ON THE MUMFORD–TATE CONJECTURE FOR 1–MOTIVES

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Abstract. We show that the statement analogous to the Mumford–Tate conjecture for abelian varieties holds for 1–motives on unipotent parts. This is done by comparing the unipotent part of the associated Hodge group and the unipotent part of the image of the absolute Galois group with the unipotent part of the motivic fundamental group.

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Introduction and overview

Let k be a field which is finitely generated over $\mathbb{Q}$, with algebraic closure $\overline{k}$. Let X be a separated scheme of finite type over k, and let $i \geq 0$ be an integer. For every embedding $\sigma : k \longrightarrow \mathbb{C}$ the cohomology group

$$V_0 = H^i(X(\mathbb{C}), \mathbb{Q})$$

carries a mixed rational Hodge structure. The fundamental group of the Tannakian subcategory of the category of mixed Hodge structures generated by V_0 is called the *Mumford–Tate group* of V_0. It is an algebraic subgroup of GL_{V_0}, which is reductive in the case X is smooth and proper. For any prime number ℓ, the ℓ–adic étale cohomology group

$$V_\ell = H^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)$$

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is a Galois representation, conjectured to be semisimple if X is smooth and proper. The vector spaces V_0 and V_ℓ both carry a *weight filtration*, and, once an extension of σ to $\overline{k}$ is chosen, there is a canonical, natural isomorphism of filtered $\mathbb{Q}_\ell$–vector spaces $V_0 \otimes \mathbb{Q}_\ell \cong V_\ell$ called *comparison isomorphism*. The general Mumford–Tate conjecture states that the image of the Galois group $\mathrm{Gal}(\overline{k}|k)$ in the group of $\mathbb{Q}_\ell$–linear automorphisms of V_ℓ contains an open subgroup which is contained and open in the $\mathbb{Q}_\ell$–points of the Mumford–Tate group associated with the Hodge structure V_0, via the comparison isomorphism. The classical Mumford–Tate conjecture is the special case where X is an abelian variety and $i = 1$.

Although a conjecture in general, the classical Mumford–Tate conjecture is known to be true in a variety of cases, see [Rib90] or the introduction of [Vas08] for overviews. For abelian varieties of complex multiplication type, the statement of the conjecture follows from Faltings's theorems, but was proven already in 1968 by Pohlmann [Poh68]. Serre proved it for elliptic curves in [Ser68], and for abelian varieties A with $\mathrm{End}_{\overline{k}} A = \mathbb{Z}$ of dimension 2, 4, 6 or an odd number in [Ser85]. Serre's results were improved by Pink in [Pin98]. Recent progress on the question is due to Vasiu who shows in [Vas08] the statement of the conjecture to be true for an abelian variety A under some conditions on the Shimura pair associated with $H_1(A(\mathbb{C}), \mathbb{Q})$.

The general Mumford–Tate conjecture fits well into the framework of motives. We will show in this note that it holds for 1–motives, provided the classical Mumford–Tate conjecture holds for abelian parts. Recall from [Del74] that a 1–motive M over k is given by a diagram of commutative group schemes over k of the form

$$M = \left[\begin{array}{ccccccccc} & & & & Y & & & & \\ & & & & \downarrow{\scriptstyle u} & & & & \\ 0 & \to & T & \to & G & \to & A & \to & 0 \end{array} \right]$$

where A is an abelian variety, T a torus and Y étale locally constant, locally isomorphic to a finitely generated free group. In other words, Y is a Galois–module which is finitely generated and free as a commutative group. We can look at tori, abelian varieties and finitely generated free groups with Galois action as 1–motives, and 1–motives come equipped with a weight filtration W such that

$$\mathrm{gr}_0^W(M) = Y \qquad \mathrm{gr}_{-1}^W(M) = A \qquad \mathrm{gr}_{-2}^W(M) = T$$

With every 1–motive M are associated ℓ–adic Galois representations $\mathrm{V}_\ell M$ and having chosen a complex embedding $k \longrightarrow \mathbb{C}$ also a mixed Hodge structure $\mathrm{V}_0 M$. There is a natural comparison isomorphism $\mathrm{V}_0 M \otimes \mathbb{Q}_\ell \cong \mathrm{V}_\ell M$ which is compatible with the weight filtration. We write $\mathfrak{l}^M$ for the Lie algebra associated with the image of $\mathrm{Gal}(\overline{k}|k)$ in $\mathrm{GL}(\mathrm{V}_\ell M)$ and $\mathfrak{h}^M$ for the Lie algebra of the Mumford–Tate group of $\mathrm{V}_0 M$. The Lie algebras $\mathfrak{l}^M \subseteq \mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M)$ and $\mathfrak{h}^M \subseteq \mathrm{End}_{\mathbb{Q}}(\mathrm{V}_0 M)$ both carry a two step filtration induced by the weight filtration on $\mathrm{V}_\ell M$ and $\mathrm{V}_0 M$ respectively which we also denote by W:

$$0 \subseteq W_{-2}\mathfrak{l}^M \subseteq W_{-1}\mathfrak{l}^M \subseteq \mathfrak{l}^M \qquad \text{and} \qquad 0 \subseteq W_{-2}\mathfrak{h}^M \subseteq W_{-1}\mathfrak{h}^M \subseteq \mathfrak{h}^M$$

The nilpotent Lie algebras $W_{-1}\mathfrak{l}^M$ and $W_{-1}\mathfrak{h}^M$ are the nilpotent radicals of $\mathfrak{l}^M$ and $\mathfrak{h}^M$ respectively, the reductive Lie algebras $\mathrm{gr}_0^W(\mathfrak{l}^M)$ and $\mathrm{gr}_0^W(\mathfrak{h}^M)$ are the ones classically associated with the abelian variety $A = \mathrm{gr}_{-1}(M)$, if $A \neq 0$. The comparison isomorphism permits us to identify $\mathfrak{h}^M \otimes \mathbb{Q}_\ell$ with a Lie subalgebra of $\mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M)$. With this identification made, we can state our first main result as follows:

Theorem 1. *Let M be a 1–motive over a finitely generated subfield k of $\mathbb{C}$. The Lie algebra $\mathfrak{l}^M$ is contained in $\mathfrak{h}^M \otimes \mathbb{Q}_\ell$, and the equality $W_{-1}\mathfrak{l}^M = W_{-1}\mathfrak{h}^M \otimes \mathbb{Q}_\ell$ holds. In particular, the Mumford–Tate conjecture holds for M if and only if it holds for the abelian variety $\mathrm{gr}_1^W(M)$.*

With every variety X over k one can naturally associate a 1–motive $M^1(X)$ over k such that there are canonical isomorphisms

$$\mathrm{V}_0 M^1(X) \cong H^1(X(\mathbb{C}), \mathbb{Q}) \qquad \text{and} \qquad \mathrm{V}_\ell M^1(X) \cong H^1_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell)$$

of Hodge structures and Galois representations respectively. For curves this is a classical construction due to Deligne, in general it is due to Barbieri–Viale and Srinivas [BVS01]. Our first theorem immediately yields:

Corollary. *Let X be a variety over k. The Mumford–Tate conjecture holds for cohomology in degree 1 of X if and only if the classical Mumford–Tate conjecture holds for the albanese of a smooth projective variety birational to X.*

It is only natural to ask for an analogue of Theorem 1 in positive characteristic, replacing k by a field which is finitely generated over a finite field. Alas, there is no Mumford–Tate group in characteristic $p > 0$. However, if we concentrate on the weight (-1)–parts, i.e. nilpotent radicals, we can do even better by constructing a motive with which we can compare $W_{-1}\mathfrak{l}^M$ and $W_{-1}\mathfrak{h}^M \otimes \mathbb{Q}_\ell$. This motive will be a semiabelian variety, and was already constructed, following Deligne, by Bertolin in [Ber03], where it is called *Lie algebra of the unipotent motivic fundamental group of M*. Our second main result is the following theorem.

Theorem 2. *With every 1–motive M over a noetherian regular scheme S is canonically associated a semiabelian scheme $P(M)$ over S, having the following properties:*

1. *For every flat morphism noetherian regular schemes $S' \longrightarrow S$, there is a natural isomorphism $P(M) \times_S S' \cong P(M \times_S S')$.*
2. *If $S = \mathrm{spec}(\mathbb{C})$, there is a canonical isomorphism of Hodge structures $W_{-1}\mathfrak{h}^M \cong \mathrm{V}_0 P(M)$, where $\mathfrak{h}^M$ is the Lie algebra of the Mumford–Tate group of $\mathrm{V}_0 M$.*
3. *If S is the spectrum of a field k which is finitely generated over its prime field, and given an algebraic closure $\overline{k}$ of k, there is a canonical isomorphism of Galois representations $W_{-1}\mathfrak{l}^M \cong \mathrm{V}_\ell P(M)$, where $\mathfrak{l}^M$ is the Lie algebra of the image of $\mathrm{Gal}(\overline{k}|k)$ in $\mathrm{GL}(\mathrm{V}_\ell M)$, upon which $\mathrm{Gal}(\overline{k}|k)$ acts by conjugation.*

To get an idea of what $P(M)$ and the isomorphisms in the theorem look like, consider a 1–motive M over a field k, where $Y = \mathbb{Z}$ and $T = 0$, so M is given by an abelian variety A over k and a rational point $a = u(1) \in A(k)$. In that case, $P(M)$ is defined to be the smallest abelian subvariety of A which contains a multiple of a. For instance, $P(M) = 0$ if and only if a is torsion, which for instance is always the case if k is finite. For a fixed prime number ℓ and an integer $i \geq 0$, consider the fields

$$k(A[\ell^i]) \qquad \text{and} \qquad k(\ell^{-i}a)$$

obtained by adjoining to k the ℓ^i–torsion points of $A(\overline{k})$, respectively all ℓ^i–division points of a in $A(\overline{k})$. So $k(\ell^{-i}a)$ is a Galois extension of $k(A[\ell^i])$, and there is a natural map

$$\vartheta : \mathrm{Gal}\left(k(\ell^{-i}a)|k(A[\ell^i])\right) \longrightarrow A[\ell^i]$$

sending σ to $\sigma(b) - b$ where $b \in A(\overline{k})$ is any point such that $\ell^i b = a$. A result of Ribet ([Rib76], see also [Hin88], Appendix 2, Lemme I,bis) states that if k is a number field, the image of the map ϑ is contained in the subgroup $P(M)[\ell^i]$ of $A[\ell^i]$ with (finite) index bounded independently of i, and even equal to $P(M)[\ell^i]$ for all but finitely many ℓ. Passing to limits over i and then passing to Lie algebras gives the isomorphism claimed in part (3) of our theorem. Let it be acknowledged that Hindry's reformulation of Ribet's result was seminal to our general construction.

An important application of 1–motives is their use as a tool in the study of the group of global sections $G(S)$ of an abelian or semiabelian scheme G over a base scheme S. This is no surprise, since to give an S–rational point on G is the same as to give a morphism $\mathbb{Z} \longrightarrow G$ over S. For instance, if S is the spectrum of a number field k, direct consequences of our theorems in the case where $M = [Y \longrightarrow A]$ for an abelian variety A over k play an important role in the proof of local–global principles for subgroups of $A(k)$, as I have shown in [Jos11]. I will give a further illustration concerning *deficient points* on semiabelian varieties over number fields, which were introduced by Jacquinot and Ribet in [JR87]. In the case where S is a curve over $\mathbb{C}$, such points have been studied recently by Bertrand in [Ber11] in connection with a relative version of the Manin–Mumford conjecture.

Overview. Section 1 is to rehearse 1–motives and related constructions. In Section 2 we show that the image of the absolute Galois group is contained in the $\mathbb{Q}_\ell$–points of the Mumford–Tate group. This is essentially a reformulation of a result of Deligne and Brylinski. In Section 3 we construct the semiabelian variety $P(M)$, that is, the Lie algebra of the unipotent part of the motivic fundamental group of a 1–motive. We then compare the Lie algebra of the unipotent motivic fundamental group with the Mumford–Tate group and with the image of Galois in sections 5 and 6 respectively, by showing that the Hodge, respectively the ℓ–adic realisation of $P(M)$ is canonically isomorphic to the Lie algebra of the unipotent part of the Mumford–Tate group, respectively to the Lie algebra of the unipotent part of the image of $\mathrm{Gal}(\overline{k}|k)$ in $\mathrm{GL}(\mathrm{V}_\ell M)$. With this we have proven the essential part of our Main Theorem. However we now have two isomorphisms between the nilpotent radicals of $\mathfrak{l}^M$ and $\mathfrak{h}^M \otimes \mathbb{Q}_\ell$, the one given in the Main Theorem, the other via comparison with the motivic fundamental group. We will check in section 7 that they are the same, and deduce our main theorems as stated above. In section 8 we give some corollaries to our main theorems, concerning deficient points. The appendix contains a comment by P. Deligne.

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1. Complements and recollections on 1–motives

We introduce some constructions related to 1–motives relevant to our goals, and also recall some standard definitions and constructions. We fix for this section a noetherian regular scheme S, serving as base scheme.

– 1.1. We start with some recollections on 1–motives. By a 1–motive M over S we understand a diagram of commutative group schemes over S of the form

$$M = \begin{bmatrix} & & Y & & \\ & & \downarrow u & & \\ 0 \to T \to & & G & \to A \to 0 \end{bmatrix}$$

where A is an abelian scheme, T a torus and Y étale locally constant, locally isomorphic to a finitely generated free group. A *morphism of 1–motives* is a morphism of diagrams. The *weight filtration* of M is the three step filtration $0 \subseteq W_{-2}M \subseteq W_{-1}M \subseteq W_0 M = M$ given by

$$W_{-2}M = \begin{bmatrix} & & 0 & & \\ & & \downarrow & & \\ 0 \to T = & & T & \to 0 \to 0 \end{bmatrix} \qquad \text{and} \qquad W_{-1}M = \begin{bmatrix} & & 0 & & \\ & & \downarrow & & \\ 0 \to T \to & & G & \to A \to 0 \end{bmatrix}$$

This filtration is functorial in M, and although the category of 1–motives is not an abelian category, the quotients $M/W_i(M)$ make sense in the obvious way and we have in particular

$$\mathrm{gr}^W_*(M) = \begin{bmatrix} & & Y & & \\ & & \downarrow 0 & & \\ 0 \longrightarrow T \to & & T \oplus A & \to A \longrightarrow 0 \end{bmatrix}$$

We will often identify 1–motives M with two term complexes $[Y \xrightarrow{u} G]$ placed in degrees 0 and 1, and accordingly morphisms of 1–motives with morphisms of complexes.

– 1.2. With every 1–motive M over $\mathbb{C}$ is associated an integral Hodge structure $\mathrm{T}_0 M$, called the *Hodge realisation* of M. The construction of $\mathrm{T}_0 M$ goes as follows: The kernel of the exponential map $\exp : \mathrm{Lie}\, G(\mathbb{C}) \longrightarrow G(\mathbb{C})$ is canonically isomorphic to the singular homology group $H_1(G(\mathbb{C}), \mathbb{Z})$. Consider then the pull–back diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(G(\mathbb{C}),\mathbb{Z}) & \longrightarrow & \mathrm{T}_0 M & \longrightarrow & Y & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow u & & \\ 0 & \longrightarrow & H_1(G(\mathbb{C}),\mathbb{Z}) & \longrightarrow & \mathrm{Lie}\, G(\mathbb{C}) & \xrightarrow{\exp} & G(\mathbb{C}) & \longrightarrow & 0 \end{array}$$

The group $\mathrm{T}_0 M$ is finitely generated and free. It depends functorially on M hence carries a weight filtration induced by the weight filtration of M. The Hodge filtration on $\mathrm{T}_0 M \otimes \mathbb{C}$ has only one nontrivial step which is determined by the Hodge filtration on $H_1(A(\mathbb{C}), \mathbb{C})$. We write $\mathrm{V}_0 M := \mathrm{T}_0 M \otimes \mathbb{Q}$ for the corresponding rational Hodge structure. The construction of $\mathrm{T}_0 M$ behaves well in families: If M is a 1–motive over a smooth complex variety X, then the family $(\mathrm{T}_0 M_x)_{x \in X}$ is a variation of mixed Hodge structures.

– **1.3.** Let V be a rational mixed Hodge structure. The *Mumford–Tate group* H^V *of* V is the fundamental group of the Tannakian category generated by V inside the Tannakian category of rational mixed Hodge structures. We identify this group with an algebraic subgroup of GL_V via its natural, faithful action on V. The *weight filtration on* H^V is the filtration given by

$$W_i(H^V) = \{f \in H^V \mid (f - \mathrm{id})(W_n V) \subseteq W_{n+i} V\}$$

Let M be a 1–motive over $\mathbb{C}$. The *Mumford–Tate group* M is the Mumford–Tate group of the Hodge–realisation $V_0 M$ of M. We write $\mathfrak{h}^M \subseteq \mathrm{End}_{\mathbb{Q}}(\mathrm{V}_0 M)$ for its Lie algebra.

– **1.4.** Let ℓ be a prime number and let M be a 1–motive over a field k of characteristic $\neq \ell$ with algebraic closure $\overline{k}$ and absolute Galois group $\Gamma := \mathrm{Gal}(\overline{k}|k)$. We consider the finite Γ–modules

$$H^0(M \otimes^{\mathbb{L}} \mathbb{Z}/\ell^i\mathbb{Z}) = \frac{\{(y,x) \in Y \times G(\overline{k}) \mid u(y) = \ell^i x\}}{\{(\ell^i y, u(y)) \mid y \in Y\}}$$

where we regard M as a two term complex of discrete Γ–modules $[Y \longrightarrow G(\overline{k})]$ placed in degrees 0 and 1. We define

$$\mathrm{T}_\ell M := \lim_{i \geq 0} H^0(M \otimes^{\mathbb{L}} \mathbb{Z}/\ell^i\mathbb{Z}) \qquad \text{and} \qquad \mathrm{V}_\ell M := \mathrm{T}_\ell M \otimes \mathbb{Q}_\ell$$

The object $\mathrm{T}_\ell M$ is a finitely generated free $\mathbb{Z}_\ell$–module equipped with a continuous action of Γ. The construction of $\mathrm{T}_\ell M$ is functorial in M, hence a weight filtration on $\mathrm{T}_\ell M$ whose graded quotients are the ordinary Tate modules of T and A, and $Y \otimes \mathbb{Z}_\ell$. The construction of behaves well for 1–motives M over a base scheme S over which ℓ is invertible. In that case, $\mathrm{T}_\ell M$ is a smooth ℓ–adic sheaf on S.

– **1.5.** Let M be a 1–motive over a number field k, and let $\rho_\ell : \mathrm{Gal}(\overline{k}|k) \longrightarrow \mathrm{GL}(\mathrm{T}_\ell M)$ be the associated Galois representation. The image of ρ_ℓ is a closed subgroup of $\mathrm{GL}(\mathrm{T}_\ell M)$, hence has the structure of an ℓ–adic Lie group. We denote by $\mathfrak{l}^M \subseteq \mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M)$ its Lie algebra.

– **1.6.** Let M be a 1–motive over a field of characteristic zero k. The deRham realisation of M is a finite dimensional vector space over k, which is constructed as follows: Among the extensions of M by vector groups there is a universal one, given by

$$\begin{array}{ccccccc} & & Y & = & Y & & \\ & & \downarrow & & \downarrow & & \\ 0 \longrightarrow \mathcal{E}xt(M, \mathbb{G}_a[-1])^* & \longrightarrow & G^\natural & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

We set $\mathrm{V}_{\mathrm{dR}}(M) = \mathrm{Lie}\, G^\natural$. This is a finite dimensional vector group over k which depends functorially on M, hence the weight filtration on M defines a weight filtration on $\mathrm{V}_{\mathrm{dR}}(M)$. We define the Hodge filtration on $\mathrm{V}_{\mathrm{dR}}(M)$ by $F^0\mathrm{V}_{\mathrm{dR}}(M) := \ker(\mathrm{Lie}\, G^\natural \longrightarrow \mathrm{Lie}\, G)$. If M is a 1–motive over a smooth variety S over k, then the deRham–realisation defines a finitely generated locally free $\mathcal{O}_S$–module. This module comes equipped with a canonical integrable connection

$$\nabla : \mathrm{V}_{\mathrm{dR}}(M) \longrightarrow \mathrm{V}_{\mathrm{dR}}(M) \otimes_{\mathcal{O}_S} \Omega^1_{S/k}$$

called the *Gauss–Manin connection* (see [AB11] §4.2 for a construction). In the case where M is given by an abelian variety A over S, then $\mathrm{V}_{\mathrm{dR}}(A)$ identifies with the dual $H_1^{\mathrm{dR}}(A/S)$ of $H^1_{\mathrm{dR}}(A/S)$, and the Gauss–Manin connection is the classical one constructed by Katz and Oda by *loc.cit.*, Lemma 4.5.

