Maximum spectral radius of graphs with given connectivity and minimum degree

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Abstract

Shiu, Chan and Chang [On the spectral radius of graphs with connectivity at most k, J. Math. Chem., 46 (2009), 340-346] studied the spectral radius of graphs of order n with $\kappa(G) \leq k$ and showed that among those graphs, the maximum spectral radius is obtained uniquely at K_k^n , which is the graph obtained by joining k edges from k vertices of K_{n-1} to an isolated vertex. In this paper, we study the spectral radius of graphs of order n with $\kappa(G) \leq k$ and minimum degree $\delta(G) \geq k$. We show that among those graphs, the maximum spectral radius is obtained uniquely at $K_k + (K_{\delta-k+1} \cup K_{n-\delta-1})$.

Key words: connectivity; spectral radius

1 Introduction

Let G be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. We denote by $\delta(G)$ the minimum degree of vertices of G. The adjacency matrix of the graph G is defined to be a matrix $A(G) = [a_{ij}]$ of order n, where $a_{ij} = 1$ if v_i is adjacent to v_j , and $a_{ij} = 0$ otherwise. Since A(G) is symmetric and real, the eigenvalues of A(G), also referred to as the eigenvalues of G, can be arranged as: $\lambda_n(G) \leq \lambda_{n-1}(G) \leq \cdots \leq \lambda_1(G)$. The largest eigenvalue $\lambda_1(G)$ is called spectral radius and also denoted by $\rho(G)$. For $k \geq 1$, we say that a graph G is k-connected if either G is a complete graph K_{k+1} , or else it has at least k + 2 vertices and contains no (k - 1)-vertex cut. The connectivity $\kappa(G)$ of G is the maximum value of k for which G is k-connected.

When G is connected, A(G) is irreducible and by the Perron-Frobenius Theorem, the spectral radius is simple and there is an unique positive unit eigenvector. We shall refer to such an eigenvector as the Perron vector of G.

The eigenvalues of a graph are related to many of its properties and key parameters. The most studied eigenvalues have been the spectral radius $\rho(G)$ (in connection with the chromatic number, the independence number and the clique number of the graph [9,11]). Brualdi and Solheid [2] proposed the following problem concerning spectral radius:

Given a set of graphs \mathcal{C} , find an upper bound for the spectral radius of graphs in \mathcal{C} and characterize the graphs in which the maximal spectral radius is attained.

If \mathscr{C} is the set of all connected graphs on n vertices with k cut vertices, Berman and Zhang [1] solved this problem. Liu et al. [8] studied this problem for \mathscr{C} to be the set of all graphs on n vertices with k cut edges. Wu et al. [12] studied this problem for \mathscr{C} to be the set of trees on n vertices with k pendent vertices. Feng, Yu and Zhang [3] studied this problem for \mathscr{C} to be the set of graphs on n vertices with matching number β .

Li, Shiu, Chan and Chang studied this question for graphs with n vertices and connectivity at most k, and obtained the following result.

Theorem 1.1 (Li, Shiu, Chan and Chang [10]) Among all the graphs with connectivity at most k, the maximum spectral radius is obtained uniquely at $K_k + (K_1 \cup K_{n-k-1})$.

Let $G_{k,\delta,n} = K_k + (K_{\delta-k+1} \cup K_{n-\delta-1})$. We denote by $\mathcal{V}_{k,\delta,n}$ the set of graphs of order n with $\kappa(G) \leq k \leq n-1$ and $\delta(G) \geq k$. Clearly, $\mathcal{V}_{k,\delta+1,n} \subseteq \mathcal{V}_{k,\delta,n}$. In this paper, we investigate the problem for the graphs in $\mathcal{V}_{k,\delta,n}$. We show that among all those graphs, the maximal spectral radius is obtained uniquely at $G_{k,\delta,n}$.

In our arguments, we need the following technical lemma.

Theorem 1.2 (Li and Feng [7]) Let G be a connected graph, and G' be a proper subgraph of G. Then $\rho(G') < \rho(G)$.

2 Main results

Theorem 2.1 Let u, v be two vertices of the connected graph G. Let $\{v_1, \ldots, v_k\} \subseteq N(v)$ and $\{v_{k+1}, \ldots, u_{k+l}\} \subseteq V(G) - N(v)$. Suppose $x = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of G, where x_i corresponds to the vertex $v_i(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges vv_i $(1 \leq i \leq k)$ and adding the edges vv_i $(k+1 \leq i \leq k+l)$. If $\sum_{i=1}^k x_i \leq \sum_{i=k+1}^{k+l} x_i$, then $\rho(G) \leq \rho(G^*)$. Furthermore, if $\sum_{i=1}^k x_i < \sum_{i=k+1}^{k+l} x_i$, then $\rho(G) < \rho(G^*)$. **Proof.** Then we have

$$x^{T}(A(G^{*}) - A(G))x = -x_{v}\sum_{i=1}^{k} x_{i} + x_{v}(\sum_{j=k+1}^{k+l} x_{j} - \sum_{i=1}^{k} x_{i}) + x_{v}\sum_{j=k+1}^{k+l} x_{j}$$
$$= 2x_{v}(\sum_{j=k+1}^{k+l} x_{j} - \sum_{i=1}^{k} x_{i}) \ge 0.$$

