LYUBEZNIK NUMBERS OF MONOMIAL IDEALS

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Abstract. Let $R = k[x_1, ..., x_n]$ be the polynomial ring in n independent variables, where k is a field. In this work we will study Bass numbers of local cohomology modules $H_I^r(R)$ supported on a squarefree monomial ideal $I \subseteq R$. Among them we are mainly interested in Lyubeznik numbers. We build a dictionary between the modules $H_I^r(R)$ and the minimal free resolution of the Alexander dual ideal $I^\vee$ that allow us to interpret Lyubeznik numbers as the obstruction to the acyclicity of the linear strands of $I^\vee$. The methods we develop also help us to give a bound for the injective dimension of the local cohomology modules in terms of the dimension of the small support.

1. Introduction

Some finiteness properties of local cohomology modules $H_I^r(R)$ were established by C. Huneke and R. Y. Sharp [24] and G. Lyubeznik [28, 29] for the case of regular local rings $(R, \mathfrak{m}, k)$ containing a field. Among these properties they proved a bound for the injective dimension

$$\mathrm{id}_R(H_I^r(R)) \leq \mathrm{dim}_R H_I^r(R)$$

and the finiteness of all the Bass numbers $\mu_p(\mathfrak{p}, H_I^r(R)) := \mathrm{dim}_{k(\mathfrak{p})}\mathrm{Ext}^p_{R_\mathfrak{p}}(k(\mathfrak{p}), H^r_{IR_\mathfrak{p}}(R_\mathfrak{p}))$ with respect to any prime ideal $\mathfrak{p} \subseteq R$. This last fact prompted G. Lyubeznik to define a new set of numerical invariants $\lambda_{p,i}(R/I) := \mu_p(\mathfrak{m}, H_I^{n-i}(R))$, where n is the dimension of R. These invariants satisfy $\lambda_{d,d} \neq 0$ and $\lambda_{p,i} = 0$ for $i > d$, $p > i$, where $d = \mathrm{dim} R/I$. Therefore we can collect them in the following table:

$$\Lambda(R/I) = \begin{pmatrix} \lambda_{0,0} & \cdots & \lambda_{0,d} \\ & \ddots & \vdots \\ & & \lambda_{d,d} \end{pmatrix}$$

Lyubeznik numbers carry some interesting topological information (see [28], [17], [9], [8]) but not too many examples can be found in the literature. We point out that a general algorithm to compute these invariants in characteristic zero has been given by U. Walther [42] using the theory of D-modules, i.e. the theory of modules over the ring of k-linear differential operators $D_{R|k}$.

The D-module approach was also used by the first author in [1, 3] to study local cohomology modules supported on monomial ideals over the polynomial ring $R = k[x_1, ..., x_n]$ and compute Lyubeznik numbers using the so-called *characteristic cycle*. Local cohomology modules supported on monomial ideals $H_I^r(R)$ have also been extensively studied

†Partially supported by MTM2010-20279-C02-01 and SGR2009-1284.

using their natural structure as $\mathbb{Z}^n$-graded modules. For example, N. Terai [40] gives a formula for its graded pieces equivalent, using local duality with monomial support (see [30, §6.2]), to the famous Hochster formula for the $\mathbb{Z}^n$-graded Hilbert function of $H^r_{\mathfrak{m}}(R/I)$ [39]. Simultaneously, M. Mustaţă [32] gives a complete description of the $\mathbb{Z}^n$-graded structure, i.e. a formula for the graded pieces of $H^r_I(R)$ and a description of the linear maps among them. This description is equivalent to the one given by H. G. Gräbe [19] to describe the module structure of $H^r_{\mathfrak{m}}(R/I)$. In the same spirit, a formula for the graded pieces of $H^r_J(R/I)$, where $J \supseteq I$ is another squarefree monomial ideal was given by V. Reiner, V. Welker and K. Yanagawa in [35].

Building on previous work on *squarefree* modules [43], K. Yanagawa develops in [44] the theory of *straight* modules to study local cohomology modules $H^r_I(R)$ and their Bass numbers. Simultaneously, E. Miller [30] also generalized squarefree modules by introducing the categories of $\mathbf{a}$-positively determined (resp. $\mathbf{a}$-determined) modules[1]. When dealing with Bass numbers, K. Yanagawa gave the following formula for Lyubeznik numbers:

$$\lambda_{p,i}(R/I) = \dim_k[\mathrm{Ext}^{n-p}_R(\mathrm{Ext}^{n-i}_R(R/I,R),R)]_{\mathbf{0}}$$

here $[\cdot]_{\mathbf{0}}$ denotes the degree 0 component of a $\mathbb{Z}^n$-graded module.

The approach we take in this work to study Lyubeznik numbers uses the fact that they can be realized as the dimension of the degree $\mathbf{1}$ part of the local cohomology modules $H^p_{\mathfrak{m}}(H^r_I(R))$. In Section 3 we compute these graded pieces and, in general, the graded pieces of $H^p_{\mathfrak{p}}(H^r_I(R))$, where $\mathfrak{p}$ is any homogeneous prime ideal. More precisely, the piece $[H^p_{\mathfrak{m}}(H^r_I(R))]_{\mathbf{1}}$ is nothing but the $p-th$ homology group of a complex of k-vector spaces we construct using the whole structure of $H^r_I(R)$, i.e. the graded pieces and the linear maps among them.

In Section 4 we build a dictionary between local cohomology modules and free resolutions of monomial ideals that gives us a very simple interpretation of Lyubeznik numbers. It turns out that the complex we use to compute the degree $\mathbf{1}$ part of $H^p_{\mathfrak{m}}(H^r_I(R))$ is the dual, as k-vector spaces, of the complex given by the scalar entries in the monomial matrices (in the sense of [30, 31]) of the r-linear strand of the Alexander dual ideal $I^\vee$. Thus, Lyubeznik numbers can be thought as a measure of the acyclicity of these linear strands.

Using the techniques we developed previously we are able to study some properties of Bass numbers of local cohomology modules in Section 5. Recall that, given a finitely generated module M, one has $\mathrm{id}_R\, M \geq \dim_R \mathrm{Supp}_R M$. This bound is a consequence of the following well-known property: Let $\mathfrak{p} \subseteq \mathfrak{q} \in \mathrm{Spec} R$ such that $\mathrm{ht}\,(\mathfrak{q}/\mathfrak{p}) = s$. Then

$$\mu_i(\mathfrak{p}, M) \neq 0 \Longrightarrow \mu_{i+s}(\mathfrak{q}, M) \neq 0.$$

For the case of local cohomology modules this property is no longer true but we can control the behavior of Bass numbers depending on the structure of $H^r_I(R)$. This control leads to

[1]Squarefree (resp. straight) modules correspond to $\mathbf{1}$-positively determined (resp. $\mathbf{1}$-determined) modules.

a sharper bound for the injective dimension of local cohomology modules supported on monomial ideals in terms of the dimension of the small support of these modules

$$\mathrm{id}_R H_I^r(R) \leq \dim_R \mathrm{supp}_R H_I^r(R).$$

We recall that the small support was introduced by H. B. Foxby [14] and consists on the prime ideals having a Bass number different from zero. For finitely generated modules the small support coincide with the support but this is no longer true for non-finitely generated modules.

In Section 6 we use a shifted version of graded Matlis duality to study dual Bass numbers. We obtain analogous results to those obtained for Bass numbers that allow us to study projective resolutions of local cohomology modules.

Acknowledgement: We would like to thank O. Fernández-Ramos for implementing our methods using the package `Macaulay 2` [20]. We also thank E. Miller and K. Yanagawa for several useful remarks and clarifications and J. Herzog for pointing us out to sequentially Cohen-Macaulay ideals as those ideals having trivial Lyubeznik numbers.

2. Local cohomology modules supported on monomial ideals

Let $R = k[x_1, ..., x_n]$ be the polynomial ring in n independent variables, where k is a field. An ideal $I \subseteq R$ is said to be a squarefree monomial ideal if it may be generated by squarefree monomials $\mathbf{x}^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha \in \{0,1\}^n$. Its minimal primary decomposition is given in terms of face ideals $\mathfrak{p}_\alpha := \langle x_i \mid \alpha_i \neq 0 \rangle$, $\alpha \in \{0,1\}^n$. For simplicity we will denote the homogeneous maximal ideal $\mathfrak{m} := \mathfrak{p}_{\mathbf{1}} = (x_1, \dots, x_n)$, where $\mathbf{1} = (1, \dots, 1)$. As usual, we denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\varepsilon_1, \dots, \varepsilon_n$ will be the natural basis of $\mathbb{Z}^n$.

A lot of progress in the study of local cohomology modules $H_I^r(R)$ supported on monomial ideals has been made based on the fact that they have a structure as $\mathbb{Z}^n$-graded modules. Another line of research uses their structure as regular holonomic modules over the ring of k-linear differential operators $D_{R|k}$, in particular the fact that they are finitely generated. The aim of this Section is to give a quick overview of both approaches. For the $\mathbb{Z}^n$-graded case we will highlight the main results obtained in [32], [40], [44] (see also [31]). The main sources for the $D_{R|k}$-module case are [4], [5]. For unexplained terminology in the theory of $D_{R|k}$-modules one may consult [7], [10].

2.1. **$\mathbb{Z}^n$-graded structure.** Local cohomology modules $H_I^r(R)$ supported on monomial ideals are $\mathbb{Z}^n$-graded modules satisfying some nice properties since they fit, modulo a shifting by $\mathbf{1}$, into the category of straight (resp. $\mathbf{1}$-determined) modules introduced by K. Yanagawa [44] (resp. E. Miller [30]). In this framework, these modules are completely described by the graded pieces $H_I^r(R)_{-\alpha}$ for all $\alpha \in \{0,1\}^n$ and the morphisms given by the multiplication by x_i:

$$\cdot x_i : H_I^r(R)_{-\alpha} \longrightarrow H_I^r(R)_{-(\alpha - \varepsilon_i)}$$

N. Terai [40] gave a description of these graded pieces as follows:

$$H_I^r(R)_{-\alpha} \cong \widetilde{H}_{n-r-|\alpha|-1}(\mathrm{link}_\alpha \Delta; k),$$

where Δ is the simplicial complex on the set of vertices $\{x_1, \dots, x_n\}$ corresponding to the squarefree monomial ideal I via the Stanley-Reisner correspondence and, given a face $\sigma_\alpha := \{x_i \mid \alpha_i = 1\} \in \Delta$, the link of σ_α in Δ is

$$\mathrm{link}_\alpha \Delta := \{\tau \in \Delta \mid \sigma_\alpha \cap \tau = \emptyset,\ \sigma_\alpha \cup \tau \in \Delta\}.$$

A different approach was given independently by M. Mustaţă [32] in terms of the restriction to σ_α that we denote $\Delta_\alpha := \{\tau \in \Delta \mid \tau \in \sigma_\alpha\}$. We have:

$$H_I^r(R)_{-\alpha} \cong \widetilde{H}^{r-2}(\Delta^\vee_{\mathbf{1}-\alpha}; k),$$

where $\Delta^\vee_{\mathbf{1}-\alpha}$ denotes the Alexander dual of $\Delta_{\mathbf{1}-\alpha}$. Both approaches are equivalent since the equality of simplicial complexes $\Delta^\vee_{\mathbf{1}-\alpha} = (\mathrm{link}_\alpha \Delta)^\vee$ induces, by Alexander duality, the isomorphism

$$\widetilde{H}_{n-r-|\alpha|-1}(\mathrm{link}_\alpha \Delta; k) \cong \widetilde{H}^{r-2}(\Delta^\vee_{\mathbf{1}-\alpha}; k).$$

Mustaţă also describes the multiplication morphism $\cdot x_i : H_I^r(R)_{-\alpha} \longrightarrow H_I^r(R)_{-(\alpha-\varepsilon_i)}$. It corresponds to the morphism

$$\widetilde{H}^{r-2}(\Delta^\vee_{\mathbf{1}-\alpha-\varepsilon_i}; k) \longrightarrow \widetilde{H}^{r-2}(\Delta^\vee_{\mathbf{1}-\alpha}; k),$$

induced by the inclusion $\Delta^\vee_{\mathbf{1}-\alpha-\varepsilon_i} \subseteq \Delta^\vee_{\mathbf{1}-\alpha}$.

2.2. ***D*-module structure.** Local cohomology modules $H_I^r(R)$ supported on monomial ideals also satisfy nice properties when viewed as $D_{R|k}$-modules since they belong to the subcategory $D^T_{v=0}$ of regular holonomic $D_{R|k}$-modules with support a normal crossing $T := \{x_1 \cdots x_n = 0\}$ and variation zero defined in [4]. An object M of this category is characterized by the existence of an increasing filtration $\{F_j\}_{0 \leq j \leq n}$ of submodules of M such that there are isomorphisms of $D_{R|k}$-modules

$$F_j/F_{j-1} \simeq \bigoplus_{|\alpha|=j} (H^{|\alpha|}_{\mathfrak{p}_\alpha}(R))^{m_\alpha},$$

for some integers $m_\alpha \geq 0$, $\alpha \in \{0,1\}^n$. We point out that in this category we have the following objects, $\forall \alpha \in \{0,1\}^n$:

- **Simple:** $H^{|\alpha|}_{\mathfrak{p}_\alpha}(R) \cong \frac{R[\frac{1}{\mathbf{x}^\alpha}]}{\sum_{\alpha_i=1} R[\frac{1}{\mathbf{x}^{\alpha-\varepsilon_i}}]} \cong \frac{D_{R|k}}{D_{R|k}(\{x_i \,|\, \alpha_i=1\}, \{\partial_j \,|\, \alpha_j=0\})}$.
- **Injective:** $E_\alpha := {}^*\mathrm{E}_R(R/\mathfrak{p}_\alpha)(\mathbf{1}) \cong \frac{R[\frac{1}{\mathbf{x}^{\mathbf{1}}}]}{\sum_{\alpha_i=1} R[\frac{1}{\mathbf{x}^{\mathbf{1}-\varepsilon_i}}]} \cong \frac{D_{R|k}}{D_{R|k}(\{x_i \,|\, \alpha_i=1\}, \{x_j\partial_j+1 \,|\, \alpha_j=0\})}$.
- **Projective:** $R_{x^\alpha} \cong \frac{D_{R|k}}{D_{R|k}(\{x_i\partial_i+1 \,|\, \alpha_i=1\}, \{\partial_j \,|\, \alpha_j=0\})}$.