– 1.7. There exist canonical isomorphisms comparing the Hodge realisation of a 1–motive with the ℓ–adic and the deRham realisation. Given a 1–motive M over a finitely generated extension k of $\mathbb{Q}$, a complex embedding $\sigma : k \longrightarrow \mathbb{C}$ and an extension of σ to an algebraic closure $\overline{k}$ of k, these are isomorphisms

$$\mathrm{T}_0(\sigma^*M)\otimes_{\mathbb{Z}}\mathbb{Z}_\ell \xrightarrow{\cong} \mathrm{T}_\ell M \qquad \text{and} \qquad \mathrm{T}_0(\sigma^*M)\otimes_{\mathbb{Z}}\mathbb{C} \xrightarrow{\cong} \mathrm{V}_{\mathrm{dR}}M\otimes_k \mathbb{C}$$

of $\mathbb{Z}_\ell$–modules and of complex vector–spaces respectively, where σ^*M is the pull–back of M to $\operatorname{spec}\mathbb{C}$ via σ. These isomorphisms are natural in M, hence in particular respect the weight filtration. These isomorphisms work of course also for families.

We now present a special family of 1–motives, which shows that two 1–motives can be smoothly deformed into into each other if they have the same graded pieces for the weight filtration, that is, if they are built from the same torus, abelian variety and lattice.

Proposition 1.8. *Let T be a torus, A be an abelian scheme and Y be a lattice over S, and set $M_0 := [Y \xrightarrow{0} (T\oplus A)]$. The fppf–presheaf on S given by*

$$(i : U \to S) \quad \longmapsto \quad \frac{\text{1–Motives } M \text{ over } U \text{ with } \mathrm{gr}^W_*(M) = i^*M_0}{\text{Isomorphisms } \alpha \text{ with } \mathrm{gr}^W_*(\alpha) = \mathrm{id}_{i^*M_0}}$$

is representable by a S–scheme $X_S(T,A,Y)$ which is smooth over S. In particular, this presheaf is a sheaf. More precisely, the S–schemes $X_S(T,A,0)$ and $X_S(0,A,Y)$ are abelian schemes over S, and $X_S(T,0,Y)$ is a torus over S, and there are isomorphisms of sheaves

$$X_S(T,A,0)\cong \mathcal{H}om(T^\vee,A^\vee) \qquad X_S(0,A,Y)\cong \mathcal{H}om(Y,A) \qquad X_S(T,0,Y)\cong \mathcal{H}om(Y,T)$$

where $T^\vee$ is the character group of T and $A^\vee$ the abelian scheme dual to A. There is a canonical morphism

$$X_S(T,A,Y)\longrightarrow X_S(T,A,0)\times_S X_S(0,A,Y)$$

which gives $X_S(T,A,Y)$ the structure of a $X_S(T,0,Y)$–torsor on $X_S(T,A,0)\times_S X_S(0,A,Y)$.

Proof. The fppf presheaf on S associating with $i : U \to S$ the group $\mathrm{Ext}^1(i^*A, i^*T)$ is representable by an abelian scheme $p' : X' \longrightarrow S$ over S, by the Barsotti–Weil formula. This means that we have natural bijections

$$\mathrm{Mor}_S(U,X') \quad \xrightarrow{\cong} \quad \frac{\text{Semiabelian schemes on } U\text{, extensions of } i^*A \text{ by } i^*T}{\text{Isomorphisms inducing the identity on } i^*T \text{ and } i^*A}$$

where by a *semiabelian scheme* we understand a a group scheme which is globally an extension of an abelian scheme by a torus[1] over S. Denoting by $\mathcal{G}'$ be the semiabelian scheme over X'

[1]I ask the reader to forgive me this nonstandard terminology. By a semiabelian scheme over S one usually understands a group scheme over S each of whose fibres is an extension of an abelian scheme by a torus. This is the right thing to consider in order to study degenerations of abelian schemes.

corresponding via this bijection to the identity map $\mathrm{id}_{X'}$, the above bijection is given by

$$(f' : U \longrightarrow X') \;\longmapsto\; f'^*\mathcal{G}'$$

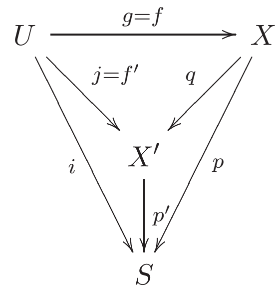
Set $\mathcal{Y}' := p'^*Y$. The fppf presheaf on X' associating with $j : U \longrightarrow X'$ the group $\mathrm{Hom}_U(j^*\mathcal{Y}', j^*\mathcal{G}')$ is representable by a semiabelian scheme $q : X \longrightarrow X'$ over X', so we have natural bijections

$$\mathrm{Mor}_{X'}(U, X) \xrightarrow{\cong} \text{Homomorphisms } j^*\mathcal{Y}' \longrightarrow j^*\mathcal{G}' \text{ of fppf–sheaves on } U$$

Define $\mathcal{Y} := q^*\mathcal{Y}'$ and $\mathcal{G} := q^*\mathcal{G}'$, and let $\mathcal{M} := [u : \mathcal{Y} \longrightarrow \mathcal{G}]$ be the 1–motive over X where u is the morphism corresponding via this bijection to the identity morphism id_X. The above bijection is then given by sending a $j : U \to X'$ to the 1–motive $j^*\mathcal{M}$. We claim that the scheme X, considered as a scheme over S via the composite $p := p' \circ q$, has the required properties. Because X' is smooth and connected over S and X is smooth and connected over X', the scheme X is smooth and connected over S. We now show that for every S–scheme $i : U \longrightarrow S$ the natural map

$$\mathrm{Mor}_S(U, X) \longrightarrow \frac{\text{1–Motives } M \text{ over } U \text{ with } \mathrm{gr}^W_*(M) = i^*M_0}{\text{Isomorphisms } \alpha \text{ with } \mathrm{gr}^W_*(\alpha) = \mathrm{id}_{i^*M_0}}$$

sending an S–morphism $(f : U \longrightarrow X)$ to the 1–motive $f^*\mathcal{M}$ over U is a bijection. Indeed, to give an S–morphism f of an S–scheme U to X is the same as to give an S–morphism $f' : U \longrightarrow X'$ and an X'–morphism $g : U \longrightarrow X$, where U is now viewed as an X'–scheme via $j = f'$:



So, to give an S–morphism $f : U \longrightarrow X$ is the same as to give an extension $G = f'^*\mathcal{G}'$ of i^*A by i^*T on U modulo appropriate isomorphisms and a homomorphism of $i^*Y = j^*\mathcal{Y}'$ to $G = j^*\mathcal{G}'$. This datum is exactly what a 1–motive M over U with $\mathrm{gr}^*_W(M) = i^*M_0$ consists of, again modulo appropriate isomorphisms. □

Remark 1.9. Consider the case where $Y = \mathbb{Z}$ and $T = \mathbb{G}_m$. Then $X_S(T, A, Y)$ is a $\mathbb{G}_m$–bundle over $A \times A^\vee$, that is, an invertible sheaf. This sheaf is the Poincaré sheaf. This shows that $X_S(T, A, Y)$ is not a group scheme, except in the degenerate cases.

Remark 1.10. Let $M = [Y \longrightarrow G]$ be a 1–motive over S, and write $M_A := M/W_{-2}M = [Y \longrightarrow A]$, notations being as in 1.1. We have already used that the fppf–sheaves $\mathcal{H}om(Y, G)$ and $\mathcal{E}xt^1(M_A, T)$ are representable by semiabelian schemes over S. If ℓ is invertible on S, there are canonical isomorphisms of ℓ–adic sheaves

$$\mathrm{T}_0\mathcal{H}om(Y, G) \cong \mathrm{Hom}_{\mathbb{Z}}(Y, \mathrm{T}_0G) \qquad \text{and} \qquad \mathrm{T}_\ell\mathcal{E}xt^1(M_A, T) \cong \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathrm{T}_\ell M_A, \mathrm{T}_\ell T)$$

on S, and similar isomorphisms of variations of Hodge–structures if S is smooth over $\mathbb{C}$. These isomorphisms are compatible with the comparison isomorphisms, meaning that the squares

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}(Y,\mathrm{T}_0G)\otimes\mathbb{Z}_\ell & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{Z}_\ell}(Y\otimes\mathbb{Z}_\ell,\mathrm{T}_\ell G) \\ \cong\updownarrow & & \cong\updownarrow \\ \mathrm{T}_0\mathcal{H}om(Y,G)\otimes\mathbb{Z}_\ell & \xrightarrow{\cong} & \mathrm{T}_\ell\mathcal{H}om(Y,G) \end{array} \qquad \begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{T}_0M_A,\mathrm{T}_0T)\otimes\mathbb{Z}_\ell & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathrm{T}_\ell M_A,\mathrm{T}_\ell T) \\ \cong\updownarrow & & \cong\updownarrow \\ \mathrm{T}_0\mathcal{E}xt(M_A,T)\otimes\mathbb{Z}_\ell & \xrightarrow{\cong} & \mathrm{T}_\ell\mathcal{E}xt(M_A,T) \end{array}$$

commute.

2. Cohomological realisation of families of 1–motives

In the previous section we have associated a $\mathbb{Q}$–Lie algebra $\mathfrak{h}^M \subseteq \mathrm{End}_{\mathbb{Q}}(\mathrm{V}_0M)$ with a 1–motive M over $\mathbb{C}$ (1.3), and a $\mathbb{Q}_\ell$–Lie algebra $\mathfrak{l}^M \subseteq \mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M)$ with a 1–motive M over a field of characteristic $\neq \ell$ (1.5). In this section we will show that if M is a 1–motive over a field which is finitely generated over $\mathbb{Q}$, then the Lie algebra $\mathfrak{l}^M$ is contained in $\mathfrak{h}^M \otimes \mathbb{Q}_\ell$ via the comparison isomorphism. By naturality of the comparison isomorphism, this inclusion is compatible with the weight filtration.

Theorem 2.1. *Let k be a field of finite transcendence degree over $\mathbb{Q}$ and let $\sigma : k \longrightarrow \mathbb{C}$ be an embedding. Let M be a 1–motive over k, and identify $\mathfrak{h}^{(\sigma^*M)} \otimes \mathbb{Q}_\ell$ with a subalgebra of $\mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M)$ via the comparison isomorphism $\mathrm{V}_0(\sigma^*M) \otimes \mathbb{Q}_\ell \cong \mathrm{V}_\ell M$. Then the Lie algebra $\mathfrak{l}^M$ is contained in $\mathfrak{h}^{(\sigma^*M)} \otimes \mathbb{Q}_\ell$.*

For abelian varieties in place of M this was shown by Deligne in [Del82] (see also [CS11]), essentially by proving that every Hodge cycle on an abelian variety is an absolute Hodge cycle. For 1–motives, the corresponding statement was proven by J.-L. Brylinski ([Bry86], Théorème 2.2.5):

Theorem 2.2 (Brylinski, Deligne)**.** *Let M be a 1–motive over k. Every Hodge cycle of M relative to some embedding $\sigma : k \longrightarrow \mathbb{C}$ is an absolute Hodge cycle.*

After recalling the notion of absolute Hodge cycles we will give a proof of theorem 2.2, and then show how the statement about Lie algebras follows from it. The proof of Brylinski's Theorem consists essentially of a deformation argument, so we will be concerned with families of 1–motives and their realisations. The idea is to show that if M_1 and M_2 are 1–motives such that M_1 can be smoothly deformed to M_2, then the statement of Theorem 2.2 holds for M_1 if it holds for M_2. We have already seen in Proposition 1.8 that every 1–motive M can be smoothly deformed to a split 1–motive. For split 1–motives the statement of the theorem 2.2 is true by Deligne's Theorem on absolute Hodge cycles on abelian varieties.

I have decided to include a proof of theorem 2.2 to make the text more self contained on one hand, and on the other hand because the proof I present here seems a little more natural to me than Brylinski's. Indeed, Brylinski's deformation process consists of using Hodge realisations in order to produce an analytic family of 1–motives deforming a given 1–motive to a 1–motive which is split up to isogeny, and then to make this family algebraic using GAGA ([Bry86], Lemme 2.2.8.6). Having Proposition 1.8 at hand, we can avoid all this.

– **2.3.** We fix for this section a field k of of finite transcendence degree over $\mathbb{Q}$ with algebraic closure $\overline{k}$. In order to handle realisations of a motive M over k simultaneously, we introduce

$$\mathbb{A}_k := k \times \Big(\mathbb{Q} \otimes \prod_{\ell \text{ prime}} \mathbb{Z}_\ell\Big) \qquad \mathrm{V}_{\mathbb{A}}(M) := \mathrm{V}_{\mathrm{dR}}M \times \Big(\mathbb{Q} \otimes \prod_{\ell \text{ prime}} \mathrm{T}_\ell M\Big)$$

So $\mathbb{A}_k$ is a commutative $\mathbb{Q}$–algebra, and $\mathrm{V}_{\mathbb{A}}(M)$ is a finitely generated $\mathbb{A}_k$–module. This works also in families: Let S be an integral, regular scheme of finite type over k and let M be a 1–motive over S. Then we can consider the sheaf $\mathbb{A}_S$ on $S_{\text{ét}}$, and so $\mathrm{V}_{\mathbb{A}}(M)$ is naturally a sheaf of $\mathbb{A}_S$–modules. We will work with tensor spaces of $\mathrm{V}_{\mathbb{A}}(M)$. For a finite index set I and integers a_i, b_i, c_i for $i \in I$ with $a_i, b_i \geq 0$ set

$$\mathrm{V}^I_{\mathbb{A}}(M) = \bigoplus_{i\in I} \Big(\mathrm{V}_{\mathbb{A}}(M)^{\otimes a_i} \otimes (V_{\mathbb{A}}(M)^*)^{\otimes b_i} \otimes \mathbb{A}(1)^{\otimes c_i}\Big)$$

where $\mathbb{A}(1) := \mathrm{V}_{\mathbb{A}}(\mathbb{G}_m)$ and $(-)^* = \mathrm{Hom}(-, \mathbb{A}_S)$. We refer to global sections of $\mathrm{V}^I_{\mathbb{A}}(M)$ as *tensors*. For every embedding $\sigma : k \longrightarrow \mathbb{C}$ there is a canonical isomorphism of $\mathbb{A}_{\sigma^*S}$–module sheaves

$$\alpha^\sigma : \mathrm{V}^I_0(\sigma^*M) \otimes_{\mathbb{Q}} \mathbb{A}_{\sigma^*S} \xrightarrow{\cong} V^I_{\mathbb{A}}(\sigma^*M)$$

on the complex variety σ^*S, where $\mathrm{V}^I_0(\sigma^*M)$ is the corresponding tensor space of variations of Hodge structures. These sheaves are local systems for the complex topology.

Definition 2.4. Let M be a 1–motive over k. A tensor $t \in \Gamma(k, V^I_{\mathbb{A}}(M))$ for some finite index set I is called *Hodge cycle relative to an embedding* $\sigma : k \longrightarrow \mathbb{C}$ if the following holds:

(1) there exists an element $t_0 \in \mathrm{V}^I_0(\sigma^*M)$ such that $t = \alpha^\sigma(t_0 \otimes 1)$.
(2) the deRham component t_{dR} of t belongs to $F^0(\mathrm{V}^I_{\mathrm{dR}}M) \cap W_0(\mathrm{V}^I_{\mathrm{dR}}M)$.

An element $t \in \mathrm{V}^I_{\mathbb{A}}(M)$ is called *absolute Hodge cycle* if it is a Hodge cycle relative to all embeddings $\sigma : k \longrightarrow \mathbb{C}$.

– **2.5.** In other words, the Hodge cycles relative to $\sigma : k \longrightarrow \mathbb{C}$ are the image of elements of bidegree $(0, 0)$ in $\mathrm{V}^I_0(M_\sigma)$ under the comparison isomorphism α^σ. They form therefore a finite dimensional $\mathbb{Q}$–linear subspace of $\mathrm{V}^I_{\mathbb{A}}(\sigma^*M)$.

Proposition 2.6 (Deligne)**.** *Let S be a connected scheme over $\overline{k}$ and let s_0, s_1 be closed points of S. Let M be a 1–motive over S, and let $t \in \Gamma(S, \mathrm{V}^I_{\mathbb{A}}(M)$ be a tensor. Suppose that t_{dR} is annihilated by the Gauss–Manin connection and that $(t_{\mathrm{dR}})_s$ is in $F^0(\mathrm{V}_{\mathrm{dR}}M_s)$ at every point $s \in S$. If t_{s_0} is an absolute Hodge cycle, then so is t_{s_1}.*

Proof. Under the assumptions of the proposition, we have to show that t_{s_1} meets the two conditions in Definition 2.4. We start with condition (1). Fix an embedding $\sigma : k \longrightarrow \mathbb{C}$. We claim that for all $s \in S$ the natural maps

$$\Gamma(\sigma^*S, \mathrm{V}^I_0(\sigma^*M)) \longrightarrow \mathrm{V}^I_0(\sigma^*M_s) \qquad \text{and} \qquad \Gamma(S, \mathrm{V}^I_{\mathbb{A}}(M)) \longrightarrow \mathrm{V}^I_{\mathbb{A}}(M_s)$$

are injective. Indeed, $\mathrm{V}^I_0(\sigma^*M)$ is a local local system of finite dimensional $\mathbb{Q}$–vector spaces on σ^*S, and $\mathrm{T}_\ell(M)$ is a locally constant ℓ–adic sheaf on S, so for any $s \in S(\overline{k})$ the global sections of these sheaves can be regarded as the fixed points of the respective fibres at s under the monodromy action of the étale fundamental group based at s:

$$\Gamma(\sigma^*S, \mathrm{V}_0(\sigma^*M)) \cong \mathrm{V}_0(\sigma^*M_s)^{\pi_1^{\text{ét}}(s,\sigma^*S)} \qquad \text{and} \qquad \Gamma(S, \mathrm{T}_\ell(M)) \cong \mathrm{T}_\ell(M_s)^{\pi_1^{\text{ét}}(s,S)}$$

The map $\Gamma(S, \mathrm{V}_{\mathrm{dR}}(M)) \longrightarrow \mathrm{V}_{\mathrm{dR}}(M_s)$ is injective because $\mathrm{V}_{\mathrm{dR}}(M)$ is a finitely generated locally free $\mathcal{O}_S$–module, so our claim follows. Following Deligne, we consider now this diagram:

$$\begin{array}{ccccc} \Gamma(\sigma^*S, \mathrm{V}_0(\sigma^*M)) & \longrightarrow & \Gamma(\sigma^*S, \mathrm{V}_0(\sigma^*M_\sigma)) \otimes \mathbb{A}) & \xrightarrow{\cong} & \Gamma(S, \mathrm{V}_{\mathbb{A}}(M)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{V}_0(\sigma^*M_s) & \longrightarrow & \mathrm{V}_0(\sigma^*M_s) \otimes \mathbb{A} & \xrightarrow{\cong} & \mathrm{V}_{\mathbb{A}}(M_s) \end{array}$$

The left hand horizontal maps are given by $x \longmapsto x \otimes 1$, and the right hand horizontal maps are the comparison isomorphisms. We have seen that the vertical maps are injective. Consider the above diagram for $s = s_0$. We are given $t \in \Gamma(S, \mathrm{V}_{\mathbb{A}}(M))$ and are told that its image in $\mathrm{V}_{\mathbb{A}}(M_{s_0})$ comes from an element in $\mathrm{V}_0(\sigma^*M_{s_0})$. The intersection of the images of $\Gamma(S, \mathrm{V}_{\mathbb{A}}(M))$ and $\mathrm{V}_0(\sigma^*M_s)$ in $\mathrm{V}_{\mathbb{A}}(M_s)$ is exactly the image of $\Gamma(\sigma^*S, \mathrm{V}_0(\sigma^*M))$ in $\mathrm{V}_{\mathbb{A}}(M_s)$ by standard linear algebra (choose a $\mathbb{Q}$–basis of $\mathbb{A}$ containing 1). So t comes from a global section t^h of $\mathrm{V}_0(\sigma^*M)$. The element $t_{s_1} \in \mathrm{V}_{\mathbb{A}}(M_{s_1})$ comes thus from an element in $\mathrm{V}_0(\sigma^*M_{s_1})$, namely from the image of t^h in $\mathrm{V}_{\mathbb{A}}(\sigma^*M_{s_1})$, and that is what condition (1) asks for.

