On the characteristic torsion of *gwistor* space

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Abstract

We give a presentation of gwistor space. Then we compute the characteristic torsion T^{c} of the G_2-twistor space of an oriented Riemannian 4-manifold with constant sectional curvature k and deduce the condition under which T^{c} is ∇^{c}-parallel; this allows for the classification of the G_2 structure with torsion and the characteristic holonomy according to known references. The case with the Einstein base manifold is envisaged.

Key Words: Einstein metric, gwistor space, characteristic torsion, G_2 structure.

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1.1 The purpose

It has now become clear that every oriented Riemannian 4-manifold M gives rise to a G_2-twistor space, as well as its celebrated twistor space. The former was discovered in [5, 6] and we shall start here by recalling how it is obtained. Often we abbreviate the name G_2-twistor for *gwistor*, as started in [3]. Briefly, given M as before, the G_2-twistor space of M consists of a natural G_2 structure on the S^3-bundle over M of unit tangent vectors

$$SM = \big\{u \in TM : \ \ \|u\| = 1\big\} \tag{1}$$

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exclusively induced by the metric $g = \langle\ ,\ \rangle$ and orientation.

We shall describe the characteristic connection ∇^{c} of SM in the case where M is an Einstein manifold. This guarantees the *gwistor* structure is cocalibrated, an equivalent condition. And hence the existence of that particular connection by [21]. Then we restrict to constant sectional curvature; we deduce the condition under which the characteristic torsion, i.e. the torsion of the characteristic connection, is parallel for ∇^{c}. Finally we are able to deduce its classification, according with the holonomy obtained and the cases in [19]. The reason why we made such restriction is that the study of the characteristic connection in the general Einstein base case is much more difficult and we wish to present the problem.

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1.2 Elements of G_2-twistor space

Let M be an oriented smooth Riemannian 4-manifold and SM its unit tangent sphere bundle. The G_2-twistor structure is constructed with the following briefly recalled techniques (cf. [3, 5, 6]).

Let $\pi : TM \to M$ denote the projection onto M, let $\nabla^{\text{L-C}}$ be the Levi-Civita connection of M and let U be the canonical vertical unit vector field over TM pointing outwards of SM. More precisely, we define U such that $U_u = u$, $\forall u \in TM$. The Levi-Civita connection of M induces a splitting $TTM \simeq \pi^*TM \oplus \pi^*TM$. The pull-back bundle on the left hand side is the horizontal subspace $\ker \pi^*\nabla^{\text{L-C}}_{\cdot} U$ isomorphic to π^*TM through $\mathrm{d}\pi$. The other π^*TM, on the right, is the vertical subspace $\ker \mathrm{d}\pi$. We are henceforth referring to the classical decomposition of TTM, as displayed in several articles and textbooks.

Restricting π to SM we have $T\,SM = H \oplus V$ where H denotes the restriction of the horizontal sub-bundle to SM and V is such that $V_u = u^\perp \subset \pi^*TM$, thus contained on the vertical side. Every vector field over SM may be written as

$$X = X^h + X^v = X^h + \pi^*\nabla^{\text{L-C}}_X U. \tag{2}$$

The tangent sphere bundle inherits a Riemannian metric, the induced metric from the metric on TM attributed to Sasaki: $\pi^*g \oplus \pi^*g$. We simply invoke this metric with the same letter g or by the brackets $\langle\ ,\ \rangle$. Then we may say that SM is the locus set of the equation $\langle U, U\rangle = 1$ and indeed (2) is confirmed: notice $\mathrm{d}\langle U, U\rangle(X) = 2\langle \pi^*\nabla^{\text{L-C}}_X U, U\rangle$. There is also a natural map

$$\theta : TTM \longrightarrow TTM \tag{3}$$

which is a $\pi^*\nabla^{\text{L-C}}$-parallel endomorphism of TTM identifying H isometrically with the vertical bundle $\pi^*TM = \ker \mathrm{d}\pi$ and defined as 0 on the vertical side. It was introduced in [3, 5, 6]. Then we define the horizontal vector field $\theta^t U$.

The tangent bundle TSM inherits a metric connection, via the pull-back connection and still preserving the splitting, which we denote by $\nabla^\star$. On tangent vertical directions, due to the geometry of the 3-sphere with the round metric, we must add a correction term to the pull-back connection. That is, for any $X, Y \in \Gamma(TSM)$:

$$\nabla^\star_Y X^v \;=\; \pi^*\nabla^{\text{L-C}}_Y X^v - \langle \pi^*\nabla^{\text{L-C}}_Y X^v, U\rangle U \;=\; \pi^*\nabla^{\text{L-C}}_Y X^v + \langle X^v, Y^v\rangle U. \tag{4}$$

We then let $\mathcal{R}^U(X,Y) = \pi^*R(X,Y)U = R^{\pi^*\nabla^{\text{L-C}}}(X,Y)U$, which is a V-valued tensor. We follow the convention $R(X,Y) = [\nabla^{\text{L-C}}_X, \nabla^{\text{L-C}}_Y] - \nabla^{\text{L-C}}_{[X,Y]}$. Notice $\mathcal{R}^U(X,Y) = \mathcal{R}^U(X^h, Y^h)$. Finally, the Levi-Civita connection ∇^g of SM is given by

$$\nabla^g{}_X Y = \nabla^\star_X Y - \frac{1}{2}\mathcal{R}^U(X,Y) + A(X,Y) \tag{5}$$

where A is the H-valued tensor defined by

$$\langle A(X,Y), Z\rangle = \frac{1}{2}\big(\langle \mathcal{R}^U(X,Z), Y\rangle + \langle \mathcal{R}^U(Y,Z), X\rangle\big), \tag{6}$$

for any vector fields X, Y, Z over SM.

