NILPOTENT p-LOCAL FINITE GROUPS

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Abstract. In this paper we prove characterizations of p-nilpotency for fusion systems and p-local finite groups that are inspired by results in the literature for finite groups. In particular, we generalize criteria by Atiyah, Brunetti, Frobenius, Quillen, Stammbach and Tate.

Introduction

This article is concerned with the concept of p-nilpotency. A finite group G is p-nilpotent if its p-Sylow subgroup S has a normal complement N in G, that is, the composition $S \to G \to G/N$ is an isomorphism. The importance of this property was highlighted by Henn and Priddy in [HP94]: for almost all finite p-groups S and for G a finite group with S as a Sylow p-subgroup, G is p-nilpotent.

In the literature, there are characterizations of p-nilpotency using different languages, such as group cohomology, Quillen categories, group theory, and fusion systems. For example, the restriction map in mod p cohomology $H^*(G, \mathbb{F}_p) \to H^*(S, \mathbb{F}_p)$ is an isomorphism if and only if G is p-nilpotent [Qui71]. The Frobenius p-nilpotency criterion in group theory, [Rob82, 10.3.2], characterizes these groups as those for which $\mathrm{Aut}_G(P)$ is a p-group for all subgroups P of S.

For our purposes, an interesting characterization of p-nilpotency is given by [Hup67, Satz IV.4.9], two subgroups of S are conjugate in G if and only if they are conjugate in S. In the terminology of fusion systems [BLO03b], this is equivalent to saying that the fusion system $\mathcal{F}_S(G)$, the category whose objects are subgroups of S and morphisms are maps induced by conjugations in G, equals $\mathcal{F}_S(S)$, the category with the same objects but whose morphisms are maps induced by conjugations in S. This characterization seems appropriate as a definition of p-nilpotency for fusion systems, and in particular, for p-local finite groups. This has already been adopted by several authors, starting with Kessar and Linckelmann in [KL08].

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The objects known as p-local finite groups were introduced by Broto, Levi, and Oliver in [BLO03b] as a mean to extract the essence of the p-local structure of a group of course, but also of a block, or of more exotic occurrences such as the Solomon groups [LO02]. For a finite group G, this p-local structure encodes the homotopical properties of the lattice of its p-subgroups under the conjugation action of G. It is natural then that most p-local properties enjoyed by finite groups also hold for p-local finite groups.

Following this motivation and the definition for nilpotence of fusion systems in [KL08], we say that a p-local finite group is nilpotent when its fusion system is that of a p-group. Note that the prime p is implicitly given when we have a fusion system or a p-local finite group, so we omit the mention of p. In fact, considering that a p-local finite group is an object created to keep track of the p-local structure only, this naming convention agrees with the well known fact that a finite group is nilpotent if and only it is p-nilpotent for all prime numbers p.

According to the previous comments, if G is a finite group with Sylow subgroup S, the associated p-local group is nilpotent if and only if G is p-nilpotent. Note that this definition does not depend on the centric linking system $\mathcal{L}$, but Proposition 2.1 in this paper shows that the saturated fusion system $\mathcal{F}_S(S)$ has a unique centric linking system $\mathcal{L}_S(S)$ up to isomorphism. In particular, a p-local finite group is nilpotent if and only if it is actually an honest p-group. Sometimes it is more convenient to check a topological condition than to stay at the level of the algebra of the fusion data, and having a centric linking system allows for topological characterizations.

We offer a list of characterizations of nilpotency in this context, inspired by the work on p-nilpotency of honest groups. Apart from the criteria we have already mentioned above, we also obtain analogues of the criteria of Tate [Tat64], Stammbach [Sta77], Atiyah [Qui71], and Quillen [Qui71].

In the last decade this question has attracted a lot of attention. Let us mention for example the work of Kessar and Linckelmann, who proved the p-nilpotency theorem of Glaubermann and Thompson for blocks in [KL03] and later for fusion systems, [KL08].

Even more recently a result of Tate has been extended to fusion systems by the four authors in [DGPS10], providing another proof of one part of our Theorem 6.1. In [DGMP09], the Frobenius nilpotency criterion, already present in [Lin07] is used to prove Thompson's result in the setting of the theory of p-local finite groups. This criterion is available in this article as Theorem 4.8.

The work carried out in this paper is also of importance because nilpotency is used to determine sparseness and extreme sparseness of fusion systems, as defined in [Gle10]. A fusion system is sparse if the only proper subfusion system over the same Sylow p-group S is nilpotent. Sometimes sparseness implies that the fusion system is constrained, a condition that reduces certain questions about fusion systems to group theoretical ones. Extremely sparse fusion systems are those for which the only proper subsystem on any subgroup Q of S is nilpotent, and they are always constrained.

Nilpotency is also used to define quasicentric subgroups, a concept heavily used in [BCG+05] and [BCG+07] A subgroup P of S is $\mathcal{F}$-quasicentric if for all $P' \leq S$ that are $\mathcal{F}$-conjugate to P and fully centralized in $\mathcal{F}$, the fusion system $C_{\mathcal{F}}(P')$ is nilpotent. Some of the characterizations that we prove in this paper can be useful to show the sparseness or extreme sparseness of a fusion system, or to determine whether certain subgroups are quasicentric.

The basic definitions of saturated fusion systems and their associated centric linking systems are given in Section 1. Section 2 is dedicated to generalities about nilpotent fusion systems and p-local finite groups. Section 3 deals with homological and cohomological characterizations of nilpotency in low degrees, and some global fusion criteria are studied in Section 4. A criterion in terms of elementary abelian subgroups is described in Section 5. Section 6 is then concerned with cohomological criteria in high degrees, as well as Morava K-theory. Finally, Section 7 contains criteria obtained by using Quillen categories.

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1. Fusion systems and associated centric linking systems

This first section is devoted to the basic definitions and properties we will use from the beautiful theory of p-local finite groups. Our main reference is the article [BLO03b] by Broto, Levi, and Oliver. The expert can happily skip this section.

Given two finite groups P, Q, let $\mathrm{Hom}(P,Q)$ denote the set of group homomorphisms from P to Q, and let $\mathrm{Inj}(P,Q)$ denote the set of monomorphisms. If P and Q are subgroups of a larger group G, then $\mathrm{Hom}_G(P,Q) \subseteq \mathrm{Inj}(P,Q)$ denotes the subset of homomorphisms induced by conjugation by elements of G, and $\mathrm{Aut}_G(P)$ the group of automorphisms of P induced by conjugation in G.

Definition 1.1. A *fusion system* $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, and whose morphism sets $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ satisfy the following conditions:

(a) $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ for all $P, Q \leq S$.
(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

If $\mathcal{F}$ is a fusion system over S and $P, Q \leq S$, then we write $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ for the set of morphisms in $\mathcal{F}$ to emphasize that all morphisms in the category $\mathcal{F}$ are homomorphisms. If $\mathrm{Iso}_{\mathcal{F}}(P,Q)$ denotes the subset of isomorphisms in $\mathcal{F}$, we see that $\mathrm{Iso}_{\mathcal{F}}(P,Q) = \mathrm{Hom}_{\mathcal{F}}(P,Q)$ if $|P| = |Q|$, and $\mathrm{Iso}_{\mathcal{F}}(P,Q) = \emptyset$ otherwise. We also write

$\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Iso}_{\mathcal{F}}(P,P)$ and $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$. Two subgroups $P, P' \leq S$ are called *$\mathcal{F}$-conjugate* if $\mathrm{Iso}_{\mathcal{F}}(P,P') \neq \emptyset$.

Here, and throughout the paper, we write $\mathrm{Syl}_p(G)$ for the set of Sylow p-subgroups of G. Also, for any $P \leq G$ and any $g \in N_G(P)$, $c_g \in \mathrm{Aut}(P)$ denotes the automorphism $c_g(x) = gxg^{-1}$. The fusion systems we consider in this paper will all be saturated in the following sense, [BLO03b, Definition 1.2].

Definition 1.2. Let $\mathcal{F}$ be a fusion system over a p-group S.

- A subgroup $P \leq S$ is *fully centralized* in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ that are $\mathcal{F}$-conjugate to P.
- A subgroup $P \leq S$ is *fully normalized* in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ that are $\mathcal{F}$-conjugate to P.
- $\mathcal{F}$ is a *saturated fusion system* if the following two conditions hold:
  (I) Any $P \leq S$ which is fully normalized in $\mathcal{F}$ is fully centralized in $\mathcal{F}$, and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.
  (II) If $P \leq S$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ are such that ϕP is fully centralized, and if we set
  $$N_\phi = \{g \in N_S(P) \mid \phi c_g \phi^{-1} \in \mathrm{Aut}_S(\phi P)\},$$
  then there is $\bar{\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_\phi, S)$ such that $\bar{\phi}|_P = \phi$.

The motivating example for this definition is the fusion system of a finite group G. For any $S \in \mathrm{Syl}_p(G)$, we let $\mathcal{F}_S(G)$ be the fusion system over S defined by setting $\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)$ for all $P, Q \leq S$.