Following the work of A. Galligo, M. Granger and Ph. Maisonobe [15, 16] one may describe this category as a quiver representation. More precisely, let $\mathcal{C}^n_{v=0}$ be the category whose objects are families $\mathcal{M} := \{\mathcal{M}_\alpha\}_{\alpha\in\{0,1\}^n}$ of finitely dimensional k-vector spaces, endowed with linear maps

$$\mathcal{M}_\alpha \xrightarrow{u_{\alpha,i}} \mathcal{M}_{\alpha+\varepsilon_i},$$

for each $\alpha \in \{0,1\}^n$ such that $\alpha_i = 0$. These maps are called canonical maps, and they are required to satisfy $u_{\alpha,i} \circ u_{\alpha+\varepsilon_i,j} = u_{\alpha,j} \circ u_{\alpha+\varepsilon_j,i}$. Such an object will be called an n-hypercube. A morphism between two n-hypercubes $\{\mathcal{M}_\alpha\}_\alpha$ and $\{\mathcal{N}_\alpha\}_\alpha$ is a set of linear maps $\{f_\alpha : \mathcal{M}_\alpha \to \mathcal{N}_\alpha\}_\alpha$, commuting with the canonical maps.

There is an equivalence of categories between $D^T_{v=0}$ and $\mathcal{C}^n_{v=0}$ given by the contravariant exact functor that sends an object M of $D^T_{v=0}$ to the n-hypercube $\mathcal{M}$ constructed as follows:

i) The vertices of the n-hypercube are the k-vector spaces $\mathcal{M}_\alpha := \mathrm{Hom}_{D_{R|k}}(M, E_\alpha)$.
ii) The linear maps $u_{\alpha,i}$ are induced by the natural epimorphisms $\pi_{\alpha,i} : E_\alpha \to E_{\alpha+\varepsilon_i}$.

The irreducibility of M is determined by the extension classes of the short exact sequences

$$\begin{gathered} 0\longrightarrow F_0\longrightarrow F_1\longrightarrow F_1/F_0\longrightarrow 0 \\ \vdots \\ 0\longrightarrow F_{n-1}\longrightarrow F_n = M\longrightarrow F_n/F_{n-1}\longrightarrow 0 \end{gathered}$$

associated to the filtration $\{F_j\}_{0\le j\le n}$ of submodules of M. It is shown in [4] and [5] that these extension classes are uniquely determined by the linear maps $u_{\alpha,i}$.

It is also worth to point out that if $CC(M) = \sum m_\alpha\, T^*_{X_\alpha}\mathbb{A}^n_k$ is the *characteristic cycle* of M, then for all $\alpha \in \{0,1\}^n$ one has the equality $\dim_k \mathcal{M}_\alpha = m_\alpha$ so the pieces of the n-hypercube of a module M are described by the characteristic cycle of M. Finally, the n-hypercube $\{[H^r_I(R)]_\alpha\}_{\alpha\in\{0,1\}^n}$ associated to a local cohomology module $H^r_I(R)$ has been computed in [5].

2.3. **Both approaches are equivalent.** The category $D^T_{v=0}$ of regular holonomic $D_{R|k}$-modules with variation zero is equivalent to the category of straight modules shifted by $\mathbf{1}$ (see [4]). Let $M \in D^T_{v=0}$ and $\mathcal{M} \in \mathcal{C}^n_{v=0}$ be the corresponding n-hypercube. The vertices and linear maps of $\mathcal{M}$ can be described from the graded pieces of M. Let $(M_{-\alpha})^*$ be the dual of the k-vector space defined by the piece of M of degree $-\alpha$, $\alpha \in \{0,1\}^n$. Then, there are isomorphisms

$$\mathcal{M}_\alpha \cong (M_{-\alpha})^*$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{M}_\alpha \arrow[r, "u_{\alpha,i}"] & \mathcal{M}_{\alpha+\varepsilon_i} \\ (M_{-\alpha})^* \arrow[u, "\cong"] \arrow[r, "(x_i)^*"] & (M_{-\alpha-\varepsilon_i})^* \arrow[u, "\cong"] \end{tikzcd}$$

where $(x_i)^*$ is the dual of the multiplication by x_i.

In this work we are going to use the D-module approach just because of the habit of the first author. In principle this approach only works for the case of fields of characteristic zero since the category $\mathcal{C}^n$ described in [15] is defined over $\mathbb{C}$ and its subcategory $\mathcal{C}^n_{v=0}$ can be extended to any field of characteristic zero (see [5]). We did not make any previous mention to the characteristic of the field because the results are also true in positive characteristic even though we do not have an analogue to the results of [15, 16]. In this case one has to define modules with variation zero via the characterization given by the existence of an increasing filtration $\{F_j\}_{0\leq j\leq n}$ of submodules of M such that

$$F_j/F_{j-1} \simeq \bigoplus\nolimits_{|\alpha|=j} (H^{|\alpha|}_{\mathfrak{p}_\alpha}(R))^{m_\alpha},$$

for some integers $m_\alpha \geq 0$, $\alpha \in \{0,1\}^n$. Finally we point out that, using the same arguments as in [4, Lemma 4.4], the n-hypercube $\mathcal{M}$ associated to a module with variation zero M should be constructed using the following variant in terms of graded morphisms

i) The vertices of the n-hypercube are the k-vector spaces $\mathcal{M}_\alpha :=^* \operatorname{Hom}_R(M, E_\alpha)$.
ii) The linear maps $u_{\alpha,i}$ are induced by the natural epimorphisms $\pi_{\alpha,i} : E_\alpha \to E_{\alpha+\varepsilon_i}$.

From now on we will loosely use the term *pieces of a module* M meaning the pieces of the n-hypercube associated to M. If the reader is more comfortable with the $\mathbb{Z}^n$-graded point of view one may also reformulate all the results in this paper using the $\mathbb{Z}^n$-graded pieces of M (with the appropriate sign). One only has to be careful with the direction of the arrows in the complexes of k-vector spaces we will construct in the next Sections.

Remark 2.1. The advantage of the D-module approach is that it is more likely to be extended to other situations like the case of hyperplane arrangements. We recall that local cohomology modules with support an arrangement of linear subvarieties were already computed in [4] and a quiver representation of $D_{R|k}$-modules with support a hyperplane arrangement is given in [25], [26].

3. Local cohomology of modules with variation zero

Let $M \in D^T_{v=0}$ be a regular holonomic $D_{R|k}$-module with variation zero. The aim of this Section is to compute the pieces of the local cohomology module $H^p_{\mathfrak{p}_\alpha}(M)$, for any given homogeneous prime ideal $\mathfrak{p}_\alpha$, $\alpha \in \{0,1\}^n$. This module also belongs to $D^T_{v=0}$ so we want to compute the pieces of the corresponding n-hypercube $\{[H^p_{\mathfrak{p}_\alpha}(M)]_\beta\}_{\beta\in\{0,1\}^n} \in \mathcal{C}^n_{v=0}$. Among these pieces we find the Bass numbers of M (see [3]). Namely, we have

$$\mu_p(\mathfrak{p}_\alpha, M) = \dim_k\, [H^p_{\mathfrak{p}_\alpha}(M)]_\alpha$$

Bass numbers have a good behavior with respect to localization so we can always assume that $\mathfrak{p}_\alpha = \mathfrak{m}$ is the maximal ideal and $\mu_p(\mathfrak{m}, M) = \dim_k\, [H^p_{\mathfrak{m}}(M)]_{\mathbf{1}}$.

Remark 3.1. Let $\mathcal{M} \in \mathcal{C}^n_{v=0}$ be an n-hypercube. The restriction of $\mathcal{M}$ to a face ideal $\mathfrak{p}_\alpha$, $\alpha \in \{0,1\}^n$ is the $|\alpha|$-hypercube $\mathcal{M}_{\leq\alpha} := \{\mathcal{M}_\beta\}_{\beta\leq\alpha} \in \mathcal{C}^{|\alpha|}_{v=0}$ (see [3]). This gives a functor that in some cases plays the role of the localization functor. In particular, to compute the Bass numbers with respect to $\mathfrak{p}_\alpha$ of a module with variation zero M we only have to consider the corresponding $|\alpha|$-hypercube $\mathcal{M}_{\leq\alpha}$ so we may assume that $\mathfrak{p}_\alpha$ is the maximal ideal.

In Section 4 we will specialize to the case of M being a local cohomology module $H^r_I(R)$.

3.1. **The degree** 1 **piece of** $H^p_{\mathfrak{m}}(M)$**.** We start with his particular case since it is more enlightening than the general one. Using the whole structure of M. i.e. the pieces of M and the linear maps between them, we want to construct a complex of k-vector spaces whose homology is $[H^p_{\mathfrak{m}}(M)]_{\mathbf{1}}$.

The degree $\mathbf{1}$ part of the hypercube corresponding to the local cohomology module $H^p_{\mathfrak{m}}(M)$ is the p-th homology of the complex of k-vector spaces

$$[\check{C}_{\mathfrak{m}}(M)]^\bullet_{\mathbf{1}} : 0 \longleftarrow [M]_{\mathbf{1}} \xleftarrow{\overline{d_0}} \bigoplus_{|\alpha|=1} [M_{\mathbf{x}^\alpha}]_{\mathbf{1}} \xleftarrow{\overline{d_1}} \cdots \xleftarrow{\overline{d_{p-1}}} \bigoplus_{|\alpha|=p} [M_{\mathbf{x}^\alpha}]_{\mathbf{1}} \xleftarrow{\overline{d_p}} \cdots \xleftarrow{\overline{d_{n-1}}} [M_{\mathbf{x}^{\mathbf{1}}}]_{\mathbf{1}} \longleftarrow 0$$

that we obtain applying the exact functor $\mathrm{Hom}_{D_{R|k}}(\cdot, E_{\mathbf{1}})$ to the Čech complex

$$\check{C}^\bullet_{\mathfrak{m}}(M) : 0 \longrightarrow M \xrightarrow{d_0} \bigoplus_{|\alpha|=1} M_{\mathbf{x}^\alpha} \xrightarrow{d_1} \cdots \xrightarrow{d_{p-1}} \bigoplus_{|\alpha|=p} M_{\mathbf{x}^\alpha} \xrightarrow{d_p} \cdots \xrightarrow{d_{n-1}} M_{\mathbf{x}^{\mathbf{1}}} \longrightarrow 0,$$

where the map between summands $M_{\mathbf{x}^\alpha} \longrightarrow M_{\mathbf{x}^{\alpha+\varepsilon_i}}$ is $\mathrm{sign}(i, \alpha+\varepsilon_i)$ times the canonical localization map[2]. On the other hand, giving the appropriate sign to the canonical maps of the hypercube $\mathcal{M} = \{[M]_\alpha\}_\alpha$ associated to M we can construct the following complex of k-vector spaces:

$$\mathcal{M}^\bullet : 0 \longleftarrow [M]_{\mathbf{1}} \xleftarrow{u_0} \bigoplus_{|\alpha|=n-1} [M]_\alpha \xleftarrow{u_1} \cdots \xleftarrow{u_{p-1}} \bigoplus_{|\alpha|=n-p} [M]_\alpha \xleftarrow{u_p} \cdots \xleftarrow{u_{n-1}} [M]_{\mathbf{0}} \longleftarrow 0$$

where the map between summands $[M]_\alpha \longrightarrow [M]_{\alpha+\varepsilon_i}$ is $\mathrm{sign}(i, \alpha+\varepsilon_i)$ times the canonical map $u_{\alpha,i}$.