There are many global differential forms on SM. Specially relevant are the 1- and a 2-forms given by

$$\mu(X) = \langle U, \theta X\rangle \qquad \text{and} \qquad \beta(X,Y) = \langle \theta X, Y\rangle - \langle \theta Y, X\rangle. \tag{7}$$

The easiest way to see other forms is by taking an orthonormal basis on a trivialised neighbourhood as follows. First we take a direct orthonormal basis $e_0, \ldots, e_3$ of H, arising from another one fixed on the trivialising open subset of M, such that $e_0 = u \in SM$ on each point u, i.e. $e_0 = \theta^t U$. Then we define

$$e_4 = \theta e_1, \qquad e_5 = \theta e_2, \qquad e_6 = \theta e_3. \tag{8}$$

This completes the desired set; we say $e_0, \ldots, e_6$ is a standard or adapted frame. Notice $\theta e_0 = U$, as if u has the gift of ubiquity. The dual co-frame is used to write

$$\mu = e^0, \qquad \mathrm{vol} = e^{0123}, \qquad \beta = e^{14} + e^{25} + e^{36}, \qquad \alpha = e^{456},$$
$$\alpha_1 = e^{156} + e^{264} + e^{345}, \qquad \alpha_2 = e^{126} + e^{234} + e^{315}, \qquad \alpha_3 = e^{123}.$$

These are all global well defined forms. They satisfy the basic structure equations, cf. [3]:

$$*\alpha = \mathrm{vol} = \mu\alpha_3 = \pi^*\mathrm{vol}_M, \qquad *\alpha_1 = -\mu\wedge\alpha_2, \qquad *\alpha_2 = \mu\wedge\alpha_1,$$
$$*\beta = -\frac{1}{2}\mu\wedge\beta^2, \qquad *\beta^2 = -2\mu\wedge\beta, \qquad \beta^3\wedge\mu = -6\mathrm{Vol}_{SM},$$
$$\alpha_1\wedge\alpha_2 = 3*\mu = -\frac{1}{2}\beta^3, \qquad \beta\wedge\alpha_i = \beta\wedge *\alpha_i = \alpha_0\wedge\alpha_i = 0,$$

$\forall i = 0, 1, 2$, where we wrote $\alpha = \alpha_0$. We use the notation $e^{ab\cdots jk} = e^a e^b \cdots e^j e^k$ and often omit the wedge product symbol, like in β^2.

1.3 The gwistor space

We have given the name G_2-*twistor* or *gwistor space* to the G_2 structure on SM defined by the stable 3-form

$$\phi = \alpha + \mu\beta - \alpha_2$$

(it is induced by the Cayley-Dickson process using the vector field U and the volume forms vol, α). Let $*$ denote the Hodge star product. Then

$$*\,\phi = \mathrm{vol} - \frac{1}{2}\beta^2 - \mu\alpha_1. \tag{9}$$

We know from [3, Proposition 2.4] that

$$\mathrm{d}\phi = \mathcal{R}^U\alpha + \underline{r}\mathrm{vol} - \beta^2 - 2\mu\alpha_1 \qquad \text{and} \qquad \mathrm{d}*\phi = -\rho\mathrm{vol}$$

where we have set

$$\mathcal{R}^U\alpha \;=\; \sum_{0\leq i<j\leq 3} R_{ij01}e^{ij56} + R_{ij02}e^{ij64} + R_{ij03}e^{ij45}, \tag{10}$$

with $R_{ijkl} = \langle R(e_i, e_j)e_k, e_l\rangle$, $\forall i, j, k, l \in \{0, 1, 2, 3\}$.

Also, $\underline{r} = r(U, U)$ is a function, with r the Ricci tensor, and ρ is the 1-form $(\mathrm{Ric}\,U)^\flat \in \Omega^0(V^*)$, vanishing on H and restricted to vertical tangent directions. One may view ρ as the vertical lift of $r(\ ,U)$. We continue considering the adapted frame $e_0, \ldots, e_6$ on SM; then

$$\rho = \sum_{i,k=1}^{3} R_{ki0k}e^{i+3} \qquad \text{and} \qquad \underline{r} = \sum_{j=1}^{3} R_{j00j}. \tag{11}$$

We also remark

$$\mathrm{d}\alpha = \mathcal{R}^U\alpha, \qquad \mathrm{d}\mu = -\beta, \qquad \mathrm{d}\alpha_2 = 2\mu\alpha_1 - \underline{r}\mathrm{vol}. \tag{12}$$

We know the gwistor space SM is never a geometric G_2 manifold. Recall that any given G_2-structure ϕ is parallel for the Levi-Civita connection if and only if ϕ is a harmonic 3-form. Indeed, our $\mathrm{d}\phi$ never vanishes. However, an auspicious result leads us forward. (SM, ϕ) is cocalibrated, ie. $\delta\phi = 0$, if and only if M is an Einstein manifold, cf. [3, 5, 6].

The curvature of the unit tangent sphere bundle has been studied, but the Riemannian holonomy group remains unknown in general (cf. [1, 10] and the references therein). From the point of view of gwistor spaces, hence just on the 4-dimensional base space, we are interested on the holonomy of the G_2 characteristic connection.

