On Thouvenot's ergodic proof of Roth's theorem

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1 Introduction

Roth's theorem says that a subset of $\mathbf{N}$ of a positive density contains an arithmetic progression of length 3. H. Furstenberg has proved that this theorem is equivalent to the following assertion: *for any invertible measure-preserving transformation T of a probability space (X,μ) and any set A of a positive measure it holds*

$$\liminf_N \frac{1}{N}\sum_{i=1}^{N}\mu(A\cap T^iA\cap T^{2i}A)>0.$$

Furstenberg also gave an ergodic proof of this fact (see [1]). In 2002 J.-P. Thouvenot communicated me an interesting modification of this proof using an observation from [2] (see also a joining proof of Marcus' theorem on multiple mixing for horocycle flows [3]) . Sometimes I included his short proof in my talks replacing a joining by an operator. Now I present this topic here (section 2) adding old remarks-proofs connected with Furstenberg's theorems on multiple progression average mixing for weakly mixing transformations (section 3).

2 Thouvenot's proof of Furstenberg's version of Roth's theorem

Let $f,g,h\in L_\infty(X,\mu)$. From any sequence $N_{k'}$ we choose a subsequence N_k such that for an operator $J: L_2\to L_2\otimes L_2$ the equality

$$\langle Jf, g\otimes h\rangle_{L_2\otimes L_2}=\lim_k\frac{1}{N_k}\sum_{i=1}^{N_k}\int fT^igT^{2i}h\ d\mu$$

holds for any $f, g, h \in L_\infty(X, \mu)$. The definition of J is correct, this follows from the ergodicity of T:

$$\lim_k \frac{1}{N_k} \sum_{i=1}^{N_k} \int T^i g T^{2i} h \; d\mu = \lim_k \frac{1}{N_k} \sum_{i=1}^{N_k} \int g T^i h \; d\mu = \int f d\mu \int g d\mu.$$

We see that $(T \otimes T^2)J = J$. So $(T \otimes T^2)Jf = Jf$. But a $(T \otimes T^2)$-invariant function belongs to $L_2(\mathcal{K} \otimes \mathcal{K}, \mu \otimes \mu)$, where $\mathcal{K}$ is a compact factor algebra (= Kronecker algebra generated by all proper functions of T). Indeed, we must only to remark that a restriction of T (and T^2 as well) onto $L_2(\mathcal{K}, \mu)^\perp$, say T', has the property $T'^i \to_w 0$ (T' has continuous spectrum), hence, $(T \otimes T^2)F = F$ implies $F \in L_2(\mathcal{K}, \mu) \otimes L_2(\mathcal{K}, \mu)$.

Denoting P for the orthogonal projection $L_2(\mu) \to L_2(\mathcal{K}, \mu)$ we obtain

$$\langle Jf, g \otimes h \rangle = \langle Jf, Pg \otimes Ph \rangle = \lim_k \frac{1}{N_k} \sum_{i=1}^{N_k} \int f \; T^i Pg \; T^{2i} Ph \; d\mu =$$

$$= \lim_k \frac{1}{N_k} \sum_{i=1}^{N_k} \int Pf \; T^i Pg \; T^{2i} Ph \; d\mu.$$

Let $f = g = h = \chi_A$, $\mu(A) > 0$. A closure of $\{T^i Pf\}$ is a compact set, for any $\varepsilon > 0$ there is L such that for any n and for at least one of $i = n+1, n+2 \ldots, n+L$ we get

$$\|T^i Pf - Pf\|_{L_2} < \varepsilon.$$

Thus, for a sufficiently small $\varepsilon' > 0$ we have

$$\liminf_N \frac{1}{N} \sum_{i=1}^{N} \mu(A \cap T^i A \cap T^{2i} A) \geq \frac{1}{L} \left(\int (P\chi_A)^3 \; d\mu \; - \varepsilon' \right) > 0.$$