Example 3.2. 3-hypercube and its associated complex

$$\begin{tikzcd}
 & M_{(0,0,0)} \arrow[dl, "u_1"'] \arrow[d, "u_2"] \arrow[dr, "u_3"] & \\
M_{(1,0,0)} \arrow[d, "u_2"'] \arrow[dr, "u_3"] & M_{(0,1,0)} \arrow[dl, "u_1"] \arrow[dr, "u_3"] & M_{(0,0,1)} \arrow[dl, "u_1"'] \arrow[d, "u_2"] \\
M_{(1,1,0)} \arrow[dr, "u_3"'] & M_{(1,0,1)} \arrow[d, "u_2"] & M_{(0,1,1)} \arrow[dl, "u_1"] \\
 & M_{(1,1,1)} &
\end{tikzcd}$$

[2]$\mathrm{sign}(i, \alpha) = (-1)^{r-1}$ if α_i is the r^{th} component of α different from zero

$$0 \longleftarrow M_{(1,1,1)} \xleftarrow{(u_3,u_2,u_1)} \begin{matrix} M_{(1,1,0)} \\ \oplus \\ M_{(1,0,1)} \\ \oplus \\ M_{(0,1,1)} \end{matrix} \xleftarrow{\begin{pmatrix} -u_2 & -u_1 & 0 \\ u_3 & 0 & -u_1 \\ 0 & u_3 & u_2 \end{pmatrix}} \begin{matrix} M_{(1,0,0)} \\ \oplus \\ M_{(0,1,0)} \\ \oplus \\ M_{(0,0,1)} \end{matrix} \xleftarrow{\begin{pmatrix} -u_1 \\ u_2 \\ -u_3 \end{pmatrix}} M_{(0,0,0)} \longleftarrow 0$$

The main result of this Section is the following

Proposition 3.3. *Let $M \in D^T_{v=0}$ be a regular holonomic $D_{R|k}$-module with variation zero and $\mathcal{M}^\bullet$ its corresponding complex associated to the n-hypercube. Then, there is an isomorphism of complexes $\mathcal{M}^\bullet \cong [\check{C}_\mathfrak{m}(M)]^\bullet_\mathbf{1}$. In particular $[H^p_\mathfrak{m}(M)]_\mathbf{1} \cong \mathrm{H}_p(\mathcal{M}^\bullet)$.*

Therefore we have the following characterization of Bass numbers:

Corollary 3.4. *Let $M \in D^T_{v=0}$ be a regular holonomic $D_{R|k}$-module with variation zero and $\mathcal{M}^\bullet$ its corresponding complex associated to the n-hypercube. Then*

$$\mu_p(\mathfrak{m}, M) = \dim_k \mathrm{H}_p(\mathcal{M}^\bullet)$$

Proof. Using [3, Prop. 3.2] one may check out that the k-vector spaces $[M_{\mathbf{x}^\alpha}]_\mathbf{1}$ and $[M]_{\mathbf{1}-\alpha}$ have the same dimension. An explicit isomorphism $\phi_\alpha : [M]_{\mathbf{1}-\alpha} \longrightarrow [M_{\mathbf{x}^\alpha}]_\mathbf{1}$ is defined as follows:

Let $f \in [M]_{\mathbf{1}-\alpha} = \mathrm{Hom}_{D_{R|k}}(M, E_{\mathbf{1}-\alpha})$, then $\phi_\alpha(f) \in [M_{\mathbf{x}^\alpha}]_\mathbf{1} = \mathrm{Hom}_{D_{R|k}}(M_{\mathbf{x}^\alpha}, E_\mathbf{1})$ is the composition

$$M_{\mathbf{x}^\alpha} \xrightarrow{f_{\mathbf{x}^\alpha}} (E_{\mathbf{1}-\alpha})_{\mathbf{x}^\alpha} \xrightarrow{\theta_\alpha^{-1}} E_{\mathbf{1}-\alpha} \xrightarrow{\pi_\alpha} E_\mathbf{1}$$

where:

- $f_{\mathbf{x}^\alpha} : M_{\mathbf{x}^\alpha} \longrightarrow (E_{\mathbf{1}-\alpha})_{\mathbf{x}^\alpha}$ is the localization of f.
- $\theta_\alpha : E_{\mathbf{1}-\alpha} \longrightarrow (E_{\mathbf{1}-\alpha})_{\mathbf{x}^\alpha}$ is the natural localization map.
- $\pi_\alpha : E_{\mathbf{1}-\alpha} \longrightarrow E_\mathbf{1}$ is the natural epimorphism.

Claim: θ_α is an isomorphism.

Proof of Claim: When $\alpha = \mathbf{1}$ we have $E_\mathbf{0} \cong R_{\mathbf{x}^1}$ so the result follows. For $\alpha \neq \mathbf{1}$, let $t \in \mathbb{Z}_{\geq 0}$ and $m = \overline{\frac{\sum_{\beta\in\mathbb{Z}^n} a_\beta \mathbf{x}^\beta}{\mathbf{x}^{\mathbf{1}\cdot t}}}$ be an element of $E_{\mathbf{1}-\alpha}$ such that $\theta_\alpha(m) = 0$. There exists $s \in \mathbb{Z}_{\geq 0}$ such that

$$0 = \mathbf{x}^{\alpha\cdot s} m = \overline{\frac{\sum_{\beta\in\mathbb{Z}^n} a_\beta \mathbf{x}^{\beta+\alpha\cdot s}}{\mathbf{x}^{\mathbf{1}\cdot t}}}$$

so, there exists i such that $\alpha_i = 0$ and $\beta_i + \alpha_i \cdot s \geq t$. Thus $\beta_i \geq t$ and $m = 0$ so θ_α is a monomorphism. Now, let $m' = \frac{\overline{\frac{\sum_{\beta\in\mathbb{Z}^n} a_\beta \mathbf{x}^\beta}{\mathbf{x}^{\mathbf{1}\cdot t}}}}{\mathbf{x}^{\alpha\cdot s}}$ be an element of $(E_{\mathbf{1}-\alpha})_{\mathbf{x}^\alpha}$. Then $m' = \theta_\alpha(m)$, where $m = \overline{\frac{\sum_{\beta\in\mathbb{Z}^n} a_\beta \mathbf{x}^{\beta+(\mathbf{1}-\alpha)\cdot s}}{\mathbf{x}^{\mathbf{1}\cdot(t+s)}}}$ $\square$

Now we check out that ϕ_α is an isomorphism. Recall that $[M]_{\mathbf{1}-\alpha}$ and $[M_{\mathbf{x}^\alpha}]_{\mathbf{1}}$ have same dimension so it is enough to prove that ϕ_α is a monomorphism. Consider $f \in [M]_{\mathbf{1}-\alpha}$ such that $\phi_\alpha(f) = 0$. There exists $t \in \mathbb{Z}_{\geq 0}$ such that $f(m) = \overline{\frac{\sum_{\beta\in\mathbb{Z}^n} a_\beta \mathbf{x}^\beta}{\mathbf{x}^{\mathbf{1}\cdot t}}} \in E_{\mathbf{1}-\alpha}$, for a given $m \in M$. Then:

$$0 = \phi_\alpha(f)(\frac{m}{\mathbf{x}^{\alpha\cdot s}}) = \pi_\alpha\theta_\alpha^{-1}(\frac{f(m)}{\mathbf{x}^{\alpha\cdot s}}) = \pi_\alpha\theta_\alpha^{-1}(\frac{\overline{\frac{\sum_{\beta\in\mathbb{Z}^n} a_\beta \mathbf{x}^\beta}{\mathbf{x}^{\mathbf{1}\cdot t}}}}{\mathbf{x}^{\alpha\cdot s}}) = \pi_\alpha(\frac{\overline{\sum_{\beta\in\mathbb{Z}^n} a_\beta \mathbf{x}^{\beta+(\mathbf{1}-\alpha)s}}}{\mathbf{x}^{\mathbf{1}\cdot(t+s)}}) =$$
$$= \overline{\frac{\sum_{\beta\in\mathbb{Z}^n} a_\beta \mathbf{x}^{\beta+(\mathbf{1}-\alpha)s}}{\mathbf{x}^{\mathbf{1}\cdot(t+s)}}}$$

Thus, there exists $1 \leq i \leq n$ such that $\beta_i + (\mathbf{1}-\alpha)_i s \geq t + s$. If we take s big enough (e.g. $s > \max\{|\beta_i - t|, i = 1,\dots,n\}$), it follows that $(\mathbf{1}-\alpha)_i = 1$ and $\beta_i \geq t$. Hence $f(m) = \overline{\frac{\sum_{\beta\in\mathbb{Z}^n} a_\beta \mathbf{x}^\beta}{\mathbf{x}^{\mathbf{1}\cdot t}}} = 0$ so $f = 0$ as desired.

To finish the proof we have to check out that the diagram

$$\begin{tikzcd}
\bigoplus_{|\alpha|=n-p}[M]_\alpha \arrow[d, "\oplus\phi_\alpha"] & \bigoplus_{|\alpha|=n-(p+1)}[M]_\alpha \arrow[l, "u_p"'] \arrow[d, "\oplus\phi_\alpha"] \\
\bigoplus_{|\alpha|=p}[M_{\mathbf{x}^\alpha}]_{\mathbf{1}} & \bigoplus_{|\alpha|=p+1}[M_{\mathbf{x}^\alpha}]_{\mathbf{1}} \arrow[l, "\overline{d_p}"]
\end{tikzcd}$$

is commutative. Restricting to the corresponding summands it is enough to consider the following diagram

$$\begin{tikzcd}
\operatorname{Hom}_{D_{R|k}}(M, E_{\mathbf{1}-\alpha}) \arrow[d, "\phi_\alpha"] & \operatorname{Hom}_{D_{R|k}}(M, E_{\mathbf{1}-(\alpha+\varepsilon_i)}) \arrow[l, "(-1)^s u_{\mathbf{1}-(\alpha+\varepsilon_i),i}"'] \arrow[d, "\phi_{\alpha+\varepsilon_i}"] \\
\operatorname{Hom}_{D_{R|k}}(M_{\mathbf{x}^\alpha}, E_{\mathbf{1}}) & \operatorname{Hom}_{D_{R|k}}(M_{\mathbf{x}^{\alpha+\varepsilon_i}}, E_{\mathbf{1}}) \arrow[l, "(-1)^s\overline{\vartheta_{\alpha,i}}"]
\end{tikzcd}$$

where $\vartheta_{\alpha,i} : M_{\mathbf{x}^\alpha} \longrightarrow M_{\mathbf{x}^{\alpha+\varepsilon_i}}$ is the natural localization map.

For $f \in \operatorname{Hom}_{D_{R|k}}(M, E_{\mathbf{1}-(\alpha+\varepsilon_i)})$ the morphisms $\phi_\alpha(u_{\mathbf{1}-(\alpha+\varepsilon_i),i}(f))$ and $\overline{\vartheta_{\alpha,i}}(\phi_{\alpha+\varepsilon_i}(f))$ are, respectively, the compositions

$$M_{\mathbf{x}^\alpha} \xrightarrow{f_{\mathbf{x}^\alpha}} (E_{\mathbf{1}-(\alpha+\varepsilon_i)})_{\mathbf{x}^\alpha} \xrightarrow{(\pi_i)_{\mathbf{x}^\alpha}} (E_{\mathbf{1}-\alpha})_{\mathbf{x}^\alpha} \xrightarrow{\theta_\alpha^{-1}} E_{\mathbf{1}-\alpha} \xrightarrow{\pi_\alpha} E_{\mathbf{1}}$$

$$M_{\mathbf{x}^\alpha} \xrightarrow{\vartheta_{\alpha,i}} M_{\mathbf{x}^{\alpha+\varepsilon_i}} \xrightarrow{f_{\mathbf{x}^{\alpha+\varepsilon_i}}} (E_{\mathbf{1}-(\alpha+\varepsilon_i)})_{\mathbf{x}^\alpha} \xrightarrow{\theta_{\alpha+\varepsilon_i}^{-1}} E_{\mathbf{1}-(\alpha+\varepsilon_i)} \xrightarrow{\pi_{\alpha+\varepsilon_i}} E_{\mathbf{1}}$$

Let $m \in M$ and $f(m) = \overline{\frac{\sum_{\beta\in\mathbb{Z}^n} a_\beta \mathbf{x}^\beta}{\mathbf{x}^{\mathbf{1}\cdot t}}} \in E_{\mathbf{1}-(\alpha+\varepsilon_i)}$, where $t \in \mathbb{Z}_{\geq 0}$. Then, for $s \in \mathbb{Z}_{\geq 0}$

$$\phi_\alpha(u_{\mathbf{1}-(\alpha+\varepsilon_i),i}(f))(\frac{m}{\mathbf{x}^{\alpha\cdot s}}) = \pi_\alpha(\theta_\alpha^{-1}((\pi_i)_{\mathbf{x}^\alpha}(f_{\mathbf{x}^\alpha})))(\frac{m}{\mathbf{x}^{\alpha\cdot s}}) = \pi_\alpha(\theta_\alpha^{-1}((\pi_i)_{\mathbf{x}^\alpha}))(\frac{f(m)}{\mathbf{x}^{\alpha\cdot s}}) =$$

$$= \pi_\alpha(\theta_\alpha^{-1})(\frac{\overline{\frac{\sum_{\beta\in\mathbb{Z}^n} a_\beta \mathbf{x}^\beta}{\mathbf{x}^{\mathbf{1}\cdot t}}}}{\mathbf{x}^{\alpha\cdot s}}) = \pi_\alpha(\frac{\overline{\sum_{\beta\in\mathbb{Z}^n} a_\beta \mathbf{x}^{\beta+(\mathbf{1}-\alpha)s}}}{\mathbf{x}^{\mathbf{1}\cdot(t+s)}}) = \frac{\overline{\sum_{\beta\in\mathbb{Z}^n} a_\beta \mathbf{x}^{\beta+(\mathbf{1}-\alpha)s}}}{\mathbf{x}^{\mathbf{1}\cdot(t+s)}}$$

on the other hand

$$\overline{\vartheta_{\alpha,i}}(\phi_{\alpha+\varepsilon_i}(f))(\frac{m}{\mathbf{x}^{\alpha\cdot s}}) = \pi_{\alpha+\varepsilon_i}(\theta_{\alpha+\varepsilon_i}^{-1}(f_{\mathbf{x}^{\alpha+\varepsilon_i}}(\vartheta_{\alpha,i})))(\frac{m}{\mathbf{x}^{\alpha\cdot s}}) = \pi_{\alpha+\varepsilon_i}(\theta_{\alpha+\varepsilon_i}^{-1}(f_{\mathbf{x}^{\alpha+\varepsilon_i}}))(\frac{x_i^s m}{\mathbf{x}^{\alpha\cdot s}}) =$$

$$= \pi_{\alpha+\varepsilon_i}(\theta_{\alpha+\varepsilon_i}^{-1})(\frac{x_i^s f(m)}{\mathbf{x}^{\alpha\cdot s}}) = \pi_{\alpha+\varepsilon_i}(\theta_{\alpha+\varepsilon_i}^{-1})(\frac{x_i^s \overline{\frac{\sum_{\beta\in\mathbb{Z}^n} a_\beta \mathbf{x}^\beta}{\mathbf{x}^{\mathbf{1}\cdot t}}}}{\mathbf{x}^{\alpha\cdot s}}) =$$

$$= \pi_{\alpha+\varepsilon_i}(\frac{\overline{\sum_{\beta\in\mathbb{Z}^n} a_\beta \mathbf{x}^{\beta+\varepsilon_i\cdot s+(\mathbf{1}-(\alpha+\varepsilon_i))s}}}{\mathbf{x}^{\mathbf{1}\cdot(t+s)}}) = \frac{\overline{\sum_{\beta\in\mathbb{Z}^n} a_\beta \mathbf{x}^{\beta+(\mathbf{1}-\alpha)s}}}{\mathbf{x}^{\mathbf{1}\cdot(t+s)}}$$

