Simple Spectrum for Tensor Products of Mixing Map Powers

V.V. Ryzhikov*

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1 Introduction

In this note we consider measure-preserving transformations of a Probability space (X, μ). We prove the existence of a mixing rank one construction T such that the product $T \otimes T^2 \otimes T^3 \otimes \ldots$ has simple spectrum. This result has been announced in [6]. It had an application in recent Thikhonov's proof [9] of the existence of mixing transformation with homogeneous spectrum of multiplicity $m > 2$ (see [3]). Let us remark that for generic non-mixing transformations the above spectral properties have been found by Ageev [2].

Rank one construction is determined by h_1 and a sequence r_j of cuttings and a sequence $\bar{s}_j$ of spacers

$$\bar{s}_j = (s_j(1), s_j(2), \ldots, s_j(r_j - 1), s_j(r_j)).$$

We recall its definition. Let our T on the step j be associated with a collection of disjoint sets (intervals)

$$E_j, TE_jT^2, E_j, \ldots, T^{h_j}E_j.$$

We cut E_j into r_j sets (subintervals) of the same measure

$$E_j = E_j^1 \bigsqcup E_j^2 \bigsqcup E_j^3 \bigsqcup \ldots \bigsqcup E_j^{r_j},$$

then for all $i = 1, 2, \ldots, r_j$ we consider columns

$$E_j^i, TE_j^i, T^2E_j^i, \ldots, T^{h_j}E_j^i.$$

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Adding $s_j(i)$ spacers we obtain

$$E_j^i, TE_j^i T^2 E_j^i, \dots, T^{h_j} E_j^i, T^{h_j+1} E_j^i, T^{h_j+2} E_j^i, \dots, T^{h_j+s_j(i)} E_j^i$$

(the above intervals are disjoint). For all $i < r_j$ we set

$$TT^{h_j+s_j(i)} E_j^i = E_j^{i+1}.$$

Now we obtain a tower

$$E_{j+1}, TE_{j+1} T^2 E_{j+1}, \dots, T^{h_{j+1}} E_{j+1},$$

where

$$E_{j+1} = E_j^1,$$
$$T^{h_{j+1}} E_{j+1} = T^{h_j+s_j(r_j)} E_j^{r_j},$$
$$h_{j+1} + 1 = (h_j + 1) r_j + \sum_{i=1}^{r_j} s_j(i).$$

So step by step we define a general rank one construction.

On notation. We denote weak operator approximations by $\approx_w$ and $\approx_s$ for strong ones. Θ is the orthogonal projection into the space of constant functions in $L_2(X, \mu)$. Thus, the expression $T^m \approx_w \Theta$ (for large m) means that T is mixing.

Stochastic Ornstein's rank one transformation. D. Ornstein has proved the mixing for almost all spesial stochastic constructions. His proof can be very shortly presented in the following manner. Let $H_j \to \infty$, $H_j << r_j$, we consider uniformly distributed stochastic variables $a_j(i) \in \{0, 1, \dots, H_j\}$ and let

$$s_j(i) = H_j + a_j(i) - a_j(i+1).$$

Then for $m \in [h_j, h_{j+1})$

$$T^m \approx_w D_1 T^{k_1} P_1 + D_2 T^{k_2} P_2 + D_3 T^{k_3} P_3,$$

where D_i are operators of multiplication by indicators of special parts of j-towers (all D_i and P_i depend on m), $|k_1| < h_{j+1}$, $|k_2|, |k_3| < h_j$, the operators P_i have the form:

$$P_1 = \sum_{n \in [-H_{j+1}, H_{j+1}]} c_{j+1}(n) T^n, \quad c_{j+1}(n) = \frac{H_{j+1} + 1 - |n|}{(H_{j+1} + 1)^2},$$

$$P_{2,3} = \sum_{n\in[-H_j,H_j]} c_j(n)T^n, \quad c_j(n) = \frac{H_j+1-|n|}{(H_j+1)^2}.$$

We have

$$\|D_i T^{k_i}\| \le 1$$

and

$$P_i \approx_s \Theta$$

since T is ergodic. Finally we get for large m

$$T^m \approx_w \Theta.$$

2 $T^{sm_j} \to \Theta$ for a given exclusive s

Weak limits $aI + (1-a)\Theta$ are well known in ergodic theory. They have been used in connection with Kolmogorov's problem [8]and for a machinery of counterexamples [4].

