SHRINKING TARGETS FOR COUNTABLE MARKOV MAPS

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Abstract. Let T be an expanding Markov map with a countable number of inverse branches and a repeller Λ contained within the unit interval. Given $\alpha \in \mathbb{R}_+$ we consider the set of points $x \in \Lambda$ for which $T^n(x)$ hits a shrinking ball of radius $e^{-n\alpha}$ around y for infinitely many iterates n. Let $s(\alpha)$ denote the infimal value of s for which the pressure of the potential $-s\log|T'|$ is below $s\alpha$. Building on previous work of Hill, Velani and Urbański we show that for all points y contained within the limit set of the associated iterated function system the Hausdorff dimension of the shrinking target set is given by $s(\alpha)$. Moreover, when $\overline{\Lambda} = [0,1]$ the same holds true for all $y \in [0,1]$. However, given $\beta \in (0,1)$ we provide an example of an expanding Markov map T with a repeller Λ of Hausdorff dimension β with a point $y \in \overline{\Lambda}$ such that for all $\alpha \in \mathbb{R}_+$ the dimension of the shrinking target set is zero.

1. Introduction

Suppose we have a dynamical system (X, T, μ) consisting of a space X together with a map $T : X \to X$ and a T-invariant ergodic probability measure μ. Let A be a subset of positive μ measure. Poincaré's recurrence theorem implies that μ almost every $x \in X$ will visit A an infinite number of times, ie. $\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} T^{-n}A$ has full μ measure. This raises the question of what happens when we allow A to shrink with respect to time. How does the size of $\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} T^{-n}A(n)$ depend upon the sequence $\{A(n)\}_{n\in\mathbb{N}}$?

We shall consider this question in the setting of hyperbolic maps. Given a Gibbs measure μ, Chernov and Kleinbock have given general conditions according to which $\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} T^{-n}A(n)$ will have full μ measure [CK]. However, when $\sum_{n=0}^{\infty} \mu(A(n))$ is finite it is clear that $\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} T^{-n}A(n)$ must be of zero μ measure. In particular, if $\{A(n)\}_{n\in\mathbb{N}}$ is a sequence of balls which shrink exponentially fast around a point, then $\bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} T^{-n}A(n)$ must be of zero Lebesgue measure. Thus, in order to understand its geometric complexity we must determine its Hausdorff dimension (see [F1] for an introduction to dimension theory).

In [HV1, HV2] Hill and Velani consider the dimension of the shrinking target set

$$\mathcal{D}_y(\alpha) := \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} \left\{x \in X : |T^n(x) - y| < e^{-n\alpha}\right\}.$$

Let $s(\alpha)$ denote the infimal value of s for which the pressure of the potential $-s\log|T'|$ is below $s\alpha$. In [HV2] it is shown that for an expanding rational maps of the Riemann sphere the dimension of $\mathcal{D}_y(\alpha)$ is given by $s(\alpha)$ for all points y contained within the Julia set. Now suppose we have a piecewise continuous map of the unit interval T with repeller Λ. When T has just finitely many inverse branches, Hill and Velani's formula for the dimension of $\mathcal{D}_y(\alpha)$ extends unproblematically. That is, for all $y \in \overline{\Lambda}$, $\dim_{\mathcal{H}}\mathcal{D}_y(\alpha) = s(\alpha)$. However when T has an infinite number of inverse branches things become more difficult, owing to the unboundedness $|T'|$. In [U] Urbański showed that for those $y \in \Lambda$ satisfying $\sup\{|(T')(T^n(y))|\}_{n\geq 0} < \infty$, the dimension of $\mathcal{D}_y(\alpha)$ is equal to $s(\alpha)$. We prove that, even for systems with an infinite number of inverse branches, this formula extends to all points $y \in \Lambda$. Moreover, when $\overline{\Lambda} = [0,1]$ we have $\dim_{\mathcal{H}}\mathcal{D}_y(\alpha) = s(\alpha)$ for all $y \in [0,1]$. However, we provide a family of examples showing that when $\dim_{\mathcal{H}}\Lambda \in (0,1)$, whilst $s(\alpha)$ is always positive, the dimension of $\mathcal{D}_y(\alpha)$ can be zero for certain members of $y \in \overline{\Lambda}\backslash\Lambda$.

2. Statement of results

Before stating our main results we shall introduce some notation and provide some further background.

