are fellow traveling during a time at least $n - C$ by definition (see Section 4.2), Theorem 4.3 shows that the first factor is bounded by $1 + C\varrho^n$ for some $\varrho < 1$. The second factor is bounded in the same way. Multiplying by $\eta(r)^{-1}\gamma_L(y)$ and summing over y, we obtain $f_r(L) \leq (1 + C\varrho^n) f_r(L')$. Since f_r is bounded, this yields $f_r(L) - f_r(L') \leq C\varrho^n$. Exchanging the role of L and L', we get $|f_r(L) - f_r(L')| \leq C\varrho^n$. This shows that the functions f_r are uniformly Hölder-continuous.

Let us now show that the functions f_r converge in the Hölder topology when $r \to R-$. It suffices to show that they converge simply: indeed, in this case, the uniform convergence follows from the uniform Hölder bounds, and this implies convergence in the Hölder topology for any exponent strictly less than the initial exponent.

Let L be any fixed geodesic segment with endpoints a and b, let us show that $f_r(L)$ converges when $r \to R-$. We have

$$\begin{aligned} f_r(L) &= \frac{1}{G_r(a,b)}\eta(r)^{-1}\sum_m \sum_{y\in S_m} G_r(1,y)^2 \gamma_{L[-K,K]}(y)\frac{G_r(a,y)}{G_r(1,y)}\frac{G_r(b,y)}{G_r(1,y)} \\ &= \frac{1}{G_r(a,b)}\eta(r)^{-1}\sum_m \left(\sum_{y\in S_m} G_r(1,y)^2\right)\int F_r(y)\,d\lambda_{r,m}(y), \end{aligned}$$

where $F_r(y) = \gamma_{L[-K,K]}(y)\frac{G_r(a,y)}{G_r(1,y)}\frac{G_r(b,y)}{G_r(1,y)}$ is a continuous function on Γ which extends continuously to $\overline{\Gamma}$. Moreover, F_r converges uniformly when $r \to R-$ to a limit F_R.

By Proposition 6.2, the measures $\lambda_{r,m}$ converge when $m \to \infty$ to a measure λ_r supported on $\partial\Gamma$, uniformly in $r \in [1,R]$. On the other hand, the influence of bounded m in the above sum tends to 0 when $r \to R-$ since $\eta(r) = \sum_m \left(\sum_{y\in S_m} G_r(1,y)^2\right)$ tends to infinity. Therefore, $f_r(L)$ converges, to $\frac{1}{G_r(a,b)}\int_{\partial\Gamma} F_R\,d\lambda_R$. □

Consequently, for a suitable constant $\xi(r) = \int f_r\,d\mu_r$, the convergence (44) follows from Corollary 6.4. That $\xi(r)$ varies continuously with r for $r \leq R$ follows from the continuous dependence of the Gibbs state μ_r with r (Theorem 6.1). It remains only to show that $\xi(R) > 0$; equivalently, one should show that, for r close enough to R and for x with $|x| = m$, then $\sum_{y\in\Gamma} G_r(1,y)G_r(y,x)/G_r(1,x) \geq Cm\eta(r)$ for some $C > 0$ independent of r. This is most conveniently established by decomposing Γ as $\bigcup \Gamma(z_j)$ for $z_j \in L(1,x)$ and showing that each long enough block of consecutive z_j contributes at least an amount $C\eta(r)$ to the sum:

Lemma 8.7. *There exist $K < \infty$ and $C > 0$ independent of $1 \leq r < R$ so that the following is true. For any geodesic segment L of length $\geq K$ corresponding to a path in the Cannon automaton, and any K consecutive vertices $z_1, z_2, \dots, z_K$ on L,*