LEMMA 1. *For any $\varepsilon > 0$, N and $s \in [1, N]$ there is a rank one $(1-\varepsilon)$-partially mixing construction T with the following property: for a sequence m_j*

$$(\mathbf{N}, \mathbf{s}) - \textbf{Property} \begin{cases} T^{sm_j} \to \Theta, \\ T^{km_j} \to (1-a_k)\Theta + a_k I \end{cases}$$

for some $a_k > 0$, $k \ne s$, $1 \le k \le N$.

We are able to work with staircase spacer arrays [1] as well as algebraic spacers [7], but we prefer stochastic constructions [5]. We do not try to construct explicit examples here and follow this simple way.

Proof. Let $s = 3$, $N = 5$. A sequence of spacers is organized as follows. A spacer vector $\bar{s_j}$ is a concatenation of arrays

$$S1, S1, A1, S2, S2, A2, S4, S4, A4, S5, S5, A5,$$

where Sk, Ak are independent arrays of spacers. Moreover let arrays Sk be stochastic Ornstein's spacer sequences of the length kL_j with an average value equals to H_j; let an array Ak be of a length $[\varepsilon^{-1}kL_j]$.

Let $m_j = (h_j+H_j)L_j$, then for a small constant $a > 0$ (we omit its calculation) and $k \neq s$, $1 \leq k \leq N$, one gets

$$T^{km_j} \approx_w ka_k I + (1 - ka_k)(D_1 T^{k_1} P_1 + D_2 T^{k_2} P_2 + D_3 T^{k_3} P_3) \approx_w$$

$$\approx_w ka_k I + (1 - ka_k)\Theta, \quad a_k > a.$$

Via Ornstein's approach the mixing is everywhere in j-tower except a part D that is situated under the second spacer array Sk. For this part we have for measurable sets B, B'

$$\mu(T^{km_j} B \cap B' \cap D) \approx \mu(D)\mu(B \cap B'),$$

so $ka_k I$ appears. However

$$T^{3m_j} \approx_w \Theta$$

since we "forget" to copy an array of the length $3L$.

3 Exclusive n for which $T^{nm_j} \to aI + bT + c\Theta$

(n, a, b)**-constructions.** Let $r_j \to \infty$. We fix positive a, b, $a + b + c = 1$, and $n > 1$. For a subsequence $r_{j'-1}$ we produce a flat part (a-part), a polynomial part (b-part) and a mixing part (c-part) (stochastic [5], algebraic [7], or staircase [1] that we use here). These parts will be now provided by the following spacer sequence $\bar{s}_{j'}$. We set (writing again j instead of j')

a-Part: for $i = 1, 2, \ldots, [ar_j]$

$$s_j(i) = H_j.$$

b-Part: for $i \in ([ar_j], [(a+b)r_j])$ if $i = ni'$ we set $s_j(i) = nH_j - 1$, otherwise $s_j(i) = 0$. So this part of spacer vector looks as

$\ldots, 0, 0, \ldots, 0, nH_j - 1,\ 0, 0, \ldots, 0, nH_j - 1,\ 0, 0, \ldots, 0, nH_j - 1,\ 0, \ldots$

Mixing c-Part: $s_j(i) = i$ for $i > [(a + b)r_j]$.

A condition for $(j-1)$**-steps.** *We define on $j-1$-step our construction to be a pure staircase and we set $H_j = h_{j-1}$ (recall that $j = j'$ is a subsequence).*

Weak limits. Let $m_j = h_j + H_j$. We get the following convergences:

$$\mathbf{n-Property} \quad \begin{cases} T^{m_j} \to aI + (b+c)\Theta, \\ T^{2m_j} \to aI + (b+c)\Theta, \\ \dots \\ T^{(n-1)m_j} \to aI + (b+c)\Theta, \\ T^{nm_j} \to aI + bT + c\Theta. \end{cases}$$

Indeed, we have

$$T^{Km_j} \approx_w aT^0 + b\left(\frac{n-K}{n}T^{KH_j} + \frac{K}{n}T^{(K-n)H_j+1}\right) + \frac{1}{r_j}\sum_{i>(a+b)r_j}^{r_j} T^{-2i-1},$$

$$T^{Km_j} \ \approx_w \ aI + b\left(\frac{n-K}{n}T^{KH_j} + \frac{K}{n}T^{(K-n)H_j+1}\right) + c\Theta.$$