Definition 2.1 (Expanding Markov Map)**.** *Let $\mathcal{V} = \{V_i\}_{i\in\mathcal{A}}$ be a countable family of disjoint subintervals of the unit interval with non-empty interior. Given $\omega = (\omega_0, \cdots, \omega_{n-1}) \in \mathcal{A}^n$ for some $n \in \mathbb{N}$ we let $V_\omega := \cap_{\nu=0}^{n-1} T^{-\nu} V_{\omega_\nu}$. We shall say that $T : \cup_{i\in\mathcal{A}} V_i \to [0,1]$ is an expanding Markov map if T satisfies the following conditions.*

1. *For each $i \in \mathcal{A}$, $T|_{V_i}$ is a C^1 map which maps the interior of V_i onto open unit interval* $(0,1)$,
2. *There exists $\xi > 1$ and $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in \cup_{\omega\in\mathcal{A}^n} V_\omega$ we have $|(T^n)'(x)| > \xi^n$,*
3. *There exists some sequence $\{\rho_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ with $lim_{n\to\infty}\rho_n = 0$ such that for all $n \in \mathbb{N}$, $\omega \in \mathcal{A}^n$, and all $x, y \in V_\omega$,*
$$e^{-n\rho_n} \leq \frac{|(T^n)'(x)|}{|(T^n)'(y)|} \leq e^{n\rho_n}.$$

We shall say that T is a finite branch expanding Markov map if $\mathcal{A}$ is a finite set.

The repeller Λ of an expanding Markov map is the set of points for which every iterate of T is well-defined, $\Lambda := \bigcap_{n\in\mathbb{N}} T^{-n}([0,1])$. We assume throughout that $\#\mathcal{A} > 1$. Otherwise Λ would either empty or contained within a single point.

Given a point $y \in \overline{\Lambda}$ in the closure of the repeller and some $\alpha \in \mathbb{R}_+$ we shall be interested in the set of points $x \in \Lambda$ for which $T^n(x)$ hits a shrinking ball of radius $e^{-n\alpha}$ around y for infinitely many iterates n,

$$\mathcal{D}_y(\alpha) := \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} \left\{x \in \Lambda : |T^n(x) - y| < e^{-n\alpha}\right\}. \tag{2.1}$$

More generally, given a function $\varphi : \Lambda \to \mathbb{R}_+$ we let $S_n(\varphi) := \sum_{i=0}^{n-1} \varphi \circ T^l$ and define

$$\mathcal{D}_y(\varphi) := \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} \left\{x \in \Lambda : |T^n(x) - y| < e^{-S_n(\varphi)(x)}\right\}. \tag{2.2}$$

Sets of the form $\mathcal{D}_y(\varphi)$ arise naturally in Diophantine approximation.

Example 2.1. *Given $\alpha \in R_+$ we let*

$$J(\alpha) := \left\{x \in [0,1] : \left|x - \frac{p}{q}\right| < \frac{1}{q^\alpha} \text{ for infinitely many } p, q \in \mathbb{N}\right\}.$$

Let $T : [0,1] \to [0,1]$ be the Gauss map $x \mapsto \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ which is an expanding Markov map on the repeller $\Lambda = [0,1]\backslash\mathbb{Q}$. We define $\psi : \Lambda \to \mathbb{R}$ by $\psi(x) = \log|T'(x)|$ and for each $\alpha > 2$ we let $\psi_\alpha := \left(\frac{\alpha}{2} - 1\right)\psi$. Then for all $2 < \alpha < \beta < \gamma$ we have,

$$\mathcal{D}_0(\psi_\alpha) \subset J(\beta) \subset \mathcal{D}_0(\psi_\gamma). \tag{2.3}$$

In [J, B] *Jarńik and Besicovitch showed that for $\alpha > 2$, $dim_{\mathcal{H}}(J(\alpha)) = \frac{2}{\alpha}$. By (2.3) this is equialent to the fact that for all $\alpha > 2$*

$$dim_{\mathcal{H}} D_0(\psi_\alpha) = \frac{2}{\alpha}.$$

As we shall see, in sufficiently well behaved settings, the Hausdorff dimension of $\mathcal{D}_y(\varphi)$ may be expressed in terms of the thermodynamic pressure.

Definition 2.2 (Tempered Distortion Property)**.** *Given a real-valued potential $\varphi : \Lambda \to \mathbb{R}$ we define the n-th level variation of φ by,*

$$var_n(\varphi) := \sup\left\{|\varphi(x) - \varphi(y)| : x, y \in V_\omega, \omega \in \mathcal{A}^n\right\}.$$

We shall say that a potential φ satisfies the tempered distortion condition if $var_1(\varphi) < \infty$ and $\lim_{n\to\infty} n^{-1} var_n(S_n(\varphi)) = 0$.