1.4 The characteristic connection

Following the theory of metric connections on a Riemannian 7-manifold (N, ϕ) with G_2 structure, cf. [2, 20, 21], the *characteristic connection* consists of a metric connection with skew-symmetric torsion for which ϕ is parallel. If it exists, then it is unique. Formally we may write

$$\langle \nabla^{c}{}_{X}Y, Z\rangle = \langle \nabla^{g}{}_{X}Y, Z\rangle + \frac{1}{2}T^{c}(X,Y,Z)$$

where g denotes the metric and ∇^g the Levi-Civita connection. If ϕ is cocalibrated, then such T^c exists; it is given by

$$T^{c} = *\mathrm{d}\phi - \frac{1}{6}\langle \mathrm{d}\phi, *\phi\rangle\phi \tag{13}$$

cf. [21, Theorems 4.7 and 4.8][1].

We recall there are three particular G_2-modules decomposing the space Λ^3 of 3-forms (cf. [9, 15, 18]). They are Λ^3_1, Λ^3_7, Λ^3_{27}, with the lower indices standing for the respective dimensions. In the same reasoning, $\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{14}$. Thus, by Hodge duality, $\mathrm{d}\phi$ has three invariant structure components and $\delta\phi$ has two. In gwistor space we have proved the latter vanish altogether, or not, with ρ, given in (11). The analysis of the tensor $\mathrm{d}\phi$ is struck with the never-vanishing component in Λ^3_{27}. It is of pure type Λ^3_{27} if and only if M is an Einstein manifold with Einstein constant -6 (see [3, Theorem 3.3]).

Apart from a Ricci tensor dependent component, the curvature tensor of M contained in $\mathrm{d}\phi = \mathcal{R}^U\alpha + \cdots$ remains much hidden in the Λ^3_{27} subspace.

We have deduced a formula for the Levi-Civita connection ∇^g of SM, shown in (5). The characteristic connection ∇^c is to be deduced here in the cocalibrated case given by a constant sectional curvature metric on M. In our opinion, this analysis corroborates the correct choice of techniques in dealing with the equations of gwistor space.

1.5 Gwistor space of a space form

Let us start by assuming M, g is an Einstein manifold with Einstein constant λ. Such condition is given by any of the following, where λ is a priori a scalar function on M:

$$r = \lambda g \quad \Leftrightarrow \quad \mathrm{Ric}\, U = \lambda U \quad \Leftrightarrow \quad \underline{r} = \lambda. \tag{14}$$

In our setting it is also equivalent to $\mathrm{d} * \phi = 0$. Then λ is a constant.

Proposition 1.1. *The characteristic connection* $\nabla^c = \nabla^g + \frac{1}{2}T^c$ *of* SM *is given by*

$$T^{c} = *(\mathcal{R}^U\alpha) + \frac{2\lambda - 6}{3}\alpha - \frac{\lambda}{3}\mu\beta + \frac{\lambda}{3}\alpha_2. \tag{15}$$

Moreover, $\delta T^c = 0$.

Proof. We have by (10) and some computations

$$\langle \mathcal{R}^U\alpha, *\phi\rangle \mathrm{Vol}_{SM} = (\mathcal{R}^U\alpha)\phi = (\mathcal{R}^U\alpha)(\mu\beta - \alpha_2) = \lambda \mathrm{Vol}_{SM}.$$

[1] Notice we use a different orientation than that in [21]. Therefore, we have to replace $*$ by $-*$ in formulas given there.

Also $\underline{r}\mathrm{vol}\phi = \lambda\mathrm{Vol}_{SM}$, $-\beta^2\phi = -\mu\beta^3 = 6\mathrm{Vol}_{SM}$, $-2\mu\alpha_1\phi = 2\mu\alpha_1\alpha_2 = 6\mathrm{Vol}_{SM}$. Hence $\langle \mathrm{d}\phi, *\phi\rangle = 2(\lambda+6)$. One finds helpful identities in (1.2). Since $*\mathrm{d}\phi = *\mathcal{R}^U\alpha + \lambda\alpha + 2\mu\beta - 2\alpha_2$, we get from (13)

$$\begin{aligned} T^{\mathrm{c}} &= *\mathrm{d}\phi - \frac{2}{6}(\lambda+6)\phi \\ &= *\mathcal{R}^U\alpha + \lambda\alpha + 2\mu\beta - 2\alpha_2 - (\frac{\lambda}{3}+2)(\alpha+\mu\beta-\alpha_2) \end{aligned}$$

and the first part of the result follows. From the first line we immediately see $\mathrm{d}{*T^{\mathrm{c}}} = 0$. ∎

Until the rest of this section we assume M has constant sectional curvature k, so that $R_{ijkl} = k(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})$ with $k \in \mathbb{R}$ a constant. Then by (10)

$$\mathcal{R}^U\alpha = -k\mu\alpha_1. \tag{16}$$

In particular,

$$\mathrm{d}\phi = 3k\mathrm{vol} - \beta^2 - (k+2)\mu\alpha_1. \tag{17}$$

Henceforth $\lambda = \underline{r} = 3k$ and $*\mathcal{R}^U\alpha = -k\alpha_2$, and the following result is immediate.

Proposition 1.2. *The characteristic torsion of the characteristic connection is given by*

$$T^{\mathrm{c}} = 2(k-1)\alpha - k\mu\beta. \tag{18}$$

Taking formulas (5) and (6), the next Propositions are the result of simple computations.