Thus $\phi_\alpha(u_{\mathbf{1}-(\alpha+\varepsilon_i),i}(f)) = \overline{\vartheta_{\alpha,i}}(\phi_{\alpha+\varepsilon_i}(f))$

□

3.2. **The pieces of $H^p_{\mathfrak{p}_\alpha}(M)$.** In general, for any given $\alpha,\beta\in\{0,1\}^n$, the degree β part of the hypercube corresponding to $H^p_{\mathfrak{p}_\alpha}(M)$ is the p-th homology of the complex of k-vector spaces $[\check{C}_{\mathfrak{p}_\alpha}(M)]^\bullet_\beta$ that we obtain applying the exact functor $\mathrm{Hom}_{D_{R|k}}(\cdot, E_\beta)$ to the Čech complex $\check{C}^\bullet_{\mathfrak{p}_\alpha}(M)$ associated to the face ideal $\mathfrak{p}_\alpha$. On the other hand, we can also associate to the n-hypercube of M the complex of k-vector spaces:

$$\mathcal{M}^\bullet_{\alpha,\beta} : 0 \longleftarrow [M]_\beta \xleftarrow{u_0} \bigoplus_{\substack{|\gamma|=1\\ \gamma\leq\alpha}} [M]_{\beta\backslash\gamma} \xleftarrow{u_1} \cdots \xleftarrow{u_{p-1}} \bigoplus_{\substack{|\gamma|=p\\ \gamma\leq\alpha}} [M]_{\beta\backslash\gamma} \xleftarrow{u_p} \cdots \xleftarrow{u_{|\alpha|-1}} [M]_{\beta\backslash\alpha} \longleftarrow 0$$

where $\beta\backslash\alpha\in\{0,1\}^n$ is the vector with components $(\beta\backslash\alpha)_i := \beta_i$ if $\alpha_i = 0$ and 0 otherwise. The maps between summands are defined by the corresponding canonical maps.

A description of the pieces of $H^p_{\mathfrak{p}_\alpha}(M)$ can be obtained using the same arguments as in the previous subsection so we will skip the details. The proofs are a little bit more involved just because of the extra notation.

Proposition 3.5. *Let $M\in D^T_{v=0}$ be a regular holonomic $D_{R|k}$-module with variation zero and, $\forall\alpha,\beta\in\{0,1\}^n$, $\mathcal{M}^\bullet_{\alpha,\beta}$ its corresponding complex associated to the n-hypercube. Then, $\mathcal{M}^\bullet_{\alpha,\beta}\cong[\check{C}_{\mathfrak{p}_\alpha}(M)]^\bullet_\beta$. In particular $[H^p_{\mathfrak{p}_\alpha}(M)]_\beta\cong \mathrm{H}_p(\mathcal{M}^\bullet_{\alpha,\beta})$.*

Corollary 3.6. *Let $M\in D^T_{v=0}$ be a regular holonomic $D_{R|k}$-module with variation zero and $\mathcal{M}^\bullet_{\alpha,\alpha}$ its corresponding complex associated to the n-hypercube. Then*

$$\mu_p(\mathfrak{p}_\alpha, M) = \dim_k \mathrm{H}_p(\mathcal{M}^\bullet_{\alpha,\alpha})$$

4. Lyubeznik numbers of monomial ideals

Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension n containing a field k and A a local ring which admits a surjective ring homomorphism $\pi : R \longrightarrow A$. G. Lyubeznik [28] defines a new set of numerical invariants of A by means of the Bass numbers $\lambda_{p,i}(A) := \mu_p(\mathfrak{m}, H_I^{n-i}(R))$, where $I = \operatorname{Ker} \pi$. This invariant depends only on A, i and p, but neither on R nor on π. Completion does not change $\lambda_{p,i}(A)$ so one can assume $R = k[[x_1, \dots, x_n]]$. These invariants satisfy $\lambda_{d,d}(A) \neq 0$ and $\lambda_{p,i}(A) = 0$ for $i > d$, $p > i$, where $d = \dim A$. Therefore we can collect them in what we refer as *Lyubeznik table*:

$$\Lambda(R/I) = \begin{pmatrix} \lambda_{0,0} & \cdots & \lambda_{0,d} \\ & \ddots & \vdots \\ & & \lambda_{d,d} \end{pmatrix}$$

It is worth to point out that for the case of monomial ideals one may always assume that $R = k[x_1, \dots, x_n]$. Then, let $\mathcal{M} = \{[H_I^r(R)]_\alpha\}_{\alpha \in \{0,1\}^n}$ be the n-hypercube of a local cohomology module $H_I^r(R)$ supported on a monomial ideal $I \subseteq R$. In this case we have a topological description of the pieces and linear maps of the n-hypercube, e.g. using M. Mustaţă's approach [32], the complex of k-vector spaces associated to $\mathcal{M}$ is:

$$\mathcal{M}^\bullet : 0 \longleftarrow \widetilde{H}^{r-2}(\Delta_{\mathbf{0}}^\vee; k) \xleftarrow{u_0} \cdots \xleftarrow{u_{p-1}} \bigoplus_{|\alpha|=p} \widetilde{H}^{r-2}(\Delta_\alpha^\vee; k) \xleftarrow{u_p} \cdots \xleftarrow{u_{n-1}} \widetilde{H}^{r-2}(\Delta_{\mathbf{1}}^\vee; k) \longleftarrow 0$$

where the map between summands $\widetilde{H}^{r-2}(\Delta_{\alpha+\varepsilon_i}^\vee; k) \longrightarrow \widetilde{H}^{r-2}(\Delta_\alpha^\vee; k)$, is induced by the inclusion $\Delta_\alpha^\vee \subseteq \Delta_{\alpha+\varepsilon_i}^\vee$. In particular, the Lyubeznik numbers of R/I are

$$\lambda_{p,n-r}(R/I) = \dim_k \mathrm{H}_p(\mathcal{M}^\bullet)$$

At this point one may wonder whether there is a simplicial complex, a regular cell complex, or a CW-complex that supports $\mathcal{M}^\bullet$ so one may get a Hochster-like formula not only for the pieces of the local cohomology modules $H_I^r(R)$ but for its Bass numbers as well. Unfortunately this is not the case in general. To check this out we will make a detour through the theory of free resolutions of monomial ideals and we refer to the work of M. Velasco [41] to find examples of free resolutions that are not supported by CW-complexes.

4.1. **Building a dictionary.** The minimal graded free resolution of a monomial ideal J is an exact sequence of free $\mathbb{Z}^n$-graded R-modules:

$$\mathbb{L}_\bullet(J) : \quad 0 \longrightarrow L_m \xrightarrow{d_m} \cdots \longrightarrow L_1 \xrightarrow{d_1} L_0 \longrightarrow J \longrightarrow 0$$

where the j-th term is of the form

$$L_j = \bigoplus_{\alpha \in \mathbb{Z}^n} R(-\alpha)^{\beta_{j,\alpha}(J)},$$

and the matrices of the morphisms $d_j : L_j \longrightarrow L_{j-1}$ do not contain invertible elements. The $\mathbb{Z}^n$-graded *Betti numbers* of J are the invariants $\beta_{j,\alpha}(J)$. Given an integer r, the r-*linear strand* of $\mathbb{L}_\bullet(J)$ is the complex:

$$\mathbb{L}_\bullet^{<r>}(J): \quad 0 \longrightarrow L_{n-r}^{<r>} \xrightarrow{d_{n-r}^{<r>}} \cdots \longrightarrow L_1^{<r>} \xrightarrow{d_1^{<r>}} L_0^{<r>} \longrightarrow 0\ ,$$

where

$$L_j^{<r>} = \bigoplus_{|\alpha|=j+r} R(-\alpha)^{\beta_{j,\alpha}(J)},$$

and the differentials $d_j^{<r>} : L_j^{<r>} \longrightarrow L_{j-1}^{<r>}$ are the corresponding components of d_j. A combinatorial description of the first linear strand was given in [34].

E. Miller [30, 31] developed the notion of *monomial matrices* to encode the structure of free, injective and flat resolutions. These are matrices with scalar entries that keep track of the degrees of the generators of the summands in the source and the target. The goal of this Section is to show that the n-hypercube of a local cohomology module $H_I^r(R)$ has the same information as the r-linear strand of the Alexander dual ideal of I. More precisely, we will see that the matrices in the complex of k-vector spaces associated to the n-hypercube of $H_I^r(R)$ are the transpose of the monomial matrices of the r-linear strand[3].

M. Mustaţă [32] already proved the following relation between the pieces of the local cohomology modules and the Betti numbers of the Alexander dual ideal

$$\beta_{j,\alpha}(I^\vee) = \dim_k [H_I^{|\alpha|-j}(R)]_\alpha$$

so the pieces of $H_I^r(R)$ for a fixed r describe the modules and the Betti numbers of the r-linear strand of $I^\vee$. To prove the following proposition one has to put together some results scattered in the work of K. Yanagawa [43, 44].

Proposition 4.1. *Let* $\mathcal{M} = \{[H_I^r(R)]_\alpha\}_{\alpha\in\{0,1\}^n}$ *be the* n*-hypercube of a fixed local cohomology module* $H_I^r(R)$ *supported on a monomial ideal* $I \subseteq R = k[x_1,\ldots,x_n]$*. Then,* $\mathcal{M}^\bullet$ *is the complex of* k*-vector spaces whose matrices are the transpose of the monomial matrices of the* r*-linear strand* $\mathbb{L}_\bullet^{<r>}(I^\vee)$ *of the Alexander dual ideal of* I*.*

In [43] K. Yanagawa develops the notion of *squarefree module*, this is a $\mathbb{N}^n$-graded module M described by the graded pieces M_α, $\alpha \in \{0,1\}^n$ and the morphisms given by the multiplication by x_i. To such a module M he constructs a chain complex $\mathbb{F}_\bullet(M)$ of free R-modules as follows:

$$\mathbb{F}_\bullet(M): 0 \longrightarrow [M]_{\mathbf{1}} \otimes_k R \xrightarrow{d_0} \cdots \xrightarrow{d_{p-1}} \bigoplus_{|\alpha|=n-p} [M]_\alpha \otimes_k R \xrightarrow{d_p} \cdots \xrightarrow{d_{n-1}} [M]_{\mathbf{0}} \otimes_k R \longrightarrow 0$$

where the map between summands $[M]_{\alpha+\varepsilon_i} \otimes_k R \longrightarrow [M]_\alpha \otimes_k R$ sends $y \otimes 1 \in [M]_{\alpha+\varepsilon_i} \otimes_k R$ to $\operatorname{sign}(i, \alpha+\varepsilon_i)\ (x_i y \otimes x_i)$. For the particular case of $M = \operatorname{Ext}_R^r(R/I, R(-\mathbf{1}))$ he proved

[3] In the language of [33] we would say that the n-hypercube has the same information as the frame of the r-linear strand

an isomorphism (after an appropiate shifting) between $\mathbb{F}_\bullet(M)$ and the r-linear strand $\mathbb{L}_\bullet^{<r>}(I^\vee)$ of the Alexander dual ideal $I^\vee$ of I.