Proposition 1.3. ([BLO03b, Proposition 1.3]) *Let G be a finite group, and let S be a Sylow p-subgroup of G. Then, the fusion system $\mathcal{F}_S(G)$ over S is saturated. Also, a subgroup $P \leq S$ is fully centralized in $\mathcal{F}_S(G)$ if and only if $C_S(P) \in \mathrm{Syl}_p(C_G(P))$, while P is fully normalized in $\mathcal{F}_S(G)$ if and only if $N_S(P) \in \mathrm{Syl}_p(N_G(P))$.*

In order to help motivate the next constructions, we recall some definitions. If G is a finite group and p is a prime, then a p-subgroup $P \leq G$ is p-centric if $C_G(P) = Z(P) \times C'_G(P)$, where $C'_G(P)$ has order prime to p. For any $P, Q \leq G$, let $N_G(P,Q)$ denote the *transporter*, that is, the set of all $g \in G$ such that $gPg^{-1} \leq Q$. For any $S \in \mathrm{Syl}_p(G)$, $\mathcal{L}^c_S(G)$ denotes the category whose objects are the p-centric subgroups of S, and where $\mathrm{Mor}_{\mathcal{L}^c_S(G)}(P,Q) = N_G(P,Q)/C'_G(P)$. By comparison, $\mathrm{Hom}_G(P,Q) \cong N_G(P,Q)/C_G(P)$. Hence there is a functor from $\mathcal{L}^c_S(G)$ to $\mathcal{F}_S(G)$ which is the inclusion on objects, and which sends the morphism corresponding to $g \in N_G(P,Q)$ to $c_g \in \mathrm{Hom}_G(P,Q)$.

Definition 1.4. Let $\mathcal{F}$ be any fusion system over a p-group S. A subgroup $P \leq S$ is *$\mathcal{F}$-centric* if P and all of its $\mathcal{F}$-conjugates contain their S-centralizers. We denote by $\mathcal{F}^c$ the full subcategory of $\mathcal{F}$ whose objects are the $\mathcal{F}$-centric subgroups of S.

Definition 1.5. Let $\mathcal{F}$ be any fusion system over a finite p-group S. A subgroup $P \leq S$ is *$\mathcal{F}$-radical* if $\mathrm{Out}_{\mathcal{F}}(P)$ is p-reduced, that is, if the maximal normal p-subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$ is $\{1\}$.

Theorem 1.6. (Alperin's fusion theorem, [BLO03b, Theorem A.10]) *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then for each $\phi \in \mathrm{Iso}_{\mathcal{F}}(P, P')$, there exist sequences of subgroups of S*

$$P = P_0, P_1, \ldots, P_k = P' \text{ and } Q_1, Q_2, \ldots Q_k$$

and elements $\phi_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$, such that

(a) *Q_i is fully normalized in $\mathcal{F}$, $\mathcal{F}$-radical, and $\mathcal{F}$-centric for each i ;*
(b) *P_{i-1}, $P_i \leq Q_i$ and $\phi_i(P_{i-1}) = P_i$ for each i ; and*
(c) *$\phi = \phi_k \circ \phi_{k-1} \circ \ldots \circ \phi_1$.*

Definition 1.7. Let $\mathcal{F}$ be a fusion system over the p-group S. A *centric linking system* associated to $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor

$$\pi : \mathcal{L} \to \mathcal{F}^c$$

and "distinguished" monomorphisms $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions:

(A) π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq \mathrm{Aut}_{\mathcal{L}}(P)$), and π induces a bijection

$$\mathrm{Mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}}(P, Q).$$

(B) For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $g \in P$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.
(C) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ {\scriptstyle \delta_P(g)}\downarrow & & \downarrow{\scriptstyle \delta_Q(\pi(f)(g))} \\ P & \xrightarrow{f} & Q. \end{array}$$

One easily checks that for any finite group G and any $S \in \mathrm{Syl}_p(G)$, $\mathcal{L}^c_S(G)$ is a centric linking system associated to the fusion system $\mathcal{F}_S(G)$. We are now ready to give the definition of a p-local finite group, as in [BLO03b, Definition 1.8].

Definition 1.8. A *p-local finite group* is a triple $(S, \mathcal{F}, \mathcal{L})$, where $\mathcal{F}$ is a saturated fusion system over the p-group S and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The classifying space of the p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is the space $|\mathcal{L}|^\wedge_p$.

Thus, for any finite group G and any $S \in \mathrm{Syl}_p(G)$, the triple $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ is a p-local finite group. Its classifying space is $|\mathcal{L}_S^c(P)|_p^\wedge$ is homotopy equivalent to $BG_p^\wedge$ by [BLO03a]. In general, the fact that S itself is an object of the centric linking system yields a "canonical inclusion" $Bi : BS \to |\mathcal{L}|$ of the Sylow subgroup S into the p-local finite group, which allows us to compare their mod p cohomologies. We will denote by $Bi_p^\wedge$ the composition of Bi with the p-completion map $|\mathcal{L}| \to |\mathcal{L}|_p^\wedge$.

2. Nilpotent fusion systems and p-local finite groups

In this section we start from the definition of nilpotency in terms of fusion data and obtain different global characterizations, at the level of linking systems and their mod p cohomology. We start by giving an elementary proof of the fact that the fusion system of a finite p-group has a unique associated centric linking system. This also follows from [BCG$^+$05, Proposition 4.2], since a nilpotent fusion system is constrained, or from Oliver's solution to the Martino-Priddy conjecture in [Oli06] and [Oli04], where it is shown that there is a unique centric linking system associated to the fusion system of a finite group.

Proposition 2.1. *If S is a finite p-group, the fusion system $\mathcal{F}_S(S)$ has a unique associated centric linking system up to isomorphism of categories.*

Proof. Note that $\mathcal{L}_S(S)$ is a centric linking system associated to $\mathcal{F}_S(S)$. Let $\mathcal{L}$ be another centric linking system associated to $\mathcal{F}_S(S)$. The categories $\mathcal{L}_S(S)$ and $\mathcal{L}$ have the same objects, the $\mathcal{F}_S(S)$-centric subgroups of S. Now consider the functor $\delta : \mathcal{L}_S(S) \to \mathcal{L}$ constructed in [BLO03b, Proposition 1.11]. Recall that $\mathrm{Mor}_{\mathcal{L}_S(S)}(P,Q) = N_S(P,Q)$ for $\mathcal{F}_S(S)$-centric P, $Q \leq S$. On morphisms, the maps $\delta_{P,Q} : N_S(P,Q) \to \mathrm{Mor}_{\mathcal{L}}(P,Q)$ are injective and $\delta_{P,P}(g) = \delta_P(g)$ for $g \in P$. Therefore for all $\mathcal{F}_S(S)$-centric P, $Q \leq S$, there are commutative diagrams:

$$\begin{array}{ccc} N_S(P,Q)/Z(P) & \longrightarrow & \mathrm{Mor}_{\mathcal{L}}(P,Q)/Z(P) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_S(P,Q) & \longrightarrow & \mathrm{Hom}_{\mathcal{F}_S(S)}(P,Q), \end{array}$$

where the two vertical maps are isomorphisms by the definition of linking centric system and the lower horizontal map is the identity. Hence the upper horizontal map must be an isomorphism as well. Since the action of $Z(P)$ is free and compatible with the maps $\delta_{P,Q}$, this implies that $\delta_{P,Q}$ is an isomorphism and therefore the categories $\mathcal{L}$ and $\mathcal{L}_S(S)$ are isomorphic. □

Following Kessar and Linckelmann's definition for fusion systems in [KL08] and the characterization of p-nilpotence for honest groups in terms of fusion stated in the introduction, we define a p-local finite group to be nilpotent when its fusion system is that of a p-group.

Definition 2.2. A p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is *nilpotent* if $\mathcal{F} = \mathcal{F}_S(S)$.

Even though this definition seems to be stated at the level of fusion systems only, it says that the p-local finite group is an honest p-group. As it is sometimes more convenient to check a topological condition (using the centric linking system) than to stay at the purely algebraic level of the fusion data, we propose the following characterization.

Proposition 2.3. *A p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is nilpotent if and only if $|\mathcal{L}|_p^\wedge \simeq BS$.*

Proof. If $\mathcal{F} = \mathcal{F}_S(S)$, then $\mathcal{L}$ is isomorphic to $\mathcal{L}_S(S)$ by Proposition 2.1 and in particular $|\mathcal{L}|_p^\wedge \simeq |\mathcal{L}_S(S)|_p^\wedge \simeq BS$.

On the other hand, if $|\mathcal{L}|_p^\wedge \simeq BS$, let $f : BS \to |\mathcal{L}|_p^\wedge$ be a homotopy equivalence. By [BLO03b, Theorem 4.4(a)], f is homotopic to $Bi_p^\wedge \circ B\rho$ for some $\rho \in \mathrm{Hom}(S, S)$. Therefore, $Bi_p^\wedge \simeq f \circ B(\rho^{-1})$ is also a homotopy equivalence.