We now come to the second condition. We have $(t_{\mathrm{dR}})_{s_1} \in F^0(V^I_{\mathrm{dR}}M_{s_1})$ by assumption. The Gauss–Manin connection is functorial, hence preserves the weight filtration ([AB11], §4.2). Since t_{dR} is horizontal and $(t_{\mathrm{dR}})_{s_0} \in W_0(V^I_{\mathrm{dR}}M_{s_0})$ we must also have $W_0(V^I_{\mathrm{dR}}M_{s_1})$ as needed. □

Corollary 2.7. *Let S be a connected scheme over $\overline{k}$ and let M be a 1–motive over S. Let V be a local subsystem of a tensor space $V_0^I(M)$ such that V_s consists of $(0,0)$–cycles for all $s \in S$ and of absolute Hodge cycles for at least one $s_0 \in S$. Then V_s consists of absolute Hodge cycles for all s.*

Proof. The proof is litterally the same as the proof of 2.15 in [Del82]. The argument is the following: If $V_0^I(M)$ is constant, every element of V_{s_0} extends to a global section of V, and we are done by Proposition 2.6. In general, observe that $\mathrm{gr}_0^W(V_0^I(M))$ has a polarisation, so there is a rational, positive definite bilinear form on $\mathrm{gr}_0^W(V_0^I(M))$ which is compatible with the action of $\pi^1(S, s_0)$. Hence the image of $\pi^1(S, s_0)$ in $\mathrm{GL}(V_{s_0})$ is finite. After passing to a finite cover of S, the local system V becomes constant, and we are done. □

Lemma 2.8. *Let $M_0 = [Y \xrightarrow{0} (T \oplus A)]$ be a split 1–motive over k. Every Hodge cycle of M_0 relative to some embedding $\sigma : k \longrightarrow \mathbb{C}$ is an absolute Hodge cycle.*

Proof. Without loss of generality we may assume that T is split and that Y is constant. We can also assume that A is not trivial. But then, all tensor spaces associated with realisations of M_0 are also tensor spaces associated with A, and we know by the main result in [Del82] that every Hodge cycle for A is an absolute Hodge cycle. □

Proof of Theorem 2.2. Let M_1 be a 1–motive over k, let $\sigma : k \longrightarrow \mathbb{C}$ be a complex embedding and let $t_1 \in \mathrm{V}_{\mathbb{A}}(M_1)$ be a Hodge cycle relative σ. We have to show that t_1 is an absolute Hodge cycle.

Set $M_0 := \mathrm{gr}^W_*(M_1) = [Y \xrightarrow{0} (T \oplus A)]$ and consider the smooth connected scheme $X := X_k(T, A, Y)$ and the universal 1–motive M on X from Proposition 1.8. The 1–motives M_0 and M_1 are isomorphic to the fibres of M in k–rational points $x_0, x_1 \in X(k)$, so we can deduce Theorem 2.2 from Corollary 2.7 and Lemma 2.8. □

It remains to deduce Theorem 2.1 from Theorem 2.2. We start with the the following proposition, analogous to Proposition 2.9.b of [Del82].

Proposition 2.9. *Let M be a 1–motive over k and define C^I_{AH} to be the subspace of absolute Hodge cycles in $\mathrm{V}^I_{\mathbb{A}}(M)$. The Galois group $\Gamma := \mathrm{Gal}(\overline{k}|k)$ leaves C^I_{AH} invariant, and the action of Γ on C^I_{AH} factors over a finite quotient of Γ.*

Proof. That Γ leaves C^I_{AH} invariant is immediate from the definition of absolute Hodge cycles. Then, observe that for any prime number ℓ the map $C^I_{AH} \longrightarrow \mathrm{V}^I_\ell(M)$ is injective, and that the subgroup N of Γ fixing C^I_{AH} is closed. The quotient Γ/N is a profinite group, and can be identified with a subgroup of the countable group $\mathrm{GL}(C^I_{AH})$. Hence Γ/N must be finite. □

– **2.10.** In what follows, we will use the following alternative description of the Mumford–Tate group. Let V be a rational mixed Hodge structure and consider a tensor space

$$V^I = \bigoplus_{i\in I}\Big(V^{\otimes a_i} \otimes (V^*)^{\otimes b_i} \otimes \mathbb{Q}(1)^{\otimes c_i}\Big)$$

The algebraic group GL_V acts naturally on the $\mathbb{Q}$–vector spaces $V^{\otimes a_i} \otimes (V^*)^{\otimes b_i}$ and the multiplicative group $\mathbb{G}_m$ acts on $\mathbb{Q}(1)^{\otimes c_i}$, so $\mathrm{GL}_V \times \mathbb{G}_m$ acts on V^I. It follows from general tannakian formalism that the Mumford–Tate group of $V \oplus \mathbb{Q}(1)$ is the largest subgroup of $\mathrm{GL}_V \times \mathbb{G}_m$ which fixes all elements of bidegree $(0, 0)$ in all tensor spaces V^I.

Lemma 2.11. *Let V be a rational mixed Hodge structure such that $\mathbb{Q}(1)$ is contained in $\langle V\rangle^\otimes$. There exists a tensor space V^I of V such that the Mumford–Tate group H^V of V is isomorphic to the stabiliser in $\mathrm{GL}_V \times \mathbb{G}_m$ of the rational elements of bidegree $(0, 0)$ of V^I.*

Proof. Because $\mathrm{GL}_V \times \mathbb{G}_m$ is a noetherian scheme over $\mathbb{Q}$, there exists an index set I such that $H^{V\oplus\mathbb{Q}(1)}$ is the largest subgroup of $\mathrm{GL}_V \times \mathbb{G}_m$ which fixes all elements of bidegree $(0, 0)$ in V^I. If the tannakian category $\langle V\rangle^\otimes$ generated by V already contains $\mathbb{Q}(1)$, then the projection on the first factor induces an isomorphism $H^{V\oplus\mathbb{Q}(1)} \cong H^V$. □

Lemma 2.12. *Let M be a 1–motive over $\mathbb{C}$. Then $\mathbb{Q}(1)$ is contained in $\langle \mathrm{V}_0M\rangle^\otimes$ if and only if M is not pure of weight zero.*

Proof. If M is pure of weight zero, then $\langle \mathrm{V}_0M\rangle^\otimes$ only consists of pure Hodge structures of weight 0, hence does not contain $\mathbb{Q}(1)$. Otherwise, either the torus $T := W_{-2}M$ or the abelian variety $A := \mathrm{gr}^W_{-1} M$ is nontrivial. If T is nontrivial, then $\mathbb{Q}(1)$ is a substructure of V_0M. If A is nonzero, then the choice of a polarisation of A yields a surjective morphism $(\mathrm{V}_0A)^{\otimes 2} \longrightarrow \mathbb{Q}(1)$. □

Proof of Theorem 2.1. If M is pure of weight zero, the Lie algebras $\mathfrak{l}^M$ and $\mathfrak{h}^{(\sigma^*M)} \otimes \mathbb{Q}_\ell$ are both trivial. We suppose that M is not pure of weight zero. Let us fix an index set I such that H^M is isomorphic, via projection on the first factor, to the stabiliser in $\mathrm{GL}_{\mathrm{V}_0(\sigma^*M)}$ of elements of bidegree $(0,0)$ in the tensor space $\mathrm{V}_0^I M$, as given by Lemma 2.11. Denote by C_{AH}^I the finite dimensional $\mathbb{Q}$–linear subspace of absolute Hodge cycles in $\mathrm{V}_{\mathbb{A}}^I(M)$. By Theorem 2.2 this subspace is equal to the image in $\mathrm{V}_{\mathbb{A}}^I(M)$ of elements in $\mathrm{V}_0^I(\sigma^*M)$ of bidegree $(0,0)$ via the comparison isomorphism

$$\mathrm{V}_0^I(\sigma^*M) \otimes_{\mathbb{Q}} \mathbb{A} \xrightarrow{\cong} \mathrm{V}_{\mathbb{A}}^I(M)$$

By Proposition 2.9 there is an open subgroup Γ' of Γ such that the action of Γ' on C_{AH}^I is trivial. In particular the image of Γ' in the automorphisms of $\mathrm{V}_\ell^I(M)$ fixes the images of elements in $\mathrm{V}_0^I(\sigma^*M)$ of bidegree $(0,0)$ under the comparison isomorphism $\mathrm{V}_0^I(M) \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{V}_\ell^I(M)$. The image of Γ' in the group of $\mathbb{Q}_\ell$–linear automorphisms of $\mathrm{V}_\ell M$ is therefore contained in the $\mathbb{Q}_\ell$–points of H^M, and because Γ' is of finite index in Γ this shows that $\mathfrak{l}^M$ is contained in $\mathfrak{h}^{(\sigma^*M)} \otimes \mathbb{Q}_\ell$ as we wanted to show. □

3. Construction of the unipotent motivic fundamental group

In this section we construct the Lie algebra of the unipotent motivic fundamental group of a 1–motive. A construction of this object in terms of biextensions and cubist symmetric torsors was proposed by P. Deligne ([Ber03]). Our construction is more elementary, but has the disadvantage that it is not a priori clear why it should produce the right thing. Deligne has remediated this, I have reproduced his comments in the appendix.

– 3.1. The idea of the motivic fundamental group of a 1–motive is the following: Let k be a field, and suppose for a moment that there exists a Tannakian category $\mathcal{M}_k$ of mixed motives over k with rational coefficients. Let $M \in \mathcal{M}_k$ be a 1-motive and write $\langle M \rangle^{\otimes}$ for the Tannakian subcategory of $\mathcal{M}_k$ generated by M. The motivic Galois group $\pi_{\mathrm{mot}}(M)$ of M is defined to be the Tannakian fundamental group of $\langle M \rangle^{\otimes}$. The weight filtration W_* on M defines a filtration on the group $\pi_{\mathrm{mot}}(M)$ and also on its Lie algebra, which we denote by the same letter W_* and also call weight filtration. The first filtration step $W_{-1}\pi_{\mathrm{mot}}(M)$ is the unipotent radical of $\pi_{\mathrm{mot}}(M)$, because pure motives are semisimple objects. We are interested in its Lie algebra

$$W_{-1}(\mathrm{Lie}\,\pi_{\mathrm{mot}}(M)) = \mathrm{Lie}\,W_{-1}(\pi_{\mathrm{mot}}(M))$$

This is a Lie algebra object in the category of motives whose underlying mixed motive has weights -1 and -2. From the point of view of 1–motives, it is a semiabelian variety, say $P(M)$, which is moreover equipped with a Lie algebra structure. We want to construct this semiabelian variety.

Our plan of action for this section is the following: Given a 1–motive M we will construct geometrically a semiabelian variety $P(M)$ and declare it to be $W_{-1}(\mathrm{Lie}\,\pi_{\mathrm{mot}}(M))$. To justify our declaration, we establish in sections 5 and 6 canonical isomorphisms

$$W_{-1}(\mathfrak{h}^M) \longrightarrow \mathrm{V}_0P(M) \qquad \text{and} \qquad W_{-1}(\mathfrak{l}^M) \longrightarrow \mathrm{V}_\ell P(M)$$

of rational Hodge structures and of Galois representations respectively. The semiabelian variety $P(M)$ comes equipped with a Lie bracket, and these isomorphisms are both compatible with Lie brackets, that is, they are isomorphisms of Lie algebra objects. This structure is important, but not for our construction. For the sake of completeness we discuss it in the next section, where we also check that our construction coincides with Deligne's up to a canonical isomorphism.

– **3.2.** Let S be a noetherian regular scheme, and let $M = [u : Y \longrightarrow G]$ be a 1–motive over S. Recall that by a semiabelian scheme over S we understand an extension of an abelian scheme by a torus over S. We start with constructing a semiabelian scheme $U(M)$ over S, which will contain $P(M)$. Write $M_A := M/W_{-1}M = [Y \longrightarrow A]$. The two semiabelian schemes $\mathcal{H}om(Y, G)$ and $\mathcal{E}xt^1(M_A, T)$ are extensions over S of the abelian schemes $\mathcal{H}om(Y, A)$ and $\mathcal{E}xt^1(A, T)$ respectively by the torus $\mathcal{H}om(Y, T)$. We define a semiabelian scheme $U(M)$ by requiring the short sequence of fppf–sheaves on S

$$0 \longrightarrow \mathcal{H}om(Y, T) \xrightarrow{(+,-)} \mathcal{H}om(Y, G) \times \mathcal{E}xt^1(M_A, T) \longrightarrow U(M) \longrightarrow 0$$

to be exact. The first arrow is given on points by sending t to the pair $(\iota_1(t), -\iota_2(t))$, where ι_1 is obtained by applying $\mathcal{H}om(Y, -)$ to the morphism $T \longrightarrow G$ and where ι_2 is obtained by applying $\mathcal{E}xt^1(-, T)$ to the map $M_A \longrightarrow Y[1]$. Representability of $U(M)$ by a semiabelian scheme is not a problem. The map u corresponds to a global section u of $\mathcal{H}om(Y, G)$, and viewing M as an extension of M_A by T we also get a global section η on $\mathcal{E}xt^1(M_A, T)$. Denote by $\overline{u}$ the image of (u, η) in $U(M)(S)$.

Definition 3.3. Let M be a 1–motive over a noetherian regular scheme S. We write $P(M)$ for the smallest semiabelian subscheme of $U(M)$ which contains $n\overline{u}$ for some nonzero $n \in \mathbb{Z}$, and name it *Lie algebra of the unipotent motivic fundamental group of M.*

Alternatively, we could declare $P(M)$ to be the connected component of the unity of the Zariski closure of $\mathbb{Z}\overline{u}$. Over a field, this is clear. In general, one has to know that the Zariski closure of of a semiabelian subvariety in the generic fibre is semiabelian[2]. We continue by checking two things: First, that the construction of $P(M)$ is compatible with flat base change, and secondly that that the realisations of $U(M)$ are canonically isomorphic to the weight (-1) part of the linear endomorphisms of the corresponding realisation of M, showing that $U(M)$ is the right habitat for $P(M)$.

Proposition 3.4. *Let $S' \longrightarrow S$ be a flat morphism between noetherian regular schemes, and let M be a 1–motive over S. There are natural isomorphisms*

$$U(M \times_S S') \cong U(M) \times_S S' \qquad \textit{and} \qquad P(M \times_S S') \cong P(M) \times_S S'$$

of group schemes over S'.

[2]Our terminology is that a semiabelian scheme over S is an extension over S of an abelian scheme by a torus over S. A possible argument is this: For an abelian scheme A the statement follows from the fact that any abelian subvariety of its generic fibre is the image of an endomorphism, and endomorphisms of the generic fibre of A uniquely extend to endomorphisms of A over S. The same works for a torus T. Then use that extensions of A by T over the generic fibre uniquely extend to extensions over S by the Barsotti–Weil formula, and because A is proper.

Proof. The construction of $U(M)$ is even compatible with arbitrary base change, because the formation of $\mathcal{H}om(Y,T)$, $\mathcal{H}om(Y,G)$ and $\mathcal{E}xt^1(M_A,T)$ is so. If $S' \longrightarrow S$ is flat, then so is the induced morphism $U(M)\times_S S' \longrightarrow U(M)$. In order to see that the construction of $P(M)$ is compatible with flat base change, we have to show that the operation of taking Zariski closures is compatible with flat base change. In fact, less than flatness is necessary, as the following general argument shows:

Let X be a scheme of finite type over S, let $R \subseteq X(S)$ be a subset of the set of S–rational points of X and denote by $\overline{R}$ the smallest closed subscheme of X whose S–rational points contain R. Let $f : S' \longrightarrow S$ be a morphism such that the generic points of S' go to generic points of S (this is weaker than flatness). Write $X' := X \times_S S'$ and denote by R' the image of R in $X(S') = X'(S')$. Again, let $\overline{R'}$ be the smallest closed subscheme of X' whose S'–rational points contain R'. Then, the equality

$$\overline{R'} = \overline{R} \times_S S'$$

holds. Indeed, we can reduce to the case that S and S' are both spectra of some fields and we can also assume that $\overline{R} = X$. If X is a variety over a field k any Zariski dense subset of $X(k)$ is also dense as a subset of $X(K)$ where K is any extension of k, proving the claim. □

Proposition 3.5. *Let M be a 1–motive over $\mathbb{C}$, or over a number field k. Respectively, there are canonical isomorphisms*

$$\alpha_0 : \mathrm{V}_0U(M) \xrightarrow{\cong} W_{-1}\,\mathrm{End}_{\mathbb{Q}}(\mathrm{V}_0M) \qquad \textit{and} \qquad \alpha_\ell : \mathrm{V}_\ell U(M) \xrightarrow{\cong} W_{-1}\,\mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M)$$

of Hodge structures and of Galois representations.

Proof. The constructions and verifications are analogous for the Hodge and the ℓ–adic realisations, so we only treat the case of ℓ–adic realisation. We identify $\mathrm{V}_\ell T$ and $\mathrm{V}_\ell G$ with subspaces of $\mathrm{V}_\ell M$, and write π_Y and π_{M_A} for the canonical projections onto $Y\otimes\mathbb{Q}_\ell$ and $\mathrm{V}_\ell M_A$. There are natural isomorphisms of Galois representations

$$\mathrm{V}_\ell\mathcal{H}om(Y,G) \cong \mathrm{Hom}_{\mathbb{Q}_\ell}(Y\otimes\mathbb{Q}_\ell, \mathrm{V}_\ell G) \qquad \text{and} \qquad \mathrm{V}_\ell\mathcal{E}xt^1(M_A,T) \cong \mathrm{Hom}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M_A, \mathrm{V}_\ell T)$$

We can therefore represent elements of $\mathrm{V}_\ell U(M)$ by pairs (f,g) where $f : Y\otimes\mathbb{Q}_\ell \longrightarrow \mathrm{V}_\ell G$ and $g : \mathrm{V}_\ell M_A \longrightarrow \mathrm{V}_\ell T$ are $\mathbb{Q}_\ell$–linear functions. We set

$$\alpha_\ell(f,g) = f\circ\pi_Y + g\circ\pi_{M_A} \qquad \in W_{-1}\,\mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M)$$

This yields a well defined map. Indeed, two pairs (f,g) and (f',g') represent the same element of $\mathrm{V}_\ell U(M)$ if and only if there exists a $\mathbb{Q}_\ell$–linear function $h : Y\otimes\mathbb{Q}_\ell \longrightarrow \mathrm{V}_\ell T$ such that $f - f' = h$ and $g - g' = -h\circ\pi_Y$. So we have

$$(f-f')\circ\pi_Y + (g-g')\circ\pi_{M_A} = h\circ\pi_Y - h\circ\pi_Y = 0$$

The map $\alpha_\ell : \mathrm{V}_\ell U \longrightarrow W_{-1}\,\mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M)$ thus defined is linear, and also Galois equivariant. An inverse to α_ℓ can be obtained as follows. Choose a $\mathbb{Q}_\ell$–linear section s of $\pi_Y : \mathrm{V}_\ell M \longrightarrow \mathrm{V}_\ell Y$ and a $\mathbb{Q}_\ell$–linear retraction r of the inclusion $\mathrm{V}_\ell T \longrightarrow \mathrm{V}_\ell M$. For $\gamma \in W_{-1}\,\mathrm{End}_{\mathbb{Q}}(\mathrm{V}_\ell M)$ we set

$$\alpha_\ell^{-1}(\gamma) = (f-h, g)$$

where f,g and h defined by $f = \gamma \circ s$, $g \circ \pi_{M_A} = r \circ \gamma$ and $h = r \circ \gamma \circ s$. This makes sense because we have

$$\gamma(\mathrm{V}_\ell M) \subseteq \mathrm{V}_\ell G \qquad \text{and} \qquad \gamma(\mathrm{V}_\ell G) \subseteq \mathrm{V}_\ell T \qquad \text{and} \qquad \gamma(\mathrm{V}_\ell T) = \{0\}$$

by definition of the weight filtration on $\mathrm{End}(\mathrm{V}_\ell M)$. To check that α_ℓ^{-1} is an inverse to α_ℓ is straightforward. □

We end this section with a technical definition, which we will use later in sections 5 and 6:

Definition 3.6. Let G and $\widetilde{G}$ be semiabelian varieties over an algebraically closed field $\overline{k}$. We say that $\widetilde{G}$ *contains all isogeny types of* G if there exists an integer $n \geq 0$ and a morphism with finite kernel $\mathrm{gr}^W_*(G) \longrightarrow \mathrm{gr}^W_*(\widetilde{G})^n$.