$$\sum_{j=1}^{K}\sum_{y\in\Gamma(z_j)} G_r(z_j,y)^2 \geq C\eta(r). \tag{47}$$

Proof. This is in essence a consequence of hyperbolicity, but is easiest to prove using symbolic dynamics. Recall that the sphere S_m of radius m in Γ is in bijection with the set $\Sigma^m(E_*)$ of paths of length m in the Cannon automaton originating from s_*. Since the shift (Σ,σ) has a connected recurrent part and positive topological entropy, there exists K so large that any path of length $\geq K$ has a *fork* in the set of recurrent vertices of $\mathcal{A}$, that is, a point where the path could be continued in an alternative fashion. Let γ be the path in the automaton corresponding to L, and let γ' be a path (possibly much longer than γ) that agrees with γ up to a fork, where it then deviates from γ. If K is sufficiently large, then the geodesic L' corresponding to γ' will be such that for every vertex $y \in L'$ the nearest vertex to y in L will be one of the K vertices $z_1, \dots, z_K$, by hyperbolicity. Denote by β' the segment of γ' following the fork from γ. Because the shift (Σ,σ) is topologically mixing on its recurrent part, the set of possible continuations β' of length m nearly coincides with the set of paths β'' such that for some short path α in $\mathcal{A}$ starting at s_* the concatenation $\alpha\beta''$ is a path in $\mathcal{A}$. Thus, the sum in (47), which (roughly) corresponds to the sum over all β', is comparable to the sum over all paths $\alpha\beta''$ in $\mathcal{A}$. □

9. Asymptotics of transition probabilities

Theorem 1.11 (giving the asymptotics of transition probabilities in a co-compact Fuchsian group) is a direct consequence of the asymptotics of the Green's function given by Theorem 1.10 and of the following general statement.

Theorem 9.1. *Consider a symmetric irreducible aperiodic random walk in a countable group* Γ. *Let* R *denote the radius of convergence of the Green's function* $G_z(x, y) = \sum z^n p^n(x, y)$. *Assume that there exists* $\beta > 0$ *such that for all* $x, y \in \Gamma$,

$$G'_r(x, y) \sim C_{x,y}/(R - r)^{\beta} \quad \text{as} \ \ r \uparrow R, \tag{48}$$

for some $C_{x,y} > 0$. *Then there exist constants* $C'_{x,y} > 0$ *such that*

$$p^n(x, y) \sim C'_{x,y} R^{-n} n^{\beta-2} \quad \text{as} \ \ n \to \infty. \tag{49}$$

For the proof, we will rely on the following tauberian theorem of Karamata (see e.g. [6], Corollary 1.7.3 or [13], Theorem XII.5.5).

Theorem 9.2. *Let* $A(z) = \sum a_n z^n$ *be a power series with nonnegative coefficients* a_n *and radius of convergence* 1. *Assume that, when* s *tends to* 1 *along the real axis,* $\sum a_n s^n \sim c/(1 - s)^{\beta}$ *with* $\beta > 0$. *Then* $\sum_{k=1}^{n} a_k \sim cn^{\beta}/\Gamma(1 + \beta)$.

The coefficients $R^n p^n(x, y)$ of the (re-parameterized) Green's function need not in general be monotone, so Karamata's Theorem cannot be applied directly. However, we will show that for a *symmetric, aperiodic* random walk the coefficients can be decomposed as $R^n p^n(x, y) = q_n(x, y) + O(e^{-\delta n})$ with $q_n(x, y)$ non-increasing and $\delta > 0$; this will allow us to deduce the local limit theorem (49) from corresponding asymptotics for $q_n(x, y)$, which in turn will follow from Karamata.

9.1. Spectral analysis of the transition probability operator.

The hypothesis that the random walk is symmetric implies that the Markov operator $\mathbb{P}$ of the random walk, acting on the space $\ell^2(\Gamma)$ by $\mathbb{P}u(x) = \sum p(x, y)u(y)$, is self-adjoint and bounded.

Theorem 9.3. *Assume that the random walk is symmetric, irreducible and aperiodic. Then there exists* $\varepsilon > 0$ *such that the spectrum of the Markov operator* $\mathbb{P}$ *is contained in the interval* $[-R^{-1}(1 + \varepsilon)^{-1}, R^{-1}]$. *Consequently, for all* $x, y \in \Gamma$ *the Green's function* $G_z(x, y)$ *extends holomorphically to the doubly slit plane* $\mathbb{C} \setminus ((-\infty, -R(1 + \varepsilon)] \cup [R, \infty))$.

Proof. It is well known (see, for instance, [35]) that the spectrum of $\mathbb{P}$ is contained in the interval $[-R^{-1}, R^{-1}]$, where R is the common radius of convergence of the Green's functions. Hence, by the spectral theorem, for any function $u \in \ell^2(\Gamma)$ there exists a nonnegative measure $\nu = \nu_u$ on $[-R^{-1}, R^{-1}]$, with total mass $\|\nu\| = \|u\|_2^2$, such that for any $n \in \mathbb{N}$

$$\langle u, \mathbb{P}^n u\rangle = \int t^n \, d\nu(t), \tag{50}$$

and for any complex number z of modulus $|z| > R^{-1}$,

$$R_u(z) := \langle u, (z - \mathbb{P})^{-1} u\rangle = \int \frac{1}{z - t} \, d\nu(t). \tag{51}$$

To prove the theorem it suffices to show that for some $\varepsilon > 0$ the spectral measures ν_u, where $u \in \ell^2(\Gamma)$, all have support contained in $[-R^{-1}(1 + \varepsilon)^{-1}, R^{-1}]$.