Now [BLO03b, Theorem 7.4] says that if $|\mathcal{L}|_p^\wedge \simeq |\mathcal{L}_S(S)|_p^\wedge$, then the p-local finite groups $(S, \mathcal{F}, \mathcal{L})$ and $(S, \mathcal{F}_S(S), \mathcal{L}_S(S))$ are isomorphic, in the sense that there are isomorphisms of groups and categories $\alpha : S \to S$, $\alpha_\mathcal{F} : \mathcal{F}_S(S) \to \mathcal{F}$ and $\alpha_\mathcal{L} : \mathcal{L}_S(S) \to \mathcal{L}$ such that $\alpha_\mathcal{F}(P) = \alpha_\mathcal{L}(P) = \alpha(P)$ and such that they commute with the projections $\mathcal{L} \to \mathcal{F}^c$ and the structure maps $\delta_Q : Q \to \mathrm{Aut}_\mathcal{L}(Q)$ for all $Q \leq S$. The proof of the theorem shows that we can choose $\alpha : S \to S$ to be any isomorphism that makes the following diagram commute up to homotopy:

$$\begin{array}{ccc} BS & \xrightarrow{id} & BS \\ {\scriptstyle B\alpha}\downarrow & & \downarrow{\scriptstyle Bi_p^\wedge} \\ BS & \xrightarrow{Bi_p^\wedge} & |\mathcal{L}|_p^\wedge. \end{array}$$

We can choose the identity map and in particular we have an isomorphism of categories $\alpha_\mathcal{F} : \mathcal{F}_S(S) \to \mathcal{F}$ such that $\alpha_\mathcal{F}(P) = P$ for all $P \leq S$. Therefore $\mathrm{Hom}_\mathcal{F}(P, Q) = \mathrm{Hom}_{\mathcal{F}_S(S)}(P, Q)$ so $\mathcal{F} = \mathcal{F}_S(S)$. □

Our first cohomological criterion is an elementary reformulation of the definition.

Proposition 2.4. *A p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is nilpotent if and only if the canonical inclusion induces an isomorphism $H^*(|\mathcal{L}|, \mathbb{F}_p) \to H^*(BS, \mathbb{F}_p)$.*

Proof. By Proposition 2.3, if $(S, \mathcal{F}, \mathcal{L})$ is nilpotent, then $Bi_p^\wedge : BS \to |\mathcal{L}|_p^\wedge$ is a homotopy equivalence and therefore the induced map $H_*(BS; \mathbb{F}_p) \to H_*(|\mathcal{L}|, \mathbb{F}_p)$ is an isomorphism. By [BS08, Theorem 2.48], the induced map in mod p cohomology is also an isomorphism.

By [BLO03b, Proposition 5.2] and [BLO03b, Theorem 5.8], the ring $H^*(|\mathcal{L}|, \mathbb{F}_p)$ is Noetherian, and in particular the groups $H^n(|\mathcal{L}|, \mathbb{F}_p) \cong H_n(|\mathcal{L}|, \mathbb{F}_p)$ are finitely generated $\mathbb{F}_p$-modules. Again by [BS08, Theorem 2.48], an isomorphism in mod p

cohomology implies an isomorphism in mod p homology, which is equivalent to a homotopy equivalence between their p-completions. □

Proposition 2.5. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and let $Bi : BS \to |\mathcal{L}|$ be the standard inclusion. Then $(S, \mathcal{F}, \mathcal{L})$ is nilpotent if and only if the map $Bi^\wedge_p$ has a retraction $r : |\mathcal{L}|^\wedge_p \to BS$.*

Proof. The existence of the retraction implies that the cohomology of the Sylow subgroup is a retract of the cohomology of $|\mathcal{L}|^\wedge_p$. But the mod p cohomology of $|\mathcal{L}|$ can be computed by stable elements by [BLO03b, Theorem 5.8], and is in particular a subalgebra of $H^*(S; \mathbb{F}_p)$. Therefore $BS \to |\mathcal{L}|^\wedge_p$ must be an equivalence.

On the other hand, if $\mathcal{F} = \mathcal{F}_S(S)$, then there is a unique centric linking system associated to $\mathcal{F}$ up to isomorphism by Proposition 2.1. Given an isomorphism of categories $\mathcal{L} \to \mathcal{L}_S(S)$ compatible with the projection maps to $\mathcal{F} = \mathcal{F}_S(S)$ and the structure maps δ_P, the induced map on classifying spaces gives such a retraction. □

Remark 2.6. The result [BLO03b, Proposition 5.5] on which the proof of the stable element formula relies is stronger than the use we make of it in the previous proof. It actually tells us that the suspension spectrum of $|\mathcal{L}|^\wedge_p$ appears as a stable summand in the suspension spectrum $\Sigma^\infty BS$. In particular $Bi^\wedge_p$ induces always an epimorphism in homology and a monomorphism in cohomology with arbitrary coefficients.

3. Cohomological characterizations in low degrees

We show in this section that a nilpotent p-local finite group can be recognized by looking at (co)homological information in low degree. For finite groups the criteria we obtain are those of Tate [Tat64], Stammbach [Sta77], and some variations. Proposition 2.5 is useful in view of the following result, which establishes a "semi-localization property" for certain maps $BS \to X$.

Proposition 3.1. *Let S be a p-group and $f : BS \to X$ be a map inducing an epimorphism $H^1(X; \mathbb{F}_p) \twoheadrightarrow H^1(BS; \mathbb{F}_p)$ and a monomorphism $H^2(X; \mathbb{F}_p) \hookrightarrow H^2(BS; \mathbb{F}_p)$. Then any map $BS \to BP$ to the classifying space of a p-group P factors through f up to homotopy. In particular BS is a retract of X.*

Proof. We proceed by induction on the order of P. If $P \cong C_p$, the cyclic group of order p, the conclusion is a direct consequence of the surjectivity of $H^1(f; \mathbb{F}_p)$. Let us thus consider a p-group P of order p^n with $n \geq 2$ and a group homomorphism $\phi : S \to P$.

The center of a p-group is never trivial, therefore there is a central extension $C_p \hookrightarrow P \overset{\pi}{\twoheadrightarrow} Q$ and by induction there exists a map $g : X \to BQ$ such that the following square commutes up to homotopy

$$\begin{array}{ccc} BS & \xrightarrow{B\phi} & BP \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle B\pi} \\ X & \xrightarrow{g} & BQ. \end{array}$$

Now we have to lift g to a map $h : X \to BP$. The central extension gives rise to a fibration $BP \to BQ \xrightarrow{k} K(\mathbb{Z}/p, 2)$. The composite map $k \circ B\pi \circ B\phi \simeq k \circ g \circ f$ is null-homotopic, and hence so is $k \circ g : X \to K(\mathbb{Z}/p, 2)$ because of the injectivity of $H^2(f; \mathbb{F}_p)$. Therefore there exists a map $h : X \to BP$ such that $B\pi \circ h \simeq g$.

In general, $h \circ f \not\simeq B\phi$ because they can differ by the action of the central C_p. This means, [BK72, IX.4.1], that there exists a map $\alpha : BS \to BC_p$ such that the composite map

$$BS \xrightarrow{\Delta} BS \times BS \xrightarrow{(h\circ f)\times\alpha} BP \times BC_p \to BP$$

is equal to $B\phi$. By the induction hypothesis again, there exists a map $\beta : X \to BC_p$ such that $\beta \circ h \simeq \alpha$. The composite map

$$X \xrightarrow{\Delta} X \times X \xrightarrow{h\times\beta} BP \times BC_p \to BP$$

is the map we are seeking. □

We emphasize that the factorization through f in the statement of the previous proposition is not unique. So f is only "weakly initial" for maps to classifying spaces of p-groups, or, in other words, BP is not quite local with respect to f (in the sense of e.g. [Far96]). This property is the key ingredient in the proof of the p-local version of Stammbach's criterion [Sta77] about the second (co)homology group and the Huppert-Thompson-Tate criterion [Tat64] about the first one. Our proof goes along the same lines as in Stammbach's note.

Theorem 3.2. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and let $Bi : BS \to |\mathcal{L}|$ be the standard inclusion. Then $(S, \mathcal{F}, \mathcal{L})$ is nilpotent if and only if one of the following four conditions is satisfied:*

(1) $Bi^* : H^1(|\mathcal{L}|; \mathbb{F}_p) \to H^1(BS; \mathbb{F}_p)$ *is an isomorphism.*
(2) $Bi^* : H^2(|\mathcal{L}|; \mathbb{F}_p) \to H^2(BS; \mathbb{F}_p)$ *is an isomorphism.*
(3) $Bi_* : H_1(BS; \mathbb{F}_p) \to H_1(|\mathcal{L}|; \mathbb{F}_p)$ *is an isomorphism.*
(4) $Bi_* : H_2(BS; \mathbb{F}_p) \to H_2(|\mathcal{L}|; \mathbb{F}_p)$ *is an isomorphism.*

Proof. The universal coefficient formula clearly implies the equivalence of the homological conditions and the cohomological ones. We follow Stammbach's strategy

[Sta77] and assume that (4) holds. Consider the universal coefficients exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2(S;\mathbb{Z})\otimes\mathbb{Z}/p & \longrightarrow & H_2(S;\mathbb{F}_p) & \longrightarrow & \mathrm{Tor}(H_1(S;\mathbb{Z});\mathbb{Z}/p) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\beta} & & \\ 0 & \longrightarrow & H_2(|\mathcal{L}|_p^\wedge;\mathbb{Z})\otimes\mathbb{Z}/p & \longrightarrow & H_2(|\mathcal{L}|_p^\wedge;\mathbb{F}_p) & \longrightarrow & \mathrm{Tor}(H_1(|\mathcal{L}|_p^\wedge;\mathbb{Z});\mathbb{Z}/p) & \longrightarrow & 0. \end{array}$$

We see that α must be a monomorphism and β an epimorphism. By Remark 2.6 we know that α is an epimorphism, so that β is actually an isomorphism.