Lemma 3.7. *Let $M = [Y \longrightarrow G]$ be a 1–motive over an algebraically closed field $\overline{k}$. The semiabelian variety G contains all isogeny types of $P(M)$.*

Proof. We have $P(M) \subseteq U(M)$ by definition, so we can as well show that $\mathrm{gr}^W_*(G)$ contains all isogeny types of $U(M)$. Write G as an extension of an abelian variety A by a torus T. We choose an isogeny $A^\vee \longrightarrow A$ and isomorphisms $Y \simeq \mathbb{Z}^r$ and $T \simeq \mathbb{G}_m^s$. These choices induce a morphism

$$\mathrm{gr}^W_{-1} U(M) = \mathcal{H}om(Y,A) \oplus \mathcal{E}xt^1(A,T) \simeq A^r \oplus (A^\vee)^s \longrightarrow A^{r+s}$$

with finite kernel. We get also an isomorphism $\mathrm{gr}^W_{-2} U(M) = \mathcal{H}om(Y,T) \simeq \mathbb{G}_m^{r+s}$, hence we can find a morphism from $\mathrm{gr}^W_*(U(M))$ to $\mathrm{gr}^W_*(G)^{r+s}$ with finite kernel as needed. □

4. Comments on Lie structures

In this section we explain the Lie algebra structure on $P(M)$ and compare our construction of $P(M)$ with the construction presented in [Ber03]. We will not use this comparison later, and the reader who is only interested in the proof of our main theorems can skip it. We fix a field k of characteristic zero.

– **4.1.** What is a Lie bracket on a semiabelian variety G over k? Naively, that should be an alternating bilinear map $G \times G \longrightarrow G$ satisfying the Jacobi identity, but there are no such maps except for the zero map. There are two ways out, one via the theory of biextensions, the other via homological algebra. We choose to formulate our constructions in terms of homological algebra, so a Lie bracket should be a morphism

$$\beta : G[-1] \otimes^{\mathbb{L}} G[-1] \longrightarrow G[-1]$$

in the derived category of fppf–sheaves[3] on $\mathrm{spec}\, k$, which is graded antisymmetric and satisfies the Jacobi–identity. Instead of β we can also give its adjoint $\mathrm{ad}_\beta : G[-1] \longrightarrow \mathbb{R}\mathcal{H}om(G,G)$. The object $\mathbb{R}\mathcal{H}om(G,G)$ is homologically concentrated in degrees 0 and 1, and given in degree 1 by the sheaf

[3]Recall that we have chosen to place the complexes $[Y \longrightarrow G]$ associated with 1–motives in degrees 0 and 1. With this convention the 1–motive $[\mathbb{Z} \longrightarrow 0]$ is a neutral object for the tensor product of complexes, as it should be.

$\mathcal{E}xt^1(A,T)$ which is representable by an abelian variety. Therefore, β is uniquely determined by a morphism of abelian varieties

$$\mathrm{ad}_\beta : A \longrightarrow \mathcal{E}xt^1(A,T)$$

Any morphism $A \longrightarrow \mathcal{E}xt^1(A,T)$ yields a morphism $G[-1] \otimes^{\mathbb{L}} G[-1] \longrightarrow G[-1]$, so it remains to express antisymmetry and the Jacobi identity for β in terms of its adjoint. The Jacobi–identity comes for free, because β factors over a map $A[-1] \otimes^{\mathbb{L}} A[-1] \longrightarrow T[-1]$, so the derived algebra is contained in the centre. This shows in particular that the Lie algebra object (G,β) is necessarily nilpotent – from the point of view of motives this was already clear for weight reasons. Denoting by $T^\vee$ the group of characters of T, the abelian variety dual to $\mathcal{E}xt(A,T)$ is canonically isomorphic to $T^\vee \otimes A$, and the morphism dual to ad_β is then given by a morphism $T^\vee \otimes A \longrightarrow A^\vee$, or equivalently by a homomorphism of Galois modules

$$\lambda : T^\vee \longrightarrow \mathrm{Hom}_{\overline{k}}(A, A^\vee)$$

From this point of view the antisymmetry condition is easy to express: The image of λ must be contained in the subgroup of selfdual homomorphisms $A \longrightarrow A^\vee$, that is, $\lambda(\chi) = \lambda(\chi)^\vee$ must hold for all $\chi \in T^\vee$. We make this our definition:

Definition 4.2. Let G be a semiabelian variety over k, extension of an abelian variety A by a torus T. Write $A^\vee$ for the abelian variety dual to A and $T^\vee$ for the group of characters of T. A *Lie algebra structure on* G is a homomorphism of Galois modules

$$\lambda : T^\vee \longrightarrow \mathrm{Hom}_{\overline{k}}(A, A^\vee)$$

such that $\lambda(\chi) = \lambda(\chi)^\vee$ holds for all $\chi \in T^\vee$.

– 4.3. This definition makes sense over any base scheme in place of k. To give a Lie algebra structure on the semiabelian variety G is the same as to give a Lie algebra on the associated split semiabelian variety $\mathrm{gr}^W_*(G) = T \oplus A$. Given a Lie algebra structure $\lambda : T^\vee \longrightarrow \mathrm{Hom}_{\overline{k}}(A, A^\vee)$ and a realisation functor, say V_ℓ, one gets a map

$$\mathrm{V}_\ell A \otimes \mathrm{V}_\ell A \longrightarrow \mathrm{V}_\ell T$$

which equips the vector space $\mathrm{V}_\ell G$ with the structure of a nilpotent Lie algebra. The derived Lie algebra $[\mathrm{V}_\ell G, \mathrm{V}_\ell G]$ is contained in $\mathrm{V}_\ell T$, and in fact equal to $\mathrm{V}_\ell(T')$, where $T' \subseteq T$ is the subtorus with character group $T^\vee/\ker(\lambda)$ modulo torsion. A polarisation $\psi : A \longrightarrow A^\vee$ defines a Lie algebra structure on $\mathbb{G}_m \oplus A$, and the Lie bracket one obtains from this is the classical Weil pairing.

– 4.4. Let M be a 1–motive over k. The semiabelian variety $U(M)$ carries a canonical Lie algebra structure. Set $U_T := \mathcal{H}om(Y,T)$ and $U_A := \mathcal{H}om(Y,A) \oplus \mathcal{H}om(T^\vee, A^\vee)$, so $U(M)$ is an extension of U_A by U_T. The dual $U_A^\vee$ of U_A is canonically isomorphic to $(Y \otimes A^\vee) \oplus (T^\vee \otimes A)$, and the character group of U_T is canonically isomorphic to $Y \otimes T^\vee$. So we can define

$$\lambda : (Y \otimes T^\vee) \longrightarrow \mathrm{Hom}_{\overline{k}}(U_A, U_A^\vee) \qquad \lambda(y \otimes \chi)(f,g) = \big(y \otimes g(\chi), \chi \otimes f(y)\big)$$

We leave it to the reader to check that we have indeed $\lambda(y\otimes\chi) = \lambda(y\otimes\chi)^\vee$ for all $y \in Y$ and $\chi \in T^\vee$, and that the induced Lie bracket on realisations, say on $\mathrm{V}_0 U(M)$, is given by

$$[(f,g),(f',g')] = (g\circ f', -f'\circ g)$$

for all $(f,g),(f',g') \in \mathrm{Hom}_\mathbb{Q}(Y\otimes\mathbb{Q}, \mathrm{V}_0 G)\oplus \mathrm{Hom}_\mathbb{Q}(\mathrm{V}_0 M_A, \mathrm{V}_0 T)$. This shows in particular that the canonical isomorphisms α_0 and α_ℓ from Proposition 3.5 are isomorphisms of Lie algebras.

– **4.5.** The abelian variety U_A contains the special rational point $\overline{v}$ coming from the 1–motive M, the image of $\overline{u}$ in U_A. We can recover U from λ, $\overline{v}$ and its graded pieces U_T and U_A. Indeed, the dual of U is given by the morphism

$$(Y\otimes T^\vee) \longrightarrow U_A^\vee \qquad\qquad (y\otimes\chi)\longmapsto \lambda(y\otimes\chi)(\overline{v}) = \big(y\otimes v^\vee(\chi), \chi\otimes v(y)\big)$$

The same formula must then hold for the subvariety $P(M)$ of $U(M)$, whose dual is a quotient of the 1–motive $[(Y\otimes T^\vee)\longrightarrow U_A^\vee]$. This is what is meant in [Ber03] by saying that the unipotent radical of the Lie algebra of $\pi_{\mathrm{mot}}(M)$ is the semiabelian variety defined by the adjoint action of the semisimplification of the Lie algebra of $W_{-1}\pi_{\mathrm{mot}}(M)$ on itself.

– **4.6.** It remains to explain why $P(M)$ is a Lie subobject of $U(M)$. This is indeed nothing special to $P(M)$ and $U(M)$, so let us consider any abelian variety A, torus T, a Lie algebra structure

$$\lambda : T^\vee \longrightarrow \mathrm{Hom}_{\overline{k}}(A, A^\vee)$$

and a rational point $a \in A(k)$. The Lie algebra structure λ and the point a define an extension G of A by T, namely the dual of the 1–motive

$$[w : T^\vee \longrightarrow A^\vee] \qquad\qquad w(\chi) = \lambda(\chi)(a)$$

In this situation, the following is true:

Proposition 4.7. *Let $A' \subseteq A$ be the connected component of the algebraic subgroup of A generated by a, and let $G' \subseteq G$ be any semiabelian subvariety whose projection to A equals A'. Then G' is a Lie subobject of G.*

For example if g is a preimage of a in G, then G' could be the connected component of the algebraic subgroup of G generated by g. This is what we have in our concrete situation $P(M)\subseteq U(M)$.

Proof of 4.7. We suppose without loss of generality that a is a rational point of A', so A' is the Zariski closure of $\mathbb{Z}a$. Denote by $[w' : T'^\vee \longrightarrow A'^\vee]$ the 1–motive dual to G'. The dual of the inclusion $G' \longrightarrow G$ is then a commutative square

$$\begin{array}{ccc} T^\vee & \xrightarrow{w} & A^\vee \\ {\scriptstyle\kappa^\vee}\downarrow & & \downarrow{\scriptstyle\iota^\vee} \\ T'^\vee & \xrightarrow{w'} & A'^\vee \end{array}$$

with surjective vertical maps dual to the inclusions $\kappa : T' \longrightarrow T$ and $\iota : A' \longrightarrow A$. To say that G' is a Lie subobject of G is to say that the arrow λ' in the diagram

$$\begin{array}{ccc} T^\vee & \xrightarrow{\lambda} & \mathrm{Hom}_{\overline{k}}(A, A^\vee) \\ {\scriptstyle\kappa^\vee}\downarrow & & \downarrow{\scriptstyle f \mapsto \iota^\vee \circ f \circ \iota} \\ T'^\vee & \overset{\lambda'}{\dashrightarrow} & \mathrm{Hom}_{\overline{k}}(A', A'^\vee) \end{array}$$

exists. Let $\chi \in T^\vee$ be a character such that $\kappa^\vee(\chi) = 0$. We must show that the endomorphism $\iota^\vee \circ \lambda(\chi) \circ \iota$ of A' is trivial. Indeed, we have

$$\iota^\vee(\lambda(\chi)(a)) = \iota^\vee(w(\chi)) = w'(\kappa(\chi)) = w'(0) = 0$$

so $a \in A'$ belongs to the kernel of this endomorphism, and because a generates A' as an algebraic group, we must have $\iota^\vee \circ \lambda(\chi) \circ \iota = 0$. □

5. Comparison of the motivic fundamental group with the Mumford–Tate group

In this section, we show that for every 1–motive M over $\mathbb{C}$ there is a canonical isomorphism of mixed rational Hodge structures $W_{-1}\mathfrak{h}^M \cong \mathrm{V}_0 P(M)$. We write Γ for the absolute Hodge group over $\mathbb{Q}$, that is, the Tannakian fundamental group of the category of mixed rational Hodge structures. So Γ is a group scheme over $\mathbb{Q}$ which acts on the underlying rational vector space of every mixed rational Hodge structure, in such a way that we have an equivalence of categories

$$\mathrm{MHS}_{\mathbb{Q}} \xrightarrow{\cong} \{\text{Finite dimensional } \mathbb{Q}\text{–linear representations of } \Gamma\}$$

which is compatible with duals and tensor products. We look at mixed Hodge structures, and in particular at Hodge realisations of 1–motives, as $\mathbb{Q}$–vector spaces together with an action of Γ. The Mumford–Tate group of a Hodge structure V is then just the image of Γ in GL_V. For a 1–motive M we write Γ_M for the maximal subgroup of Γ acting trivially on $\mathrm{V}_0 M$, in particular if notations are as in 1.1, then $\Gamma_{\mathrm{gr}^W_* M} = \Gamma_{T \oplus A \oplus Y}$ is the largest subgroup of Γ acting trivially on all pure subquotients of $\mathrm{V}_0 M$.

– 5.1. Let $M = [u : Y \longrightarrow G]$ be a 1–motive over $\mathbb{C}$ and set $U := U(M)$. The action of Γ on $\mathrm{V}_0 M$ is given by a group homomorphism $\rho_0 : \Gamma \longrightarrow \mathrm{GL}_{\mathrm{V}_0 M}$, whose image is by definition the Mumford–Tate group of M. The subgroup $\Gamma_{\mathrm{gr}^W_* M}$ acts on $\mathrm{V}_0 M$ by unipotent automorphisms, and we have

$$\log(\rho_0(\gamma)) = (\rho_0(\gamma) - 1) - \tfrac{1}{2}(\rho_0(\gamma) - 1)^2 \qquad \in W_{-1}\,\mathrm{End}(\mathrm{V}_0 M)$$

We have constructed a canonical isomorphism $\alpha_0 : \mathrm{V}_0 U \longrightarrow W_{-1}\,\mathrm{End}(\mathrm{V}_0 M)$, and by composing we get a map $\vartheta_0 := \alpha_0^{-1} \circ \log \circ \rho_0$. The image of $\vartheta_0 : \Gamma_{\mathrm{gr}^W_* M} \longrightarrow \mathrm{V}_0 U$ is a Lie subalgebra of $\mathrm{V}_0 U$, isomorphic via α_0 to the Lie algebra $W_{-1}\mathfrak{h}^M$. Here is the picture:

$$\begin{array}{ccccc} & & \Gamma_{\mathrm{gr}_*^W M} & \xrightarrow{\ \subseteq\ } & \Gamma \\ & \swarrow^{\vartheta_0} & \downarrow{\scriptstyle \log\circ\rho_0} & & \downarrow{\scriptstyle \rho_0} \\ \mathrm{V}_0P(M) \subseteq \mathrm{V}_0U(M) & \xrightarrow{\alpha_0} & W_{-1}\operatorname{End}(\mathrm{V}_0M) & & \mathrm{GL}_{\mathrm{V}_0M} \end{array}$$

More explicitly, the map ϑ_0 is given as follows: Choose a section $s : Y\otimes\mathbb{Q} \longrightarrow \mathrm{V}_0M$ and a retraction $r : \mathrm{V}_0M \longrightarrow \mathrm{V}_0T$. Then, $\vartheta(\gamma)$ is represented by the pair

$$(*) \qquad (f-h-\tfrac{1}{2}e, g) \quad \in \operatorname{Hom}_{\mathbb{Q}}(Y\otimes\mathbb{Q}, \mathrm{V}_0G)\times\operatorname{Hom}_{\mathbb{Q}}(\mathrm{V}_0M_A, \mathrm{V}_0T)$$

where f, g, h and e are given by

$$(**) \quad f(y) = \gamma s(y) - s(y) \qquad g(a) = r(\gamma\widetilde{a}-\widetilde{a}) \qquad h = r\circ f \qquad e(y) = \gamma^2 s(y) - 2\gamma s(y) + s(y)$$

for all $y\in Y$ and $a\in\mathrm{V}_0M_A$. In the second equality, $\widetilde{a}$ is any element of V_0M mapping to $a\in\mathrm{V}_0M_A$ and V_0T is understood to be contained in V_0G. The class of $(f-h-\frac{1}{2}e, g)$ in $\mathrm{V}_0U(M)$ is independent of the choice of s and r. The main result of this section is:

Theorem 5.2. *The image of the map $\vartheta_0 : \Gamma_{\mathrm{gr}_*^W M} \longrightarrow \mathrm{V}_0U(M)$ is equal to $\mathrm{V}_0P(M)$. In other words, the map α_0 induces an isomorphism*

$$\mathrm{V}_0P(M) \xrightarrow{\ \cong\ } W_{-1}\mathfrak{h}^M$$

of rational Hodge structures.

– **5.3.** We begin with an auxiliary construction. Let G be a semiabelian variety over $\mathbb{C}$, and let us construct a $\mathbb{Q}$–linear map

$$\kappa_0 : G(\mathbb{C})\otimes\mathbb{Q} \longrightarrow H^1(\Gamma, \mathrm{V}_0G)$$

as follows: Given a complex point $x\in G(\mathbb{C})$ we consider the 1–motive $M_x := [\mathbb{Z} \xrightarrow{1\longmapsto x} G]$. The weight filtration on M_x induces a long exact sequence of rational vector spaces starting with

$$0 \longrightarrow (\mathrm{V}_0G)^\Gamma \longrightarrow (\mathrm{V}_0M_x)^\Gamma \longrightarrow \mathbb{Q} \xrightarrow{\ \partial\ } H^1(\Gamma, \mathrm{V}_0G) \longrightarrow \cdots$$

and we set $\kappa_0(x\otimes 1) = \partial(1)$. Explicitly, elements of the integral Hodge realisation T_0M_x are pairs $(v,n)\in\operatorname{Lie}G(\mathbb{C})\times\mathbb{Z}$ with $\exp(v) = nx$. We define $\kappa_0(x\otimes 1)$ to be the class of the cocycle $\gamma\longmapsto\gamma(v,1)-(v,1)$ where $v\in\operatorname{Lie}G(\mathbb{C})$ is any element such that $\exp(v) = x$.

Proposition 5.4. *The map κ_0 constructed in 5.3 is injective and natural in G.*

Proof. Let $x\in G(\mathbb{C})$ be a complex point such that the cocycle $c : \gamma\longmapsto\gamma(v,1)-(v,1)$ is a coboundary, where $v\in\operatorname{Lie}G(\mathbb{C})$ is such that $\exp(v) = x$. We have to show that x is a torsion point. Indeed, since c is a cocycle, there exists an element $w\in\mathrm{V}_0G\subseteq\operatorname{Lie}G(\mathbb{C})$ such that $c(\gamma) = \gamma w - w$ for all $\gamma\in\Gamma$. Let $n > 0$ be an integer such that $nw\in\mathrm{T}_0G = \ker(\exp)$. Then $(nv-nw, n)$ is a

Γ–invariant element of $\mathrm{T}_0 M_x$, hence the linear map $\mathbb{Z} \longrightarrow \mathrm{T}_0[n\mathbb{Z} \longrightarrow G]$ sending 1 to $(nv - nw, n)$ is a morphism of Hodge structures. From this we get a morphism of 1–motives

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & 0 \\ {\scriptstyle \cdot n}\downarrow & & \downarrow \\ n\mathbb{Z} & \xrightarrow{n \mapsto nx} & G \end{array}$$

because integral Hodge realisation is a fully faithful functor by [Del74], 10.1.3. This shows that $nx = 0$. Naturality of κ_0 follows from naturality of the Hodge realisation. □

Proposition 5.5. *Let $M = [\mathbb{Z} \xrightarrow{u} G]$ be a 1–motive over $\mathbb{C}$. The class $\kappa_0(u(1) \otimes 1)$ restricts to zero in $H^1(\Gamma_M, \mathrm{V}_0 G)$.*

Proof. Choose $v \in \operatorname{Lie} G(\mathbb{C})$ such that $\exp(v) = u(1)$, so that the pair $(v, 1) \in \operatorname{Lie} G(\mathbb{C}) \times \mathbb{Z}$ defines an element of $\mathrm{V}_0 M$. By definition of Γ_M we have $\gamma(v, 1) = (v, 1)$ and hence $\kappa_0(\gamma) = 0$ for all $\gamma \in \Gamma_M$. □

Proposition 5.6. *Let $M = [u : Y \longrightarrow G]$ be a 1–motive over $\mathbb{C}$ and consider the 1–motives*

$$M_U := [\mathbb{Z} \xrightarrow{1 \mapsto \overline{u}} U(M)] \qquad \text{and} \qquad M_P := [n\mathbb{Z} \xrightarrow{n \mapsto n\overline{u}} P(M)]$$

where $n \geq 1$ is an integer such that the point $n\overline{u}$ of $U(M)$ belongs to $P(M)$ and $\overline{u}$ is as in Definition 3.3. The inclusions $\Gamma_M = \Gamma_{M_U} \subseteq \Gamma_{M_P}$ hold in Γ.