In [44] he proves that the categories of squarefree modules and straight modules are equivalent. Therefore one may also construct the chain complex $\mathbb{F}_\bullet(M)$ for any straight module M. The squarefree module $\mathrm{Ext}^r_R(R/I, R(-\mathbf{1}))$ corresponds[4] to the local cohomology modules $H^r_I(R)(-\mathbf{1})$ so there is an isomorphism between $\mathbb{F}_\bullet(H^r_I(R)(-\mathbf{1}))$ and the r-linear strand $\mathbb{L}_\bullet^{<r>}(I^\vee)$ after an appropriate shifting. Taking a close look to the construction of $\mathbb{F}_\bullet(M)$ one may check that the scalar entries in the corresponding monomial matrices are obtained by transposing the scalar entries in the one associated to the hypercube of $H^r_I(R)$ with the appropriate shift. More precisely, if

$$\mathbb{L}_\bullet^{<r>}(I^\vee): \quad 0 \longrightarrow L^{<r>}_{n-r} \longrightarrow \cdots \longrightarrow L^{<r>}_1 \longrightarrow L^{<r>}_0 \longrightarrow 0\,,$$

is the r-linear strand of the Alexander dual ideal $I^\vee$ then we transpose its monomial matrices to obtain a complex of k-vector spaces indexed as follows:

$$\mathbb{F}_\bullet^{<r>}(I^\vee)^*: \quad 0 \longleftarrow K^{<r>}_0 \longleftarrow \cdots \longleftarrow K^{<r>}_{n-r-1} \longleftarrow K^{<r>}_{n-r} \longleftarrow 0$$

Corollary 4.2. *Let $\mathbb{F}_\bullet^{<r>}(I^\vee)^*$ be the complex of k-vector spaces obtained from the r-linear strand of the minimal free resolution of the Alexander dual ideal $I^\vee$ transposing its monomial matrices. Then*

$$\lambda_{p,n-r}(R/I) = \dim_k H_p(\mathbb{F}_\bullet^{<r>}(I^\vee)^*)$$

It follows that one may think Lyubeznik numbers of a squarefree monomial I as a measure of the acyclicity of the r-linear strand of the Alexander dual $I^\vee$.

Remark 4.3. As a summary of the dictionary between local cohomology modules and free resolutions we have:

- The graded pieces $[H^r_I(R)]_\alpha$ correspond to the Betti numbers $\beta_{|\alpha|-r,\alpha}(I^\vee)$
- The n-hypercube of $H^r_I(R)$ corresponds to the r-linear strand $\mathbb{L}_\bullet^{<r>}(I^\vee)$

Given a free resolution $\mathbb{L}_\bullet$ of a finitely generated graded R-module M, D. Eisenbud, G. Fløystad and F.O. Schreyer [13] defined its *linear part* as the complex $\mathrm{lin}(\mathbb{L}_\bullet)$ obtained by erasing the terms of degree ≥ 2 from the matrices of the differential maps. To measure the acyclicity of the linear part, J. Herzog and S. Iyengar [23] introduced the *linearity defect* of M as $\mathrm{ld}_R(M) := \sup\{p \mid H_p(\mathrm{lin}(\mathbb{L}_\bullet))\}$. Therefore we also have:

- The n-hypercubes of $H^r_I(R)$, $\forall r$ correspond to the linear part $\mathrm{lin}(\mathbb{L}_\bullet(I^\vee))$
- The Lyubeznik table of R/I can be viewed as a generalization of $\mathrm{ld}_R(I^\vee)$

[4]In the terminology of E. Miller [30] one states that the Čech hull of $\mathrm{Ext}^r_R(R/I, R(-\mathbf{1}))$ is $H^r_I(R)(-\mathbf{1})$

4.2. **Examples.** It is well-known that Cohen-Macaulay squarefree monomial ideals have a trivial Lyubeznik table

$$\Lambda(R/I) = \begin{pmatrix} 0 & \cdots & 0 \\ & \ddots & \vdots \\ & & 1 \end{pmatrix}$$

because they only have one non-vanishing local cohomology module. Recall that its Alexander dual has a linear resolution (see [12]) so its acyclic. In general, there are non-Cohen-Macaulay ideals with trivial Lyubeznik table. Some of them are far from having only one local cohomology module different from zero.

Example 4.4. Consider the ideal in $k[x_1, \dots, x_9]$:

$$I = (x_1, x_2) \cap (x_3, x_4) \cap (x_5, x_6) \cap (x_7, x_8) \cap (x_9, x_1) \cap (x_9, x_2) \cap (x_9, x_3) \cap (x_9, x_4) \cap (x_9, x_5) \cap \\ \cap (x_9, x_6) \cap (x_9, x_7) \cap (x_9, x_8)$$

The non-vanishing local cohomology modules are $H^r_I(R)$, $r = 2, 3, 4, 5$ but the Lyubeznik table is trivial.

One may characterize ideals with trivial Lyubeznik table using a weaker condition than being Cohen-Macaulay, the class of *sequentially Cohen-Macaulay* ideals given by R. Stanley [39]. J. Herzog and T. Hibi [22] introduced the class of *componentwise linear* ideals and proved that their Alexander dual are sequentially Cohen-Macaulay. The following result is a direct consequence of [43, Prop. 4.9], [36, Thm. 3.2.8] where componentwise linear ideals are characterized as those having acyclic linear strands.

Proposition 4.5. *Let $I \subseteq R = k[x_1, \dots, x_n]$ be a squarefree monomial ideal. Then, the following conditions are equivalent:*

i) *R/I is sequentially Cohen-Macaulay.*
ii) *R/I has a trivial Lyubeznik table.*

The simplest examples of ideals with non-trivial Lyubeznik table are minimal non-Cohen-Macaulay squarefree monomial ideals (see [27])

Example 4.6. The unique minimal non-Cohen-Macaulay squarefree monomial ideal of pure height two in $R = k[x_1, \dots, x_n]$ is:

$$\mathfrak{a}_n = (x_1, x_3) \cap \cdots \cap (x_1, x_{n-1}) \cap (x_2, x_4) \cap \cdots \cap (x_2, x_n) \cap (x_3, x_5) \cap \cdots \cap (x_{n-2}, x_n).$$

- $\mathfrak{a}_4 = (x_1, x_3) \cap (x_2, x_4)$.

We have $H^2_{\mathfrak{a}_4}(R) \cong H^2_{(x_1,x_3)}(R) \oplus H^2_{(x_2,x_4)}(R)$ and $H^3_{\mathfrak{a}_4}(R) \cong E_{\mathbf{1}}$. Thus its Lyubeznik table is

$$\Lambda(R/\mathfrak{a}_4) = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 2 \end{pmatrix}$$

- $\mathfrak{a}_5 = (x_1, x_3) \cap (x_1, x_4) \cap (x_2, x_4) \cap (x_2, x_5) \cap (x_3, x_5)$.

We have $H^3_{\mathfrak{a}_5}(R) \cong E_{\mathbf{1}}$ and the hypercube associated to $H^2_{\mathfrak{a}_5}(R)$ satisfy $[H^2_{\mathfrak{a}_5}(R)]_\alpha \cong k$ for

- $\alpha = (1,0,1,0,0), (1,0,0,1,0), (0,1,0,1,0), (0,1,0,0,1), (0,0,1,0,1)$
- $\alpha = (1,1,0,1,0), (1,0,1,1,0), (1,0,1,0,1), (0,1,1,0,1), (0,1,0,1,1)$

The complex associated to the hypercube is

$$0 \longleftarrow 0 \longleftarrow 0 \longleftarrow k^5 \xleftarrow{u_2} k^5 \longleftarrow 0 \longleftarrow 0 \longleftarrow 0$$

where the matrix corresponding to u_2 is the rank 4 matrix:

$$\begin{pmatrix} 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 \end{pmatrix}$$

Thus its Lyubeznik table is

$$\Lambda(R/\mathfrak{a}_5) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 \\ & & & 1 \end{pmatrix}$$

One should notice that $H^2_{\mathfrak{a}_5}(R)$ is irreducible since all the extension problems associated to it are non-trivial.

Remark 4.7. In general one gets

$$\Lambda(R/\mathfrak{a}_n) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & 0 & & 0 & 0 & 1 \\ & & & \ddots & & 0 & 0 \\ & & & & & \vdots & \vdots \\ & & & & & 0 & 0 \\ & & & & & & 1 \end{pmatrix}$$

and the result agrees with [37, Cor. 5.5]

It is well-know that local cohomology modules as well as free resolutions depend on the characteristic of the base field, the most recurrent example being the Stanley-Reisner ideal associated to a minimal triangulation of $\mathbb{P}^2_{\mathbb{R}}$. Thus, Lyubeznik numbers also depend on the characteristic.

Example 4.8. Consider the ideal in $R = k[x_1, \ldots, x_6]$:

$$I = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_2x_4x_5, x_3x_4x_5, x_2x_3x_6, x_1x_4x_6, x_3x_4x_6, x_1x_5x_6, x_2x_5x_6)$$

The Lyubeznik table in characteristic zero and two are respectively:

$$\Lambda_{\mathbb{Q}}(R/I) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 1 \end{pmatrix} \quad \Lambda_{\mathbb{Z}/2\mathbb{Z}}(R/I) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 \\ & & 0 & 1 \\ & & & 1 \end{pmatrix}$$

5. Injective dimension of local cohomology modules

Let $(R, \mathfrak{m}, k)$ be a local ring and let M be an R-module. The small support of M introduced by H. B. Foxby [14] is defined as

$$\mathrm{supp}_R M := \{\mathfrak{p} \in \mathrm{Spec} R \mid \mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty\},$$

where $\mathrm{depth}_R M := \inf\{i \in \mathbb{Z} \mid \mathrm{Ext}^i_R(R/\mathfrak{m}, M) \neq 0\} = \inf\{i \in \mathbb{Z} \mid \mu_i(\mathfrak{m}, M) \neq 0\}$. In terms of Bass numbers we have that $\mathfrak{p} \in \mathrm{supp}_R M$ if and only if there exists some integer $i \geq 0$ such that $\mu_i(\mathfrak{p}, M) \neq 0$. It is also worth to point out that $\mathrm{supp}_R M \subseteq \mathrm{Supp}_R M$, and equality holds when M is finitely generated.

Bass numbers of finitely generated modules are known to satisfy the following properties:

1) $\mu_i(\mathfrak{p}, M) < +\infty$, $\forall i$, $\forall \mathfrak{p} \in \mathrm{Supp}_R M$
2) Let $\mathfrak{p} \subseteq \mathfrak{q} \in \mathrm{Spec} R$ such that $\mathrm{ht}\,(\mathfrak{q}/\mathfrak{p}) = s$. Then
$$\mu_i(\mathfrak{p}, M) \neq 0 \Longrightarrow \mu_{i+s}(\mathfrak{q}, M) \neq 0.$$
3) $\mathrm{id}_R M := \sup\{i \in \mathbb{Z} \mid \mu_i(\mathfrak{m}, M) \neq 0\}$
4) $\mathrm{depth}_R M \leq \dim_R M \leq \mathrm{id}_R M$

When M is not finitely generated, similar properties for Bass numbers are known for some special cases. A. M. Simon [38] proved that properties 2) and 3) are still true for complete modules and M. Hellus [21] proved that $\dim_R M \leq \mathrm{id}_R M$ for cofinite modules.

For the case of local cohomology modules, C. Huneke and R. Sharp [24] and G. Lyubeznik [28, 29], proved that for a regular local ring $(R, \mathfrak{m}, k)$ containing a field k:

1) $\mu_i(\mathfrak{p}, H^r_I(R)) < +\infty$, $\forall i$, $\forall r$, $\forall \mathfrak{p} \in \mathrm{Supp}_R H^r_I(R)$

4') $\mathrm{id}_R H^r_I(R) \leq \dim_R H^r_I(R)$

In this Section we want to study property 2) for the particular case of local cohomology modules supported on monomial ideals and give a sharper bound to $4')$ in terms of the small support. We start with the following well-known general result on the minimal primes in the support of local cohomology modules.

Proposition 5.1. *Let $(R, \mathfrak{m})$ be a regular local ring containing a field k, $I \subseteq R$ be any ideal and $\mathfrak{p} \in \mathrm{Supp}_R\, H^r_I(R)$ be a minimal prime. Then we have $\mu_0(\mathfrak{p}, H^r_I(R)) \neq 0$, $\mu_i(\mathfrak{p}, H^r_I(R)) = 0$ $\forall i > 0$.*

Proof. $\dim H^r_I(R)_\mathfrak{p} = 0$ so $H^r_I(R)_\mathfrak{p} \cong E(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})^{\mu_0(\mathfrak{p}, H^r_I(R))}$ by [28, Thm 3.4] □

Corollary 5.2. *Let $(R, \mathfrak{m}, k)$ be a regular local ring containing a field k and $I \subseteq R$ be any ideal. If $\mathfrak{p} \in \operatorname{Supp}_R H^r_I(R)$ is minimal then $\mathfrak{p} \in \operatorname{supp}_R H^r_I(R)$. Thus, $\operatorname{Supp}_R H^r_I(R)$ and $\operatorname{supp}_R H^r_I(R)$ have the same minimal primes.*

The converse statement in Proposition 5.1 does not hold true.

Example 5.3. Consider the monomial ideal $I = (x_1, x_2, x_5) \cap (x_3, x_4, x_5) \cap (x_1, x_2, x_3, x_4)$. The support of the corresponding local cohomology modules are:

$\operatorname{Supp}_R H^3_I(R) = V(x_1, x_2, x_5) \cup V(x_3, x_4, x_5)$.
$\operatorname{Supp}_R H^4_I(R) = V(x_1, x_2, x_3, x_4)$.