Now the short exact sequence $K = \mathrm{Ker}(H_1(Bi;\mathbb{Z})) \hookrightarrow H_1(S;\mathbb{Z}) \twoheadrightarrow H_1(|\mathcal{L}|_p^\wedge;\mathbb{Z})$ induces an exact sequence

$$0 \to \mathrm{Tor}(K;\mathbb{Z}/p) \to \mathrm{Tor}(H_1(S;\mathbb{Z});\mathbb{Z}/p) \xrightarrow{\beta} \mathrm{Tor}(H_1(|\mathcal{L}|_p^\wedge;\mathbb{Z});\mathbb{Z}/p),$$

so that $\mathrm{Tor}(K;\mathbb{Z}/p) = 0$. Since the homology of S is p-torsion, K must be trivial. Therefore $H_1(Bi;\mathbb{Z})$ is an isomorphism, and in particular (3) holds.

It remains to prove that condition (3) – or equivalently condition (1) – implies that $(S,\mathcal{F},\mathcal{L})$ is nilpotent. We assume therefore that $H^1(Bi;\mathbb{F}_p)$ is an isomorphism. Since $H^2(Bi;\mathbb{F}_p)$ is always a monomorphism, Proposition 3.1 applies and we deduce that BS is a retract of $|\mathcal{L}|_p^\wedge$. This means by Proposition 2.5 that the p-local finite group is nilpotent. □

Using the appropriate universal coefficients theorems for the $\mathbb{Z}_p^\wedge$-module $\mathbb{F}_p$, one can easily replace the $\mathbb{F}_p$ coefficients by the p-adic integers.

Corollary 3.3. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group and let $Bi : BS \to |\mathcal{L}|$ be the standard inclusion. Then $(S,\mathcal{F},\mathcal{L})$ is nilpotent if and only if one of the following two conditions is satisfied:*

(1) $Bi_* : H_1(|\mathcal{L}|;\mathbb{Z}_p^\wedge) \to H_1(BS;\mathbb{Z}_p^\wedge)$ *is an isomorphism.*
(2) $Bi^* : H^2(|\mathcal{L}|;\mathbb{Z}_p^\wedge) \to H^2(BS;\mathbb{Z}_p^\wedge)$ *is an isomorphism.* □

4. Global fusion criteria

This section is devoted to criteria which allow to recognize a nilpotent p-local finite group from its global fusion characteristics. We first need some facts about the fundamental group of a p-local finite group. The following definition is due to Puig [Pui01]. The notation $O^p(-)$ stands for the subgroup generated by elements of order prime to p.

Definition 4.1. For any saturated fusion system $\mathcal{F}$ over a p-group S, the *hyperfocal subgroup $Hyp(\mathcal{F})$* is the normal subgroup $\langle g^{-1}\alpha(g) \mid g \in P \leq S, \alpha \in O^p(\mathrm{Aut}_{\mathcal{F}}(P))\rangle$. The *focal subgroup $Foc(\mathcal{F})$* is the normal subgroup $\langle g^{-1}\alpha(g) \mid g \in P \leq S, \alpha \in \mathrm{Aut}_{\mathcal{F}}(P)\rangle$.

Theorem 4.2. ([BCG+07, Theorem B]) *Let $(S, \mathcal{F}, \mathcal{L})$ a p-local finite group. Then $\pi_1(|\mathcal{L}|_p^\wedge) \cong S/Hyp(\mathcal{F})$.* □

We have seen in the previous section that the group $H_1(|\mathcal{L}|_p^\wedge; \mathbb{Z}_p^\wedge) \cong \pi_1(|\mathcal{L}|_p^\wedge)_{ab}$ plays an important role in establishing the nilpotence of a given p-local finite group. The computation in the next proposition (compare with [Gor80, Theorem 7.3.4]) explains the relation between the focal and the hyperfocal subgroups.

Proposition 4.3. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Then $H_1(|\mathcal{L}|_p^\wedge; \mathbb{Z}_p^\wedge) \cong S/Foc(\mathcal{F})$.*

Proof. The group $Foc(\mathcal{F})$ is generated by $Hyp(\mathcal{F})$ and elements $g^{-1}\alpha(g)$ where $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \setminus O^p(\mathrm{Aut}_{\mathcal{F}}(P))$ for some $P \leq S$. By Alperin's fusion theorem, Theorem 1.6, it is enough to consider those subgroups P that are fully normalized in $\mathcal{F}$.

Assume then that P is fully normalized in $\mathcal{F}$. Since $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$, any elements of p-power order in $\mathrm{Aut}_{\mathcal{F}}(P)$ can be conjugated to a conjugation in S via an element of $O^p(\mathrm{Aut}_{\mathcal{F}}(P))$. Therefore $Foc(\mathcal{F})$ is generated by $Hyp(\mathcal{F})$ and commutators in $[S, S]$. In view of Theorem 4.2, we conclude that the abelianization of $\pi_1(|\mathcal{L}|_p^\wedge) \cong S/Hyp(\mathcal{F})$ is $S/Foc(\mathcal{F})$. □

Recall the following definitions from [BLO03b, Definition A.3].

Definition 4.4. The *normalizer fusion system* $N_{\mathcal{F}}(Q)$ of a subgroup $Q \leq S$ in $\mathcal{F}$ is the fusion system defined over $N_S(Q)$ whose morphisms are given by:

$$\mathrm{Hom}_{N_{\mathcal{F}}(Q)}(P, P') = \{\phi \in \mathrm{Hom}_{\mathcal{F}}(P, P') \mid \exists \psi \in \mathrm{Hom}_{\mathcal{F}}(PQ, P'Q),\ \psi|_P = \phi,\ \psi(Q) \leq Q\}.$$

The *centralizer fusion system* $C_{\mathcal{F}}(Q)$ of a subgroup $Q \leq S$ in $\mathcal{F}$ is the fusion system defined over $C_S(Q)$ whose morphisms are given by:

$$\mathrm{Hom}_{C_{\mathcal{F}}(Q)}(P, P') = \{\phi \in \mathrm{Hom}_{\mathcal{F}}(P, P') \mid \exists \psi \in \mathrm{Hom}_{\mathcal{F}}(PQ, P'Q),\ \psi|_P = \phi,\ \psi|_Q = \mathrm{id}\}.$$

These definitions extend to centric linking systems, see [BLO03b, Definition 2.4] and [BLO03b, Definition 6.1].

Definition 4.5. ([BCG+05, Definition 1.5]) A subgroup of S is *$\mathcal{F}$-weakly closed* if no other subgroup is $\mathcal{F}$-conjugate to it.

Lemma 4.6. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and let $V \leq Z(S)$ be $\mathcal{F}$-weakly closed. Then V is normal in $(S, \mathcal{F}, \mathcal{L})$, that is, $(S, \mathcal{F}, \mathcal{L}) = (N_S(V), N_{\mathcal{F}}(V), N_{\mathcal{L}}(V))$.*

Proof. By Alperin's fusion theorem for saturated fusion systems, Theorem 1.6, every morphism in $\mathcal{F}$ is the composition of automorphisms of $\mathcal{F}$-centric subgroups P_i. More precisely, every morphism $\alpha \in \mathrm{Mor}_{\mathcal{F}}(P, Q)$ is the composition of appropriate restrictions of certain $\alpha_i \in \mathrm{Aut}_{\mathcal{F}}(P_i)$. Notice that $V \leq P_i$ since P_i is centric and $\alpha_i(V) \leq V$ since V is weakly closed in $(S, \mathcal{F}, \mathcal{L})$. Thus α can be extended to a morphism from VP to VQ. □

A straightforward consequence is the following p-local version of Grün's theorem [Gor80, Theorem 7.5.2].

Proposition 4.7. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and let $A \leq Z(S)$ be weakly closed in $\mathcal{F}$. Then $Foc(\mathcal{F}) = Foc(N_{\mathcal{F}}(A))$.*

Proof. By Lemma 4.6, A is normal in $\mathcal{F}$, so the fusion systems $\mathcal{F}$ and $N_{\mathcal{F}}(A)$ actually coincide. □

We are now ready to give our "global fusion criteria". At the level of honest groups, the first criterion is due to Huppert and is proven in [Hup67, Satz IV.4.9] by means of the abelian transfer. We propose also a variation. The third one is called Frobenius p-nilpotency criterion in the literature, see for example [Rob82, 10.3.2], and the fourth criterion is a stronger form of this criterion.