For $K = n$

$$b\left(\frac{n-K}{n}T^{KH_j} + \frac{K}{n}T^{(K-n)H_j+1}\right) = bT.$$

For $K = 1, 2, \dots, n-1$ we use $T^{KH_j}, T^{KH_j+1} \approx_w \Theta$ and obtain

$$b\left(\frac{n-K}{n}T^{KH_j} + \frac{K}{n}T^{(K-n)H_j+1}\right) \approx_w \ b\Theta.$$

4 Main result

LEMMA 2. *Let for $m = 2, 3, \dots, n$ and all $s \le m$ a transformation T have m-Properies and (s, m)-Properties. Then $T \otimes T^2 \otimes \dots T^n$ has simple spectrum.*

Proof. A cyclic vector for T in $H = L_2^0$ is denoted by f. We shall prove that a cyclic space C_F is $H^{\otimes n}$, where $F = f^{\otimes n}$ and $T \otimes T^2 \otimes \dots T^n$ is restricted to $H^{\otimes n}$. For $S = T \otimes T^2 \otimes \dots T^{n-1}$ we assume it has simple spectrum by induction.

From n-property we get

$$b^{n-1} f^{\otimes n-1} \otimes (aI + bT)f \in C_F,$$

hence, $f^{\otimes n-1} \otimes Tf \in C_F$, thus, for all k

$$f^{\otimes n-1} \otimes T^k f \in C_F.$$

This implies

$$f^{\otimes n-1}\otimes H\subset C_F,\quad S^i f^{\otimes n-1}\otimes H\subset C_F,\quad H^{\otimes n-1}\otimes H\subset C_F.$$

To see that $T\otimes T^2\otimes\dots T^n$ has a simple spectrum in $L_2^{\otimes n}$ we note that all different products $T^{n_1}\otimes\dots\otimes T^{n_k}$ are spectrally disjoint. This follows directly from (s,n)-Properties (see Lemma 1). For example, if $s=2$ (in Lemma 2), then in $H^{\otimes 3}$

$$(T\otimes T^2\otimes\ T^5)^{m_j}\ \to_w\ 0,$$

but

$$(T\otimes T^3\otimes\ T^5)^{m_j}\ \to_w\ a_1a_3a_5 I > a^3 I.$$

THEOREM. *There is a mixing transformation T such that $T\otimes T^2\otimes\ T^3\dots$ has simple spectrum.*

Proof. We construct rank one transformations T_p with n-Properties and (s,N)-Properties $(n,N\le p)$. We make these constructions c_p-partially mixing with $T_p\otimes T_p^2\otimes\ T_p^3\dots$ of simple spectrum (Lemma 2). Then c_p tends very slowly to 1, and we force a limit mixing construction T to have the desired spectral property via standard technique (see [3], [6]). In [6] we define $T_p\to T$ $(p\to\infty)$ to have simple spectrum for all $T^{\odot n}$. Replacing this aim by another one we provide simple spectrum of $T\otimes T^2\otimes\dots T^n$ via the same methods.

Finally let us formulate a similar problem on flows:

Conjecture. *There is a mixing flow T_t such that for all collections of different $t_i>0$ the products $T_{t_1}\otimes T_{t_2}\otimes T_{t_3}\dots$ have simple spectrum. Moreover the same is true for*

$$exp(T_{t_1})\otimes exp(T_{t_2})\otimes exp(T_{t_3})\dots$$

(here T_{t_i} are now treated as unitary operators restricted onto H).

The main difficulty is not to find a solution but is to find an elegant one. It seems that the following lemma could be useful.

LEMMA 3. *If for a flow T_t with simple spectrum and any positive different $s,t_1,t_2,\dots,t_n$ there is $m_j\to\infty$ and positive $a_1,a_2,\dots,a_n$ such that*

$$T_{t_i m_j}\to a_i I+(1-a_i)\Theta,\ \ i=1,2,\dots,n-1,$$

$$T_{t_n m_j} \to a_n T_s + (1 - a_n)\Theta,$$

then the corresponding (Gaussian) automorphisms

$$exp(T_{t_1}) \otimes exp(T_{t_2}) \otimes exp(T_{t_3}) \ldots$$

have simple spectrum.

Remark. There is a weakly mixing flow T_t possessing the following property: given $a \in [0,1]$ there is a sequence m_i such that for any real $s > 0$ there is a subsequence $m_{i(k)}$ (it depends on s) providing

$$T_{sm_{i(k)}} \to aI + (1 - a)\Theta.$$

(It is not possible to have $T_{sm_i} \to aI + (1-a)\Theta$ for a set of s of a positive measure.)

Hint: let us consider rank one flows with $r_j >>> h_j$.

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E-mail: vryz@mail.ru