Note that by condition (3) in definition 2.1 the potential $\psi(x) := \log|T'(x)|$ satisfies the tempered distortion condition.

Given a potential $\varphi : \Lambda \to \mathbb{R}$ and a word $\omega \in \mathcal{A}^n$ for some $n \in \mathbb{N}$ we define $\varphi(\omega) := \sup\left\{\varphi(x) : x \in V_\omega\right\}$.

Definition 2.3. *Given a potential $\varphi : \Lambda \to \mathbb{R}$, satisfying the tempered distortion condition, we define the pressure by*

$$P(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in\mathcal{A}^n} \exp(S_n(\varphi)(\omega)).$$

This definition of pressure is essentially the same as that given by Mauldin and Urbański in [MU1, MU2]. We note that the limit always exists, but may be infinite. Recall that we defined $\psi(x)$ to be the log-derivative, $\psi(x) := \log|T'(x)|$. Given $\alpha > 0$ we define $s(\alpha)$ by,

$$s(\alpha) := \inf\{s : P(-s\psi) \leq s\alpha\}. \tag{2.4}$$

More generally, given a non-negative positive potential $\varphi : \overline{\Lambda} \to \mathbb{R}_{\geq 0}$, satisfying the tempered distortion condition, we define,

$$s(\varphi) := \inf\{s : P(-s(\psi + \varphi)) \leq 0\}. \tag{2.5}$$

The project of trying to determine the Hausdorff dimension of $\mathcal{D}_y(\varphi)$ began with a series of articles due to Hill and Velani [HV1, HV2, HV3]. Whilst Hill and Velani gave the dimension of $\mathcal{D}_y(\varphi)$ for an expanding rational map of the Riemann sphere, the result extends unproblematically to any expanding Markov map with finitely many inverse branches.

Theorem 1 (Hill, Velani)**.** *Let T be a finite branch expanding Markov map with repeller Λ and let $\varphi : \Lambda \to \mathbb{R}$ a non-negative potential which satisfies the tempered distortion condition. Then, for all $y \in \overline{\Lambda}$ we have $dim_{\mathcal{H}}\mathcal{D}_y(\varphi) = s(\varphi)$.*

Given the neat connection between Diophantine approximation and shrinking target sets for the Gauss map it is natural to try to generalise Theorem 1 to the setting of expanding Markov maps with an infinite number of inverse branches. However, for such maps things can become much more delicate.

Note that we always have $\Lambda_\circ \subseteq \Lambda \subseteq \overline{\Lambda}$. Indeed, when T is a finite branch Markov map $\Lambda_\circ = \Lambda = \overline{\Lambda}$, up to a countable set. However, for Markov maps with infinitely many inverse branches both of these containments may be strict.

In [U] Urbański proves the following extention of Theorem 1 to points $y \in \Lambda_\circ$ for an infinite branch expanding Markov map.

Theorem 2 (Urbański)**.** *Let T be an expanding Markov map with repeller Λ and let $\varphi : \Lambda \to \mathbb{R}$ a non-negative potential which satisfies the tempered distortion condition. Then, for every $y \in \Lambda_\circ$ we have $dim_{\mathcal{H}}\mathcal{D}_y(\varphi) = s(\varphi)$.*

In terms of dimension $\Lambda_\circ$ is a large set, with $\dim_{\mathcal{H}}\Lambda_\circ = \dim_{\mathcal{H}}\Lambda$ [MU1]. However, it follows from Bowen's equation combined with the strict monotonicity of the pressure function for finite iterated function systems (see [F2, Chapter 5]) that for any T ergodic measure with $\dim_{\mathcal{H}}\mu = \dim_{\mathcal{H}}\Lambda$, $\mu(\Lambda_\circ) = 0$. For example, when T is the Gauss map and $\mathcal{G}$ the Gauss measure, which is ergodic and equivalent to Lebesgue measure $\mathcal{L}$, then $\Lambda_\circ$ is the set of badly approximable numbers with $\dim_{\mathcal{H}}\Lambda_\circ = 1$ and $\mathcal{L}(\Lambda_\circ) = \mathcal{G}(\Lambda_\circ) = 0$.

Our main theorem extends the above result to all $y \in \Lambda$.