Theorem 4.8. *A p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is nilpotent if and only if one of the following four conditions is satisfied:*

(1) *Two elements $a, b \in S$ are $\mathcal{F}$-conjugate if and only if they are S-conjugate.*
(2) *Two n-tuples of commuting elements of S are $\mathcal{F}$-conjugate if and only if they are S-conjugate, $n > 1$.*
(3) *For every subgroup $P \leq S$, $\mathrm{Aut}_{\mathcal{F}}(P)$ is a p-group.*
(4) *For every $\mathcal{F}$-centric subgroup $P \leq S$, $\mathrm{Aut}_{\mathcal{F}}(P)$ is a p-group.*

Proof. It is clear that all four conditions hold for a nilpotent p-local finite group. Let us thus assume that condition (2) holds. In particular for any two elements a and b in S which are $\mathcal{F}$-conjugate, the n-tuples $(a, \ldots, a)$ and $(b, \ldots, b)$ are also $\mathcal{F}$-conjugate. Therefore they must be S-conjugate, which proves that a and b are S-conjugate. This shows that (2) implies (1).

Assume now that condition (1) holds, so that the image of any element under an $\mathcal{F}$-automorphism in S is a conjugate of that element by some element in S, that is both the focal and hyperfocal subgroups are contained in the commutator subgroup $[S, S]$. Therefore, according to Proposition 4.3, the identity of $S/[S, S]$ factors through the map $H_1(Bi; \mathbb{Z}_p^\wedge) : S/[S, S] \to H_1(|\mathcal{L}|_p^\wedge; \mathbb{Z}_p^\wedge) \cong S/Foc(\mathcal{F})$. We have seen in Remark 2.6 that this map is always an epimorphism. It is hence an isomorphism and we have proven that the p-adic characterization (1) in Corollary 3.3 holds.

It is obvious that (3) implies (4). Finally, if condition (4) holds, and $P \lneq S$ is an $\mathcal{F}$-centric subgroup of S, then $P \lneq N_S(P)$ and so $\mathrm{Inn}(P) \lneq \mathrm{Aut}_S(P)$. In particular, $\mathrm{Out}_S(P)$ is a non-trivial subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$ and so P is not $\mathcal{F}$-radical. Therefore the only $\mathcal{F}$-radical $\mathcal{F}$-centric subgroup is S. By Alperin's fusion theorem, Theorem 1.6, every morphism in $\mathcal{F}$ is the restriction of an $\mathcal{F}$-automorphism of S. On the other hand, since S is fully normalized in $\mathcal{F}$, $\mathrm{Inn}(S) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(S))$, which implies $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Inn}(S)$. Hence every morphism in $\mathcal{F}$ is induced by S-conjugation, that is, $\mathcal{F} = \mathcal{F}_S(S)$. □

The following corollary is a very simple criterion to determine nilpotency of p-local finite groups with an abelian p-Sylow.

Corollary 4.9. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group with S abelian. Then $(S, \mathcal{F}, \mathcal{L})$ is nilpotent if and only if one of the following two conditions is satisfied:*

(1) *Two elements $a, b \in S$ are $\mathcal{F}$-conjugate if and only if they are equal.*
(2) *$\mathrm{Aut}_{\mathcal{F}}(S)$ is the trivial group.* □

Theorem 4.8 allows us to obtain for example a p-local version of Huppert's result [Hup67, Satz III.12.1].

Proposition 4.10. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group such that every element of order $p^n > 2$ in S is central in $\mathcal{F}$. Then $(S, \mathcal{F}, \mathcal{L})$ is nilpotent.*

Proof. We use the Frobenius type characterization (4) in Theorem 4.8 of nilpotent p-local finite groups. Let K be the subgroup generated by all the elements of order p^n in S. This abelian subgroup has exponent smaller than or equal to p^n and it is maximal with respect to these two conditions. Let $P \leq S$ be an $\mathcal{F}$-centric subgroup, and let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$. Since every element of order $p^n > 2$ is central in $\mathcal{F}$, α acts trivially on every element of order p^n in P. In particular, it acts trivially on K and, by [Bla66], α is a p-element, i.e., $\mathrm{Aut}_{\mathcal{F}}(P)$ is a p-group. □

5. Quillen's first criterion

The criterion we offer in this section is an extension of Quillen's [Qui71, Theorem 1.5], which is of course done in the setting of honest groups. The following lemma was originally [Qui71, Proposition 4.1], in which Quillen only considers elementary abelian subgroups of the Sylow p-subgroup. Here the exponent is arbitrary.

Lemma 5.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and let A be a subgroup of S maximal subject to being normal abelian and of exponent $p^n > 2$. Then A is also maximal subject to being abelian and of exponent p^n in $(S, \mathcal{F}, \mathcal{L})$, that is, any $\mathcal{F}$-conjugate of A is maximal subject to being abelian and of exponent p^n in S.*

Proof. Assume that there exist abelian subgroups $A' \leq W' \leq S$ of exponent $p^n > 2$ such that A and A' are $\mathcal{F}$-conjugate by the morphism $\varphi : A' \to A$. Since $W' \leq C_S(A') \leq N_\varphi$, then W' is $\mathcal{F}$-conjugate to another abelian subgroup W that contains A. But according to [Alp64] (see also [Hup67, Satz III.12.1]) A is maximal in S subject to being abelian and of exponent p^n, hence $W = A$, $W' = A'$ and the result follows. □

The following result provides a way to show that certain subgroups are central in a p-local finite group.

Proposition 5.2. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and let $V \leq Z(S)$ be an $\mathcal{F}$-weakly closed subgroup such that $\mathrm{Aut}_{\mathcal{F}}(V)$ is a p-group. Then V is central in $(S, \mathcal{F}, \mathcal{L})$, that is, $(S, \mathcal{F}, \mathcal{L}) = (C_S(V), C_{\mathcal{F}}(V), C_{\mathcal{L}}(V))$.*

Proof. By Lemma 4.6 we know that $(S, \mathcal{F}, \mathcal{L}) = (S, N_{\mathcal{F}}(V), N_{\mathcal{L}}(V))$. Note that V is $\mathcal{F}$-weakly closed, and so it is fully normalized and fully centralized. Therefore $\mathrm{Aut}_S(V) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(V))$. Since $\mathrm{Aut}_{\mathcal{F}}(V)$ is a p-group, it must be $\mathrm{Aut}_S(V) = N_S(V)/C_S(V) = S/S = \{1\}$. We will show that the two p-local finite groups $(S, C_{\mathcal{F}}(V), C_{\mathcal{L}}(V))$ and $(S, N_{\mathcal{F}}(V), N_{\mathcal{L}}(V))$ are equal.

Clearly $\mathrm{Hom}_{C_{\mathcal{F}}(V)}(P, Q) \subseteq \mathrm{Hom}_{N_{\mathcal{F}}(V)}(P, Q)$ for all P, $Q \leq S$. Let $\phi : P \to Q$ be a morphism in $N_{\mathcal{F}}(V)$, then there is $\psi : PV \to QV$ in $\mathcal{F}$ such that $\psi|_P = \phi$ and $\psi(V) \leq V$. Since $\psi|_V$ is a morphism in $\mathcal{F}$, it must be the identity, hence ϕ is a morphism in $C_{\mathcal{F}}(V)$. Now, if P and Q are $\mathcal{F}$-centric, then the morphisms for $C_{\mathcal{L}}(V)$ and $N_{\mathcal{L}}(V)$ as defined in [BLO03b, Definition 2.4] and [BLO03b, Definition 6.1] are given by $\mathrm{Mor}_{C_{\mathcal{L}}(V)}(P, Q) = \pi^{-1}(\mathrm{Hom}_{C_{\mathcal{F}}(V)}(P, Q))$ and $\mathrm{Mor}_{N_{\mathcal{L}}(V)}(P, Q) = \pi^{-1}(\mathrm{Hom}_{N_{\mathcal{F}}(V)}(P, Q))$, where $\pi : \mathcal{L} \to \mathcal{F}^c$ is the projection functor from 1.7. This shows that $C_{\mathcal{L}}(V) = N_{\mathcal{L}}(V)$. □

Before stating Quillen's characterization, we start with a special case.

Lemma 5.3. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group at an odd prime p. Assume that there exists a maximal elementary abelian normal subgroup $V \lhd S$ which is $\mathcal{F}$-normal and such that $\mathrm{Aut}_{\mathcal{F}}(V)$ is a p-group. Then $(S, \mathcal{F}, \mathcal{L})$ is nilpotent.*

Proof. We show that for every $P \leq S$, $\mathrm{Aut}_{\mathcal{F}}(P)$ is a p-group, thereby verifying condition (3) in Theorem 4.8. Let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ be an element of order q prime to p. Then it is the restriction of some element $\tilde{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(PV)$. Let the order of $\tilde{\alpha}$ be ar where a is a multiple of q that is prime to r and p. In particular, r is a prime to q and so there is a positive integer l prime to q such that lr is congruent to 1 mod q. The order of $\tilde{\alpha}^{lr}$ divides a, in particular it is prime to p, and the restriction to P is α again. Therefore, we may assume that $\tilde{\alpha}$ has order prime to p.