The Bass numbers of $H^3_I(R)$ and $H^4_I(R)$ are respectively

$\mathfrak{p}_\alpha$	μ_0	μ_1	μ_2
(x_1, x_2, x_5)	1	-	-
(x_3, x_4, x_5)	1	-	-
(x_1, x_2, x_i, x_5)	-	1	-
(x_i, x_3, x_4, x_5)	-	1	-
$(x_1, x_2, x_3, x_4, x_5)$	-	-	2

$\mathfrak{p}_\alpha$	μ_0	μ_1	μ_2
(x_1, x_2, x_3, x_4)	1	-	-
$(x_1, x_2, x_3, x_4, x_5)$	1	-	-

In particular, its Lyubeznik table is

$$\Lambda(R/I) = \begin{pmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 2 \end{pmatrix}$$

Notice that $\mathfrak{m} = (x_1, x_2, x_3, x_4, x_5)$ is not a minimal prime in the support of $H^4_I(R)$ but $\mu_0(\mathfrak{m}, H^4_I(R)) \neq 0$, $\mu_i(\mathfrak{m}, H^4_I(R)) = 0\ \forall i > 0$. We have to point out that this module is not irreducible[5]

$$H^4_I(R) \cong E_{(1,1,1,1,0)} \oplus E_{(1,1,1,1,1)}$$

From now on we will stick to the case of local cohomology modules supported on squarefree monomial ideals. The methods developed in the previous Sections allow us to describe the Bass numbers in the minimal *injective resolution of a module with variation zero M. That is:

$$\mathbb{I}^\bullet(M): \quad 0 \longrightarrow I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{m-1}} I^m \xrightarrow{d^m} \cdots,$$

where the j-th term is

$$I^j = \bigoplus_{\alpha \in \{0,1\}^n} E_\alpha^{\mu_j(\mathfrak{p}_\alpha, M)} = \bigoplus_{\alpha \in \{0,1\}^n} {}^*E(R/\mathfrak{p}_\alpha)(\mathbf{1})^{\mu_\mathbf{j}(\mathfrak{p}_\alpha, \mathbf{M})}.$$

In particular we are able to compute the injective dimension of M in the category of $\mathbb{Z}^n$-graded R-modules that we denote $^*\mathrm{id}_R M$. We can also define the $\mathbb{Z}^n$-graded small support that we denote $^*\mathrm{supp}_R M$ as the set of face ideals in the support of M that at least have a Bass number different from zero.

[5]It is enough to check out the corresponding n-hypercubes.

If we want to compute the Bass numbers with respect to any prime ideal, the injective dimension of M as R-module and the small support we have to refer to the result of S. Goto and K. I. Watanabe [18, Thm. 1.2.3]. Namely, given any prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, let $\mathfrak{p}_\alpha$ be the largest face ideal contained in $\mathfrak{p}$. If $\operatorname{ht}(\mathfrak{p}/\mathfrak{p}_\alpha) = s$ then $\mu_p(\mathfrak{p}_\alpha, M) = \mu_{p+s}(\mathfrak{p}, M)$. Notice that in general we have ${}^*\mathrm{id}_R M \leq \mathrm{id}_R M$.

To compare the injective dimension and the dimension of a local cohomology module $M = H^r_I(R)$ we are going to consider chains of prime face ideals $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{m}$ in the support of M such that $\mathfrak{p}_0$ is minimal. The Bass numbers with respect to $\mathfrak{p}_0$ are completely determined and, even though property 2) is no longer true, we have some control on the Bass numbers of $\mathfrak{p}_i$ depending on the structure of the corresponding n-hypercube. For simplicity, assume that $\mathfrak{p}_i$ is a face ideal $\mathfrak{p}_\alpha \subseteq \mathfrak{m}$ of height $n-1$ and $x_n \in \mathfrak{m} \setminus \mathfrak{p}_\alpha$ and that the Bass numbers with respect to $\mathfrak{p}_\alpha$ are known. We are going to compute the Bass numbers with respect to $\mathfrak{m}$ using the degree $\mathbf{1}$ part of the exact sequence of Čech complexes

$$0 \longrightarrow \check{C}^\bullet_{\mathfrak{p}_\alpha}(M_{x_n})[-1] \longrightarrow \check{C}^\bullet_{\mathfrak{m}}(M) \longrightarrow \check{C}^\bullet_{\mathfrak{p}_\alpha}(M) \longrightarrow 0$$

Let $\mathcal{M}^\bullet$ be the complex associated to the n-hypercube of M that is isomorphic to $[\check{C}^\bullet_{\mathfrak{m}}(M)]_{\mathbf{1}}$. For any $\beta \in \{0,1\}^n$, let $\mathcal{M}^\bullet_{\leq\beta}$ (resp. $\mathcal{M}^\bullet_{\geq\beta}$) be the subcomplex of $\mathcal{M}^\bullet$ with pieces of degree $\leq \beta$ (resp. $\geq \beta$). Using the techniques of Section 3 one may see that

$$0 \longleftarrow [\check{C}^\bullet_{\mathfrak{p}_\alpha}(M_{x_n})[-1]]_{\mathbf{1}} \longleftarrow [\check{C}^\bullet_{\mathfrak{m}}(M)]_{\mathbf{1}} \longleftarrow [\check{C}^\bullet_{\mathfrak{p}_\alpha}(M)]_{\mathbf{1}} \longleftarrow 0$$

is isomorphic to the short exact sequence

$$0 \longleftarrow \mathcal{M}^\bullet_{\leq\alpha} \longleftarrow \mathcal{M}^\bullet \longleftarrow \mathcal{M}^\bullet_{\geq \mathbf{1}-\alpha} \longleftarrow 0$$

Example 5.4. The short exact sequence $0 \longleftarrow \mathcal{M}^\bullet_{\leq(1,1,0)} \longleftarrow \mathcal{M}^\bullet \longleftarrow \mathcal{M}^\bullet_{\geq(0,0,1)} \longleftarrow 0$ can be visualized from the corresponding 3-hypercube as follows:

$$\begin{tikzcd}
& M_{(0,0,0)} \arrow[dl, "u_1"'] \arrow[d, "u_2"] \\
M_{(1,0,0)} \arrow[d, "u_2"'] & M_{(0,1,0)} \arrow[dl, "u_1"] \\
M_{(1,1,0)} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& M_{(0,0,0)} \arrow[dl, "u_1"'] \arrow[d, "u_2"] \arrow[dr, "u_3"] & \\
M_{(1,0,0)} \arrow[d, "u_2"'] \arrow[dr, "u_3"] & M_{(0,1,0)} \arrow[dl, "u_1"] \arrow[dr, "u_3"] & M_{(0,0,1)} \arrow[dl, "u_1"'] \arrow[d, "u_2"] \\
M_{(1,1,0)} \arrow[dr, "u_3"'] & M_{(1,0,1)} \arrow[d, "u_2"] & M_{(0,1,1)} \arrow[dl, "u_1"] \\
& M_{(1,1,1)} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& M_{(0,0,1)} \arrow[dl, "u_1"'] \arrow[d, "u_2"] \\
M_{(1,0,1)} \arrow[d, "u_2"'] & M_{(0,1,1)} \arrow[dl, "u_1"] \\
M_{(1,1,1)} &
\end{tikzcd}$$

At this point we should notice the following key observations that we will use throughout this Section:

i) We have $\mathcal{M}^\bullet_{\leq\alpha} \cong [\check{C}^\bullet_{\mathfrak{p}_\alpha}(M_{x_n})[-1]]_{\mathbf{1}} \cong [\check{C}^\bullet_{\mathfrak{p}_\alpha}(M)]_\alpha$, thus $\mu_p(\mathfrak{p}_\alpha, M) = \dim_k H_p(\mathcal{M}^\bullet_{\leq\alpha})$.

ii) Consider the long exact sequence

$$\cdots \longrightarrow H^{p-1}_{\mathfrak{p}_\alpha}(M_{x_n}) \longrightarrow H^p_{\mathfrak{m}}(M) \longrightarrow H^p_{\mathfrak{p}_\alpha}(M) \xrightarrow{\delta^p} H^p_{\mathfrak{p}_\alpha}(M_{x_n}) \longrightarrow H^{p+1}_{\mathfrak{m}}(M) \longrightarrow \cdots$$

associated to the short exact sequence of Čech complexes. Its degree $\mathbf{1}$ part is

$$\cdots \longleftarrow H_{p-1}(\mathcal{M}^{\bullet}_{\leq\alpha}) \longleftarrow H_p(\mathcal{M}^{\bullet}) \longleftarrow H_p(\mathcal{M}^{\bullet}_{\geq \mathbf{1}-\alpha}) \xleftarrow{\delta^p} H_p(\mathcal{M}^{\bullet}_{\leq\alpha}) \longleftarrow H_{p+1}(\mathcal{M}^{\bullet}) \longleftarrow \cdots$$

but it might be useful to view it as

$$\cdots \longleftarrow k^{\mu_{p-1}(\mathfrak{p}_\alpha,M)} \longleftarrow [H^p_{\mathfrak{m}}(M)]_{\mathbf{1}} \longleftarrow [H^p_{\mathfrak{p}_\alpha}(M)]_{\mathbf{1}} \xleftarrow{\delta^p} k^{\mu_p(\mathfrak{p}_\alpha,M)} \longleftarrow [H^{p+1}_{\mathfrak{m}}(M)]_{\mathbf{1}} \longleftarrow \cdots$$

or even as the complex

$$\cdots \longleftarrow k^{\mu_{p-1}(\mathfrak{p}_\alpha,M)} \longleftarrow k^{\mu_p(\mathfrak{m},M)} \longleftarrow [H^p_{\mathfrak{p}_\alpha}(M)]_{\mathbf{1}} \xleftarrow{\delta^p} k^{\mu_p(\mathfrak{p}_\alpha,M)} \longleftarrow k^{\mu_{p+1}(\mathfrak{m},M)} \longleftarrow \cdots$$

Notice that the connecting morphisms δ^p are the classes, in the corresponding homology groups, of the canonical morphisms $u_{\alpha,n}$ that describe the n-hypercube of M.

iii) The 'difference' between $\mu_p(\mathfrak{p}_\alpha, M)$ and $\mu_{p+1}(\mathfrak{m}, M)$, i.e. the 'difference' between $H_p(\mathcal{M}^{\bullet}_{\leq\alpha})$ and $H_{p+1}(\mathcal{M}^{\bullet})$, comes from the homology of the complex $\mathcal{M}^{\bullet}_{\geq\mathbf{1}-\alpha}$. Roughly speaking, it comes from the contribution of other chains of prime face ideals $\mathfrak{q}_0 \subseteq \mathfrak{q}_1 \subseteq \cdots \subseteq \mathfrak{m}$ in the support of a local cohomology module $M = H^r_I(R)$ such that $\mathfrak{q}_0$ is minimal and not containing $\mathfrak{p}_\alpha$[6].

Discussion 1: Consider the case where $\mathfrak{p}_\alpha \subseteq \mathfrak{m}$ is a minimal prime of height $n-1$ in the support of M. We have

$$0 \longleftarrow [H^0_{\mathfrak{m}}(M)]_{\mathbf{1}} \longleftarrow [H^0_{\mathfrak{p}_\alpha}(M)]_{\mathbf{1}} \xleftarrow{\delta^0} k \longleftarrow [H^1_{\mathfrak{m}}(M)]_{\mathbf{1}} \longleftarrow [H^1_{\mathfrak{p}_\alpha}(M)]_{\mathbf{1}} \longleftarrow 0$$

and $[H^i_{\mathfrak{m}}(M)]_{\mathbf{1}} \cong [H^i_{\mathfrak{p}_\alpha}(M)]_{\mathbf{1}}$, for all $i \geq 2$, so $\mathfrak{p}_\alpha$ contributes to $\mu_p(\mathfrak{m}, M)$ for $p = 0, 1$. In particular, we have:

- $\mu_0(\mathfrak{m}, M) = 0$ if and only if $[H^0_{\mathfrak{p}_\alpha}(M)]_{\mathbf{1}} = 0$ or $[H^0_{\mathfrak{p}_\alpha}(M)]_{\mathbf{1}} = k$ and $\delta^0 \neq 0$.
- $\mu_1(\mathfrak{m}, M) = 0$ if and only if $[H^1_{\mathfrak{p}_\alpha}(M)]_{\mathbf{1}} = 0$ and $\delta^0 \neq 0$.

The non-vanishing of the 0-th Bass number is related to the decomposability of the local cohomology module. One should compare the following result with Prop 5.1 and check out the local cohomology module $H^4_I(R)$ in Example 5.3.