Theorem 3. *Let T be an expanding Markov map with repeller Λ and let $\varphi : \Lambda \to \mathbb{R}$ be a non-negative potential which satisfies the tempered distortion condition. Then, for every $y \in \Lambda$ we have $dim_{\mathcal{H}}\mathcal{D}_y(\varphi) = s(\varphi)$.*

Note that in Example 2.1 $0 \notin \Lambda = \mathbb{R}\backslash\mathbb{Q}$, so it is clear that for certain maps $\dim_{\mathcal{H}} D_y(\varphi) = s(\varphi)$ holds for $y \in \overline{\Lambda}\backslash\Lambda$. The following theorem shows that this holds whenever Λ is dense in the unit interval.

Theorem 4. *Let T be an expanding Markov map with a repeller Λ satisfying $\overline{\Lambda} = [0,1]$ and let $\varphi : \Lambda \to \mathbb{R}$ a non-negative potential which satisfies the tempered distortion condition. Then, for every $y \in [0,1]$ we have $\dim_{\mathcal{H}} D_y(\varphi) = s(\varphi)$.*

Returning to Example 2.1 we let T denote the Gauss map and $\psi_\alpha := \left(\frac{\alpha}{2} - 1\right)\psi$ and let $\alpha > 2$. By the Jarńik Besicovitch theorem [J, B] we have $\dim_{\mathcal{H}} D_0(\psi_\alpha) = \frac{2}{\alpha}$. It follows from Theorem 2 [U] that $\dim_{\mathcal{H}} \mathcal{D}_y(\psi_\alpha) = \frac{2}{\alpha}$ also holds for all badly approximable numbers y. By Theorem 4 we see that $\dim_{\mathcal{H}} \mathcal{D}_y(\psi_\alpha) = \frac{2}{\alpha}$ for all $y \in [0,1]$.

We remark that Bing Li, BaoWei Wang, Jun Wu, Jian Xu have independently obtained a proof of Theorem 4 in the special case in which T is the Gauss map, as well some interesting results concerning targets which shrink at a super-exponential rate [BBJJ]. However, the methods used in [BBJJ] rely upon certain properties of continued fractions which do not hold in full generality.

Now suppose that $\overline{\Lambda} \neq [0,1]$ and $y \in \overline{\Lambda}\backslash\Lambda$. It might seem reasonable to conjecture that again $\dim_{\mathcal{H}} \mathcal{D}_y(\varphi) = s(\varphi)$. However this is not always the case and, as the following theorem demonstrates, this conjecture fails in rather a dramatic way.

Given $\Phi : \mathbb{N} \to \mathbb{R}_+$ we define,

$$\mathcal{S}_y(\Phi) := \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \{x \in X : d(T^n(x), y) < \Phi(n)\}.$$

Theorem 5. *Let $\Phi : \mathbb{N} \to \mathbb{R}_{>0}$ be any strictly decreasing function satisfying $\lim_{n\to\infty} \Phi(n) = 0$. Then, for each $\beta \in (0,1)$ there exists an expanding Markov map T with a repeller Λ with $\dim_{\mathcal{H}} \Lambda = \beta$ together with a point $y \in \overline{\Lambda}$ satisfying $\dim_{\mathcal{H}} \mathcal{S}_y(\Phi) = 0$.*

Thus, even for Φ which approaches zero at a subexponential rate we can have $\dim_{\mathcal{H}} \mathcal{S}_y(\Phi) = 0$. We remark that $s(\alpha)$ is always strictly positive.

We begin In Section 4 we prove the upper bound in Theorems 3 and 4 simultaneously with an elementary covering argument. In Section 5 we introduce and prove a technical proposition which implies the lower bounds in both Theorems 3 and 4. In Section 6 we prove Theorem 5. We conclude in Section 7 with some remarks.

3. Infinite iterated function systems

In order to make the proof more transparent we shall employ the language of iterated function systems.

Let $T : \cup_{i \in \mathcal{A}} V_i \to [0,1]$ be a countable Markov map. We associate an iterated function system $\{\phi_i\}_{i \in \mathcal{A}}$ corresponding to T in the following way.

For each $i \in \mathcal{A}$ we let $\phi_i : [0,1] \to \overline{V}_i$ denote the unique C^1 map satisfying $\phi_i \circ T(x) = x$ for all $x \in V_i$.