Formula (51) exhibits the resolvent function $R_u(z)$ as the *Stieltjes transform* of the measure $\nu = \nu_u$. Since ν is nonnegative and has finite total mass, its Stieltjes transform $R_u(z)$ extends holomorphically to $\mathbb{C} \setminus [-R^{-1}, R^{-1}]$, and it satisfies the reflection identity $R_u(\overline{z}) = \overline{R_u(z)}$. According to the *Stieltjes inversion theorem* (see, e.g., [32]), for any two real numbers $x_1 < x_2$, neither of which is an atom of ν,

$$\nu[x_1, x_2] = -\lim_{y\to 0} \Im \frac{1}{\pi} \int_{x_1}^{x_2} R_u(x + iy)\, dy \tag{52}$$

where $\Im$ denotes imaginary part. Suppose that $R_u(z)$ is analytic in a neighborhood of $[x_1, x_2]$. Since the function R_u satisfies the reflection identity, it must be real-valued on $[x_1, x_2]$, and so it follows from the inversion formula (52) that $\nu[x_1, x_2] = 0$. Thus, to complete the proof of the theorem it suffices to show that there is some $\delta > 0$ such that for every $u \in \ell^2(\Gamma)$, the resolvent function $R_u(z)$ has an analytic continuation to $[-R^{-1}, -R^{-1} + \delta]$.

Observe that in the special case where $u = I_{\{1\}}$ is the indicator function of the group identity, $z^{-1}R_u(z^{-1}) = G_z(x, x)$ is the Green's function, and in the case where $u = I_{\{x,y\}}$ is the indicator of a two-point set, $z^{-1}R_u(z^{-1}) = 2G_z(x, x) + 2G_z(x, y)$. Hence, the Green's functions extend holomorphically to $\mathbb{C} \setminus ((-\infty, -R] \cup [R, \infty)$. According to a theorem of Cartwright [12], for any aperiodic, symmetric random walk on a countable group the only singularity of the Green's functions $G_z(x, y)$ on the circle $|z| = R$ is $z = R$. In fact, his proof shows that there is an open neighborhood U of $\{z : |z| = R\} \setminus \{R\}$ to which all of the functions $G_z(x, y)$ extend holomorphically. It follows that for $\delta > 0$ sufficiently small, the resolvent function $R_u(z)$ extends holomorphically to $[-R^{-1}, -R^{-1} + \delta]$ for all functions u that are indicators of either one-point or two-point sets, and so the corresponding spectral measures attach zero mass to this interval. Since the spectral measure ν_u of any function u with finite support can be written as a linear combination of one-point or two-point indicators, it follows that they likewise attach zero mass to the interval $[-R^{-1}, -R^{-1} + \delta]$. Finally, for any $u \in \ell^2$ there is a sequence of finitely supported functions u_n that converge to u in ℓ^2, and for any such sequence the corresponding spectral measures ν_{u_n} converge weakly to ν_u. (This follows, for instance, from (50), which implies convergence of moments.) Therefore, for any $u \in \ell^2$ the spectral measure has support contained in $[-R^{-1} + \delta, R^{-1}]$. □

Corollary 9.4. *Consider a symmetric irreducible aperiodic random walk on a countable group, and let R be the radius of convergence of the Green's function. Then $R^n p^n(1, 1) = q_n + O(e^{-\delta n})$, where q_n is nonincreasing and $\delta > 0$. Furthermore, for every $x \neq 1$, $R^n p^n(1, 1) + R^n p^n(1, x) = q_n(1, x) + O(e^{-\delta n})$, where $q_n(1, x)$ is nonincreasing.*