Now, $\tilde{\alpha}|_V$ is an element in the p-group $\mathrm{Aut}_{\mathcal{F}}(V)$, it must be the identity. As $p > 2$, Lemma 5.1 applies and so V is a maximal elementary abelian subgroup of S and hence of PV. We deduce from [Bla66] that the order of $\tilde{\alpha}$ is a power of p, hence $\tilde{\alpha} = 1$ and therefore $\alpha = 1$. □

Theorem 5.4. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group at an odd prime p. Then $(S, \mathcal{F}, \mathcal{L})$ is nilpotent if and only if every elementary abelian normal subgroup $V \lhd S$ is $\mathcal{F}$-weakly closed and $\mathrm{Aut}_{\mathcal{F}}(V)$ is a p-group.*

Proof. This condition is necessary in view of Theorem 4.8. Let thus $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group such that every elementary abelian normal subgroup $V \lhd S$ is $\mathcal{F}$-weakly closed and $\mathrm{Aut}_{\mathcal{F}}(V)$ is a p-group. We show that it must then be nilpotent.

Consider first the elementary abelian normal subgroup $V_0 = {}_pZ(S)$. It is weakly closed in $(S, \mathcal{F}, \mathcal{L})$ by hypothesis, thus normal in $\mathcal{F}$ by Lemma 4.6. According to Proposition 5.2, $(S, \mathcal{F}, \mathcal{L}) = (S, C_{\mathcal{F}}(V_0), C_{\mathcal{L}}(V_0))$, hence V_0 is central in $(S, \mathcal{F}, \mathcal{L})$.

Let $V \leq S$ be a maximal normal elementary abelian subgroup of S containing V_0. If $V = V_0$, then it is $\mathcal{F}$-normal and by Lemma 5.3, $(S, \mathcal{F}, \mathcal{L})$ is nilpotent. Let us thus assume that V_0 is strictly contained in V.

Consider the quotient p-local finite group $(S_1, \mathcal{F}_1, \mathcal{L}_1) = (S/V_0, \mathcal{F}/V_0, \mathcal{L}/V_0)$, as in [BLO03b, Lemma 5.6]. From the fibration

$$BV_0 \to |\mathcal{L}|^\wedge_p \to |\mathcal{L}_1|^\wedge_p,$$

we see that $(S, \mathcal{F}, \mathcal{L})$ is nilpotent if and only if $(S_1, \mathcal{F}_1, \mathcal{L}_1)$ is so. Let V_1 be the only subgroup satisfying $V_0 < V_1 \leq V$ and $V_1/V_0 = V/V_0 \cap Z(S_1)$.

By construction V_1 is elementary abelian and normal in S. It is also strictly larger than V_0 because V/V_0 is a non-trivial normal subgroup of S/V_0 and therefore intersects non-trivially its center, [Gor80, Theorem 2.6.4]. By the hypothesis, V_1 is weakly closed in $\mathcal{F}$, so V_1/V_0 is weakly closed in $\mathcal{F}_1$. By Lemma 4.6, V_1/V_0 is normal in $(S_1, \mathcal{F}_1, \mathcal{L}_1)$.

Let $f : P/V_0 \to Q/V_0$ be a map in $\mathcal{F}_1$. Since V_1/V_0 is normal in $\mathcal{F}_1$, f is the restriction of a map $g : PV_1/V_0 \to QV_1/V_0$ in $\mathcal{F}_1 = \mathcal{F}/V_0$. This map must be induced by a morphism $h : PV_1 \to QV_1$ in $\mathcal{F}$, which must satisfy $h(P) \leq Q$ and $h(V_1) = V_1$, since both P and Q contain V_0 and V_1 is $\mathcal{F}$-weakly closed. This shows that $\mathcal{F}_1 = N_\mathcal{F}(V_1)/V_0$, and so there is a fibration

$$BV_0 \to |N_\mathcal{L}(V_1)|^\wedge_p \to |\mathcal{L}_1|^\wedge_p,$$

from where we deduce that $(S_1, \mathcal{F}_1, \mathcal{L}_1)$ is nilpotent if and only if $(S, \mathcal{F}^1, \mathcal{L}^1) = (S, N_\mathcal{F}(V_1), N_\mathcal{L}(V_1))$ is nilpotent. The fusion system $\mathcal{F}^1$ is a subcategory of $\mathcal{F}$ with the same objects, so it also verifies that every elementary abelian normal subgroup $W \triangleleft S$ is $\mathcal{F}^1$-weakly closed and $\mathrm{Aut}_{\mathcal{F}^1}(W)$ is a p-group. Moreover, V_1 is normal in $(S, \mathcal{F}^1, \mathcal{L}^1)$.

If $V_1 \neq V$ we can iterate the process by defining V_2 to be the only subgroup satisfying $V_1 < V_2 \leq V$ and $V_2/V_1 = V/V_1 \cap Z(S/V_1)$. The p-local finite group $(S, \mathcal{F}^1, \mathcal{L}^1)$ is nilpotent if and only if a new p-local finite group $(S, \mathcal{F}^2, \mathcal{L}^2)$, which verifies that every elementary abelian normal subgroup $W \triangleleft S$ is $\mathcal{F}^2$-weakly closed and $\mathrm{Aut}_{\mathcal{F}^2}(W)$ is a p-group, and normalizes V_2, is nilpotent.

Iterating this process a finite number of times, we end up with a p-local finite group $(S, \tilde{\mathcal{F}}, \tilde{\mathcal{L}})$ for which V is $\tilde{\mathcal{F}}$-normal and such that $(S, \mathcal{F}, \mathcal{L})$ is nilpotent if and only if $(S, \tilde{\mathcal{F}}, \tilde{\mathcal{L}})$ is so. We conclude by Lemma 5.3. □

Note that $p > 2$ is a necessary condition, since the 2-local finite group induced by the semidirect product $Q_8 \rtimes 3$ provides a counterexample at the prime 2.

6. Cohomological characterizations in high degrees

In contrast to the results in Section 3, we look now at cohomological characterizations of nilpotent p-local finite groups in high degrees. The proofs follow the lines of Quillen's arguments in the case of finite groups [Qui71]. Quillen attributes the first criterion to Atiyah. Let $\mathcal{F}$ be a saturated fusion system. We will use the ring of stable elements:

$$H^*(\mathcal{F}; \mathbb{F}_p) = \varprojlim_{\mathcal{O}(\mathcal{F})} H^*(-; \mathbb{F}_p),$$

as introduced in [BLO03b, Section 5]. It only depends on the fusion system, but the main theorem of the cited section identifies it with the mod p cohomology of $|\mathcal{L}|$ when a centric linking system $\mathcal{L}$ associated to $\mathcal{F}$ exists.

Theorem 6.1. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group and let $Bi : BS \to |\mathcal{L}|$ be the standard inclusion. Then $(S,\mathcal{F},\mathcal{L})$ is nilpotent if and only if one of the following three conditions is satisfied:*

1. *$Bi^* : H^n(|\mathcal{L}|;\mathbb{F}_p) \to H^n(BS;\mathbb{F}_p)$ is an isomorphism for all sufficiently large n.*
2. *$Bi^* : H^n(|\mathcal{L}|;\mathbb{Z}_p^\wedge) \to H^n(BS;\mathbb{Z}_p^\wedge)$ is an isomorphism for all sufficiently large n.*
3. *For each $x \in H^{even}(BS;\mathbb{F}_p)$ there exists a power q of p such that $x^q \in \mathrm{Im}(Bi^*)$ ($p > 2$).*

Proof. Conditions (1) and (2) are equivalent by a universal coefficient theorem argument. We work therefore with p-adic coefficients and assume now that the p-local finite group $(S,\mathcal{F},\mathcal{L})$ satisfies condition (2). By Castellana and Morales' theorem [CM10], $K^0(|\mathcal{L}|;\mathbb{Z}_p^\wedge)$ is a free $\mathbb{Z}_p^\wedge$-module and its rank is the number of $\mathcal{F}$-conjugacy classes of elements in S. However, the map $Bi : BS \to |\mathcal{L}|$ induces a morphism of Atiyah-Hirzebruch spectral sequences converging to the p-completed K-theory rings $K^*(|\mathcal{L}|;\mathbb{Z}_p^\wedge) \to K^*(BS;\mathbb{Z}_p^\wedge)$. This map has finite kernel and cokernel by assumption, so that $K^0(|\mathcal{L}|;\mathbb{Z}_p^\wedge)$ and $K^0(BS;\mathbb{Z}_p^\wedge)$ must have the same rank. Two elements in S are hence $\mathcal{F}$-conjugate if and only if they are S-conjugate and this is the characterization of nilpotence given by part (1) of Theorem 4.8.