Proof. We write $\langle V \rangle$ for the Tannakian subcategory of the category of rational Hodge structures generated by a Hodge structure V. We have to show

$$\text{a)} \quad \mathrm{V}_0 M \in \langle \mathrm{V}_0 M_U \rangle \qquad \text{b)} \quad \mathrm{V}_0 M_U \in \langle \mathrm{V}_0 M \rangle \qquad \text{c)} \quad \mathrm{V}_0 M_P \in \langle \mathrm{V}_0 M_U \rangle$$

For a), consider the morphisms of 1–motives given by the following commutative squares:

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{1 \mapsto \overline{u}} & U(M) \\ \| & & \downarrow{\scriptstyle \mathrm{proj}} \\ \mathbb{Z} & \xrightarrow[1 \mapsto u]{} & \mathcal{H}om(Y, G) \end{array} \qquad\qquad \begin{array}{ccc} Y \otimes \mathbb{Z} & \xrightarrow{y \otimes 1 \mapsto y \otimes u} & Y \otimes \mathcal{H}om(Y, G) \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle y \otimes f \mapsto f(y)} \\ Y & \xrightarrow[u]{} & G \end{array}$$

Both morphisms induce surjective morphisms of Hodge structures. The left hand diagram shows that $\mathrm{V}_0[\mathbb{Z} \longrightarrow \mathcal{H}om(Y, G)]$ belongs to $\langle \mathrm{V}_0 M_U \rangle$, hence also the Hodge structure

$$(Y \otimes \mathbb{Q}) \otimes \mathrm{V}_0[\mathbb{Z} \longrightarrow \mathcal{H}om(Y, G)] \cong \mathrm{V}_0[Y \otimes \mathbb{Z} \longrightarrow Y \otimes \mathcal{H}om(Y, G)]$$

The right hand morphism shows that also $\mathrm{V}_0 M$ belongs to $\langle \mathrm{V}_0 M_U \rangle$. The verification of b) is similar, here we consider the morphism of 1–motives given by

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{1 \mapsto (u, \eta)} & \mathcal{H}om(Y, G) \oplus \mathcal{E}xt^1(M_A, T) \\ {\scriptstyle 1 \mapsto (\mathrm{id}, \mathrm{id})}\downarrow & & \| \\ \mathcal{H}om(Y, Y) \oplus \mathcal{H}om(T, T) & \xrightarrow[(f, g) \mapsto (u \circ f, g_* \eta)]{} & \mathcal{H}om(Y, G) \oplus \mathcal{E}xt^1(M_A, T) \end{array}$$

where $\eta \in \operatorname{Ext}^1_k(M_A, T)$ is the extension class defined by M. This diagram induces an injection of Hodge structures. The Hodge structure associated with the lower row is isomorphic to the direct sum of $\operatorname{Hom}_{\mathbb{Q}}(Y \otimes \mathbb{Q}, \mathrm{V}_0 M)$ and $\operatorname{Hom}_{\mathbb{Q}}(\mathrm{V}_0 M, \mathrm{V}_0 T)$, hence belongs to $\langle \mathrm{V}_0 M \rangle$. Hence also the Hodge structure associated with the upper row belongs to $\langle \mathrm{V}_0 M \rangle$, and $\mathrm{V}_0 M_U$ is a quotient of this Hodge structure by definition of $U(M)$. Finally, c) is obvious since $\mathrm{V}_0 M_P$ is a substructure of $\mathrm{V}_0 M_U$. □

Corollary 5.7. *Let $M = [u : Y \longrightarrow G]$ be a 1–motive over $\mathbb{C}$ and set $\widetilde{G} := \mathrm{gr}^W_* G = T \oplus A$. Let $n \geq 1$ be an integer such that the point $n\overline{u}$ of $U(M)$ belongs to $P(M)$. For every $\psi \in \mathrm{Hom}_{\mathbb{C}}(P(M), \widetilde{G})$, the cohomology class $\kappa_0(\psi(n\overline{u}) \otimes 1) \in H^1(\Gamma, \mathrm{V}_0\widetilde{G})$ restricts to zero in $H^1(\Gamma_M, \mathrm{V}_0\widetilde{G})$.*

Proof. Set $P := P(M)$, define 1–motives $M_P := [\mathbb{Z} \xrightarrow{1 \longmapsto n\overline{u}} P]$ and $M_\psi := [\mathbb{Z} \xrightarrow{1 \longmapsto \psi(n\overline{u})} \widetilde{G}]$. The pair $(\mathrm{id}_{\mathbb{Z}}, \psi)$ defines a morphism of 1–motives $M_P \longrightarrow M_\psi$, and we find a commutative diagram

$$\begin{array}{ccccc} P(\mathbb{C}) \otimes \mathbb{Q} & \xrightarrow{\kappa_0} & H^1(\Gamma, \mathrm{V}_0 P) & \xrightarrow{\mathrm{res}} & H^1(\Gamma_M, \mathrm{V}_0 P) \\ \downarrow{\scriptstyle \psi\otimes 1} & & \downarrow & & \downarrow \\ \widetilde{G}(\mathbb{C}) \otimes \mathbb{Q} & \xrightarrow{\kappa_0} & H^1(\Gamma, \mathrm{V}_0 \widetilde{G}) & \xrightarrow{\mathrm{res}} & H^1(\Gamma_M, \mathrm{V}_0 \widetilde{G}) \end{array}$$

where all vertical maps are induced by ψ. The restriction map $H^1(\Gamma, \mathrm{V}_0 P) \longrightarrow H^1(\Gamma_M, \mathrm{V}_0 P)$ factors over $H^1(\Gamma_{M_P}, \mathrm{V}_0 P)$ by Proposition 5.6, hence $\kappa_0(n\overline{u} \otimes 1)$ maps to zero in $H^1(\Gamma_M, \mathrm{V}_0 P)$ by Proposition 5.5. □

– 5.8. With the help of the map κ_0 and its properties we established so far, we can show that the image of ϑ_0 is contained in $\mathrm{V}_0 P(M)$. Let $M = [u : Y \longrightarrow G]$ be a 1–motive over $\mathbb{C}$ and write $M_A := M/W_{-2}M = [Y \longrightarrow A]$ and $U := U(M)$. Let

$$\pi : U \longrightarrow U_A := \mathcal{H}om(Y, A) \oplus \mathcal{E}xt^1(A, T)$$

be the projection onto the abelian quotient U_A of U and let ι be the inclusion of $\Gamma_{G \oplus M_A}$ into $\Gamma_{\mathrm{gr}^W_* M}$. We consider the two composition maps

$$U(\mathbb{C}) \otimes \mathbb{Q} \xrightarrow{\kappa_0} H^1(\Gamma_{\mathrm{gr}^W_* M}, \mathrm{V}_0 U) \xrightarrow{(\mathrm{V}_0\pi)_*} H^1(\Gamma_{\mathrm{gr}^W_* M}, \mathrm{V}_0 U_A) \cong \mathrm{Hom}(\Gamma_{\mathrm{gr}^W_* M}, \mathrm{V}_0 U_A)$$

and

$$U(\mathbb{C}) \otimes \mathbb{Q} \xrightarrow{\kappa_0} H^1(\Gamma_{\mathrm{gr}^W_* M}, \mathrm{V}_0 U) \xrightarrow{\iota^*} H^1(\Gamma_{G \oplus M_A}, \mathrm{V}_0 U) \cong \mathrm{Hom}(\Gamma_{G \oplus M_A}, \mathrm{V}_0 U)$$

These send $\overline{u} \otimes 1$ to the homomorphisms $(\mathrm{V}_0\pi) \circ \kappa_0(\overline{u} \otimes 1)$ and $\kappa_0(\overline{u} \otimes 1) \circ \iota$ respectively. Here we have used that $\Gamma_{\mathrm{gr}^W_* M}$ acts trivially on $\mathrm{V}_0 U_A$ and that $\Gamma_{G \oplus M_A}$ acts trivially on $\mathrm{V}_0 U$. The following lemma explains the relation between ϑ_0 and κ_0:

Lemma 5.9. *Notations being as in 5.8, the equalities*

$$(\mathrm{V}_0\pi) \circ \vartheta_0 = (\mathrm{V}_0\pi) \circ \kappa_0(\overline{u} \otimes 1) \qquad \textit{and} \qquad \vartheta_0 \circ \iota = \kappa_0(\overline{u} \otimes 1) \circ \iota$$

hold in $\mathrm{Hom}(\Gamma_{\mathrm{gr}^W_* M}, \mathrm{V}_0 U_A)$ and in $\mathrm{Hom}(\Gamma_{G \oplus M_A}, \mathrm{V}_0 U)$ respectively.

Proof. Write $M_U = [\mathbb{Z} \xrightarrow{1 \longmapsto \overline{u}} U]$ and choose $x \in \mathrm{Lie}\, U$ with $\exp(x) = \overline{u}$. We can represent x by a pair

$$(x_f, x_g) \in \mathrm{Lie}\, \mathcal{H}om(Y, G) \oplus \mathrm{Lie}\, \mathcal{E}xt^1(M_A, T)$$

with $\exp x_f = u$ and $\exp x_g = \eta$, the class of M in $\mathrm{Ext}^1(M_A, T)$. Regarding x_f as a group homomorphism $x_f : Y \longrightarrow \mathrm{Lie}\, G$, we get a section $s : Y \otimes \mathbb{Q} \longrightarrow \mathrm{V}_0 M$ defined by $s(y) = (x_f(y), y)$. Regarding x_g as a group homomorphism $x_g : T^\vee \longrightarrow \mathrm{Lie}(M_A^\vee)$ we get a section $T^\vee \otimes \mathbb{Q} \longrightarrow \mathrm{V}_0 M^\vee$ defined by $\chi \longmapsto (x_g(\chi), \chi)$. The linear dual of this section is a retraction $r : \mathrm{V}_0 M \longrightarrow \mathrm{V}_0 T$. With the help of the section s and the retraction r we can describe ϑ_0 explicitly as in 5.1, so $\vartheta_0(\gamma)$ is the class of $(f - h - \frac{1}{2}e, g)$ where f, g, h and e depend on s, r and γ according to the equations $(**)$ in 5.1.

Having this explicit description at hand, we can verify the left hand formula of the lemma. Applying its left hand term to $\gamma \in \Gamma_{\mathrm{gr}^W_* M}$ we get the element of $\mathrm{V}_0 U_A$ given by the pair

$$\mathrm{V}_0\pi\big(f - h - \tfrac{1}{2}e, g\big) = \big(\mathrm{V}_0\pi \circ f, \mathrm{V}_0\pi \circ g\big) \in \mathrm{Hom}(Y \otimes \mathbb{Q}, \mathrm{V}_0 A) \oplus \mathrm{Hom}(\mathrm{V}_0 A, \mathrm{V}_0 T)$$

On the right hand side we find, using the definition of κ_0

$$(\mathrm{V}_0\pi) \circ \kappa_0(\overline{u} \otimes 1)(\gamma) = (\mathrm{V}_0\pi)(\gamma(x,1) - (x,1)) = \gamma(\pi x, 1) - (\pi x, 1) \in \mathrm{V}_0 U_A$$

where πx is the image of x in $\mathrm{Lie}\, U_A$. This element is given by the pair

$$\big(\gamma(\pi x_f, 1) - (\pi x_f, 1), \gamma(\pi x_g, 1) - (\pi x_g, 1)\big) \in \mathrm{V}_0\mathcal{H}om(Y, A) \oplus \mathrm{V}_0\mathcal{E}xt^1(M_A, T)$$

which if viewed as an element of $\mathrm{Hom}(Y \otimes \mathbb{Q}, \mathrm{V}_0 A) \times \mathrm{Hom}(\mathrm{V}_0 A, \mathrm{V}_0 T)$ equals $(\mathrm{V}_0\pi \circ f, \mathrm{V}_0\pi \circ g)$.

We now come to the second formula. Let us fix an element $\gamma \in \Gamma_{G \oplus M_A}$, so γ acts trivially on $\mathrm{V}_0 G$ and on $\mathrm{V}_0 M_A$. Hence we have $f = g = h$ and $e = 0$, so $\vartheta_0(\gamma)$ is given by the homomorphism

$$h \in \mathrm{Hom}(Y \otimes \mathbb{Q}, \mathrm{V}_0 T) \qquad h(y) = \gamma s_f(y) - s_f(y)$$

Under the canonical isomorphism $\mathrm{V}_0\mathcal{H}om(Y, G) \cong \mathrm{Hom}(Y \otimes \mathbb{Q}, \mathrm{V}_0 G)$ this homomorphism h corresponds to the element

$$\gamma(x_f, 1) - (x_f, 1) = \gamma(x, 1) - (x, 1)$$

which equals $\kappa_0(\overline{u} \otimes 1)(\gamma)$ by definition of κ_0. □

Lemma 5.10. *Let $0 \longrightarrow T \longrightarrow G \xrightarrow{\pi} A \longrightarrow 0$ be a semiabelian variety over $\mathbb{C}$, let $G' \subseteq G$ be a semiabelian subvariety and let $V \subseteq \mathrm{V}_0 G$ be a Hodge substructure. If the inclusions $\pi(V) \subseteq \mathrm{V}_0\pi(G')$ and $V \cap \mathrm{V}_0 T \subseteq \mathrm{V}_0 G'$ hold, then V is contained in $\mathrm{V}_0 G'$. (Think of it as exactness of gr^W_*).*

Proof. Consider the following diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V \cap \mathrm{V}_0 T & \longrightarrow & V & \longrightarrow & \pi(V) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathrm{V}_0(T/(T \cap G')) & \longrightarrow & \mathrm{V}_0(G/G') & \longrightarrow & \mathrm{V}_0(A/\pi(G')) & \longrightarrow & 0
\end{array}$$

The left and right vertical maps are zero by hypothesis, and we have to show that the middle vertical map is zero as well. This follows by diagram chase, using that there are no nonzero morphisms of Hodge structures $\pi(V) \longrightarrow \mathrm{V}_0(T/(T \cap G'))$. Indeed, $\pi(V)$ is pure of weight -1 and $\mathrm{V}_0(T/(T \cap G'))$ is pure of weight -2. □

Proposition 5.11. *The image of the map $\vartheta_0 : \Gamma_{\mathrm{gr}^W_* M} \longrightarrow \mathrm{V}_0 U$ is contained in $\mathrm{V}_0 P$.*

Proof. This is a consequence of Lemmas 5.9 and 5.10. Indeed, by Lemma 5.10 it is enough to check that the inclusions

$$(\mathrm{V}_0\pi)(\mathrm{im}(\vartheta_0)) \subseteq \mathrm{V}_0\pi(P) \qquad \text{and} \qquad \mathrm{im}(\vartheta_0) \cap \mathrm{V}_0 U_T \subseteq \mathrm{V}_0 P$$

hold. An element $\gamma \in \Gamma_{\mathrm{gr}^W_* M}$ belongs to $\Gamma_{G \oplus M_A}$ if and only if it acts trivially on $\mathrm{V}_0 U_A$. Hence by Lemma 5.9, these inclusions are the same as

$$\mathrm{im}((\mathrm{V}_0\pi) \circ \kappa_0(\overline{u} \otimes 1)) \subseteq \mathrm{V}_0\pi(P) \qquad \text{and} \qquad \mathrm{im}(\kappa_0(\overline{u} \otimes 1) \circ \iota) \subseteq \mathrm{V}_0 P$$

But since $n\overline{u} \in P$ for some nonzero n, the class $\kappa_0(\overline{u} \otimes 1)$ comes from a cocycle which takes values in $\mathrm{V}_0 P$. □

Lemma 5.12. *Let $M = [Y \longrightarrow G]$ be a 1–motive over $\mathbb{C}$ and set $\widetilde{G} := \mathrm{gr}^*_W G = T \oplus A$. The map*

$$H^1(\mathfrak{h}^M, \mathrm{V}_0\widetilde{G}) \longrightarrow \mathrm{Hom}_{\mathfrak{h}^M}(W_{-1}\mathfrak{h}^M, \mathrm{V}_0\widetilde{G})$$

given by restriction of cocycles is an isomorphism.

Proof. The Hochschild–Serre spectral sequence for Lie algebra cohomology associated with the Lie algebra extension $0 \longrightarrow W_{-1}\mathfrak{h}^M \longrightarrow \mathfrak{h}^M \longrightarrow \mathfrak{h}^{\widetilde{G}} \longrightarrow 0$ yields the following exact sequence in low degrees

$$H^1(\mathfrak{h}^{\widetilde{G}}, \mathrm{V}_0\widetilde{G}) \longrightarrow H^1(\mathfrak{h}^M, \mathrm{V}_0\widetilde{G}) \longrightarrow \mathrm{Hom}_{\mathfrak{h}^M}(W_{-1}\mathfrak{h}^M, \mathrm{V}_0\widetilde{G}) \longrightarrow H^2(\mathfrak{h}^{\widetilde{G}}, \mathrm{V}_0\widetilde{G})$$

so it suffices to show that the first and last term in this sequence vanish. To do so, it suffices by Sah's lemma to show that there exists a central element $x \in \mathfrak{h}^{\widetilde{G}}$ which acts as an automorphism on $\mathrm{V}_0\widetilde{G}$. But this is clear since $\mathfrak{h}^{\widetilde{G}}$ contains an element which acts as the identity on $\mathrm{V}_0 A$ and as multiplication by 2 on $\mathfrak{h}^T$. □

Lemma 5.13. *Let $M = [u : Y \longrightarrow G]$ be a 1–motive over $\mathbb{C}$ and set $\widetilde{G} := \mathrm{gr}_*^W G = T \oplus A$. The map $\alpha_0^* : \mathrm{Hom}_\Gamma(\mathrm{V}_0 P(M), \mathrm{V}_0\widetilde{G}) \longrightarrow \mathrm{Hom}_\Gamma(W_{-1}\mathfrak{h}^M, \mathrm{V}_0\widetilde{G})$ given by $\alpha_0^*(f) = f \circ \alpha_0^{-1}$ is injective.*

Proof. Set $P := P(M)$ for brevity. We will construct in a first step another injective map $\beta_0 : \mathrm{Hom}_\Gamma(\mathrm{V}_0 P, \mathrm{V}_0\widetilde{G}) \longrightarrow \mathrm{Hom}_\Gamma(W_{-1}\mathfrak{h}^M, \mathrm{V}_0\widetilde{G})$, and prove in a second step that the equality $\alpha_0^* = \beta_0$ holds. For the construction of β_0 we use the following diagram

$$\begin{array}{ccccccc}
 & & & & \mathrm{Hom}_{\mathbb{C}}(P, \widetilde{G}) \otimes \mathbb{Q} & & \\
 & & & \swarrow & \downarrow (*) & & \\
0 & \longrightarrow & H^1(\mathfrak{h}^M, \mathrm{V}_0\widetilde{G}) & \longrightarrow & H^1(\Gamma, \mathrm{V}_0\widetilde{G}) & \xrightarrow{\mathrm{res}} & H^1(\Gamma_M, \mathrm{V}_0\widetilde{G}) \\
 & & \cong\downarrow & & & & \\
 & & \mathrm{Hom}_\Gamma(W_{-1}\mathfrak{h}^M, \mathrm{V}_0\widetilde{G}) & & & &
\end{array}$$

where the map $(*)$ is $\mathbb{Q}$–linear and sends $\psi \otimes 1$ to $\kappa_0(\psi(n\overline{u}) \otimes 1)$. By Corollary 5.7 the map $(*)$ lifts to $H^1(\mathfrak{h}^M, \mathrm{V}_0\widetilde{G})$ as indicated. The isomorphism on the left is given by Lemma 5.12. Let β_0 be the composition

$$\beta_0 : \mathrm{Hom}_\Gamma(\mathrm{V}_0 P, \mathrm{V}_0\widetilde{G}) \cong \mathrm{Hom}_{\mathbb{C}}(P, \widetilde{G}) \otimes \mathbb{Q} \longrightarrow \mathrm{Hom}_\Gamma(W_{-1}\mathfrak{h}^M, \mathrm{V}_0\widetilde{G})$$

The map β_0 is injective because $(*)$ is so. Indeed, let $\psi \in \mathrm{Hom}(P, \widetilde{G})$ be a homomorphism such that $\kappa_0(\psi(n\overline{u}) \otimes 1) = 0$. Then, since κ_0 is injective by Proposition 5.4 we have $\psi(n\overline{u}) = 0$ and hence $\ker \psi$ is a subgroup of P containing $n\overline{u}$. We must then have $\ker \psi = P$ by definition of P, so $\psi = 0$. It remains to check that we have $\alpha_0^* = \beta_0$. Because all maps are $\mathbb{Q}$–linear, we only have to check that for every $\psi \in \mathrm{Hom}(P, \widetilde{G})$ and every $\gamma \in \Gamma_{\mathrm{gr}_*^W M}$ the equality

$$\alpha_0^*(\mathrm{V}_0\psi)(\log \rho_0(\gamma)) = \beta_0(\mathrm{V}_0\psi)(\log \rho_0(\gamma))$$

holds in $\mathrm{V}_0\widetilde{G}$. The left hand side is equal to $\mathrm{V}_0\psi(\vartheta_0(\gamma))$. Let $w \in \operatorname{Lie} P(\mathbb{C})$ be an element with $\exp(w) = n\overline{u}$, and set $v := \psi(w) \in \operatorname{Lie}\widetilde{G}$. We then have $\exp(v) = \psi(n\overline{u})$ and using Lemma 5.9

$$\beta_0(\mathrm{V}_0\psi)(\log\rho_0(\gamma)) = \kappa_0(\psi(n\overline{u}))(\gamma) = \gamma(v,1) - (v,1) = \mathrm{V}_0\psi(\gamma(w,1) - (w,1)) = \mathrm{V}_0\psi(\vartheta_0(\gamma))$$

as we wanted to show. □

Lemma 5.14. *Let G be a semiabelian variety over $\mathbb{C}$ and let $\widetilde{G}$ be a split semiabelian variety containing all isogeny types of G (Definition 3.6). Let V be a Hodge substructure of V_0G. If the restriction map*

$$\operatorname{Hom}_\Gamma(\mathrm{V}_0G, \mathrm{V}_\ell\widetilde{G}) \longrightarrow \operatorname{Hom}_\Gamma(V, \mathrm{V}_0\widetilde{G})$$

is injective, then V is equal to V_0G.