Proposition 1.3. *For any $X, Y \in TSM$:*

1. $\mathcal{R}^U(X,Y) = k(\langle\theta Y, U\rangle\theta X - \langle\theta X, U\rangle\theta Y)$; *or simply* $\mathcal{R}^U = k\theta\wedge\mu$
2. $A(X,Y) = \frac{k}{2}\big(\langle\theta X, Y\rangle\theta^t U + \langle\theta Y, X\rangle\theta^t U - \mu(X)\theta^t Y - \mu(Y)\theta^t X\big)$.

We also omit the proof of the next formulas. These are the application of the general case treated in [3, Proposition 2.2] to our situation with $\mathcal{R}^U$ and A given just previously.

Proposition 1.4. *For any $X \in TSM$ we have:*

1. $\nabla^g{}_X\theta^t U = \frac{2-k}{2}\theta^t X - \frac{k}{2}(\theta X - \mu(X)U)$
2. $\nabla^g{}_X\mathrm{vol} = A_X\cdot\mathrm{vol} = \frac{k}{2}\big(\mu(X)\mu\wedge\alpha_2 - (\theta X)^\flat\wedge\alpha_3 - (X^\flat\circ\theta)\wedge\alpha_3\big)$
3. $\nabla^g{}_X\alpha = \frac{k}{2}\big(\mu\wedge(\theta X)\lrcorner\alpha - \mu(X)\alpha_1\big)$
4. $\nabla^g{}_X\mu = \frac{2-k}{2}X^\flat\circ\theta - \frac{k}{2}(\theta X)^\flat$
5. $\nabla^g{}_X\beta = \frac{k}{2}\mu\wedge\big((X^v)^\flat - (X^h)^\flat\big)$
6. $\nabla^g{}_X\alpha_1 = k\mu(X)\big(\frac{3}{2}\alpha - \alpha_2\big) + \mu\wedge\big(\frac{k-2}{2}X\lrcorner\alpha + \frac{k}{2}(\theta X)\lrcorner\alpha_1\big)$
7. $\nabla^g{}_X\alpha_2 = k\mu(X)\big(\alpha_1 - \frac{3}{2}\alpha_3\big) + \mu\wedge\big(\frac{k-2}{2}X^v\lrcorner\alpha_1 + \frac{k}{2}X\lrcorner\alpha_3\big)$
8. $\nabla^g{}_X\alpha_3 = \frac{2-k}{2}(\theta^t X)\lrcorner\mathrm{vol} + (\theta^t U)\lrcorner A_X\cdot\mathrm{vol} = \frac{2-k}{2}(\theta^t X)\lrcorner\mathrm{vol} + \frac{k}{2}\mu(X)\alpha_2$.

We may now deduce:

$$\begin{aligned}\nabla^g{}_X\phi &= \nabla^g{}_X\alpha+\nabla^g{}_X\mu\wedge\beta+\mu\wedge\nabla^g{}_X\beta-\nabla^g{}_X\alpha_2\\ &= \frac{k}{2}\mu\wedge(\theta X)\lrcorner\alpha-\frac{3k}{2}\mu(X)\alpha_1+\big(\frac{2-k}{2}X^\flat\circ\theta-\frac{k}{2}(\theta X)^\flat\big)\wedge\beta\\ &\qquad+\frac{3k}{2}\mu(X)\alpha_3-\frac{k}{2}\mu\wedge X\lrcorner\alpha_3-\frac{k-2}{2}\mu\wedge X^v\lrcorner\alpha_1.\end{aligned}\tag{19}$$

A computation confirms that $\nabla^{\mathrm{c}}\phi=0$ with $\nabla^{\mathrm{c}}=\nabla^g+\frac12T^{\mathrm{c}}$ and T^{c} given by (18).

We are now in position to compute $\nabla^{\mathrm{c}}T^{\mathrm{c}}$.

Theorem 1.1. *Let M be an oriented Riemannian 4-manifold of constant sectional curvature k. The characteristic connection ∇^{c} of the associated gwistor space satisfies*

$$\nabla^{\mathrm{c}}{}_XT^{\mathrm{c}} = k(k-1)X^v\lrcorner(\mu\wedge\alpha_1-\frac12\beta^2).$$

In particular, SM has parallel torsion if and only if $k=0$ or $k=1$.

Proof. For any direction $X\in TSM$ and using the cyclic sum in three vectors,

$$\nabla^{\mathrm{c}}{}_XT^{\mathrm{c}}=\nabla^g{}_XT^{\mathrm{c}}-\ \underset{}{\circlearrowright\!\!\!\!\bigoplus}\, T^{\mathrm{c}}(\frac12T^{\mathrm{c}}_X\ ,\ ,\ )=\nabla^g{}_XT^{\mathrm{c}}-\frac12\sum_{j=0}^{6}T^{\mathrm{c}}(X,\ ,e_j)\wedge T^{\mathrm{c}}(e_j,\ ,\ ).$$

Since $T^{\mathrm{c}}=2(k-1)\alpha-k\mu\beta$, we get

$$\nabla^{\mathrm{c}}{}_XT^{\mathrm{c}} = \nabla^g{}_XT^{\mathrm{c}}-\frac{k^2}{2}\beta(X,\ )\wedge\beta+k(k-1)X\lrcorner(\mu\wedge\alpha_1).$$

Now we have from Proposition 1.4

$$\nabla^g{}_XT^{\mathrm{c}}=(k-1)k\big(\mu\wedge(\theta X)\lrcorner\alpha-\mu(X)\alpha_1\big)+\big(k\frac{k-2}{2}X^\flat\circ\theta+\frac{k^2}{2}(\theta X)^\flat\big)\wedge\beta$$

and then we see easily that $\nabla^{\mathrm{c}}{}_XT^{\mathrm{c}}=0$ for $X\in H$. Taking a vertical direction X, the desired formula for the covariant derivative of T^{c} is achieved. ∎