To prove that condition (3) characterizes nilpotency for $p > 2$, we show that it implies the condition appearing in Theorem 5.4. Let $V \lhd S$ be an elementary abelian normal subgroup of S and consider the ideal

$$\mathfrak{p}_V = \{u \in H^{\mathrm{even}}(BS;\mathbb{F}_p) \mid u|_{H^{\mathrm{even}}(BV;\mathbb{F}_p)} \text{ is nilpotent}\}$$

in $H^{\mathrm{even}}(BS;\mathbb{F}_p)$. It is a prime ideal since restriction to V induces an injection $H^{\mathrm{even}}(BS;\mathbb{F}_p)/\mathfrak{p}_V \to H^{\mathrm{even}}(BV;\mathbb{F}_p)/\sqrt{0} = S(V^\#)$, where $S(V^\#)$ is the symmetric algebra on $V^\# = \mathrm{Rep}(V,\mathbb{Z}/p)$ with the elements of $V^\#$ in degree 2 [Qui71]. For simplicity of notation, let us denote by $Bi^{-1}(X)$ the set $(Bi^*)^{-1}(X)$ for any $X \subseteq H^{\mathrm{even}}(BS;\mathbb{F}_p)$.

Let A be an elementary abelian subgroup of S which is $\mathcal{F}$-isomorphic to V. Then we have $Bi^{-1}(\mathfrak{p}_A) = Bi^{-1}(\mathfrak{p}_V)$, since $H^*(|\mathcal{L}|;\mathbb{F}_p)$ is computed by stable elements. Given $u \in \mathfrak{p}_V$, condition (3) implies that $u^q \in \mathrm{Im}(Bi^*)$ for some q, power of p. Say $Bi^*(v) = u^q$. That means $v \in Bi^{-1}(\mathfrak{p}_V) = Bi^{-1}(\mathfrak{p}_A)$ and so $u^q \in \mathfrak{p}_A$. But $\mathfrak{p}_A$ is a prime ideal of $H^{\mathrm{even}}(BS;\mathbb{F}_p)$, so $u \in \mathfrak{p}_A$. We conclude that $\mathfrak{p}_A = \mathfrak{p}_V$. Applying [Qui71, Theorem 2.7] we obtain that A and V must be conjugate subgroups in S. Since $V \lhd S$, $A = V$ and V is $\mathcal{F}$-weakly closed.

It remains to prove that $\mathrm{Aut}_\mathcal{F}(V)$ is a p-group. We will actually show that $\mathrm{Aut}_\mathcal{F}(V) = \mathrm{Aut}_S(V)$. We proceed as Quillen in [Qui71, Theorem 2.10]. Consider the maps $H^{\text{even}}(|\mathcal{L}|;\mathbb{F}_p)/Bi^{-1}(\mathfrak{p}_V) \longrightarrow H^{\text{even}}(BS;\mathbb{F}_p)/\mathfrak{p}_V \longrightarrow S(V^{\#})$ and the associated extensions of quotient fields

$$k(Bi^{-1}(\mathfrak{p}_V)) \to k(\mathfrak{p}_V) \to k(V).$$

Note that the groups of automorphisms of the extensions $k(\mathfrak{p}_V) \to k(V)$ and $k(Bi^{-1}(\mathfrak{p}_V)) \to k(V)$ are $\mathrm{Aut}_S(V)$ and $\mathrm{Aut}_{\mathcal{F}}(V)$, respectively. The proof of [BLO03b, Proposition 5.2] shows that the ring $H^{\text{even}}(BS;\mathbb{F}_p)$ is integral over $H^{\text{even}}(|\mathcal{L}|;\mathbb{F}_p)$ and thus $k(Bi^{-1}(\mathfrak{p}_V)) \to k(\mathfrak{p}_V)$ is an algebraic extension. Since any element in $H^{\text{even}}(BS;\mathbb{F}_p)$ is in $H^{\text{even}}(\mathcal{L};\mathbb{F}_p)$ after being raised to a certain power of p, the extension $k(Bi^{-1}(\mathfrak{p}_V)) \to k(\mathfrak{p}_V)$ is purely inseparable, [Isa94, Theorem 19.10]. Therefore, $\mathrm{Aut}_{\mathcal{F}}(V) = \mathrm{Aut}_S(V)$. □

We also offer a characterization of nilpotency in terms of the Morava K-theory $K(n)$, for any $n \geq 1$. For honest groups this criterion has been discovered by Brunetti in [Bru98]. His proof is based on the beautiful result [HKR00] of Hopkins, Kuhn, and Ravenel on generalized characters of finite groups.

Theorem 6.2. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group and let $Bi : BS \to |\mathcal{L}|$ be the standard inclusion. Then $(S,\mathcal{F},\mathcal{L})$ is nilpotent if and only if $Bi^* : K(n)^*(|\mathcal{L}|) \to K(n)^*(BS)$ is an isomorphism.*

Proof. By [Wil99], if we have a map of spaces inducing an isomorphism on the nth Morava K-theory, then it also induces an isomorphism on the first Morava K-theory. By [CM10], an isomorphism in $K(1)$ implies that two elements are conjugate in S if and only if they are conjugate in $\mathcal{F}$, that is, condition (1) in Theorem 3.2.

Conversely, if a p-local finite group is nilpotent, we have seen in Proposition 2.4 that Bi induces an isomorphism in mod p cohomology. It must be then a $K(n)$-equivalence as well, see for example [Mis78, Corollary 1.5]. □

7. Quillen categories

For a finite p-group S, let ε_S denote the category whose objects are the elementary abelian p-subgroups of S and whose morphisms are given by conjugation. Similarly, for a p-local finite group $(S,\mathcal{F},\mathcal{L})$, let $\varepsilon_{\mathcal{F}}$ be the category with the same objects considered as a full subcategory of $\mathcal{F}$. They are the Quillen categories of S and $\mathcal{F}$, respectively. Recall that a ring homomorphism $\gamma : B \to A$ is called an F-isomorphism if each element in $\mathrm{Ker}(\gamma)$ is nilpotent and if for all $a \in A$ there is some $k > 0$ such that $a^k \in \mathrm{Im}(\gamma)$. Following [BLO03b], we define

$$H^*_{\varepsilon}(S;\mathbb{F}_p) = \lim_{\overleftarrow{\varepsilon_S}} H^*(-;\mathbb{F}_p).$$

$$H^*_{\varepsilon}(\mathcal{F};\mathbb{F}_p) = \lim_{\overleftarrow{\varepsilon_{\mathcal{F}}}} H^*(-;\mathbb{F}_p).$$

Definition 7.1. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories equipped with functors $\mathcal{C} \xrightarrow{\gamma} Gr$ and $\mathcal{D} \xrightarrow{\delta} Gr$ to the category of groups. A functor $\Psi : \mathcal{C} \to \mathcal{D}$ is *isotypical* if γ is naturally isomorphic to $\delta \circ \Psi$.

Theorem 7.2. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group, $p > 2$, and let $Bi : BS \to |\mathcal{L}|$ be the standard inclusion. Then $(S, \mathcal{F}, \mathcal{L})$ is nilpotent if and only if one of the following conditions is satisfied:*

1. *The Quillen categories of S and $\mathcal{F}$ are isotypically equivalent.*
2. *$Bi^* : H^*(|\mathcal{L}|; \mathbb{F}_p) \to H^*(BS; \mathbb{F}_p)$ is an F-isomorphism.*

Proof. By [BLO03b, Proposition 5.1], there are F-isomorphisms $\lambda_S : H^*(BS) \to H^*_\varepsilon(S)$ and $\lambda_\mathcal{F} : H^*(\mathcal{F}) \to H^*_\varepsilon(\mathcal{F})$. The stable element formula [BLO03b, Theorem 5.8] shows that the natural map $R_\mathcal{L} : H^*(|\mathcal{L}|) \to H^*(\mathcal{F})$ is an isomorphism. So in fact, we have a commutative diagram:

$$\begin{array}{ccc} H^*(|\mathcal{L}|) & \xrightarrow{\lambda_\mathcal{F} R_\mathcal{L}} & H^*_\varepsilon(\mathcal{F}) \\ {\scriptstyle Bi^*}\downarrow & & \downarrow{\scriptstyle j} \\ H^*(BS) & \xrightarrow{\lambda_S} & H^*_\varepsilon(S). \end{array}$$

If the Quillen categories of S and $\mathcal{F}$ are isotypically equivalent, then the map j is an isomorphism and therefore the map Bi^* is an F-isomorphism. Assume now that Bi^* is an F-isomorphism. Then condition (3) in Theorem 6.1 holds and so $(S, \mathcal{F}, \mathcal{L})$ is nilpotent. □

Let $\mathcal{C}$ be a class of finite subgroups of S. We say that a subgroup H of S *controls fusion* of $\mathcal{C}$-groups in $\mathcal{F}$ if the following conditions hold:

- Any $\mathcal{C}$-subgroup of S is $\mathcal{F}$-conjugate to a subgroup of H.
- For any $\mathcal{C}$-subgroup P of S and any $f : P \to S$ in $\mathcal{F}$ such that $f(P) \leq H$, there exists $h \in H$ such that $f(x) = hxh^{-1}$ for all $x \in P$.