Let Σ denote symbolic space $\mathcal{A}^{\mathbb{N}}$ endowed with the product topology and let $\sigma : \Sigma \to \Sigma$ denote the left shift operator. Given an infinite string $\omega = (\omega_\nu)_{\nu \in \mathbb{N}} \in \Sigma$ and $m, n \in \mathbb{N}$ we let $m|\omega|n$ denote the word $(\omega_\nu)_{\nu=m+1}^{n} \in \mathcal{A}^{n-m}$. Given $\tau = (\tau_1, \cdots, \tau_n) \in \mathcal{A}^n$ for some $n \in \mathbb{N}$ we let $\phi_\tau := \phi_{\tau_1} \circ \cdots \circ \phi_{\tau_n}$. Sets of the form $\phi_\tau([0,1])$ are referred to as cylinder sets.

Take $\omega \in \mathcal{A}^{\mathbb{N}}$. Note that by definition 2.1 (2) we have $\operatorname{diam}(\phi_{\omega_n}([0,1])) \leq \xi^{-n}$ for all $n \geq N$. Thus, we may define,

$$\pi(\omega) := \bigcap_{n \in \mathbb{N}} \phi_{\omega|n}([0,1]).$$

This defines a continuous map $\pi : \Sigma \to [0,1]$.

Since the intervals $\{V_i\}_{i \in \mathcal{A}}$ have disjoint interiors the iterated function system $\{\phi_i\}_{i \in \mathcal{A}}$ satisfies the open set condition (see [F1, Section 9.2]) and $\pi(\Sigma) \backslash \Lambda$ is countable. By definition 2.1 (1) we have $T \circ \pi(\omega) = \pi \circ \sigma(\omega)$ for all $\omega \in \pi^{-1}(\Lambda)$. Thus, $T : \Lambda \to \Lambda$ and $\sigma : \Sigma \to \Sigma$ are conjugate up to a countable set.

In Definition 2.3 we have used a slightly modified version of the definition given in [MU2, (2.1)]. Nevertheless, the following theorems may be proved in essentially the same way as the proofs given in [MU2].

Theorem 6 (Mauldin, Urbański)**.** *Given a countable Markov map T with repeller Λ we have $\dim_{\mathcal{H}} \Lambda = \inf\{s : P(-s\psi) \leq 0\}$.*

When T has finitely many branches there is a unique $s(\Lambda)$ such that $P(-s(\Lambda)\psi) = 0$ and $\dim_{\mathcal{H}} \Lambda = s(\Lambda)$. However, Mauldin and Urbański have shown that when T has countably many inverse branches we can have $P(-t\psi) < 0$ for all $t \geq \inf\{s : P(-s\psi) \leq 0\}$ and consequently there is no such $s(\Lambda)$ (see [MU1, Example 5.3]). Similar examples show that in general there need not be any s satisfying $P(-s(\psi + \varphi)) = 0$ and consequently we must take $s(\varphi) := \inf\{s : P(-s(\psi + \varphi)) \leq 0\}$ in Theorems 3 and 4.

The pressure P has the following finite approximation property.

Theorem 7 (Mauldin, Urbański)**.** *Let T be a countable Markov map and $\varphi : \Lambda \to \mathbb{R}$ a potential satisfying the tempered distortion condition. Then $P(\varphi) = \sup\{P_{\mathcal{F}}(\varphi) : \mathcal{F} \subseteq \mathcal{A}$ is a finite set$\}$.*

Corollary 1. *Let $\varphi : \Lambda \to \mathbb{R}$ be a non-negative potential satisfying the tempered distortion condition. Then $P(-s(\varphi)(\psi + \varphi)) \leq 0$.*

Proof. Suppose $P(-s(\varphi)(\psi+\varphi)) > 0$. Then, by Theorem 7. $P_{\mathcal{F}}(-s(\varphi)(\psi + \varphi)) > 0$ for some finite set $\mathcal{F} \subset \mathcal{A}$. However $\psi + \varphi$ is bounded on $\mathcal{F}^{\mathbb{N}}$ as $\operatorname{var}_1(\psi), \operatorname{var}_1(\varphi) < \infty$, and hence $s \mapsto P_{\mathcal{F}}(-s(\varphi)(\psi + \varphi))$ is continuous. Thus, there exists $t > s(\varphi)$ for which

$$P(-t(\psi + \varphi)) > 0 \geq P_{\mathcal{F}}(-t(\psi + \varphi)) > 0.$$

Since $\psi + \varphi \geq 0$, $s \mapsto P(-s(\psi + \varphi))$ is non-increasing and hence, $t \leq \inf\{s : P(-s(\psi + \varphi)) \leq 0\}$. Since $s(\varphi) < t$ this is a contradiction. □