Proposition 5.5. *Let $\mathfrak{p}_\alpha \in \operatorname{Supp}_R H^r_I(R)$ be a prime ideal such that $\mu_0(\mathfrak{p}_\alpha, H^r_I(R)) \neq 0$. Then, $E_\alpha^{\mu_0(\mathfrak{p}_\alpha, H^r_I(R))}$ is a direct summand of $H^r_I(R)_{\mathfrak{p}_\alpha}$.*

Proof. We assume that $\mathfrak{p}_\alpha = \mathfrak{m}$ and we denote $\mu_0 := \mu_0(\mathfrak{m}, H^r_I(R))$. The first terms of the complex $\mathcal{M}^{\bullet}$ associated to the n-hypercube of $H^r_I(R)$ have the form $0 \longleftarrow k^{m_0} \xleftarrow{u_0} k^{m_1}$ with $\mu_0 = m_0 - \operatorname{rk} u_0 > 0$. The linear map $u_0 = \oplus_{|\alpha|=n-1} u_{\alpha,i}$ determine the extension classes of the short exact sequence $0 \longrightarrow F_{n-1} \longrightarrow H^r_I(R) \longrightarrow E_{\mathbf{1}}^{m_0} \longrightarrow 0$ associated to the filtration $\{F_j\}_{0\leq j\leq n}$ of $H^r_I(R)$. Thus, we have a decomposition $H^r_I(R) \cong E_{\mathbf{1}}^{\mu_0} \oplus M$, where M corresponds to the extension $0 \longrightarrow F_{n-1} \longrightarrow M \longrightarrow E_{\mathbf{1}}^{\operatorname{rk} u_0} \longrightarrow 0$. ☐

[6] We use the fact that for local cohomology modules $M = H^r_I(R)$ the degree $\mathbf{0}$ part of the n-hypercube is always zero, i.e. its minimal primes have height > 0

Discussion 2: In general, let $s = \max\{i \in \mathbb{Z}_{\geq 0} \mid \mu_i(\mathfrak{p}_\alpha, M) \neq 0\}$, then we have

$$\cdots \longleftarrow [H^s_{\mathfrak{m}}(M)]_{\mathbf{1}} \longleftarrow [H^s_{\mathfrak{p}_\alpha}(M)]_{\mathbf{1}} \xleftarrow{\delta^s} k^{\mu_s(\mathfrak{p}_\alpha, M)} \longleftarrow [H^{s+1}_{\mathfrak{m}}(M)]_{\mathbf{1}} \longleftarrow [H^{s+1}_{\mathfrak{p}_\alpha}(M)]_{\mathbf{1}} \longleftarrow 0$$

and $[H^i_{\mathfrak{m}}(M)]_{\mathbf{1}} \cong [H^i_{\mathfrak{p}_\alpha}(M)]_{\mathbf{1}}$, for all $i \geq s+2$, so $\mathfrak{p}_\alpha$ contributes to $\mu_p(\mathfrak{m}, M)$ for $p \leq s+1$. Again, we can describe conditions for the vanishing of $\mu_s(\mathfrak{m}, M)$ and $\mu_{s+1}(\mathfrak{m}, M)$ in terms of the connecting morphism δ^s. One can find examples where any situation is possible.

- The local cohomology module $H^3_I(R)$ in Example 5.3 satisfies for any $\mathfrak{p}_\alpha \subseteq \mathfrak{m}$ such that $\mathrm{ht}\,(\mathfrak{m}/\mathfrak{p}_\alpha) = 1$, $\mu_s(\mathfrak{p}_\alpha, M) \neq 0$, $\mu_{s+1}(\mathfrak{m}, M) \neq 0$ and $\mu_s(\mathfrak{m}, M) = 0$ for $s = 1$.
- The local cohomology module $H^4_I(R)$ in Example 5.3 satisfies $\mu_s(\mathfrak{p}_{(1,1,1,1,0)}, M) = \mu_s(\mathfrak{m}, M) = 1$ and $\mu_{s+1}(\mathfrak{m}, M) = 0$ for $s = 0$.
- The local cohomology module $H^3_I(R)$ in Example 5.7 satisfies $\mu_s(\mathfrak{p}_{(1,1,1,0,0)}, M) = 1$ and $\mu_s(\mathfrak{p}_{(1,1,1,1,0)}, M) = \mu_{s+1}(\mathfrak{p}_{(1,1,1,1,0)}, M) = 0$ for $s = 0$.
- The local cohomology module $H^2_{\mathfrak{a}_5}(R)$ in Example 4.4 satisfies $\mu_s(\mathfrak{p}_{(1,1,1,0,1)}, M) = \mu_s(\mathfrak{m}, M) = \mu_{s+1}(\mathfrak{m}, M) = 1$ for $s = 2$.

Remark 5.6. One might be tempted to think that the condition $\mu_s(\mathfrak{p}_\alpha, M) \neq 0$ and $\mu_s(\mathfrak{m}, M) \neq 0$ is related to the decomposability of the corresponding module M. This is not the case as it shows Example 3.5 where we have a short exact sequence

$$0 \longleftarrow [H^2_{\mathfrak{m}}(M)]_{\mathbf{1}} \longleftarrow [H^2_{\mathfrak{p}_{(1,1,1,0,1)}}(M)]_{\mathbf{1}} \cong k \xleftarrow{\delta^2} k \longleftarrow [H^3_{\mathfrak{m}}(M)]_{\mathbf{1}} \longleftarrow 0$$

where the connecting morphism δ^2 is zero even though the local cohomology module $H^2_{\mathfrak{a}_5}(R)$ is indecomposable, i.e. the canonical morphisms $u_{\alpha,i}$ are not trivial but their classes in homology make the connecting morphism trivial.

Discussion 3: In the case that there exists a prime ideal $\mathfrak{p}_\alpha \notin \mathrm{supp}_R M$, then we have $[H^i_{\mathfrak{m}}(M)]_{\mathbf{1}} \cong [H^i_{\mathfrak{p}_\alpha}(M)]_{\mathbf{1}}$, for all i. Therefore, the contribution to the Bass numbers $\mu_p(\mathfrak{m}, M)$ comes from other chains of prime face ideals $\mathfrak{q}_0 \subseteq \mathfrak{q}_1 \subseteq \cdots \subseteq \mathfrak{m}$. This is what happens in the following example:

Example 5.7. Consider the ideal $I = (x_1, x_4) \cap (x_2, x_5) \cap (x_1, x_2, x_3)$. The non-vanishing pieces of the hypercube associated to the corresponding local cohomology modules are:

$[H^2_I(R)]_\alpha = k$ for $\alpha = (1,0,0,1,0), (0,1,0,0,1)$.
$[H^3_I(R)]_\alpha = k$ for $\alpha = (1,1,1,0,0), (1,1,1,1,0), (1,1,1,0,1), (1,1,0,1,1), (1,1,1,1,1)$.

Notice that $H^2_I(R) \cong H^2_{(x_1,x_4)}(R) \oplus H^2_{(x_2,x_5)}(R)$ and the complex associated to the hypercube of $H^3_I(R)$ is:

$$0 \longleftarrow k \xleftarrow{(1\ \ 1\ \ 1)} k^3 \xleftarrow{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}} k \longleftarrow 0$$

Thus, the Bass numbers of $H^2_I(R)$ and $H^3_I(R)$ are respectively

$\mathfrak{p}_\alpha$	μ_0	μ_1	μ_2	μ_3
(x_1, x_4)	1	-	-	-
(x_2, x_5)	1	-	-	-
(x_1, x_4, x_i)	-	1	-	-
(x_2, x_5, x_i)	-	1	-	-
(x_1, x_4, x_i, x_j)	-	-	1	-
(x_2, x_5, x_i, x_j)	-	-	1	-
$(x_1, x_2, x_3, x_4, x_5)$	-	-	-	2

,

$\mathfrak{p}_\alpha$	μ_0	μ_1	μ_2
(x_1, x_2, x_3)	1	-	-
(x_1, x_2, x_3, x_4)	-	-	-
(x_1, x_2, x_3, x_5)	-	-	-
(x_1, x_2, x_4, x_5)	1	-	-
$(x_1, x_2, x_3, x_4, x_5)$	-	1	-

Its Lyubeznik table is:

$$\Lambda(R/I) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 \\ & & 0 & 0 \\ & & & 2 \end{pmatrix}$$

We also have:

$\cdot\ {}^*\mathrm{id}_R H^2_I(R) = \dim H^2_I(R) = 3.$

$\cdot\ \mathrm{Min}_R(H^2_I(R)) = \mathrm{Ass}_R(H^2_I(R)).$

$\cdot\ \mathrm{supp}_R(H^2_I(R)) = \mathrm{Supp}_R(H^2_I(R)).$

$\cdot\ 1 = {}^*\mathrm{id}_R H^3_I(R) < \dim H^3_I(R) = 2.$

$\cdot\ \mathrm{Min}_R(H^3_I(R)) = \mathrm{Ass}_R(H^3_I(R)).$

$\cdot\ \mathrm{supp}_R(H^3_I(R)) \subset \mathrm{Supp}_R(H^3_I(R)).$

In particular (x_1, x_2, x_3, x_4) and (x_1, x_2, x_3, x_5) do not belong to $\mathrm{supp}_R(H^3_I(R))$.

It follows from the previous discussions that the length of the injective resolution of the local cohomology module $H^r_I(R)$ has a controlled growth when we consider chains of prime face ideals $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{m}$ starting with a minimal prime ideal $\mathfrak{p}_0$.

Proposition 5.8. *Let $I \subseteq R = k[x_1, \dots, x_n]$ be a squarefree monomial ideal and set*

$$s := \max\{i \in \mathbb{Z}_{\geq 0} \mid \mu_i(\mathfrak{p}_\alpha, H^r_I(R)) \neq 0\}$$

for all prime ideals $\mathfrak{p}_\alpha \in \mathrm{Supp}_R\, H^r_I(R)$ such that $|\alpha| = n - 1$. Then $\mu_t(\mathfrak{m}, H^r_I(R)) = 0$ $\forall t > s + 1$.

Therefore we get the main result of this Section:

Theorem 5.9. *Let $I \subseteq R = k[x_1, \dots, x_n]$ be a squarefree monomial ideal. Then, $\forall r$ we have*

$${}^*\mathrm{id}_R H^r_I(R) \leq \dim_R {}^*\mathrm{supp}_R H^r_I(R)$$

Remark 5.10. Using [18, Thm. 1.2.3] we also have

$$\mathrm{id}_R H^r_I(R) \leq \dim_R \mathrm{supp}_R H^r_I(R)$$

but one must be careful with the ring R we consider. In the example above we have:

$\cdot\ {}^*\mathrm{id}_R H^3_I(R) = \mathrm{id}_R H^3_I(R) < \dim_R \mathrm{supp}_R H^3_I(R)$ if $R = k[[x_1, \dots, x_n]]$

$\cdot\ {}^*\mathrm{id}_R H^3_I(R) < \mathrm{id}_R H^3_I(R) = \dim_R \mathrm{supp}_R H^3_I(R)$ if $R = k[x_1, \dots, x_n]$.

Remark 5.11. Consider the largest chain of prime face ideals $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n$ in the small support of a local cohomology module $H^r_I(R)$. In these best case scenario we have a version of property 2) that we introduced at the beginning of this Section that reads off as:

· $\mu_0(\mathfrak{p}_0, H^r_I(R)) = 1$ and $\mu_j(\mathfrak{p}_0, H^r_I(R)) = 0\ \forall j > 0$.

· $\mu_i(\mathfrak{p}_i, H^r_I(R)) \neq 0$ and $\mu_j(\mathfrak{p}_i, H^r_I(R)) = 0\ \forall j > i$, for all $i = 1, ..., n$.

Then:

i) $\mathrm{id}_R H^r_I(R) = \dim_R(\mathrm{supp}_R H^r_I(R))$ if and only if this version of property 2) is satisfied.

ii) $\mathrm{id}_R H^r_I(R) = \dim_R H^r_I(R)$ if and only if this version ofproperty 2) is satisfied and $\mathfrak{m} \in \mathrm{supp}_R H^r_I(R)$.

This sheds some light on the examples treated in [21] where the question whether the equality $\mathrm{id}_R H^r_I(R) = \dim_R H^r_I(R)$ holds is considered. On the other end of possible cases we may have:

· $\mu_0(\mathfrak{p}_0, H^r_I(R)) = \mu_0(\mathfrak{p}_n, H^r_I(R)) = 1$ and $\mu_j(\mathfrak{p}_0, H^r_I(R)) = \mu_j(\mathfrak{p}_n, H^r_I(R)) = 0\ \forall j > 0$.

Notice that in this case the same property holds for any prime ideal $\mathfrak{p}_i$ in the chain. In particular all the primes in the chain are associated primes of $H^r_I(R)$.

6. Matlis dual of local cohomology modules

The minimal projective resolution of a regular holonomic $D_{R|k}$-module with variation zero M is in the form:

$$\mathbb{P}^\bullet(\mathcal{M}) : \quad \cdots \xrightarrow{d^m} P^m \xrightarrow{d^{m-1}} \cdots \xrightarrow{d^1} P^1 \xrightarrow{d^0} P^0 \longrightarrow 0 \ ,$$

where the j-th term is

$$P^j = \bigoplus_{\alpha \in \{0,1\}^n} {R_{\mathbf{x}^\alpha}}^{\pi_j(\mathfrak{p}_\alpha, M)}.$$

The dual Bass numbers of M with respect to the face ideal $\mathfrak{p}_\alpha \subseteq R$ are the invariants defined by $\pi_j(\mathfrak{p}_\alpha, M)$. These invariants can be computed using the following form of Matlis duality introduced in [3]:

$$M^* := \mathrm{Hom}_{D_{R|k}}(M, E_{\mathbf{1}})$$

This is a shift by $\mathbf{1}$ of the usual Matlis duality of $\mathbb{Z}^n$-graded modules but it has the advantage of being a duality in the lattice $\{0,1\}^n$, i.e. is a duality of the type $\alpha \to \mathbf{1} - \alpha$ instead of a duality of the type $\alpha \to -\alpha$ among its graded pieces. In particular, the n-hypercube $\mathcal{M}^*$ corresponding to M^* satisfy:

· $\mathcal{M}^*_\alpha = \mathcal{M}_{\mathbf{1}-\alpha}$

· The map $u^*_{\alpha,i} : \mathcal{M}^*_\alpha \longrightarrow \mathcal{M}^*_{\alpha+\varepsilon_i}$ is the dual of $u_{\mathbf{1}-\alpha-\varepsilon_i,i} : \mathcal{M}_{\mathbf{1}-\alpha-\varepsilon_i} \longrightarrow \mathcal{M}_{\mathbf{1}-\alpha}$.