Let us now see the decompositions under G_2 representations referred in section 1.4, in the case under appreciation. Recall $\mathrm{d}\phi$ has no Λ^3_7 component. Since $\delta\phi=0$, there are no $\Lambda^2_7,\Lambda^2_{14}$ components either. By results in [3, Proposition 3.6], the $\Lambda^3_1,\Lambda^3_{27}$ parts are

$$\mathrm{d}\phi=\frac67(k+2)*\phi+*\frac17\big((15k-12)\alpha+(2-6k)\mu\beta-(k+2)\alpha_2\big).$$

Now T^{c} is coclosed. A characteristic connection with closed torsion is called a strong G_2 with torsion, denoted $\mathrm{SG_2T}$ in [16]. We have the decompositions $T^{\mathrm{c}}=-\frac{k+2}{7}\phi+\tau_3^{T^{\mathrm{c}}}$ and

$$\mathrm{d}T^{\mathrm{c}}=k\beta^2-2k(k-1)\mu\alpha_1=\frac67k(k-2)*\phi+*\tau_3^{\mathrm{d}T^{\mathrm{c}}},$$

where $\tau_3^{T^{\mathrm{c}}},\tau_3^{\mathrm{d}T^{\mathrm{c}}}$, sitting in $\Lambda^3_{27}=\ker(\cdot\wedge\phi)\cap\ker(\cdot\wedge*\phi)$, are given by

$$\begin{aligned}\tau_3^{T^{\mathrm{c}}} &= \frac17\big((15k-12)\alpha+(2-6k)\mu\beta-(k+2)\alpha_2\big),\\ \tau_3^{\mathrm{d}T^{\mathrm{c}}} &= \frac{2k}{7}\big((6-3k)\alpha-(1+3k)\mu\beta+(1-4k)\alpha_2\big).\end{aligned}$$

In conclusion, our special geometry induced from constant curvature on M has an $\mathrm{SG_2T}$ connection if and only if $k=0$.

1.6 Results on the Stiefel manifold $V_{l,2}$

Theorem 1.1 leads to the consideration of two distinct cases. We start with $k = 1$.

Since our results so far are local, we assume M is simply-connected and complete. As it is well known, SM with $M = S^4_1$, the radius 1 sphere, agrees with the Stiefel manifold $SO(5)/SO(3) = V_{5,2}$. Recall that transitivity of the action by isometries induced on the tangent sphere bundle of a Riemannian symmetric space is exclusive to all rank 1 spaces, cf. [9, Proposition 10.80]. In particular, in dimension 4, we are left with S^4, $\mathbb{P}^2(\mathbb{C})$, the real hyperbolic space H^4 and the hyperbolic Hermitian space $\mathbb{C}H^2$.

We thus study briefly the space $V_{l,2}$, the unit tangent sphere bundle of S^{l-1} with $l > 2$. In the sequel, we let the name Stiefel manifold refer just to $V_{l,2}$ (with the index 2 fixed). Firstly, the Stiefel manifolds are simply-connected for $l \geq 5$. The following results are due to Stiefel and to Borel, cf. [11, Proposition 10.1]:

$$\begin{cases} H^*(V_{l,2},\mathbb{Z}) = H^*(S^{l-1}\times S^{l-2},\mathbb{Z}) \qquad \text{if } l \text{ is even} \\ H^0(V_{l,2},\mathbb{Z}) = H^{2l-3}(V_{l,2},\mathbb{Z}) = \mathbb{Z}, \quad H^{l-1}(V_{l,2},\mathbb{Z}) = \mathbb{Z}_2 \qquad \text{if } l \text{ is odd} \\ H^*(V_{l,2},\mathbb{Z}_2) = H^*(S^{l-1}\times S^{l-2},\mathbb{Z}_2) = \wedge\{x_{l-1},x_{l-2}\}. \end{cases} \tag{20}$$

$\wedge$ stands for the free multiplicative exterior algebra generated on the given x_j of degree j. We also have the additive isomorphism $H^*(V_{l,2},\mathbb{Z}_2) = H^*(S^{l-1},\mathbb{Z}_2)\otimes H^*(S^{l-2},\mathbb{Z}_2)$. Moreover, $V_{l,2}$ is a rational homology sphere for l odd, cf. [12]. Now, we may deduce that

$$w(SS^{l-1}) = \sum \pi^* w_i^2$$

where S^{l-1} is the base manifold and π is the projection. There is a general formula in [4]. It is well known that $w(S^k) = \sum_{i\geq 0} w_i = 1$ for all k. Hence the following result for which we do not know a reference.

Proposition 1.5. *The total Stiefel-Whitney class of $V_{l,2}$ is 1. In particular, this space is orientable and admits a spin structure.*

Now, regarding the Riemannian structure from a slightly general picture, let us see how we are driven to $V_{l,2} = SS^{l-1}$ with the metric induced from the Sasaki metric of the tangent bundle, cf. section 1.2.