Proof. In the case where G is an abelian variety or a torus, this is clear by semisimplicity of the category of pure polarisable rational Hodge structures. The general case can be proved by dévissage, writing G as an extension of an abelian variety by a torus. We give a detailed proof for the analogous statement about Galois representations (Lemma 6.14). □

Proof of Theorem 5.2. The image $W_{-1}\mathfrak{h}^M$ of the map ϑ_0 is contained in V_0P by Proposition 5.11. By Lemma 5.13, the restriction map $\operatorname{Hom}_\Gamma(\mathrm{V}_0P, \mathrm{V}_0\widetilde{G}) \longrightarrow \operatorname{Hom}_\Gamma(\operatorname{im}\vartheta_0, \mathrm{V}_0\widetilde{G})$ is injective, and we know by Lemma 3.7 that G contains all isogeny of P. Hence the equality $\operatorname{im}\vartheta_0 = \mathrm{V}_0P$ must hold by Lemma 5.14. □

6. Comparison of the motivic fundamental group with the image of Galois

Let k be field which is finitely generated over its prime field, and let $\overline{k}$ be an algebraic closure of k. Let ℓ be a prime number different from the characteristic of k. In this section, we show that for every 1–motive M over k there is a canonical isomorphism of ℓ–adic Galois representations $W_{-1}\mathfrak{l}^M \cong \mathrm{V}_\ell P(M)$. In analogy with the previous section we write Γ for the absolute Galois group $\operatorname{Gal}(\overline{k}|k)$, and Γ_M for the subgroup of Γ consisting of those elements which act trivially on $\mathrm{V}_\ell M$. For a commutative group C, we introduce the notation

$$C \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell := \lim_{i\geq 0} C/\ell^i C \qquad \text{and} \qquad C \mathbin{\widehat{\otimes}} \mathbb{Q}_\ell := (C \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

There is a canonical map $C \longrightarrow C \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell$ whose kernel consists of the ℓ–divisible elements of C, and we write $c \mathbin{\widehat{\otimes}} 1$ for the image of $c \in C$ under this map.

– **6.1.** Let $M = [u : Y \longrightarrow G]$ be a 1–motive over k and set $U := U(M)$. The action of Γ on $\mathrm{V}_\ell M$ is given by a group homomorphism $\rho_\ell : \Gamma \longrightarrow \mathrm{GL}_{\mathrm{V}_\ell M}$. The group $\Gamma_{\mathrm{gr}^W_* M}$ consisting of those elements of Γ which act trivially on $\mathrm{gr}^W_*\mathrm{V}_\ell M$ acts on $\mathrm{V}_\ell M$ by unipotent automorphisms, and we have

$$\log\rho_\ell(\gamma) = (\rho_\ell(\gamma) - 1) - \tfrac{1}{2}(\rho_\ell(\gamma) - 1)^2 \qquad \in W_{-1}\operatorname{End}(\mathrm{V}_\ell M)$$

We have constructed a canonical isomorphism $\alpha_\ell : \mathrm{V}_0U \longrightarrow W_{-1}\,\mathrm{End}(\mathrm{V}_0M)$, and by composing we get a map $\vartheta_\ell := \alpha_\ell^{-1} \circ \log \circ \rho_\ell$. The image of $\vartheta_\ell : \Gamma_{\mathrm{gr}_*^W M} \longrightarrow \mathrm{V}_0U$ is a $\mathbb{Z}_\ell$ submodule of $\mathrm{V}_\ell U$ whose $\mathbb{Q}_\ell$–linear span is a Lie subalgebra of V_0U, isomorphic via α_ℓ to the Lie algebra $W_{-1}\mathfrak{l}^M$. The overall picture is similar to that in 5.1:

$$\begin{array}{ccccc} & & \Gamma_{\mathrm{gr}_*^W M} & \xrightarrow{\subseteq} & \Gamma \\ & \swarrow^{\vartheta_\ell} & \downarrow^{\log\circ\rho_\ell} & & \downarrow^{\rho_\ell} \\ \mathrm{V}_\ell P(M) \subseteq \mathrm{V}_\ell U(M) & \xrightarrow{\alpha_\ell} & W_{-1}\,\mathrm{End}(\mathrm{V}_\ell M) & & \mathrm{GL}_{\mathrm{V}_\ell M} \end{array}$$

For the record, the map ϑ_ℓ is explicitly given as follows: Choose a section $s : Y \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{V}_\ell M$ and a retraction $r : \mathrm{V}_\ell M \longrightarrow \mathrm{V}_\ell T$. Then, $\vartheta_\ell(\gamma)$ is represented by the pair

$$(*) \qquad (f - h - \tfrac{1}{2}e, g) \quad \in \mathrm{Hom}_{\mathbb{Q}_\ell}(Y \otimes \mathbb{Q}_\ell, \mathrm{V}_\ell G) \times \mathrm{Hom}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M_A, \mathrm{V}_\ell T)$$

where f, g, h and e are given by

$$(**) \quad f(y) = \gamma s(y) - s(y) \qquad g(a) = r(\gamma\widetilde{a} - \widetilde{a}) \qquad h = r \circ f \qquad e(y) = \gamma^2 s(y) - 2\gamma s(y) + s(y)$$

for all $y \in Y$ and $a \in \mathrm{V}_\ell M_A$. In the second equality, $\widetilde{a}$ is any element of $\mathrm{V}_\ell M$ mapping to $a \in \mathrm{V}_\ell M_A$ and $\mathrm{V}_\ell T$ is understood to be contained in $\mathrm{V}_\ell G$. The main result of this section is the following theorem, which in the case of a 1–motive of the form $[Y \longrightarrow A]$ for some abelian variety A specialises to a Theorem of Ribet ([Rib76], see also the appendix of [Hin88]).

Theorem 6.2. *The image of the map $\vartheta_\ell : \Gamma_{\mathrm{gr}_*^W M} \longrightarrow \mathrm{V}_\ell U(M)$ contained and open in $\mathrm{V}_\ell P(M)$. In other words, the map α_ℓ induces an isomorphism*

$$\mathrm{V}_\ell P(M) \xrightarrow{\cong} W_{-1}\mathfrak{l}^M$$

of Galois representations.

– **6.3.** We start with the construction of a map κ_ℓ analogous to κ_0 in the previous section. Let $K|k$ be a Galois extension contained in $\overline{k}$ and let G be a semiabelian variety over K. We construct the map

$$\kappa_\ell : G(K) \mathbin{\widehat{\otimes}} \mathbb{Z}_\ell \longrightarrow H^1(K, \mathrm{T}_\ell G)$$

as follows: The multiplication–by–ℓ^i map on $G(\overline{k})$ induces a long exact cohomology sequence, from where we can cut out the injection $G(K)/\ell^i G(K) \longrightarrow H^1(K, G[\ell^i])$. Taking limits over i and taking into account that $H^1(K, -)$ commutes with limits of finite $\mathrm{Gal}(\overline{k}|K)$–modules, we get the map κ_ℓ.

Proposition 6.4. *The map κ_ℓ constructed in 6.3 is injective and natural in G and K.*

Proof. Injectivity of κ_ℓ follows from injectivity of $G(K)/\ell^i G(K) \longrightarrow H^1(K, G[\ell^i])$ and left exactness of limits. Naturality in G and K is obvious from the construction. □

Proposition 6.5. *Let $M = [u : \mathbb{Z} \longrightarrow G]$ be a 1–motive over k given by $u(1) = p \in G(k)$. The class $\kappa_\ell(p \mathbin{\widehat{\otimes}} 1) \in H^1(\Gamma, \mathrm{V}_\ell G)$ restricts to zero in $H^1(\Gamma_M, \mathrm{V}_\ell G)$.*

Proof. We must show that Γ_M leaves all ℓ^i–division points of p fixed. The action of Γ_M on the finite quotients

$$\frac{\{(n,x)\in\mathbb{Z}\times G(\overline{k})\mid \ell^i x = np\}}{\{(\ell^i n, np)\mid n\in\mathbb{Z}\}}$$

of $\mathrm{T}_\ell M$ is trivial by definition of Γ_M. Hence if $x\in G(\overline{k})$ is an ℓ^i–division point of p, that is $\ell^i x = p$, then there exists for every $\gamma\in\Gamma_M$ an integer $n\in\mathbb{Z}$ with

$$\gamma(1,x)-(1,x) = (0,\gamma x - x) = (\ell^i n, np)$$

The only possibility is $n=0$, hence $\gamma x = x$ as desired. □

Proposition 6.6. *Let $M=[u:Y\longrightarrow G]$ be a 1–motive over k and consider the 1–motives*

$$M_U := [\mathbb{Z}\xrightarrow{1\mapsto\overline{u}} U(M)] \qquad \textit{and} \qquad M_P := [n\mathbb{Z}\xrightarrow{n\mapsto n\overline{u}} P(M)]$$

where $n\geq 1$ is an integer such that the point $n\overline{u}$ of $U(M)$ belongs to $P(M)$ and $\overline{u}$ is as in Definition 3.3. The inclusions $\Gamma_M\subseteq\Gamma_{M_U}\subseteq\Gamma_{M_P}$ hold in Γ, and Γ_M has finite index in Γ_{M_U}.

Proof. The Galois representation $\mathrm{V}_\ell M_P$ is a subrepresentation of $\mathrm{V}_\ell M_U$, so $\Gamma_{M_U}\subseteq\Gamma_{M_P}$ holds trivially. As for the other inclusion, after replacing k by a finite extension over which Y is constant, even the equality $\Gamma_M=\Gamma_{M_U}$ holds. The proof consists of recognising $\mathrm{V}_\ell M$ and $\mathrm{V}_\ell M_U$ as subquotients of products of each other, as in the proof of 5.6. □

Corollary 6.7. *Let $M=[u:Y\longrightarrow G]$ be a 1–motive over k and set $\widetilde{G}:=\mathrm{gr}^W_* G = T\oplus A$. Let $n\geq 1$ be an integer such that the point $n\overline{u}$ of $U(M)$ belongs to $P(M)$. For every $\psi\in\mathrm{Hom}_k(P(M),\widetilde{G})$, the cohomology class $\kappa_\ell(\psi(n\overline{u})\mathbin{\widehat{\otimes}}1)\in H^1(\Gamma,\mathrm{V}_\ell\widetilde{G})$ restricts to zero in $H^1(\Gamma_M,\mathrm{V}_\ell\widetilde{G})$.*

Proof. This is a consequence of 6.5 and 6.6, the same way 5.7 was a consequence of 5.5 and 5.6. □

– 6.8. We now come to the relation between κ_ℓ and ϑ_ℓ. Let $M=[u:Y\longrightarrow G]$ be a 1–motive over k and write $M_A := M/W_{-2}M = [Y\longrightarrow A]$ and $U:=U(M)$. Let

$$\pi: U\longrightarrow U_A := \mathcal{H}om(Y,A)\oplus\mathcal{H}om(T^\vee,A^\vee)$$

be the projection onto the abelian quotient U_A of U and let ι be the inclusion of $\Gamma_{G\oplus M_A}$ into $\Gamma_{\mathrm{gr}^W_* M}$. We consider the two composition maps

$$U(k)\mathbin{\widehat{\otimes}}\mathbb{Q}_\ell\xrightarrow{\kappa_\ell} H^1(\Gamma_{\mathrm{gr}^W_* M},\mathrm{V}_\ell U)\xrightarrow{(\mathrm{V}_\ell\pi)_*} H^1(\Gamma_{\mathrm{gr}^W_* M},\mathrm{V}_\ell U_A)\cong\mathrm{Hom}(\Gamma_{\mathrm{gr}^W_* M},\mathrm{V}_\ell U_A)$$

and

$$U(k)\mathbin{\widehat{\otimes}}\mathbb{Q}_\ell\xrightarrow{\kappa_\ell} H^1(\Gamma_{\mathrm{gr}^W_* M},\mathrm{V}_\ell U)\xrightarrow{\iota^*} H^1(\Gamma_{G\oplus M_A},\mathrm{V}_\ell U)\cong\mathrm{Hom}(\Gamma_{G\oplus M_A},\mathrm{V}_\ell U)$$

These send $\overline{u}\mathbin{\widehat{\otimes}}1$ to the homomorphisms $(\mathrm{V}_\ell\pi)\circ\kappa_\ell(\overline{u}\mathbin{\widehat{\otimes}}1)$ and $\kappa_\ell(\overline{u}\mathbin{\widehat{\otimes}}1)\circ\iota$ respectively. Here we have used that $\Gamma_{\mathrm{gr}^W_* M}$ acts trivially on $\mathrm{V}_\ell U_A$ and that $\Gamma_{G\oplus M_A}$ acts trivially on $\mathrm{V}_\ell U$. The following lemma is analogous to Lemma 5.9:

Lemma 6.9. *Notations being as in 6.8, the equalities*

$$(\mathrm{V}_\ell\pi)\circ\vartheta_\ell=(\mathrm{V}_\ell\pi)\circ\kappa_\ell(\overline{u}\,\widehat{\otimes}\,1)\qquad\textit{and}\qquad\vartheta_\ell\circ\iota=\kappa_\ell(\overline{u}\,\widehat{\otimes}\,1)\circ\iota$$

hold in $\mathrm{Hom}(\Gamma_{\mathrm{gr}_*^W M},\mathrm{V}_\ell U_A)$ *and in* $\mathrm{Hom}(\Gamma_{G\oplus M_A},\mathrm{V}_\ell U)$ *respectively.*

Proof. Let us choose an ℓ–division sequence of $\overline{u}$, which we may represent by two sequences of points

$$(u_i)_{i=0}^\infty\quad\text{in }\mathrm{Hom}_{\overline{k}}(Y,G)\qquad\text{and}\qquad(\eta_i)_{i=0}^\infty\quad\text{in }\mathrm{Ext}^1_{\overline{k}}(M_A,T)$$

with $u_0=u$ and $\ell u_i=u_{i-1}$, and with $\eta_0=\eta$ and $\ell\eta_i=\eta_{i-1}$. Here $\eta\in\mathrm{Ext}^1_{\overline{k}}(M_A,T)$ is as usual the class given by M. The η_i's define extensions G_i of A by T together with maps $m_i:G\longrightarrow G_i$. With the help of these division sequences we construct a section $s:Y\otimes\mathbb{Q}_\ell\longrightarrow\mathrm{V}_\ell M$ and a retraction $r:\mathrm{V}_\ell M\longrightarrow\mathrm{V}_\ell T$ as follows:

$$s((y_i)_{i=0}^\infty)=(y_i,u_i(y_i))_{i=0}^\infty\qquad\text{and}\qquad r(y_i,x_i)_{i=0}^\infty=m_i(x_i)_{i=0}^\infty$$

Using the section s and the retraction r we can write down the map ϑ_ℓ as in 6.1, equations $(*)$ and $(**)$. The remainder of the proof of 6.9 is then litterally the same as the proof of 5.9. □

Lemma 6.10. *Let* $0\longrightarrow T\longrightarrow G\xrightarrow{\pi}A\longrightarrow 0$ *be a semiabelian variety over* k*, let* G' *be a semiabelian subvariety of* G *and let* V *be a Galois invariant linear subspace of* $\mathrm{V}_\ell G$*. If the inclusions* $\pi(V)\subseteq\mathrm{V}_\ell\pi(G')$ *and* $V\cap\mathrm{V}_\ell T\subseteq\mathrm{V}_\ell G'$ *hold, then* V *is contained in* $\mathrm{V}_\ell G'$*.*

Proof. The proof is analogous to the proof of 5.10. That there are no nonzero Galois equivariant maps $\pi(V)\longrightarrow\mathrm{V}_\ell(T/(T\cap G'))$ can be seen for example by looking at absolute values of eigenvalues of Frobenius elements. □

Proposition 6.11. *The image of the map* $\vartheta_\ell:\Gamma_{\mathrm{gr}_*^W M}\longrightarrow\mathrm{V}_\ell U$ *is contained in* $\mathrm{V}_\ell P$*.*

Proof. Follows from 6.9, 6.10 and naturality of the map κ_ℓ, the same way 5.11 follows from 5.9, 5.10 and naturality of the map κ_0. □

Lemma 6.12. *Let* $M=[Y\longrightarrow G]$ *be a 1–motive over* k *and set* $\widetilde{G}:=\mathrm{gr}_*^W G=T\oplus A$*. The map* $H^1(\mathfrak{l}^M,\mathrm{V}_\ell\widetilde{G})\longrightarrow\mathrm{Hom}_{\mathrm{gr}_0^W\mathfrak{l}^M}(W_{-1}\mathfrak{l}^M,\mathrm{V}_\ell\widetilde{G})$ *that sends the class of a cocycle* c *to the restriction of* c *to* $W_{-1}\mathfrak{l}^M$ *is an isomorphism.*

Proof. The Lie subalgebra $W_{-1}\mathfrak{l}^M$ of $\mathfrak{l}^M$ is the is the largest subalgebra acting trivially on $\mathrm{V}_\ell\widetilde{G}$ by construction of the weight filtration, so we have $\mathrm{gr}_0^W\mathfrak{l}^M=\mathfrak{l}^{\widetilde{G}}$. In low degrees, the Hochschild–Serre spectral sequence associated with the Lie algebra extension $0\longrightarrow W_{-1}\mathfrak{l}^M\longrightarrow\mathfrak{l}^M\longrightarrow\mathfrak{l}^{\widetilde{G}}\longrightarrow 0$ yields an exact sequence

$$H^1(\mathfrak{l}^{\widetilde{G}},\mathrm{V}_\ell\widetilde{G})\longrightarrow H^1(\mathfrak{l}^M,\mathrm{V}_\ell\widetilde{G})\xrightarrow{*}H^1(W_{-1}\mathfrak{l}^M,\mathrm{V}_\ell\widetilde{G})^{\mathfrak{l}^{\widetilde{G}}}\longrightarrow H^2(\mathfrak{l}^{\widetilde{G}},\mathrm{V}_\ell\widetilde{G})$$

We can identify $H^1(W_{-1}L^M,\mathrm{V}_\ell\widetilde{G})$ with $\mathrm{Hom}_{\mathbb{Q}_\ell}(W_{-1}L^M,\mathrm{V}_\ell\widetilde{G})$, and under this identification the map $(*)$ is given by restricting cocycles as in the statement of the lemma. Hence, to finish the proof it suffices to show that the first and last term in the above sequence is trivial. Indeed, it follows from an adaptation of Serre's vanishing criterion given in [Ser71] that $H^i(\mathfrak{l}^{\widetilde{G}},\mathrm{V}_\ell\widetilde{G})$ is zero for all $i\geq 0$. □

Lemma 6.13. *The map* $\alpha_\ell^* : \mathrm{Hom}_\Gamma(\mathrm{V}_\ell P, \mathrm{V}_\ell \widetilde{G}) \longrightarrow \mathrm{Hom}_\Gamma(W_{-1}\mathfrak{l}^M, \mathrm{V}_\ell \widetilde{G})$ *given by* $\alpha_\ell^*(f) = f \circ \alpha^{-1}$ *is injective.*