Corollary 2. *Let T be a countable Markov map. Then for all potentials $\varphi : \Lambda \to \mathbb{R}$, satisfying the tempered distortion condition, $s(\varphi) > 0$.*

Proof. Since $\psi + \varphi \geq 0$ and $\#\mathcal{A} \geq 2$ it follows from Defintion 2.3 that $P(-s(\psi + \varphi)) \geq \log 2 > 0$ for all $s \leq 0$. If, however, $s(\varphi) \leq 0$ then by Corollary 1 there exists some $s \leq 0$ with $P(-s(\psi + \varphi)) \leq 0$, which is a contradiction. $\square$

4. Proof of the upper bound in Theorems 3 and 4

In this section we use a standard covering argument to prove a uniform upper bound on the dimension of $D_y(\varphi)$, which entails the upper bounds in Theorems 3 and 4.

Throughout the proof we shall let ρ_n denote

$$\rho_n := \max\left\{\mathrm{var}_n(A_n(\psi)), \mathrm{var}_n(A_n(\varphi))\right\}.$$

Since both ψ and φ satisfy the tempered distortion condition, $\lim_{n\to\infty} \rho_n = 0$.

Proposition 4.1. *For every $y \in [0,1]$ we have $dim_{\mathcal{H}} D_y(\varphi) \leq s(\varphi)$.*

Proof. For each $n \in \mathbb{N}$ and $\omega \in \mathcal{A}^n$ we define,

$$V_\omega^{\varphi,n} := \left\{x \in V_\omega : |T^n(x) - y| < e^{-\inf_{z\in V_\omega} S_n(\varphi)(z)}\right\}. \tag{4.1}$$

Clearly every $x \in \mathcal{D}_y(\varphi)$ is in $V_\omega^{\varphi,n}$ for infinitely many $n \in \mathbb{N}$ and $\omega \in \mathcal{A}^n$. Moreover, by the mean value theorem we have,

$$\begin{aligned} \mathrm{diam}(V_\omega^{\varphi,n}) &\leq e^{-\inf_{z\in V_\omega} S_n(\phi)(z) - \inf_{z\in V_\omega} S_n(\varphi)(z)} \\ &\leq e^{-\inf_{z\in V_\omega} S_n(\phi)(z) - \inf_{z\in V_\omega} S_n(\varphi)(z)} \\ &\leq e^{\sup_{z\in V_\omega} S_n(-(\phi+\varphi))(z) + 2n\rho_n} \\ &\leq e^{S_n(-(\phi+\varphi))(\omega) + 2n\rho_n}. \end{aligned} \tag{4.2}$$

Choose $s > s(\varphi)$, so there exists some $t < s$ with $P(-t(\phi + \varphi)) \leq 0$. By condition (2) in definition 2.1 together with $\varphi \geq 0$ we have $S_n(\phi + \varphi) \geq n \log \xi$ for all sufficiently large n and hence $P(-s(\phi + \varphi)) < 0$. Take $\epsilon > 0$ with $\epsilon < -P(-s(\phi + \varphi))$. Since $\lim_{n\to\infty} \rho_n = 0$ there exists some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have,

$$\sum_{\omega\in\mathcal{A}^n} \{\exp(S_n(-s(\phi + \varphi))(\omega))\} < e^{-n\epsilon - 2ns\rho_n}. \tag{4.3}$$

Now choose some $\delta > 0$. Since $\rho_n \to 0$ and $S_n(\phi + \varphi) \geq n \log \xi$ for all sufficiently large n, it follows from (4.2) that we may choose $n_1 \geq n_0$ so that for all $n \geq n_1$ $\mathrm{diam}(V_\omega^{\varphi,n}) < \delta$. Moreover, $\bigcup_{n\geq n_1} \{V_\omega^{\varphi,n}\}_{\omega\in\mathcal{A}^n}$ forms a countable cover of $\mathcal{D}_y(\varphi)$. Applying (4.2) together with (4.3) we see that for all $n_1 \geq n_0$,

$$\sum_{n\geq n_1}\sum_{\omega\in\mathcal{A}^n}\mathrm{diam}(V_\omega^{\varphi,n})^s \leq \sum_{n\geq n_1}\sum_{\omega\in\mathcal{A}^n} e^{\sup_{z\in V_\omega} S_n(-s(\varphi+\phi))(z)+2ns\rho_n} \leq \sum_{n\geq n_1} e^{-n\epsilon} \leq \sum_{n\geq n_0} e^{-n\epsilon} < \infty.$$