First we recall from [2, 12, 14, 21] what is the natural geometric notion concerned with a Riemannian reduction from the Lie group $SO(2n+1)$ to the structure group $U(n)$. A metric almost contact manifold consists of a Riemannian manifold $(\mathcal{S},\tilde{g})$ together with a 1-form η, a vector field ξ and an endomorphism $\varphi \in \Gamma(\mathrm{End}\,T\mathcal{S})$ satisfying the relations: $\forall X,Y \in T\mathcal{S}$

$$\begin{gathered} \eta(\xi) = 1, \qquad \varphi^2 = -1 + \eta\otimes\xi, \\ \tilde{g}(\varphi X,\varphi Y) = \tilde{g}(X,Y) - \eta(X)\eta(Y), \qquad \varphi(\xi) = 0. \end{gathered} \tag{21}$$

If furthermore $\mathrm{d}\eta = 2F$, where $F(X,Y) = \tilde{g}(X,\varphi Y)$, then we have a metric contact structure. If the CR-structure defined by the distribution $\mathcal{D} = \ker\eta$ is integrable, then we have a so called normal contact structure. The integrability condition is the vanishing of a certain Nijenhuis tensor of the almost complex structure $J=\varphi_{|\mathcal{D}}$. If ξ is a Killing vector field, i.e. $\mathcal{L}_\xi\tilde{g}=0$, then we say we have a K-contact structure. Since on a contact structure we have $\mathcal{L}_\xi F=0$, the K-contact equation is assured equivalently by $\mathcal{L}_\xi\varphi=0$. A normal K-contact structure is known as a Sasakian structure; then $\mathcal{S}$ is called a Sasakian manifold.

The K-contact condition is equivalent to $\nabla^g{}_X\xi=-\varphi(X)$, $\forall X\in T\mathcal{S}$. A K-contact structure is normal (and thence the manifold is Sasakian) if furthermore (cf. [22])

$$(\nabla^g{}_X\varphi)(Y)=\tilde{g}(X,Y)\xi-\eta(Y)X. \tag{22}$$

Now let M be a Riemannian manifold of dimension $m=n+1$. Y. Tashiro has shown the unit tangent sphere bundle SM (of dimension $2n+1$) has a metric contact structure. It is given, in present notation, by $\tilde{g}=\frac{1}{4}g$, $\eta=\frac{1}{2}\mu$, $\xi=2\theta^tU$ and $\varphi=\theta-U\mu-\theta^t$. Notice g is the Sasaki metric and θ is the map in (3). We have, by (12) easily generalized to any dimension,

$$F(X,Y):=\frac{1}{4}g(X,\varphi Y)=\frac{1}{4}(\langle X,\theta Y\rangle-\langle\theta X,Y\rangle)=-\frac{1}{4}\beta(X,Y),$$

so $\mathrm{d}\eta=\mathrm{d}\frac{1}{2}\mu=-\frac{1}{2}\beta=2F$ as expected. Tashiro also proved the following [10, Theorem 9.3]: the contact metric structure on SM is a K-contact structure if and only if (M,g) has constant sectional curvature 1. And then deduces SM is Sasakian. The proof goes as follows: notice $\nabla^g=\nabla^{\tilde{g}}$ is given in (5). Then we find

$$\langle\nabla^g{}_X\xi,Y^v\rangle=-\frac{1}{2}\langle\mathcal{R}^U{}_{X,\xi},Y^v\rangle=-\langle R_{X^h,\theta^tU}U,Y^v\rangle.$$

So, just looking at the vertical part of the equation $\nabla^g{}_X\xi=-\varphi(X)$, on the base manifold it reads $\langle R(X,u)u,Y\rangle=\langle X,Y\rangle$, $\forall X,Y\in TM\cap u^\perp$. Clearly, this means constant sectional curvature 1. The horizontal part of the equation gives the same result. The reciprocal is also easy, and the Sasakian condition follows. Moreover, in this case the Sasakian equation (22) alone implies the round curvature 1.

A contact manifold $(\mathcal{S},\tilde{g},\eta,\xi,\varphi)$ is said to be η-Einstein if its Ricci tensor can be written as $\mathrm{Ric}_{\tilde{g}}(X,Y)=\lambda\tilde{g}+\nu\eta\otimes\eta$ with λ,ν constants (cf. [14, 26]).

We compute, with methods as found in [1], that the contact manifold $(SM,\tilde{g},\eta,\xi,\varphi)$ verifies equalities

$$\begin{gathered}\mathrm{Ric}_{\tilde{g}}(X,Y)=\mathrm{Ric}_g(X,Y)=\\ =((m-1)k-\frac{k^2}{2})\langle X^h,Y^h\rangle+(m-2+\frac{k^2}{2})\langle X^v,Y^v\rangle+\frac{k^2}{2}(2-m)\mu(X)\mu(Y)\end{gathered} \tag{23}$$

if M has constant sectional curvature k.

Proposition 1.6 ([17]). *Assuming constant sectional curvature k, the contact manifold SM is η-Einstein if and only if $k=1$ or $k=m-2$.*

This result was also deduced by [17]. In the Sasakian case $k = 1$ notice the formula $\lambda + \nu = 2n$, as theoretically expected ([14, Lemma 7]).

In [21, 26] we have the notion of contact connection on a contact manifold, i.e. a linear connection on $\mathcal{S}$ such that

$$\nabla\tilde{g} = 0, \qquad \nabla\eta = 0, \qquad \nabla\varphi = 0. \tag{24}$$

[21, Theorem 8.4, case 1] guarantees that any Sasakian manifold admits a contact connection with totally skew-symmetric torsion given by

$$T = \eta \wedge \mathrm{d}\eta. \tag{25}$$

Moreover, T is parallel for such $\nabla = \nabla^g + \frac{1}{2}T$, which is unique — so it is called *the* characteristic connection of the normal contact structure. In general, cf. [21, Theorem 8.2], this contact connection with skew-symmetric torsion exists if and only if the Nijenhuis tensor is skew-symmetric and ξ is a Killing vector field.