Proof. We will again construct another injective map $\beta_\ell : \mathrm{Hom}_\Gamma(\mathrm{V}_\ell P, \mathrm{V}_\ell \widetilde{G}) \longrightarrow \mathrm{Hom}_\Gamma(W_{-1}\mathfrak{l}^M, \mathrm{V}_\ell \widetilde{G})$ and prove in a second step that the equality $\alpha_\ell^* = \beta_\ell$ holds. For the construction of β_ℓ we use the following commutative diagram.

$$\begin{array}{ccccccc} & & & & \mathrm{Hom}_k(P, \widetilde{G}) \otimes \mathbb{Q}_\ell & & \\ & & & \swarrow & \downarrow (*) & & \\ 0 & \longrightarrow & H^1(L^M, \mathrm{V}_\ell \widetilde{G}) & \longrightarrow & H^1(k, \mathrm{V}_\ell \widetilde{G}) & \longrightarrow & H^1(k_M, \mathrm{V}_\ell \widetilde{G}) \\ & & \cong \downarrow & & & & \\ & & \mathrm{Hom}_\Gamma(W_{-1}\mathfrak{l}^M, \mathrm{V}_\ell \widetilde{G}) & & & & \end{array}$$

The map $(*)$ sends a homomorphism ψ to the element $\kappa_\ell(\psi(nu) \mathbin{\widehat{\otimes}} 1)$. This map is injective by minimality of $P := P(M)$ and injectivity of the Kummer map κ_ℓ, and its composite with the restriction is zero by Corollary 6.7, hence the dashed arrow. The lower vertical isomorphism is obtained by taking Γ–fixed points of the isomorphism of Lemma 6.12 and taking into account that $H^1(L^M, \mathrm{V}_\ell \widetilde{G}) = H^1(\mathfrak{l}^M, \mathrm{V}_\ell \widetilde{G})^\Gamma$ by Lazard's theorem comparing Lie group cohomology with Lie algebra cohomology ([Laz65], V.2.4.10). Let β_ℓ be the induced injection

$$\beta_\ell : \mathrm{Hom}_\Gamma(\mathrm{V}_\ell P, \mathrm{V}_\ell \widetilde{G}) \longrightarrow H^1(L^M, \mathrm{V}_\ell \widetilde{G})$$

The equality $\beta_\ell = \alpha_\ell^*$ as in the proof of 5.13, from Lemma 6.9, so α_ℓ^* is injective. □

Lemma 6.14. *Let* G *be a semiabelian variety and let* $\widetilde{G}$ *be a split semiabelian variety over* k *containing (over* $\overline{k}$*) all isogeny types of* G *(Definition 3.6). Let* V *be a Galois invariant* $\mathbb{Q}_\ell$*–linear subspace of* $\mathrm{V}_\ell G$*. If the restriction map*

$$\mathrm{Hom}_{\Gamma'}(\mathrm{V}_\ell G, \mathrm{V}_\ell \widetilde{G}) \longrightarrow \mathrm{Hom}_{\Gamma'}(V, \mathrm{V}_\ell \widetilde{G})$$

is injective for all open subgroups Γ' *of* Γ*, then* V *is equal to* $\mathrm{V}_\ell G$*.*

Proof. Write G as an extension of an abelian variety A by a torus T, and $\widetilde{G}$ as a sum of an abelian variety $\widetilde{A}$ and a torus $\widetilde{T}$. Without loss of generality we can replace k by a finite Galois extension of k, so that there exists an integer n and morphisms $A \longrightarrow \widetilde{A}^n$ and $T \longrightarrow \widetilde{T}^n$ with finite kernels defined over k. Then, consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & V \cap \mathrm{V}_\ell T & \to & V & \longrightarrow & \pi V & \longrightarrow & 0 \\ & & \subseteq \downarrow & & \subseteq \downarrow & & \subseteq \downarrow & & \\ 0 & \longrightarrow & \mathrm{V}_\ell T & \longrightarrow & \mathrm{V}_\ell G & \xrightarrow{\pi} & \mathrm{V}_\ell A & \longrightarrow & 0 \end{array}$$

We apply $\mathrm{Hom}_\Gamma(-, \mathrm{V}_\ell \widetilde{A})$ to this diagram and get

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_\Gamma(\pi V, \mathrm{V}_\ell \widetilde{A}) & \xrightarrow{\cong} & \mathrm{Hom}_\Gamma(V, \mathrm{V}_\ell \widetilde{A}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_\Gamma(\mathrm{V}_\ell A, \mathrm{V}_\ell \widetilde{A}) & \xrightarrow{\cong} & \mathrm{Hom}_\Gamma(\mathrm{V}_\ell G, \mathrm{V}_\ell \widetilde{A}) & \longrightarrow & 0 \end{array}$$

using that $\mathrm{Hom}(\mathrm{V}_\ell T_1, \mathrm{V}_\ell \widetilde{A}) = 0$ and $\mathrm{Hom}(V \cap \mathrm{V}_\ell T_1, \mathrm{V}_\ell \widetilde{A}) = 0$ for weight reasons. The vertical maps are injective by hypothesis. The Galois representations $\mathrm{V}_\ell A$ and $\mathrm{V}_\ell \widetilde{A}$ are semisimple as Galois modules (by Faltings in characteristic zero, and by Tate, Zahrin and Mori in positive characteristic), and all simple factors appearing in $\mathrm{V}_\ell A$ also appear in $\mathrm{V}_\ell \widetilde{A}$ by hypothesis. Hence injectivity of the left hand vertical map implies the equality $\pi V = \mathrm{V}_\ell A$. Next, we apply $\mathrm{Hom}_\Gamma(-, \mathrm{V}_\ell \widetilde{T})$ instead, and find

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_\Gamma(V, \mathrm{V}_\ell \widetilde{T}) & \longrightarrow & \mathrm{Hom}_\Gamma(V \cap \mathrm{V}_\ell T, \mathrm{V}_\ell \widetilde{T}) & \longrightarrow & \mathrm{Ext}^1_\Gamma(\pi V, \mathrm{V}_\ell \widetilde{T}) \\ & & \uparrow & & \uparrow & & \| \\ 0 & \longrightarrow & \mathrm{Hom}_\Gamma(\mathrm{V}_\ell G, \mathrm{V}_\ell \widetilde{T}) & \longrightarrow & \mathrm{Hom}_\Gamma(\mathrm{V}_\ell T, \mathrm{V}_\ell \widetilde{T}) & \longrightarrow & \mathrm{Ext}^1_\Gamma(\mathrm{V}_\ell A, \mathrm{V}_\ell \widetilde{T}) \end{array}$$

using that $\mathrm{Hom}(\mathrm{V}_\ell A_1, \mathrm{V}_\ell \widetilde{T}) = 0$ for weight reasons. The right hand side vertical map is injective by hypothesis, and we have already shown that the left hand vertical map is the identity. The middle vertical map is therefore injective. The Galois representation $\mathrm{V}_\ell T$ is semisimple because $\mathrm{V}_\ell T \otimes \mathbb{Q}_\ell(-1)$ is so by Maschke's Theorem, hence the middle vertical map can only be injective if the equality $V \cap \mathrm{V}_\ell T = \mathrm{V}_\ell T$ holds, and so we are done. □

Proof of Theorem 6.2. The image $W_{-1}\mathfrak{l}^M$ of the map ϑ_ℓ is contained in $\mathrm{V}_\ell P$ by Proposition 6.11. By Lemma 6.13, the restriction map $\mathrm{Hom}_\Gamma(\mathrm{V}_\ell P, \mathrm{V}_\ell \widetilde{G}) \longrightarrow \mathrm{Hom}_\Gamma(\mathrm{im}\,\vartheta_\ell, \mathrm{V}_\ell \widetilde{G})$ is injective, and we know by Lemma 3.7 that G contains all isogeny types of P. Hence the equality $\mathrm{im}\,\vartheta_\ell = \mathrm{V}_\ell P$ must hold by Lemma 6.14. □

7. Conclusions and compatibility of the comparison isomorphisms

In this section we prove Theorems 1 and 2 as stated in the introduction, and also show that the comparison isomorphisms between realisations of M and of $P(M)$ are compatible. We start with the proof of Theorem 2, where we only have to assemble results:

Proof of Theorem 2. For every 1–motive M over a noetherian, regular scheme S we have constructed a semiabelian variety $P(M)$ in 3.3. The construction is compatible with flat base change by Proposition 3.4, hence satisfies statement (1) of the theorem. Statement (2) is the content of Theorem 5.2, and statement (3) the content of Theorem 6.2. □

– **7.1.** We now come to the compatibility problem: Let M be a 1–motive over a number field k, choose a complex embedding $k \longrightarrow \mathbb{C}$ and a prime number ℓ, and denote by $\overline{k}$ the algebraic closure of k in $\mathbb{C}$. The comparison isomorphism $\mathrm{V}_0 M \otimes \mathbb{Q}_\ell \longrightarrow \mathrm{V}_\ell M$ induces an isomorphism

$$\mathrm{End}_\mathbb{Q}(\mathrm{V}_0 M) \otimes \mathbb{Q}_\ell \xrightarrow{\cong} \mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M)$$

and we have identified the Lie algebra $\mathfrak{l}^M$ with a subalgebra of $\mathfrak{h}^M \otimes \mathbb{Q}_\ell$ via this isomorphism. In sections 5 and 6 we constructed canonical maps

$$W_{-1}\mathfrak{h}^M \longrightarrow \mathrm{V}_0 P(M) \qquad \text{and} \qquad W_{-1}\mathfrak{l}^M \longrightarrow \mathrm{V}_\ell P(M)$$

and proved that they are isomorphisms. The object $P(M)$ is a semiabelian variety, and we also have the comparison isomorphism $\mathrm{V}_0P(M)\otimes\mathbb{Q}_\ell \longrightarrow \mathrm{V}_\ell P(M)$. Putting things together, we get the following diagram

$$(*)\qquad \begin{tikzcd}[column sep=small] & \mathrm{End}_{\mathbb{Q}}(\mathrm{V}_0M)\otimes\mathbb{Q}_\ell \arrow[r, "\cong"] & \mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M) & \\ W_{-1}\mathfrak{h}^M\otimes\mathbb{Q}_\ell \arrow[ur, "\subseteq"] \arrow[dr, "\text{Theorem 5.2}", "\cong"'] & & & W_{-1}\mathfrak{l}^M \arrow[ul, "\subseteq"'] \arrow[dl, "\text{Theorem 6.2}"', "\cong"] \\ & \mathrm{V}_0P(M)\otimes\mathbb{Q}_\ell \arrow[r, "\cong"] & \mathrm{V}_\ell P(M) & \end{tikzcd}$$

We next check:

Proposition 7.2. *The diagram* $(*)$ *commutes.*

Proof. We consider the group scheme $\Gamma^{\mathrm{Hdg}}_{\mathrm{gr}^*M}$ over $\operatorname{spec}\mathbb{Q}$ and the profinite group $\Gamma^{\mathrm{Gal}}_{\mathrm{gr}^*M}$, given by

$$\Gamma^{\mathrm{Hdg}}_{\mathrm{gr}^*M} = \{\gamma\in\Gamma^{\mathrm{Hdg}} \mid \gamma|_{\mathrm{V}_0(\mathrm{gr}^*M)} = \mathrm{id}\} \qquad\text{and}\qquad \Gamma^{\mathrm{Gal}}_{\mathrm{gr}^*M} = \{\gamma\in\Gamma^{\mathrm{Gal}} \mid \gamma|_{\mathrm{V}_\ell(\mathrm{gr}^*M)} = \mathrm{id}\}$$

respectively, where Γ^{Hdg} denotes the absolute Hodge group, and Γ^{Gal} the absolute Galois group of k. In the following diagram, the triangles on the left and on the right commute by definition of ϑ_0 and ϑ_ℓ:

$$\begin{tikzcd}[column sep=small] & \mathrm{End}_{\mathbb{Q}}(\mathrm{V}_0M)\otimes\mathbb{Q}_\ell \arrow[r, "\cong"] & \mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M) & \\ \Gamma^{\mathrm{Hdg}}_{\mathrm{gr}^*M} \arrow[r, "\log\circ\rho_0"] \arrow[dr, "\vartheta_0"'] & W_{-1}\,\mathrm{End}_{\mathbb{Q}}(\mathrm{V}_0M)\otimes\mathbb{Q}_\ell \arrow[u, "\subseteq"] \arrow[r, "\cong"] & W_{-1}\,\mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M) \arrow[u, "\subseteq"'] & \Gamma^{\mathrm{Gal}}_{\mathrm{gr}^*M} \arrow[l, "\log\circ\rho_\ell"'] \arrow[dl, "\vartheta_\ell"] \\ & \mathrm{V}_0U(M)\otimes\mathbb{Q}_\ell \arrow[u, "\alpha_0\otimes 1", "\cong"'] \arrow[r, "\cong"] & \mathrm{V}_\ell U(M) \arrow[u, "\alpha_\ell", "\cong"'] & \\ & \mathrm{V}_0P(M)\otimes\mathbb{Q}_\ell \arrow[u, "\subseteq"] \arrow[r, "\cong"] & \mathrm{V}_\ell P(M) \arrow[u, "\subseteq"'] & \end{tikzcd}$$

By definition, $W_{-1}\mathfrak{h}^M\otimes\mathbb{Q}_\ell$ is the $\mathbb{Q}_\ell$–linear span of $\mathrm{im}(\log\circ\rho_0)$, and $W_{-1}\mathfrak{l}^M$ is the $\mathbb{Q}_\ell$–linear span of $\mathrm{im}(\log\circ\rho_\ell)$. The top and the bottom square in the above diagram commute by naturality of the comparison isomorphisms, and all that remains is to show that the central square commutes. By construction of the semiabelian variety $U(M)$, see 3.2, and naturality of comparison isomorphisms, this follows from the commutativity of the two squares pictured in 1.10. □

Proof of Theorem 1. In the situation of Theorem 1, we know by Theorem 2.1 that $\mathfrak{l}^M$ is contained in $\mathfrak{h}^M\otimes\mathbb{Q}_\ell$ once we identify $\mathrm{End}(\mathrm{V}_\ell M)$ with $\mathrm{End}(\mathrm{V}_0M)\otimes\mathbb{Q}_\ell$ via the comparison isomorphism $\mathrm{V}_\ell M \cong \mathrm{V}_0M\otimes\mathbb{Q}_\ell$. By Proposition 7.2, the inclusion $\mathfrak{l}^M \subseteq \mathfrak{h}^M\otimes\mathbb{Q}_\ell$ is compatible with the comparison isomorphism $\mathrm{V}_\ell P(M) \cong \mathrm{V}_0(M)\otimes\mathbb{Q}_\ell$, hence an equality (for dimension reasons, this would also follow without referring to 7.2). □

8. Corollaries

We propose in this last section to illustrate how our theorems can be used to aggress questions about the geometry and arithmetic of semiabelian varieties. We use our them here to continue some work started by Ribet and Jacquinot in [JR87] about so–called *deficient points* on semiabelian varieties. We stick here to semiabelian varieties over number fields. Motivated by [Ber11], it would be equally interesting to consider semiabelian varieties defied over smooth curves over $\mathbb{C}$.

– **8.1.** Let k be a number field with algebraic closure $\overline{k}$, let $M = [Y \xrightarrow{u} G]$ be a 1–motive over k, and denote by $k_M \subseteq \overline{k}$ the fixed field of the pointwise stabiliser of $\mathrm{T}_\ell M$ in $\mathrm{Gal}(\overline{k}|k)$. Equivalently, k_M is the smallest subfield of $\overline{k}$ over which Y is constant and all ℓ–division points of $u(Y)$ are defined.

Definition 8.2. A point $Q \in G(k)$ is called *deficient* if it is ℓ–divisible in the group $G(k_M)$. We write $D_M(k)$ for the subgroup of $G(k)$ of deficient points.

– **8.3.** Our goal is to describe the group $D_M(k)$, which has been studied in [JR87] in the case of a 1–motive of the form $[0 \longrightarrow G]$ where G is an extension of an abelian variety by $\mathbb{G}_m$. We will show here that it is independent of the prime ℓ, finitely generated of rank $\leq r$ where r only depends on the dimension of $\mathrm{V}_\ell M$, say. To do so, we will give a geometrical construction of $D_M(k)$, roughly in the following way: Recall that the semiabelian variety $P := P(M)$ associated with M in 3.3 is supposed to be a Lie algebra object, acting on G. So we can, in a sense yet to be clarified, consider derivations of P with values in G. The deficient points will, up to multiplying by integers, be the images of $n\overline{u}$ under such derivations, were $n\overline{u}$ is a multiple of the rational $\overline{u}$ point on $U(M)$ defined by M, see 3.2. That derivations play a key role in the construction of deficient points is already visible in [JR87], the construction there being attributed to Breen. We first state an immediate corollary to Theorem 6.2, which shows that the notion of deficiency does not depend on the prime ℓ. This corollary can also be utilised to extend the definition of deficient points to 1–motives over an arbitrary base.

Corollary 8.4 (To Theorem 6.2)**.** *Let $M = [Y \xrightarrow{u} G]$ be a 1–motive over k, let $Q \in G(k)$ be a rational point and define $M^+ = [Y \oplus \mathbb{Z} \xrightarrow{u^+} G]$ by $u^+(y, n) = u(y) + nQ$. The following are equivalent:*

1. *The point $Q \in G(k)$ is deficient for one (or for all) primes ℓ.*
2. *The map $P(M^+) \longrightarrow P(M)$ induced by the canonical morphism $M \longrightarrow M^+$ is an isogeny.*

Proof. The morphisms of 1–motives $M \longrightarrow M^+ \longrightarrow [\mathbb{Z} \longrightarrow 0]$ induce a short exact sequence of ℓ–adic representations $0 \longrightarrow \mathrm{V}_\ell M \longrightarrow \mathrm{V}_\ell M^+ \longrightarrow \mathbb{Q}_\ell \longrightarrow 0$. Define fields k_M and k_{M^+} as in 8.1, so k_{M^+} is a Galois extension of k_M. The Galois group $\mathrm{Gal}(k_{M^+}|k_M)$ identifies canonically with a compact subgroup of $\mathrm{V}_\ell M$, hence is commutative and has the structure of a finitely generated, free $\mathbb{Z}_\ell$–module.
The point $Q \in G(k)$ is deficient if and only the field extension $k_{M^+}|k_M$ is trivial. This in turn is the case if and only if $\mathrm{Gal}(k_{M^+}|k_M)$ is trivial, or, yet in other words, if the Lie algebra morphism

$\mathfrak{l}^{M^+} \longrightarrow \mathfrak{l}^M$ is an isomorphism. The graded quotients of weight 0 of $\mathfrak{l}^{M^+}$ and $\mathfrak{l}^M$ are the same, hence, by Theorem 6.2, the map $\mathfrak{l}^{M^+} \longrightarrow \mathfrak{l}^M$ is an isomorphism precisely if $\mathrm{V}_\ell P(M^+) \longrightarrow \mathrm{V}_\ell P(M)$ is an isomorphism. □

– **8.5.** We now explain what we mean with derivations of P with values in G. Let P be a semiabelian variety, extension of an abelian variety A_P by a torus T_P, equipped with a Lie structure λ, and let G be a semiabelian variety, extension of A by T, equipped with an action α of P. The Lie structure and the action are given by maps of Galois modules

$$\lambda : T_P^\vee \longrightarrow \mathrm{Hom}_{\overline{k}}(A_P, A_P^\vee) \qquad \text{and} \qquad \alpha : T^\vee \longrightarrow \mathrm{Hom}_{\overline{k}}(A_P, A^\vee)$$

as we have explained in 4.1. Along the same lines as in 4.1, we figure that a *derivation*[4] of P into G corresponds to a pair of morphisms of $\overline{k}$–group schemes $\partial = (\partial_T, \partial_A)$ from T_P to T and from A_P to A respectively, such that the Leibniz rule

$$(\partial_A^\vee \circ \alpha(\chi)) + (\alpha(\chi)^\vee \circ \partial_A) = \lambda(\partial_T^\vee(\chi))$$

holds for all $\chi \in T^\vee$. This is an equality in $\mathrm{Hom}_{\overline{k}}(A_P, A_P^\vee)$. We denote by $\mathrm{Der}_{\overline{k}}(P, G)$ the group of all such derivations from P to G. It is a subgroup of $\mathrm{Hom}_{\overline{k}}(T_P, T) \times \mathrm{Hom}_{\overline{k}}(A_P, A)$, hence it is finitely generated and free, and comes equipped with an action of $\mathrm{Gal}(\overline{k}|k)$.