Thus, $\mathcal{H}^s_\delta(\mathcal{D}_y(\varphi)) \leq \sum_{n\geq n_0} e^{-n\epsilon}$ for all $\delta > 0$ and hence $\mathcal{H}^s(\mathcal{D}_y(\varphi)) \leq \sum_{n\geq n_0} e^{-n\epsilon} < \infty$. Thus, $\dim_{\mathcal{H}}(\mathcal{D}_y(\varphi)) \leq s$ and since this holds for all $s > s(\varphi)$ we have $\dim_{\mathcal{H}}(\mathcal{D}_y(\varphi)) \leq s(\varphi)$. □

5. Proof of the lower bound in Theorems 3 and 4

In order to prove the lower bound to Theorems 3 and 4 we shall introduce the positive upper cylinder density condition. The condition essentially says that there is a sequence of arbitrarily small balls, surrounding a point $y \in [0,1]$, such that each ball contains a collection of disjoint cylinder sets who's total length is comparable to the diameter of the ball. As we shall see, given any countable Markov map T with repeller Λ this condition is satisfied for all $y \in \Lambda$, and if $\overline{\Lambda} = \Lambda$, this condition is satisfied for all $y \in [0,1]$. The substance of the proof lies in showing that for any point $y \in [0,1]$, for which the positive upper cylinder density condition is satisfied, we have $\dim_{\mathcal{H}}\mathcal{D}_y(\varphi) \geq s(\varphi)$.

Definition 5.1 (Positive upper cylinder density)**.** *Suppose we have an expanding Markov map with a corresponding iterated function system* $\{\phi_i\}_{i\in\mathcal{A}}$. *Given* $y \in \overline{\Lambda}$, $n \in \mathbb{N}$ *and* $r > 0$ *we define,*

$$C(y,n,r) := \{\phi_\tau([0,1]) : \tau \in \mathcal{A}^n, \phi_\tau([0,1]) \subset B(y,r)\}.$$

We shall say that the iterated function system $\{\phi_i\}_{i\in\mathcal{A}}$ *has positive upper cylinder density at* y *if there is a family of natural numbers* $(\lambda_r)_{r\in\mathbb{R}_+}$ *with* $\lim_{r\to 0}\lambda_r = \infty$ *and* $\limsup_{r\to 0}\lambda_r^{-1}\log r < 0$*, for which*

$$\limsup_{r\to 0} r^{-1}\sum_{A\in C(y,\lambda_r,r)} diam(A) > 0.$$

Proposition 5.1. *Let* T *be an expanding Markov map with associated iterated function system* $\{\phi_i\}_{i\in\mathcal{A}}$. *Suppose that* $\{\phi_i\}_{i\in\mathcal{A}}$ *has positive upper cylinder density at* $y \in \overline{\Lambda}$. *Then for each non-negative potential* $\varphi : \Lambda \to \mathbb{R}$ *which satisfies the tempered distortion condition we have* $dim_{\mathcal{H}}\mathcal{D}_y(\varphi) \geq s(\varphi)$.

Combining Proposition 5.1 with Lemmas 5.1 and 5.2 completes the proof of the lower bound in Theorems 3 and 4, respectively.

Lemma 5.1. *Let* T *be an expanding Markov map. Then the corresponding iterated function system* $\{\phi_i\}_{i\in\mathcal{A}}$ *has positive upper cylinder density at every* $y \in \Lambda$.

Proof. Suppose that $y \in \Lambda$. Then there exists some $\omega \in \Sigma$ such that $y \in \phi_{\omega|n}([0,1])$ for all $n \in \mathbb{N}$. We shall define $(\lambda_r)_{r \in \mathbb{R}_+}$ by

$$\lambda_r := \min\left\{n \in \mathbb{N} : 2\text{diam}\left(\phi_{\omega|n}([0,1])\right) \leq r\right\}.$$

Clearly $\lim_{r\to 0} \lambda_r = \infty$. Moreover,

$$r < 2\text{diam}\left(\phi_{\omega|\lambda_r - 1}([0,1])\right) \leq 2\zeta^{-\lambda_r+1},$$

so $\limsup_{r\to\infty} \lambda_r^{-1} \log r \leq -\log \xi < 0$.