In sum, Tashiro's results on SM led us to the case of integrable geometries, the homogeneous Sasakian space $V_{l,2}$, where $l = m + 1 = n + 2$, with metric $\frac{1}{4}g$ and Ricci curvature tensor $\mathrm{Ric}_g = (m - \frac{3}{2})g + \frac{2-m}{2}\mu \otimes \mu$. This space admits a characteristic contact connection (T is the same viewed as a $(2, 1)$-tensor) $\nabla = \nabla^g - \frac{1}{2}\mu\beta$. And there is no simply-connected Riemannian manifold besides S^4 whose unit tangent sphere bundle admits a characteristic contact connection. For the existence assures the manifold is K-contact.

To complete the picture, the characteristic foliation $\mathcal{F}_\xi$ determined by ξ, hence with 1 dimensional leaves, gives a projection onto the Grassmannian of oriented 2-planes in $\mathbb{R}^l$, a complex quadric, $V_{l,2} \to \tilde{\mathrm{Gr}}_{l,2}$.

Starting from a Kähler-Einstein manifold $(X^{2n}, \overline{g}, \overline{J})$ of scalar curvature $4n(n + 1)$, it is shown in [8, pag. 83] how to construct Einstein-Sasakian metrics on an associated S^1-bundle $\pi : \mathcal{S} \to X^{2n}$: the bundle whose first Chern class is $c_1 = \frac{1}{A}c_1(X^{2n})$ where A is the maximal integer such that $\frac{1}{A}c_1(X^{2n})$ is an integral cohomology class. Moreover, $\mathcal{S}$ is simply connected and admits a spin structure (cf. Proposition 1.5). The 1-form η is induced by the associated $U(1)$-connection, so that $\mathrm{d}\eta$ is essentially the Kähler form of X^{2n}.

The example of the Stiefel manifold is already mentioned in [8], as noticed by [12].

1.7 The characteristic holonomy of $V_{l,2}$

We may now continue our study of the gwistor space of the 4-sphere with the canonical Sasaki metric.

Proposition 1.7. *The characteristic connection ∇^{c} of the G_2-twistor space $(V_{5,2}, g, \phi)$ is given by the torsion $T^{\mathrm{c}} = -\mu\beta$ and its holonomy is contained in $SU(3)$. Thus coincides with the contact metric connection. The torsion is parallel.*

Proof. By the results given in (25) we find a contact connection with skew-symmetric torsion $T=-\mu\beta$ (contracting now with the metric g). Additionally we have that T is ∇-parallel. We remark that $\mu\nabla\beta=0$. Computing $\nabla\alpha=(\nabla^g+\frac{1}{2}T)\alpha$, applying Proposition 1.4 and the usual technique, we find α is parallel. Since $\nabla\varphi=0$ and

$$\alpha_2=\frac{1}{2}\alpha\circ(\theta\wedge\theta\wedge 1)=\frac{1}{2}\alpha\circ(\varphi\wedge\varphi\wedge 1),$$

(cf. [3] for this notation and remarks on differentiation) we get

$$\nabla\alpha_2=0.$$

Hence $\nabla\phi=\nabla(\alpha+\mu\beta-\alpha_2)=0$ and therefore the (unique) $SU(3)\subset G_2$ connection ∇ with totally skew-symmetric torsion is the characteristic connection of the gwistor structure, $\nabla=\nabla^{\mathrm{c}}$. ■

Of course $T^{\mathrm{c}}=-\mu\beta$ agrees with the result found in (18) for sectional curvature 1.

Now, the formulas from [21] for the curvature of the characteristic connection are quite long to exhibit in our case. They are combined with the Riemannian curvature. So it is important to recall the references on the latter. There is a long literature on results about the sectional, Ricci and scalar curvatures of the Sasaki or other g-natural metrics on the tangent sphere bundle of any given Riemannian manifold. The techniques are those from e.g. [1] and several other references therein, where Einstein metrics are found (interesting enough, for $V_{l,2}$ we also have $SO(l)$-invariant Einstein metrics given in [7], which use the method below and recur to results of Wang).

We follow homogeneous space theory to solve the problem of finding the holonomy of the characteristic connection.

Let n,m be integers such that $l=m+1=n+2$ (as in section 1.6), let $K=SO(l),\ H=SO(n),\ \mathfrak{g}=\mathfrak{so}(l),\mathfrak{h}=\mathfrak{so}(n)$. Now we consider the trivial embedding $H\subset K$. So we may decompose $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ with $\mathfrak{m}$ the subspace of matrices having 0 where $\mathfrak{h}$ falls. Since $[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m}$ and H is connected, we have a reductive homogeneous space $V_{l,2}=K/H$. Then the tangent vector bundle of K/H arises from the canonical principal H-bundle, associated to $\mathfrak{m}$. Let D_{ij} be the matrix with 0 everywhere except in position (i,j) where it has a 1. We have a canonical basis of $\mathfrak{g}$ given by

$$E_{ij}=D_{ij}-D_{ji},\qquad 1\leq i<j\leq l.$$

The vectors $e_0=E_{m,l}$ and $e_i=E_{i,l},\ e_{i+n}=E_{i,m},\ 1\leq i\leq n$ constitute a basis of $\mathfrak{m}$, which we may take to be an orthonormal basis of a K-invariant Riemannian metric, cf. [24, 27]. Compare also with formula (8), i.e. the adapted frame of G_2-twistor space.