So we have

$$\rho(G^*) = \max_{\|y\|=1} y^T A(G^*) y \ge x^T A(G^*) x \ge x^T A(G) x = \rho(G).$$
(1)

If $\rho(G^*) = \rho(G)$, then the equalities in (1) hold. Thus

$$x^T A(G^*)x = x^T A(G)x.$$

Hence $\sum_{i=k}^{k} x_i = \sum_{i=k+1}^{k+l} x_i$. This completes the proof.

With above proof, we obtain the following result.

Corollary 2.2 Suppose G^* in Theorem 2.1 is connected, and $y = (y_1, y_2, \ldots, y_n)^T$ is the Perron vector of G^* , then $\sum_{i=k+1}^{k+l} y_i \ge \sum_{i=1}^k y_i$. Furthermore, if $\sum_{i=k+1}^{k+l} x_i > \sum_{i=1}^k x_i$, then $\sum_{i=k+1}^{k+l} y_i > \sum_{i=1}^k y_i$.

Proof. Suppose that $\sum_{i=k+1}^{k+l} y_i < \sum_{i=1}^k y_i$, by Theorem 2.1, we have $\rho(G^*) < \rho(G)$, a contradiction.

Since $\sum_{i=k+1}^{k+l} x_i > \sum_{i=1}^k x_i$, by Theorem 2.1, we have $\rho(G^*) > \rho(G)$. If $\sum_{i=k+1}^{k+l} y_i \le \sum_{i=1}^k y_i$, by Theorem 2.1 then we have $\rho(G^*) \le \rho(G)$, a contradiction. This completes the proof.

Lemma 2.3 If $\delta(G) > \frac{n+k}{2} + 1$, then G is (k+1)-connected.

Theorem 2.4 Let n, k and δ be three positive integers. Among all the connected graphs of order n with connectivity at most k and minimum degree δ , the maximal spectral radius is obtained uniquely at $G_{k,\delta,n}$.

Proof. By Lemma 2.3, we have $2\delta \leq n + k + 2$. If n = k + 1, then K_{k+1} is an unique k-connected graph with order n. So we can assume that $n \geq k + 2$. Now we have to prove that for every $G \in \mathcal{V}_{k,\delta,n}$, then $\rho(G) \leq \rho(G_{k,\delta,n})$, where the equality holds if and only if $G = G_{k,\delta,n}$. Let $G^* \in \mathcal{V}_{k,\delta,n}$ with $V(G^*) = \{v_1, \ldots, v_n\}$ be the graph with maximum spectral radius in $\mathcal{V}_{k,\delta,n}$, that is, $\rho(G) \leq \rho(G^*)$ for all $G \in \mathcal{V}_{k,\delta,n}$.

Denote the Perron vector with $x = (x_1, \ldots, x_n)$, where x_i corresponding to v_i for $i = 1, \ldots, n$. Since $G^* \in \mathcal{V}_{k,\delta,n}$ and it is not a complete graph, then G^* has a k-vertex cut, say $S = \{v_1, \ldots, v_k\}$. In the following, we will prove the following three claims.

Claim 1. G^* contains exactly two components.

Suppose contrary that $G^* - S$ contains three components G_1 , G_2 and G_3 . Let $u \in G_1$ and $v \in G_2$. It is obvious that S is also an k-vertex cut of G + uv; i.e. $G^* + uv \in \mathcal{V}_{k,\delta,n}$. By Theorem 1.2, we have $\rho(G^*) < \rho(G^* + uv)$. This contradicts the definition of G^* .

Therefore, $G^* - S$ has exactly two components G_1 and G_2 .

Claim 2. Each subgraph of G^* induced by vertices $V(G_i) \cup S$, for i = 1, 2, is a clique.

Suppose contrary that there is a pair of non-adjacent vertices $u, v \in V(G_i) \cup S$ for i = 1 or 2. Again, $G^* + uv \in \mathcal{V}_{k,\delta,n}$. By Theorem 1.2, we have $\rho(G^*) < (G^* + uv)$. This contradicts the definition of G^* .

From Claim 2, it is clear that all G_1 and G_2 are cliques too. Then we write K_{n_i} instead of G_i , for i = 1, 2, in the rest of the proof, where $n_i = |G_i|$. Since $\delta(G) \ge k$, we have $n_i \ge \delta - k$ for i = 1, 2.