– **8.6.** Back to our concrete situation, let $M = [u : Y \longrightarrow G]$ be a 1–motive, where G is an extension of A by T. Let $P = P(M)$ be the semiabelian variety defined in 3.3, and write T_P for the torus part and A_P for the abelian quotient of P. Recall that A_P is an abelian subvariety of $\mathcal{H}om(Y, A) \times \mathcal{H}om(T^\vee, A^\vee)$ and T_P a subtorus of $\mathcal{H}om(Y, T)$. We identify $A_P^\vee$ with a quotient of $(Y \otimes A^\vee) \times (T^\vee \otimes A)$ and $T_P^\vee$ with a quotient of $Y \otimes T^\vee$. The dual of P is the morphism

$$w : T_P^\vee \longrightarrow A_P^\vee \qquad w(y \otimes \chi) = w(y \otimes \chi) = (y \otimes v^\vee(\chi), \chi \otimes v(y))$$

where $v : Y \longrightarrow A$ is the composite of u with the projection $G \longrightarrow A$, and $v^\vee : T^\vee \longrightarrow A^\vee$ is the corresponding map in the 1–motive dual to M. The maps λ, defining the Lie algebra structure of $P = P(M)$, and α, defining the action of P on G are given by

$$\lambda : Y \otimes T^\vee \longrightarrow \mathrm{Hom}(A_P, A_P^\vee) \qquad \lambda(y \otimes \chi)(f, g) = (y \otimes g(\chi), \chi \otimes f(y))$$

and

$$\alpha : T^\vee \longrightarrow \mathrm{Hom}(A_P, A^\vee) \qquad \alpha(\chi)(f, g) = g(\chi)$$

respectively. Observe that in the special case where Y is trivial and G a non–isotrivial extension of a simple abelian variety A by $\mathbb{G}_m$, we have $P(M) = \mathcal{H}om(\mathbb{Z}, A^\vee) = A^\vee$, and $\mathrm{Der}_k(P(M), A)$ consists of homomorphisms $\partial_A \in \mathrm{Hom}(A^\vee, A)$ with the property $\partial_A + \partial_A^\vee = 0$. So the group of derivations is isomorphic to the quotient of $\mathrm{Hom}(A^\vee, A)$ modulo the Néron–Severi group of A.

Lemma 8.7. *Let y^+ be an element of Y. The pair of homomorphisms (∂_T, ∂_A) given by*

$$\partial_A : A_P \xrightarrow{\subseteq} A_U \xrightarrow{(f,g)\longmapsto f(y^+)} A \qquad \text{and} \qquad \partial_T : T_P \xrightarrow{\subseteq} T_U \xrightarrow{h\longmapsto h(y^+)} T$$

is a derivation of P to G.

[4] actually: a $\pi_1^{\mathrm{mot}}(M)$–equivariant derivation

Proof. The duals of ∂_A and ∂_P are given by $\partial_A^\vee : a \longmapsto (y^+ \otimes a, 0)$ and $\partial_T^\vee : \chi \longmapsto y^+ \otimes \chi$. For $(f,g) \in A_P$, we compute:

$$\begin{aligned}\partial_A^\vee(\alpha(\chi)(f,g)) + \alpha(\chi)^\vee(\partial_A(f,g)) = \partial_A^\vee(g(\chi)) + \alpha(\chi)^\vee(f(y^+)) = \\ = (y^+ \otimes g(\chi), 0) + (0, \chi \otimes f(y^+)) = (y^+ \otimes g(\chi), \chi \otimes f(y^+)) = \\ = \lambda(y^+ \otimes \chi)(f,g) = \lambda(\partial_T^\vee(\chi))(f,g)\end{aligned}$$

so the equality $(\partial_A^\vee \circ \alpha(\chi)) + (\alpha(\chi)^\vee \circ \partial_A) = \lambda(\partial_T^\vee(\chi))$ holds, as demanded. □

Lemma 8.8. *Let $(\partial_T, \partial_A) \in \mathrm{Der}_k(P(M), G)$ be a derivation. If $\partial_A = 0$, then $w \circ \partial_T^\vee = 0$. Reciprocally, if $\partial_T : T_P \longrightarrow T$ is any morphism such that $w \circ \partial_T^\vee = 0$, then $(\partial_T, 0)$ is a derivation.*

Proof. By definition of w and λ, the equality

$$w(\partial_T^\vee(\chi)) = \lambda(\partial_T(\chi))(v, v^\vee)$$

holds for all $\chi \in T^\vee$. Hence, if $(\partial_T, 0)$ is a derivation, then we have $\lambda(\partial_T(\chi)) = 0$ for all $\chi \in T^\vee$ by the Leibniz rule, therefore $w \circ \partial_T^\vee = 0$. On the other hand, if $w(\partial_T^\vee(\chi)) = 0$ then we have $\lambda(\partial_T(\chi))(v, v^\vee)$, and since $(v, v^\vee)$ generates P_A, we get $\lambda(\partial_T^\vee(\chi)) = 0$. □

– 8.9. We will now construct a linear map

$$\Phi : D_M(k) \otimes \mathbb{Q} \longrightarrow \mathrm{Der}_k(P(M), G) \otimes \mathbb{Q}$$

which will eventually turn out to be an isomorphism, as follows: Given a deficient point $Q \in D_M(k)$, define a 1–motive $M^+ = [Y \oplus \mathbb{Z} \xrightarrow{u} G]$ by $u^+(y,n) = u(y) + nQ$. Let

$$r : P(M) \longrightarrow P(M^+)$$

be a morphism whose composition with the morphism $P(M^+) \longrightarrow P(M)$ induced by $M \longrightarrow M^+$ is multiplication by some nonzero integer n. Such a morphism exists by Corollary 8.4. We get morphisms

$$\begin{aligned}\partial_T &: T_{P(M)} \xrightarrow{r_T} T_{P(M^+)} \xrightarrow{\subseteq} \mathcal{H}om(Y \oplus \mathbb{Z}, T) \xrightarrow{f \longmapsto f(0,1)} T \\ \partial_A &: A_{P(M)} \xrightarrow{r_A} A_{P(M^+)} \xrightarrow{\subseteq} \mathcal{H}om(Y \oplus \mathbb{Z}, A) \oplus \mathcal{H}om(T^\vee, A^\vee) \xrightarrow{(f,g) \longmapsto f(0,1)} A\end{aligned}$$

and set $\Phi(Q \otimes 1) = (\partial_T, \partial_A) \otimes n^{-1}$. By Lemma 8.7 the pair (∂_T, ∂_A) is indeed a derivation, and $(\partial_T, \partial_A) \otimes n^{-1}$ does not depend on the choice of the isogeny r, hence Φ is a well–defined $\mathbb{Q}$–linear map.

Theorem 8.10. *The map Φ constructed in 8.9 is an isomorphism.*

Proof. Let us write T_0 for the torus dual to $\ker w$, where $w : T_P^\vee \longrightarrow A_P^\vee$ is the map defined by P. So T_0 is the largest torus quotient of P via the canonical projection $\pi_0 : P \longrightarrow T_0$. The torus T_0 comes equipped with a special point $u_0 := \pi(\overline{u})$. Every subtorus of T_0 that contains u_0 is already equal to T_0.
We now check injectivity and surjectivity of the map Φ, starting with injectivity. Let $q \in D_M(k)$ be a deficient point such that $\Phi(q \otimes 1) = 0$. Replacing M by M^+, we can suppose without loss

of generality that $q = u(y^+)$ for some $y^+ \in Y$. Because $\mathrm{Der}_k(P(M), G)$ is a finitely generated free group, the relation $\Phi(q \otimes 1) = 0$ means that the maps

$$\partial_A : A_P \xrightarrow{\subseteq} A_U \xrightarrow{(f,g) \longmapsto f(y^+)} A \qquad \text{and} \qquad \partial_T : T_P \xrightarrow{\subseteq} T_U \xrightarrow{h \longmapsto h(y^+)} T$$

are both zero. We have $\pi_A(q) = v(y^+) = \partial_A(v, v^\vee) = 0$, so q must be an element of $T(k)$. But then we have $q = u_0(y^+) = \partial_T(u_0) = 0$, using Lemma 8.8. This shows injectivity of Φ.
To show that Φ is surjective, let $\partial = (\partial_T, \partial_A)$ be a derivation, and let us construct a deficient point q with $\Phi(q \otimes 1) = \partial \otimes 1$. Define $a := \partial_A(v, v^\vee) \in A(k)$ and let $\widetilde{q} \in G$ be any point with $\pi(\widetilde{q}) = a$. Define $M^+ = [Y \oplus \mathbb{Z} \xrightarrow{u^+} G]$ by $u^+(y, n) = u(y) + n\widetilde{q}$ and let

$$\rho : P(M^+) \longrightarrow P(M)$$

be the induced morphism of semiabelian varieties. The kernel of ρ is contained in $T_{P(M^+)}$. We get two derivations $\widetilde{\partial} = (\widetilde{\partial}_T, \widetilde{\partial}_A)$ and $\partial \circ \rho = (\partial_T \circ \rho_T, \partial_A \circ \rho_A)$ on $P(M^+)$ with values in G. Their difference is the derivation

$$\widetilde{\partial} - \partial \circ \rho = (\widetilde{\partial}_T - \partial_T \circ \rho_T, 0)$$

so $\delta := \widetilde{\partial}_T - \partial_T \circ \rho_T$ is a morphism from the maximal torus quotient T_0^+ of $P(M^+)$ to T. The point $q := \widetilde{q} - \delta(\pi(\overline{u}^+))$ has the required property. □

Corollary 8.11. *Let $M = [Y \xrightarrow{u} G]$ be a 1–motive over k. The group of deficient points $D_M(k)$ is finitely generated, and its rank is the same as the rank of* $\mathrm{Der}_k(P(M), G)$.

Proof. We have an isomorphism $\Phi : D_M(k) \otimes \mathbb{Q} \longrightarrow \mathrm{Der}_k(P(M), G) \otimes \mathbb{Q}$, so all we have to show is that the group $D_M(k)$ is finitely generated. One can show (by dévissage) that the group $G(k)$ is abstractly isomorphic to a direct sum of a finite group and a free group. So any subgroup of $G(k)$, in particular $D_M(k)$, also is isomorphic to a direct sum of a finite group and a free group. □

Appendix: The construction of $P(M)$ from the Tannakian point of view

I reproduce here in almost unaltered form a comment of P. Deligne, explaining our construction of the Lie algebra of the unipotent motivic fundamental group $P(M)$ of a 1–motive M in terms of Tannakian formalism. I alone am to blame for mistakes.

The starting point is that in definition 3.3 the point $\overline{u}$ should be seen as an extension of $W_{-1}\mathcal{E}nd(M)$ by the unit object $\mathbb{Z}$. This extension can be constructed in a general setting as follows:

– **A.1.** Let $\mathcal{T}$ be a tannakian category in characteristic zero with unit object $\mathbb{1}$. Suppose each object of $\mathcal{T}$ has a functorial exhaustive filtration W compatible with tensor products and duals, the functor gr^W_* being exact. For any object M, the object $\mathcal{E}nd(M) = M^\vee \otimes M$ contains the subobject $W_{-1}\mathcal{E}nd(M)$. The filtration of $\mathcal{E}nd(M)$ is deduced from the filtration of M, hence

$$W_{-1}\mathcal{E}nd(M) = \mathrm{im}\Big(\bigoplus_p \mathcal{H}om(M/W_pM, W_pM) \longrightarrow \mathcal{E}nd(M)\Big)$$

As $\mathrm{Ext}^1(M/W_pM, W_pM) = \mathrm{Ext}^1(\mathbb{1}, \mathcal{H}om(M/W_pM, W_pM))$, we can take the class of M in each of the vector spaces $\mathrm{Ext}^1(M/W_pM, W_pM)$, interpret it as a class in $\mathrm{Ext}^1(\mathbb{1}, \mathcal{H}om(M/W_pM, W_pM))$, take the sum of those and push to $W_{-1}\mathcal{E}nd(M)$ by functoriality of Ext^1. We get a class

$$\mathrm{cl}(M) \in \mathrm{Ext}^1(\mathbb{1}, W_{-1}\mathcal{E}nd(M))$$

which is natural in M under taking subobjects and quotients.

– **A.2.** Let G be the fundamental group of $\mathcal{T}$. It has an invariant unipotent subgroup $W_{-1}(G)$ with Lie algebra $W_{-1}(\mathrm{Lie}\, G)$. Thus, for all objects M, the group G acts on M respecting the filtration W, and the unipotent subgroup $W_{-1}(M)$ acts trivially on $\mathrm{gr}^W_* M$. Let $w : \mathbb{G}_m \longrightarrow G/W_{-1}(G)$ be the cocharacter defined by the numbering of the filtration of W of M, for any M. This means that for $a \in \mathbb{G}_m$, the automorphism $w(a)$ of $\mathrm{gr}^W_*(M)$ acts as multiplication with a^n on $\mathrm{gr}^W_n(M)$. Let us consider the inverse image of this $\mathbb{G}_m$ in G, or more precisely $\mathbb{G}_m \times_{G/W_{-1}G} G$. Its Lie algebra $\mathfrak{L}$ is an extension of $\mathrm{Lie}(\mathbb{G}_m) = \mathbb{1}$ by $\mathrm{Lie}\, W_{-1}(G)$.

For any object M of $\mathcal{T}$, let $\mathfrak{u}_M$ be the image in $W_{-1}\mathcal{E}nd(M)$ of $\mathrm{Lie}\, W_{-1}(G)$. The extension $\mathfrak{L}$ gives us by functoriality a quotient $\mathfrak{L}_M$ of $\mathfrak{L}$ acting on M, which is an extension of $\mathbb{1}$ by $\mathfrak{u}_M$.

In the setting of section 3, $U(M)$ corresponds to $W_{-1}\mathcal{E}nd(M)$, and the subobject $P(M)$ of $U(M)$ corresponds to the subobject $\mathfrak{u}_M$ of $W_{-1}\mathcal{E}nd(M)$. We have a 1–motive $[\mathbb{Z} \longrightarrow U(M)]$ given by the point $\overline{u}$, which has to be seen as an extension of $[0 \longrightarrow U(M)]$ by $[\mathbb{Z} \longrightarrow 0]$, and $P(M)$ was defined to be the smallest subobject of $U(M)$ from which this extension comes, say after passing to $\mathbb{Q}$–coefficients. Exactly in this way the two stories A.1 and A.2 are related:

Proposition A.3. (1) *The extension* $\mathrm{cl}(M)$ *is the push out of the extension* $\mathfrak{L}_M$ *via the inclusion* $\mathfrak{u}_M \longrightarrow W_{-1}\mathcal{E}nd(M)$.

(2) *If* $\mathfrak{v}$ *is a subobject of* $\mathfrak{u}_M$ *such that* $\mathrm{cl}(M)$ *is the image of a class in* $\mathrm{Ext}^1(\mathbb{1}, \mathfrak{v})$*, then* $\mathfrak{v} = \mathfrak{u}_M$.

I claim in (1) an equality in $\mathrm{Ext}^1(\mathbb{1}, W_{-1}\mathcal{E}nd(M))$. As $\mathrm{Hom}(\mathbb{1}, W_{-1}\mathcal{E}nd(M))$ is 0 for weight reasons, it is the same as an (unique) isomorphism of actual extensions.

A first computation, which was helpful for understanding but now has disappeared from the proof, was to consider the category of graded representations of a graded Lie algebra $\mathfrak{g}$ with degrees < 0, and wondering what cocycle $c : \mathfrak{g} \longrightarrow W_{-1}\mathcal{E}nd(M)$ was giving me $\mathrm{cl}(M)$ in $H^1(\mathfrak{g}, W_{-1}\mathcal{E}nd(M))$. The answer is the composite map $\mathfrak{g} \longrightarrow \mathfrak{g} \longrightarrow W_{-1}\mathcal{E}nd(M)$, where the first map is multiplication by n in degree $-n$, and the second map is given by the action of $\mathfrak{g}$ on M.

Proof of Proposition A.3. (1) For each integer n we have a map

$$e_n : \mathbb{1} \longrightarrow \mathcal{E}nd(\mathrm{gr}^W_n M) \longrightarrow \mathrm{gr}^W_0 \mathcal{E}nd(M) = \bigoplus_p \mathcal{E}nd(\mathrm{gr}^W_p M)$$

The push–out by $\mathfrak{u}_M \longrightarrow W_{-1}\mathcal{E}nd(M)$ of the extension $0 \longrightarrow \mathfrak{u}_M \longrightarrow \mathfrak{L}_M \longrightarrow \mathbb{1} \longrightarrow 0$ is the pull–back of the extension

$$(*) \qquad 1 \longrightarrow W_{-1}\mathcal{E}nd(M) \longrightarrow W_0\mathcal{E}nd(M) \longrightarrow \mathrm{gr}^W_0 \mathcal{E}nd(M) \longrightarrow 1$$

by $\sum_n ne_n$. Indeed, $\sum_n ne_n$ gives the action of $\mathbb{1} = \mathrm{Lie}(\mathbb{G}_m)$ on $\mathrm{gr}^W_* M$ corresponding to the grading. Define $E_p := \sum_{n>p} e_n$ and let $A \leq B$ be integers such that $W_A M = 0$ and $W_B M = M$.

We have then

$$\sum_n ne_n = \sum_{A\le p\le B} E_p + m\sum_n e_n$$

for some integer m depending on the choice of A and B. The map $\sum_n e_n$ lifts to the identity morphism of M, viewed as a map $\mathbb{1} \longrightarrow \mathcal{E}nd(M)$ which factors over $W_0\mathcal{E}nd(M)$. The pull–back of $(*)$ by $m\sum_n e_n$ is hence trivial, and the push–out of $\mathfrak{L}_M$ by the inclusion $\mathfrak{u}_M \longrightarrow W_{-1}\mathcal{E}nd(M)$ is the sum of the pull–backs of $(*)$ by the E_p. This pull–back by E_p comes from an extension of $\mathbb{1}$ by $\mathcal{H}om(M/W_pM, W_pM)$, which I would like to call *Endomorphisms of M respecting $W_p(M)$, inducing a multiple of the identity on M/W_pM and 0 on W_pM*. At least, that is what it becomes in any realisation. This is the extension already considered in A.1, corresponding to the class of M in $\mathrm{Ext}^1(M/W_pM, W_pM)$. The sum of those extension classes, pushed to $W_{-1}\mathcal{E}nd(M)$, had been defined to be $\mathrm{cl}(M)$.

(2) On objects N of weights < 0, i.e. such that $N = W_{-1}N$, the functor $\mathrm{Ext}^1(\mathbb{1}, -)$ is left exact. For any class α in $\mathrm{Ext}^1(\mathbb{1}, N)$, there is hence a smallest sub-object N_0 of N such that α comes from a class in $\mathrm{Ext}^1(\mathbb{1}, N_0)$. Indeed, if α comes from N' and from N'', the short exact sequence

$$0 \longrightarrow N'\cap N'' \longrightarrow N'\oplus N'' \longrightarrow N$$

shows after applying $\mathrm{Ext}^1(\mathbb{1}, -)$ that α also comes from $N'\cap N''$. It remains to show that the class of $\mathfrak{L}_M$ in $\mathrm{Ext}^1(\mathbb{1}, \mathfrak{u}_M)$ does not come from any $\mathfrak{v} \subsetneq \mathfrak{u}_M$. Indeed, any subobject of $\mathfrak{L}_M$ is stable by the action of $\mathfrak{L}_M$ because G and $\mathrm{Lie}\, G$ act on everything. □

Proposition A.4. *Any Lie–subobject of $\mathfrak{L}_M$ mapping onto $\mathbb{1}$ is equal to $\mathfrak{L}_M$. In other words, the extension $0 \longrightarrow \mathfrak{u}_M \longrightarrow \mathfrak{L}_M \longrightarrow \mathbb{1} \longrightarrow 0$ is essential.*

Proof. The bracket of $\mathfrak{L}_M$ passes to the quotients by W and defines brackets

$$W_p\mathfrak{L}_M/W_{p-1}\mathfrak{L}_M \otimes \mathfrak{L}_M/W_{-1}\mathfrak{L}_M \longrightarrow W_p\mathfrak{L}_M/W_{p-1}\mathfrak{L}_M$$

and this bracket $\mathrm{gr}_p^W \mathfrak{L}_M \otimes \mathbb{1} \longrightarrow \mathrm{gr}_p^W \mathfrak{L}_M$ is the multiplication by p. For any subobject $\mathfrak{L}'$ of $\mathfrak{L}_M$ mapping onto the quotient $\mathbb{1}$, the stability of $\mathfrak{L}'$ by the action of $W_p\mathfrak{L}_M$ hence gives a surjectivity

$$\mathrm{gr}_p^W \mathfrak{L}' \longrightarrow \mathrm{gr}_p^W \mathfrak{L}_M$$

from which the equality $\mathfrak{L}' = \mathfrak{L}_M$ follows. □

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