We recall the canonical connection ∇ of K/H is given by $\nabla_{e_a}e_b=0,\ \forall a,b$ such that $0\leq a,b\leq 2n+1$. Its torsion satisfies $T^{\nabla}(X,Y)=-[X,Y]_{\mathfrak{m}}$, where the index denotes the component in $\mathfrak{m}$, cf. [25].

The new metric corresponds with the Sasaki metric of SS^m introduced in section 1.2 and generalised to any dimension. Indeed, the embedding $SO(n) \subset SO(m) \subset SO(l)$ induces the respective decomposition of $\mathfrak{h}$, to which the Levi-Civita connection of the sphere corresponds. The horizontal and vertical subspace decomposition is clear.

Theorem 1.2. *The characteristic contact connection $\nabla^{\mathrm{c}} = \nabla^g - \frac{1}{2}\mu\beta$ on $V_{l,2}$ coincides with the invariant canonical connection. Moreover, ∇^{c} is complete and its holonomy group is $SO(n)$.*

Proof. Here we refer just to Chapter X of [25, Volume II]. First recall from [25, Proposition 2.7] that every K-invariant tensor is parallel for the (invariant) canonical connection. By the way they were defined, the tensors $g, \alpha, \theta, \varphi, \mu, \beta, \xi$ are all clearly K-invariant. Also the torsion $T^\nabla(X, Y, Z) = -g([X, Y], Z) = g(Y, [X, Z])$ is totally skew-symmetric. Hence the result follows by uniqueness of the characteristic connection. The theory says the canonical connection ∇ is complete and what its holonomy Lie subalgebra is. ■

Interesting enough, one may check the identity on a triple of vectors on $\mathfrak{m}$

$$(\mu\beta)(X, Y, Z) = \langle [X, Y], Z\rangle. \tag{26}$$

Finally, the conclusion on the holonomy of the characteristic connection allows us to look for the classification of the G_2-twistor space $V_{5,2}$ according to the holonomy algebra $\mathfrak{hol}(\nabla^{\mathrm{c}}) \subset \mathfrak{g}_2$ corresponding to parallel skew-symmetric torsion, as described in [19]. We arrive precisely to the case of Theorem 7.1 (with a certain c in that reference equal to 1/7), which comes form the Lie subalgebra $\mathfrak{so}(3) \subset \mathfrak{su}(3) \subset \mathfrak{g}_2$.

The characteristic curvature tensor is given by $R^{\mathrm{c}}(X, Y)Z = -[[X, Y]_{\mathfrak{h}}, Z]$ or by

$$R^{\mathrm{c}} = -\frac{1}{2}(2S_1 \otimes S_1 + S_2 \otimes S_2 + S_3 \otimes S_3) \tag{27}$$

as results from [25] or [19], with the S_i being generators of $\mathfrak{so}(3) \subset \mathfrak{g}_2$. Formulas for Ric_g found in (23) match precisely with those given in the new reference.

1.8 The flat case

As proved in Theorem 1.1, the characteristic connection on the G_2-twistor space of a flat 4-dimensional space also has parallel torsion. The G_2-twistor structure verifies

$$\mathrm{d}\phi = -\beta^2 - 2\mu\alpha_1. \tag{28}$$

The space was described in [3] with canonical flat coordinates, which are easily complemented globally to coordinates on $SM = \mathbb{R}^4 \times S^3$. Recall from (18) that the torsion of the characteristic G_2 connection is $T^{\mathrm{c}} = -2\alpha$. Thence $\nabla^{\mathrm{c}} = \nabla^g - \alpha$.

Proposition 1.8. *Let M be an oriented flat Riemannian 4-manifold. Then the characteristic G_2 connection ∇^{c} on the gwistor space SM is flat.*

Proof. The Levi-Civita connection of TM is the flat connection $\pi^*\nabla^{\mathrm{L\text{-}C}} = \mathrm{d}$ duplicated for $TTM = \pi^*TM \oplus \pi^*TM$. Then the Levi-Civita connection of SM is just the connection $\nabla^\star = \nabla^g$ written in (4). By Gauss formula, $R^g(X,Y,Z,W) = \langle X^v, W^v\rangle\langle Y^v, Z^v\rangle - \langle X^v, Z^v\rangle\langle Y^v, W^v\rangle$. Using an adapted frame, cf. (8),

$$R^g = -(e^{45}\otimes e^{45} + e^{56}\otimes e^{56} + e^{64}\otimes e^{64}).$$

On the other hand, a formula in [21] says

$$R^{\mathrm{c}} = R^g + \frac{1}{4}\sum(e_i\lrcorner T^{\mathrm{c}})\otimes(e_i\lrcorner T^{\mathrm{c}}) + \frac{1}{4}\sum(e_i\lrcorner T^{\mathrm{c}})\wedge(e_i\lrcorner T^{\mathrm{c}}).$$

Thence, since $T^{\mathrm{c}} = -2e^{456}$, we have $\sum(e_i\lrcorner T^{\mathrm{c}})\wedge(e_i\lrcorner T^{\mathrm{c}}) = 0$ and

$$\frac{1}{4}\sum(e_i\lrcorner T^{\mathrm{c}})\otimes(e_i\lrcorner T^{\mathrm{c}}) = e^{45}\otimes e_6 + e^{64}\otimes e_5 + e^{56}\otimes e_4.$$

■

Notice both the connections $\nabla^g, \nabla^{\mathrm{c}}$ on the hypersurface preserve the Riemannian splitting. On the vertical side, the connection ∇^{c} is the invariant $SO(3)$-connection with skew-symmetric torsion -2α, described e.g. in [2, Remark 2.1].

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