Claim 3. Either $n_1 = \delta - k + 1$ or $n_2 = \delta - k + 1$.

Otherwise, we have $n_1 > \delta - k + 1$ and $n_2 > \delta - k + 1$. Let $v \in G_1$ and $u \in G_2$. Suppose

$$N_{G^*}(v) = \{v_1, v_2, \dots, v_{n_2-1}, v_1, v_2, \dots, v_k\}$$

and

$$N_{G^*}(u) = \{u_1, u_2, \dots, u_{n_1-1}, v_1, v_2, \dots, v_k\}.$$

Partition the vertex set of G into three parts: the vertices of S; the vertices of G_1 ; the vertices of G_2 . This is an equitable partition of G with quotient matrix

$$Q = \begin{pmatrix} k-1 & n_1 & n_2 \\ k & n_1-1 & 0 \\ k & 0 & n_2-1 \end{pmatrix}$$

By Perron-Frobenius Theorem, Q has a Perron-vector $x = \{x_1, x_2, x_3\}$. Now we show that $x_2 < x_3$ if $n_1 < n_2$. Let $\rho(Q)$ denotes the largest eigenvalue of Q. Then we have

$$kx_1 + (n_1 - 1)x_2 = \rho(Q)x_2 \tag{2}$$

$$kx_1 + (n_2 - 1)x_3 = \rho(Q)x_3 \tag{3}$$

By (2) and (3), we have

$$(n_2 - 1)x_3 - (n_1 - 1)x_2 = \rho(Q)(x_3 - x_2).$$

Hence

$$(\rho(Q) - n_2 + 1)(x_3 - x_2) = (n_2 - n_1)x_2 > 0.$$

Since $\rho(Q)$ is also the largest eigenvalue of G^* , we have $\rho(Q) > \rho(K_{n_2}) = n_2 - 1$. Hence $x_3 > x_2$. The eigenvector x can be extended to an eigenvector of $A(G^*)$, say

$$y = (x_{11}, \dots, x_{1k}, x_{21}, \dots, x_{2n_1}, x_{31}, \dots, x_{3n_2}),$$

where $x_{i1} = \ldots = x_{in_i} = x_i$ for i = 2, 3 and $x_{11} = \cdots = x_{1k} = x_1$. Let $z = \frac{1}{\sqrt{kx_1^2 + n_1x_2^2 + n_3x_3^2}}y$. We have $zz^T = 1$ and so z is a Perron-vector of G^* . Let $G = G^* - \{vv_1, vv_2, \ldots, vv_k\} + \{vv_{k+1}, \ldots, vv_{k+l}\}$ and we have $G \in \mathcal{V}_{k,\delta,n}$. Since $n_2x_3 > (n_1 - 1)x_2$, by Theorem 2.1, $\rho(G^*) < \rho(G)$, which is a contradiction. This completes claim 3.

By Claim 3, we have $n_1 = \delta + 1 - k$. Hence $G^* = G_{k,\delta,n}$. This completes the proof. \Box

Theorem 2.5 The spectral radius of $G_{k,\delta,n}$ is the largest root of the following equation $x^3 + (3-n)x^2 + (n\delta - \delta^2 - n - kn + k + k\delta + 2 - 2\delta)x + (kn\delta + k^2 + n\delta + k^2\delta - k\delta - k^2n - k\delta^2 - 2\delta - \delta^2) = 0.$

Proof. Let G_1 be the subgraph of $G_{k,\delta,n}$ induced by k vertices of all the vertices of degree n-1, G_2 be the subgraph induced by all the vertices of degree δ vertices and G_3 be the subgraph induced by the remaining $n-\delta-1$ vertices. Also, let G_{ij} be the bipartite subgraph induced by $V(G_i)$ and $V(G_j)$ and let e_{ij} be the size of G_{12} . A theorem of Haemers [4] shows that eigenvalues of the quotient matrix of the partition interlace the eigenvalues of the adjacency matrix of G. The quotient matrix Q is the following

$$Q = \begin{pmatrix} \frac{2e_1}{n_1} & \frac{e_{12}}{n_1} & \frac{e_{13}}{n_1} \\ \frac{e_{21}}{n_2} & \frac{2e_2}{n_2} & \frac{e_{23}}{n_2} \\ \frac{e_{31}}{n_3} & \frac{e_{32}}{n_3} & \frac{2e_{33}}{n_3} \end{pmatrix} = \begin{pmatrix} k-1 & \delta-k+1 & n-\delta-1 \\ k & \delta-k & 0 \\ k & 0 & n-\delta-2 \end{pmatrix}.$$

Applying eigenvalue interlacing to the greatest eigenvalue of G, we get

$$\lambda_1(H) \ge \lambda_1(Q),\tag{4}$$

with the equality if the partition is equitable [[5], p.202]. Note that the partition is equitable, so the equality hold. This completes the proof. \Box

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