Proof. Let u be the indicator function of the singleton $\{1\}$, and let $\nu = \nu_u$ be the corresponding spectral measure. Then by (50),

$$p^n(1, 1) = \langle u, \mathbb{P}^n u\rangle = \int t^n\, d\nu(t) = \int t^n\, d\nu^+(t) + \int t^n\, d\nu^-(t),$$

where ν^+ and ν^- are the restrictions of ν to the positive and nonpositive reals, respectively. The sequence $q_n = R^n \int t^n\, d\nu^+(t)$ is clearly nonincreasing in n, since ν^+ is supported by the interval $(0, R^{-1}]$. By Theorem 9.3, the support of ν^- is contained in $[-R^{-1}(1 + \varepsilon)^{-1}, 0]$ for some $\varepsilon > 0$. Hence,

$$\int t^n\, d\nu^-(t) = O(R^{-n}(1 + \varepsilon)^{-n}).$$

This proves the first assertion. A similar argument proves the second assertion, since

$$2p^n(1,1) + 2p^n(1,x) = \int t^n \, d\nu_{1,x}(t)$$

where $\nu_{1,x}$ is the spectral measure of the indicator function of the two-point set $\{1,x\}$. □

9.2. **Proof of Theorem 9.1.** Consider first the case $x = y = 1$. By Corollary 9.4, we may write $R^k p^k(1,1) = q_k + r_k$ where r_k is exponentially small and q_k is non-increasing. Thus, to prove the asymptotic formula (49) for $x = y = 1$, it suffices to prove that

$$q_n \sim Cn^{\beta-2} \quad \text{as} \quad n \to \infty.$$

For $s \in [0,1)$, let $r = Rs$ and

$$A(s) = rG_r(1,1)' = \sum nr^n p^n(1,1) = \sum s^n \cdot nR^n p^n(1,1).$$

By hypothesis, $A(s) \sim C/(1-s)^\beta$ when $s \uparrow 1$. Since $R^k p^k(1,1) = q_k + r_k$ with r_k exponentially decaying in k, it follows that $\sum_k q_k s^k \sim C/(1-s)^\beta$ as $s \uparrow 1$. Therefore, Karamata's theorem gives

$$\sum_{k=1}^{n} kq_k \sim Cn^\beta. \tag{53}$$

The desired result now follows from the next lemma.

Lemma 9.5. *Let q_n be a nonnegative sequence that satisfies* (53) *for some $\beta > 0$. If q_n is non-increasing, then as $n \to \infty$,*

$$q_n \sim C\beta n^{\beta-2}.$$

Proof. Fix $\varepsilon > 0$. Writing $S_n = \sum_{k=0}^{n-1} kq_k$, we have

$$\varepsilon n(1-\varepsilon)nq_n \le \sum_{k=(1-\varepsilon)n}^{n-1} kq_n \le \sum_{k=(1-\varepsilon)n}^{n-1} kq_k = S_n - S_{(1-\varepsilon)n} = C(n^\beta - (1-\varepsilon)^\beta n^\beta + o(n^\beta)).$$

Therefore,

$$q_n \le Cn^{\beta-2}\frac{1-(1-\varepsilon)^\beta + o(1)}{(1-\varepsilon)\varepsilon}.$$

Letting ε tend to 0, we obtain $\limsup q_n/n^{\beta-2} \le C\beta$. Using the interval $k \in [n,(1+\varepsilon)n]$, we control the inferior limit in the same way, and so we obtain $q_n \sim C\beta n^{\beta-2}$. □

Finally, consider the general case $x, y \in \Gamma$. Since we have already proved the formula (49) in the special case $x = y$, we may assume that $x \neq y$, and by homogeneity, $x = 1$. By Corollary 9.4, there is a non-increasing sequence $q_k(1,y)$ and an exponentially decaying sequence $r_k(1,y)$ such that $R^k p^k(1,1) + R^k p^k(1,y) = q_k(1,y) + r_k(1,y)$. Since the formula (49) holds for $x = y = 1$, to prove it for $x = 1 \neq y$ it will suffice to show that

$$q_k(1,y) \sim (C'_{1,y} + C'_{1,1})k^{\beta-2} \tag{54}$$

for some constant $C'_{1,y}$. (Note: By the Harnack inequality, the sequences $p^k(1,1)$ and $p^k(1,y)$ are comparable, so if this holds then $C'_{1,y}$ must be positive.)

By hypothesis (48),

$$\sum_k s^k k R^{k-1}(p^k(1,1) + p^k(1,y)) \sim \frac{C_{1,1} + C_{1,y}}{(1-s)^\beta}$$

as $s \uparrow 1$, and so, as in the special case considered earlier, the generating function $\sum_k kq_k(1,y)s^k$ satisfies the hypotheses of Karamata's theorem. Thus,

$$\sum_{k=1}^{n} kq_k(1,y) \sim Cn^\beta,$$

and so the relation (54) follows from Lemma 9.5. □

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