It is easy to check out that the Matlis dual of an injective $D^T_{v=0}$-module is projective, more precisely we have $E^*_\alpha = R_{\mathbf{x}^{\mathbf{1}-\alpha}}$ and the Matlis dual of a simple $D^T_{v=0}$-module is simple, namely we have $(H^{|\alpha|}_{\mathfrak{p}_\alpha}(R))^* = H^{|\mathbf{1}-\alpha|}_{\mathfrak{p}_{\mathbf{1}-\alpha}}(R)$.

Proposition 6.1. [3, Prop. 5.3] *With the previous notation*

$$\pi_p(\mathfrak{p}_\alpha, M) := \mu_p(\mathfrak{p}_{\mathbf{1}-\alpha}, M^*).$$

To compute the later Bass number we may assume $\mathfrak{p}_{\mathbf{1}-\alpha} = \mathfrak{m}$ is the maximal ideal just using localization so it boils down to compute the homology of the degree $\mathbf{1}$ part of the Čech complex:

$$[\check{C}_{\mathfrak{m}}(M^*)]^\bullet_{\mathbf{1}} : 0 \longleftarrow [M^*]_{\mathbf{1}} \xleftarrow{\overline{d_0}} \bigoplus_{|\alpha|=1} [M^*_{\mathbf{x}^\alpha}]_{\mathbf{1}} \xleftarrow{\overline{d_1}} \cdots \xleftarrow{\overline{d_{p-1}}} \bigoplus_{|\alpha|=p} [M^*_{\mathbf{x}^\alpha}]_{\mathbf{1}} \xleftarrow{\overline{d_p}} \cdots \xleftarrow{\overline{d_{n-1}}} [M^*_{\mathbf{x}^{\mathbf{1}}}]_{\mathbf{1}} \longleftarrow 0$$

On the other hand, we can also construct the following complex of k-vector spaces from the n-hypercube associated to M:

$$\mathcal{M}^{*\bullet} : 0 \longleftarrow [M]_{\mathbf{0}} \xleftarrow{u_0^*} \bigoplus_{|\alpha|=1} [M]_\alpha \xleftarrow{u_1^*} \cdots \xleftarrow{u_{p-1}^*} \bigoplus_{|\alpha|=p} [M]_\alpha \xleftarrow{u_p^*} \cdots \xleftarrow{u_{n-1}^*} [M]_{\mathbf{1}} \longleftarrow 0$$

where the map between summands $[M]_\alpha \longrightarrow [M]_{\alpha-\varepsilon_i}$ is $\mathrm{sign}(i, \alpha - \varepsilon_i)$ times the dual of the canonical map $u_{\alpha-\varepsilon_i,i}$. Namely, $\mathcal{M}^{*\bullet}$ is the dual, as k-vector spaces, of $\mathcal{M}^\bullet$. We can mimic what we did for Bass numbers to obtain:

Proposition 6.2. *Let $M \in D^T_{v=0}$ be a regular holonomic $D_{R|k}$-module with variation zero and $\mathcal{M}^{*\bullet}$ its corresponding complex associated to the n-hypercube. Then, there is an isomorphism of complexes $\mathcal{M}^{*\bullet} \cong [\check{C}_{\mathfrak{m}}(M^*)]^\bullet_{\mathbf{1}}$.*

Therefore we have the following characterization of Bass numbers:

Corollary 6.3. *Let $M \in D^T_{v=0}$ be a regular holonomic $D_{R|k}$-module with variation zero and $\mathcal{M}^{*\bullet}$ its corresponding complex associated to the n-hypercube. Then*

$$\pi_p(\mathfrak{p}_0, M) = \dim_k \mathrm{H}_p(\mathcal{M}^{*\bullet})$$

6.1. **The local cohomology case.** Let $\mathcal{M} = \{[H^r_I(R)]_\alpha\}_{\alpha\in\{0,1\}^n}$ be the n-hypercube of a local cohomology module $H^r_I(R)$ supported on a monomial ideal. As in the case of Lyubeznik numbers we can also relate dual Bass numbers of $\mathfrak{p}_0$ to the r-linear strand of the Alexander dual ideal $I^\vee$. In this case, if

$$\mathbb{L}^{<r>}_\bullet(I^\vee) : \quad 0 \longrightarrow L^{<r>}_{n-r} \longrightarrow \cdots \longrightarrow L^{<r>}_1 \longrightarrow L^{<r>}_0 \longrightarrow 0\,,$$

is the r-linear strand of $I^\vee$ then we consider its monomial matrices to obtain a complex of k-vector spaces indexed as follows:

$$\mathbb{F}^{<r>}_\bullet(I^\vee) : \quad 0 \longrightarrow K^{<r>}_n \longrightarrow \cdots \longrightarrow K^{<r>}_{r+1} \longrightarrow K^{<r>}_r \longrightarrow 0$$

Therefore we obtain analogous results to those in Section 4.1

Proposition 6.4. *The complex of k-vector spaces $\mathcal{M}^{*\bullet}$ associated to the n-hypercube of a fixed local cohomology module $H^r_I(R)$ is isomorphic to the complex $\mathbb{F}^{<r>}_\bullet(I^\vee)$ obtained from the r-linear strand $\mathbb{L}^{<r>}_\bullet(I^\vee)$ of the Alexander dual ideal of I.*

Corollary 6.5. *Let $\mathbb{F}^{<r>}_\bullet(I^\vee)$ be the complex of k-vector spaces obtained from the r-linear strand of the minimal free resolution of the Alexander dual ideal $I^\vee$. Then*

$$\pi_p(\mathfrak{p}_0, H^r_I(R)) = \dim_k H_p(\mathbb{F}^{<r>}_\bullet(I^\vee))$$

Example 6.6. Consider the ideal $I = (x_1, x_4) \cap (x_2, x_5) \cap (x_1, x_2, x_3)$ in Example 5.7. The non-vanishing pieces of the hypercube associated to the Matlis dual of the corresponding local cohomology modules are:

$[(H^2_I(R))^*]_\alpha = k$ for $\alpha = (0,1,1,0,1), (1,0,1,1,0)$.
$[(H^3_I(R))^*]_\alpha = k$ for $\alpha = (0,0,0,1,1), (0,0,0,0,1), (0,0,0,1,0), (0,0,1,0,0), (0,0,0,0,0)$.

In this case we have $(H^2_I(R))^* \cong H^3_{(x_2,x_3,x_5)}(R) \oplus H^3_{(x_1,x_3,x_4)}(R)$ and the complex associated to the hypercube of $(H^3_I(R))^*$ is:

$$0 \longleftarrow k \xleftarrow{(-1 \quad 1 \quad 0)} k^3 \xleftarrow{\begin{pmatrix}1\\1\\1\end{pmatrix}} k \longleftarrow 0$$

Then, the dual Bass numbers are:

$\pi_0(\mathfrak{p}_\alpha, H^2_I(R)) = 1$ for $\alpha = (1,0,0,1,0), (0,1,0,0,1)$.
$\pi_1(\mathfrak{p}_\alpha, H^2_I(R)) = 1$ for $\alpha = (1,0,0,0,0), (0,0,0,1,0), (0,1,0,0,0), (0,0,0,0,1)$.
$\pi_2(\mathfrak{p}_\alpha, H^2_I(R)) = 2$ for $\alpha = (0,0,0,0,0)$.

$\pi_0(\mathfrak{p}_\alpha, H^3_I(R)) = 1$ for $\alpha = (1,1,1,1,1)$.
$\pi_1(\mathfrak{p}_\alpha, H^3_I(R)) = 1$ for $\alpha = (1,1,0,0,1), (1,1,0,1,0), (0,1,1,1,1), (1,0,1,1,1)$.
$\pi_2(\mathfrak{p}_\alpha, H^3_I(R)) = 1$ for $\alpha = (1,1,0,0,0), (1,0,0,1,0), (1,0,0,0,1), (0,1,0,1,0), (0,1,0,0,1), (0,0,1,1,1)$.
$\pi_3(\mathfrak{p}_\alpha, H^3_I(R)) = 1$ for $\alpha = (1,0,0,0,0), (0,1,0,0,0), (0,0,0,1,0), (0,0,0,0,1)$.
$\pi_4(\mathfrak{p}_\alpha, H^3_I(R)) = 1$ for $\alpha = (0,0,0,0,0)$.

Remark 6.7. It is worth to point out that one may find examples of modules having the same Bass numbers but different dual Bass numbers. For example, consider $H^3_I(R)$ in the previous example and $H^4_{(x_1,x_2,x_4,x_5)}(R) \oplus E_{(1,1,1,0,0)}$. The non-vanishing parts of their corresponding hypercubes are respectively

$$\begin{tikzcd} k \arrow[d, "1"'] \arrow[dr, "1"] & & \\ k \arrow[dr, "1"'] & k \arrow[d, "1"] & k \arrow[dl, "1"] \\ & k & \end{tikzcd} \qquad \begin{tikzcd} k \arrow[d, "1"'] \arrow[dr, "1"] & & \\ k \arrow[dr, "1"'] & k \arrow[d, "1"] & k \arrow[dl, "0"] \\ & k & \end{tikzcd}$$

Notice that these modules are not isomorphic.

7. Questions

The approach we take in this work to study Lyubeznik numbers opens up a number of questions just because of its relation to free resolutions. We name here a few and we hope that they will be addressed elsewhere.

• **Topological description of Lyubeznik numbers:** A recurrent topic in recent years has been to attach a cellular structure to the free resolution of a monomial ideal. In general this can not be done as it is proved in [41] but there are large families of ideals having a cellular resolution. Using the dictionary described in Section 4 we can translate the same questions to Lyubeznik numbers. In particular we would be interested in finding cellular structures on the linear strands of a free resolution so one can give a topological description of Lyubeznik numbers.

Another question that immediately pops up is the behavior of Lyubeznik numbers with respect to the characteristic of the field. In [11] it is proved that the Betti table in characteristic zero is obtained from the positive characteristic Betti table by a sequence of consecutive cancelations, i.e. cancelation of terms in two different linear strands as it can be seen in Example 4.6. In our situation not only the cancelation affects the behavior, we also have to put into the picture the acyclicity of the linear strands.

We do not know whether it is possible to find an example where the Betti table depends on the characteristic but the Lyubeznik table does not. Such an example would require that the Betti table in characteristic zero is obtained from the positive characteristic Betti table by at least two consecutive cancelations.

• **Injective resolution of local cohomology modules:** When R/I is Cohen-Macaulay, a complete description of the injective resolution of $H_I^r(R)$, i.e. Bass numbers and maps between injective modules, was given in [44]. The question on how to find a general description for any ideal might be too difficult so we turn our attention to some nice properties of the resolution.

The injective resolution can be decomposed in linear strands. Namely, if

$$\mathbb{I}_\bullet(M): \quad 0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_m \longrightarrow 0$$

is the injective resolution of a module with variation zero then, given an integer r, the *r-linear strand* of $\mathbb{I}_\bullet(M)$ is the complex:

$$\mathbb{I}_\bullet^{<r>}(M): \quad 0 \longrightarrow I_0^{<r>} \longrightarrow I_1^{<r>} \longrightarrow \cdots \longrightarrow I_m^{<r>} \longrightarrow 0\ ,$$

where

$$I_j^{<r>} = \bigoplus_{|\alpha|=j+r} E_\alpha^{\mu_j(\mathfrak{p}_\alpha, M)},$$

When R/I is Cohen-Macaulay, we have $\mu_p(\mathfrak{p}_\alpha, H_I^{\operatorname{ht} I}(R)) = \delta_{p,n-|\alpha|}$ for all face ideals in the support of R/I, so the injective resolution of $H_I^{\operatorname{ht} I}(R)$ behaves like the injective resolution of a Gorenstein ring. In particular, this resolution is linear. When we turn our attention to minimal non-Cohen-Macaulay we see that the injective resolution of $H_I^{\mathrm{ht}\, I}(R)$ behaves like that of a Gorenstein ring except for the Bass number with respect to the maximal ideal. Notice that the module $H^2_{\mathfrak{a}_4}(R)$ in Example 4.4 has a 2-linear injective resolution with $\mu_2(\mathfrak{m}, H^2_{\mathfrak{a}_4}(R)) = 2$ but the resolution of $H^2_{\mathfrak{a}_5}(R)$ has two linear strands since $\mu_2(\mathfrak{m}, H^2_{\mathfrak{a}_5}(R)) = \mu_3(\mathfrak{m}, H^2_{\mathfrak{a}_5}(R)) = 1$. As in the case of free resolutions it would be interesting to study the different linear strands in the injective resolution of $H_I^r(R)$ and how these linear strands depend on the other local cohomology modules $H_I^s(R)$, $s \neq r$.

• **Projective resolution of local cohomology modules:** The same questions we posted above for injective resolutions can be asked for projective resolutions. We have to point out that F. Barkats [6] gave an algorithm to compute a presentation of the local cohomology modules $H_I^r(R)$ using in an implicit way a projective resolution of these modules with variation zero. However she was only able to compute effectively examples in the polynomial ring $k[x_1, ..., x_6]$.

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