When $H = S$, the first condition is a tautology, so S itself controls fusion if any $\mathcal{F}$-morphism $P \to S$ from a group $P \in \mathcal{C}$ is conjugation by an element in S.

Definition 7.3. Let G be a group. We say $x \in G$ is a $\mathcal{C}_p$-*element* when $x^p = 1$ if p is odd or $x^4 = 1$ if $p = 2$. A subgroup of G generated by a $\mathcal{C}_p$-element is called a $\mathcal{C}_p$-*subgroup* and we denote by $\mathcal{C}_p$ the class of $\mathcal{C}_p$-subgroups.

Given $K \leq S$ and $f : K \to S$, let us denote by $[K, f]$ the subgroup of S generated by the elements of the form $[a, f] = af(a)$ with $a \in K$. We also denote by $[K, f, f]$ the subgroup $[[K, f], f]$.

Theorem 7.4. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Then $(S, \mathcal{F}, \mathcal{L})$ is nilpotent if and only if S controls fusion of $\mathcal{C}_p$-subgroups.*

Proof. We follow the strategy in [GS10, Theorem 2]. We will show that the control of fusion condition implies condition (3) in Theorem 4.8. Let $P \leq S$ and $f \in \mathrm{Aut}_{\mathcal{F}}(P)$ an automorphism of order prime to p. There exists some $l \geq 1$ such that P is contained in a subgroup $Z_l(S)$ of the upper central series. We will prove by induction on l that f is the identity map.

Suppose first that $P \leq Z(S)$ and consider a $\mathcal{C}_p$-element $a \in P$. Since S controls fusion of $\mathcal{C}_p$-subgroups, there is an $s \in S$ such that $f(a) = sas^{-1}$. But $a \in Z(S)$, so $f(a) = a$. By [Gor80, Theorem 5.2.4], $f = 1_P$.

Now consider $P \leq Z_l(S)$ and suppose that the result is known for subgroups of $Z_{l-1}(S)$. Let K be the subgroup of P generated by $\mathcal{C}_p$-elements. Both K and $[K, f]$ are stabilized by f.

Consider a $\mathcal{C}_p$-element $a \in P$. Since S controls fusion of $\mathcal{C}_p$-subgroups, there is $s \in S$ such that $f(a) = sas^{-1}$ and so $[a, f] = [a, s] \in Z_{l-1}(S)$. Note that $[K, f]$ is generated by elements of the form $b[a, f]b^{-1}$, where a, $b \in K$ and a is a $\mathcal{C}_p$-element. Therefore $[K, f] \leq Z_{l-1}(S)$, so that the induction hypothesis applies: f restricted to $[K, f]$ must be the identity. Since f stabilizes $[K, f]$, $[K, f, f] = 1$. But by [Gor80, Theorem 5.3.6], $[K, f, f] = [K, f]$ under these circumstances, so f restricted to K is the identity as well. In particular, f fixes the $\mathcal{C}_p$-elements of P. Now [Hup67, Satz IV.5.12] implies that f is the identity. □

Definition 7.5. ([BCG+05]) For any saturated fusion system $\mathcal{F}$ over a finite p-group S, the *center* of $\mathcal{F}$ is the subgroup

$$Z_{\mathcal{F}}(S) = \{x \in Z(S) \mid f(x) = x \ \forall f \in \mathrm{Mor}(\mathcal{F}^c)\} = \varprojlim_{\mathcal{F}^c} Z(-)$$

The characteristic subgroup $\Omega_i(S)$ is the subgroup generated by all elements x such that $x^{p^i} = 1$. The following result is now a straightforward consequence of Theorem 7.4.

Corollary 7.6. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. If the $\mathcal{C}_p$-elements of S are in the center of $\mathcal{F}$, then $(S, \mathcal{F}, \mathcal{L})$ is nilpotent. In particular, if p is odd and $\Omega_1(S) \subseteq Z_{\mathcal{F}}(S)$ (respectively $p = 2$ and $\Omega_2(S) \subseteq Z_{\mathcal{F}}(S)$), then $(S, \mathcal{F}, \mathcal{L})$ is nilpotent.*

References

[Alp64] J. L. Alperin. Centralizers of abelian normal subgroups of p-groups. *J. Algebra*, 1:110–113, 1964.

[BCG+05] C. Broto, N. Castellana, J. Grodal, R. Levi, and B. Oliver. Subgroup families controlling p-local finite groups. *Proc. London Math. Soc. (3)*, 91(2):325–354, 2005.

[BCG+07] C. Broto, N. Castellana, J. Grodal, R. Levi, and B. Oliver. Extensions of p-local finite groups. *Trans. Amer. Math. Soc.*, 359(8):3791–3858 (electronic), 2007.

[BK72] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations.* Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 304.

[Bla66] N. Blackburn. Automorphisms of finite p-groups. *J. Algebra*, 3:28–29, 1966.

[BLO03a] C. Broto, R. Levi, and B. Oliver. Homotopy equivalences of p-completed classifying spaces of finite groups. *Invent. Math.*, 151(3):611–664, 2003.

[BLO03b] C. Broto, R. Levi, and B. Oliver. The homotopy theory of fusion systems. *J. Amer. Math. Soc.*, 16(4):779–856 (electronic), 2003.

[Bru98] M. Brunetti. A new cohomological criterion for the p-nilpotence of groups. *Canad. Math. Bull.*, 41(1):20–22, 1998.

[BS08] David J. Benson and Stephen D. Smith. *Classifying spaces of sporadic groups*, volume 147 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2008.

[CM10] N. Castellana and D. Morales. Generalized cohomology theories for p-local finite groups. Preprint, 2010.

[DGMP09] A. Díaz, A. Glesser, N. Mazza, and S. Park. Glauberman's and Thompson's theorems for fusion systems. *Proc. Amer. Math. Soc.*, 137(2):495–503, 2009.

[DGPS10] A. Díaz, A. Glesser, S. Park, and R. Stancu. Tate's and yoshida's theorem on control of transfer for fusion systems. Preprint available at http://arxiv.org/abs/1002.4343, 2010.

[Far96] E. D. Farjoun. *Cellular spaces, null spaces and homotopy localization*, volume 1622 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996.

[Gle10] A. Glesser. Sparse fusion system. Preprint available at http://arxiv.org/abs/1005.5503, 2010.

[Gor80] D. Gorenstein. *Finite groups*. Chelsea Publishing Co., New York, second edition, 1980.

[GS10] J. Gonzalez-Sanchez. A p-nilpotency criterion. *Arch. Math.*, 94(3):201–205, 2010.

[HKR00] M. J. Hopkins, Nicholas J. Kuhn, and D. C. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.*, 13(3):553–594 (electronic), 2000.

[HP94] H.W. Henn and S. Priddy. p-nilpotence, classifying space indecomposability, and other properties of almost all finite groups. *Comment. Math. Helv.*, 69(3):335–350, 1994.

[Hup67] B. Huppert. *Endliche Gruppen. I*. Die Grundlehren der Mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin, 1967.

[Isa94] I. Martin Isaacs. *Algebra*. Brooks/Cole Publishing Co., Pacific Grove, CA, 1994. A graduate course.

[KL03] R. Kessar and M. Linckelmann. A block theoretic analogue of a theorem of Glauberman and Thompson. *Proc. Amer. Math. Soc.*, 131(1):35–40 (electronic), 2003.

[KL08] R. Kessar and M. Linckelmann. ZJ-theorems for fusion systems. *Trans. Amer. Math. Soc.*, 360(6):3093–3106, 2008.

[Lin07] M. Linckelmann. Introduction to fusion systems. In *Group representation theory*, pages 79–113. EPFL Press, Lausanne, 2007.

[LO02] R. Levi and B. Oliver. Construction of 2-local finite groups of a type studied by Solomon and Benson. *Geom. Topol.*, 6:917–990 (electronic), 2002.

[Mis78] G. Mislin. Localization with respect to K-theory. *J. Pure Appl. Algebra*, 10(2):201–213, 1977/78.

[Oli04] B. Oliver. Equivalences of classifying spaces completed at odd primes. *Math. Proc. Cambridge Philos. Soc.*, 137(2):321–347, 2004.

[Oli06] B. Oliver. Equivalences of classifying spaces completed at the prime two. *Mem. Amer. Math. Soc.*, 180(848):vi+102, 2006.

[Pui01] Ll. Puig. Full frobenius systems and their localizing categories. Preprint, 2001.

[Qui71] D. Quillen. A cohomological criterion for p-nilpotence. *J. Pure Appl. Algebra*, 1(4):361–372, 1971.

[Rob82] Derek John Scott Robinson. *A course in the theory of groups*, volume 80 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[Sta77] U. Stammbach. Another homological characterisation of finite p-nilpotent groups. *Math. Z.*, 156(2):209–210, 1977.

[Tat64] J. Tate. Nilpotent quotient groups. *Topology*, 3(suppl. 1):109–111, 1964.

[Wil99] W. S. Wilson. $K(n+1)$ equivalence implies $K(n)$ equivalence. In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 375–376. Amer. Math. Soc., Providence, RI, 1999.

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