3 Remarks to Furstenberg's theorems on weakly mixing transformations

Furstenberg [1] proved the following theorem: If T is weakly mixing, then

$$\frac{1}{N} \sum_{i=1}^{N} \prod_{p=1}^{m} T^{pi} f_p \;\to_{L_2}\; \prod_{p=1}^{m} \int f_p \; d\mu \quad (N \to \infty) \qquad (2, m)$$

holds for any collection of $f_i \in L_\infty$. Let $f, g, h \in L_\infty(X, \mu)$ and T be weakly mixing, let us show

$$\frac{1}{N}\sum_{i=1}^{N}\int fT^igT^{2i}h \to \int fd\mu\int gd\mu\int hd\mu, \tag{1,2}$$

$$\frac{1}{N}\sum_{i=1}^{N}T^ifT^{2i}gT^{3i}h \to_{L_2} \int fd\mu\int gd\mu\int hd\mu. \tag{2,3}$$

Proof of (1,2). We define a joining

$$\nu(f\otimes g\otimes h) = \lim_k \frac{1}{N_k}\sum_{i=1}^{N_k}\int fT^igT^{2i}h.$$

We have $(I\otimes T\otimes T^2)\nu = \nu$, but $T\otimes T^2$ is ergodic.

$$\nu(f\otimes g\otimes h) = \nu\left(f\otimes\left(\frac{1}{N}\sum_{i=1}^{N}T^ig\otimes T^{2i}h\right)\right) =$$

$$\nu(f\otimes\mathbf{1}\otimes\mathbf{1})\int gd\mu\int hd\mu = \int fd\mu\int gd\mu\int hd\mu.$$

(1) is proved. Here we can use also that Id and an ergodic transformation $S = T\otimes T^2$ are disjoint, so our joining has to be a direct product of its projections, see [2], [3].

Proof of (2,3). We define a joining η setting

$$\eta(f\otimes g\otimes h\otimes f'\otimes g'\otimes h') = \lim_k \frac{1}{N_k^2}\int\sum_{i=1}^{N_k}T^if\ T^{2i}g\ T^{3i}h\ \ \sum_{j=1}^{N_k}\ T^jf'\ T^{2j}g'\ T^{3j}h'\ d\mu.$$

From the above definition it follows an invariance

$$\eta = (I\otimes I\otimes I\otimes T\otimes T^2\otimes T^3)\eta,$$

but $T\otimes T^2\otimes T^3$ is ergodic. Again our joining will be a product: $\eta = \mu^3\otimes\mu^3$. Here we have made use of (1,2): the projections of η are equal to μ^3, indeed

$$\eta(f\otimes g\otimes h\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}) = \lim_N \frac{1}{N}\sum_{i=1}^{N}\int T^if\ T^{2i}g\ T^{3i}h\ d\mu =$$

$$= \lim_N \frac{1}{N}\sum_{i=1}^{N}\int f\ T^ig\ T^{2i}h\ d\mu = \int fd\mu\int gd\mu\int h\ d\mu.$$

Let $\int f d\mu = 0$, then

$$\lim_k \|\frac{1}{N_k}\sum_{i=1}^{N_k} T^i f T^{2i} g T^{3i} h\|^2_{L_2} = \eta(f\otimes g\otimes h\otimes f\otimes g\otimes h) = 0. \qquad (2',3)$$

To prove (2,3) we have to say only that for any sequence $N_{k'}$ one can choose a subsequence N_k for which (2',3) holds.

Now let's remark that (2,3) implies

$$\frac{1}{N}\sum_{i=1}^{N}\int f_0 T^i f_1\ T^{2i} f_2\ T^{3i} f_3\ d\mu\ \to \prod_{p=0}^{3}\int f_p\ d\mu\ \ (N\to\infty). \qquad (1,3)$$

From (1-3) we deduce as above

$$\frac{1}{N}\sum_{i=1}^{N} T^i f_1\ T^{2i} f_2\ T^{3i} f_3\ T^{4i} f_4\ \to_{L_2}\ \prod_{p=1}^{4}\int f_p\ d\mu, \qquad (2,4)$$

and so on: (2,m) implies (1, m), from (1,m) we get

$$\frac{1}{N}\sum_{i=1}^{N}\prod_{p=1}^{m+1} T^{pi} f_p\ \to_{L_2}\ \prod_{p=1}^{m+1}\int f_p\ d\mu \qquad (2,m+1)$$

as $N\to\infty$.

References

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