Given any $n \in \mathbb{N}$ choose $r_n := 2\text{diam}\left(\phi_{\omega|n}([0,1])\right)$. Clearly $\lambda_{r_n} = n$ and $\phi_{\omega|n}([0,1]) \in C(y, n, r_n)$. Hence,

$$\limsup_{r\to 0} r^{-1} \sum_{A \in C(y,\lambda_r,r)} \text{diam}(A) \geq \frac{1}{2}.$$

□

Lemma 5.2. *Suppose T is an expanding Markov map with $\overline{\Lambda} = [0,1]$. Then the corresponding iterated function system $\{\phi_i\}_{i\in\mathcal{A}}$ has positive upper cylinder density at every $y \in [0,1]$.*

Proof. Suppose T satisfies $\overline{\Lambda} = [0,1]$. Then for any $n \in \mathbb{N}$ we have

$$[0,1] \subseteq \overline{\Lambda} \subseteq \overline{\Lambda} \subseteq \overline{\bigcup_{\omega \in \mathcal{A}^n} \phi_\omega(\Lambda)} \subseteq \overline{\bigcup_{\omega\in\mathcal{A}^n} \phi_\omega([0,1])}. \tag{5.1}$$

We define $(\lambda_r)_{r\in\mathbb{R}_+}$ by

$$\lambda_r := \left\lceil \frac{-\log r + \log 2}{\log \xi} \right\rceil.$$

Clearly $\lim_{r\to 0} \lambda_r = \infty$ and $\limsup_{r\to 0} \lambda_r^{-1} \log r = -\log \xi < 0$.

Suppose $y \in [0, \frac{1}{2}]$. Given any $r < \frac{1}{2}$ and any $\omega \in \mathcal{A}^{\lambda_r}$ we have

$$\text{diam}\left(\phi_\omega([0,1])\right) \leq \xi^{-\lambda_r} < r/2. \tag{5.2}$$

Now $C(y, n, r)$ contains all but the right most member of

$$\mathcal{I} := \left\{\phi_\omega([0,1]) : \phi_\omega([0,1]) \cap [y, y+r) \neq \emptyset\right\},$$

if such a member exists. By (5.1) $\sum_{A\in\mathcal{I}} \text{diam}(A) \geq r$, so by (5.2) we have,

$$\sum_{A\in C(y,\lambda_r,r)} \text{diam}(A) \geq r/2. \tag{5.3}$$

By symmetry 5.3 also holds for $y \in [\frac{1}{2}, 1]$.

Letting $r \to 0$ proves the lemma. □

Before going into details we shall give a brief outline of the proof of Proposition 5.1. We begin by taking $s < s(\varphi)$ and extracting a certain finite set of words $\mathcal{B}$ such that $P_{\mathcal{B}}(-s(\phi+\varphi)) > 0$. In addition, we take a Bernoulli measure μ supported on $\mathcal{B}^{\mathbb{N}}$ with $h(\mu) = t\int(\phi+\varphi)d\mu$ for some $t > s$. We then construct a tree structure, iteratively, in the following way. Let Γ_{q-1} be the finite collection of words in the tree at stage $q-1$ and

γ_{q-1} denote the length of those words. At stage q we take α_q so large that $\alpha_q^{-1} \max\left\{S_{\gamma_{q-1}}(\psi)(\omega), S_{\gamma_{q-1}}(\varphi)(\omega) : \omega \in \Gamma_q\right\}$ is negligible. We then take a ball of radius $B(y, r_q)$ so that $r_q < \exp(-\alpha_q \int \varphi d\mu)$ and $B(y, r_q)$ contains a collection of disjoint cylinder sets who's total width is comparable to r_q, corresponding to a finite collection of words $\mathcal{R}_q$ of length λ_q. This is made possible by the upper cylinder density condition. We then choose β_q so that $\exp(-\beta_q \int \varphi d\mu)$ is greater than, but comparable with, r_q. Γ_q consists of all continuations of Γ_{q-1} of length $\gamma_q := \beta_q + \lambda_q$ so that $\beta_q|\omega \in \mathcal{R}_q$ and ω_ν is chosen freely from $\mathcal{B}$ for all $\gamma_{q-1} < \nu \leq \beta_q$. Having constructed our tree we shall define S to be a certain subset of its limit points for which $\omega|\beta_q$ behaves "typically" with respect to μ for each q. Given $\omega \in S$ we have $S_{\beta_q}(\varphi)(\pi(\omega)) \approx \beta_q \int \varphi d\mu < -\log r_q$ so $\beta_q|\omega|\gamma_q \in \mathcal{R}_q$ implies $|T^{\beta_q}(\pi(\omega)) - y| < \exp(-S_{\beta_q}(\varphi)(\pi(\omega)))$. Hence $\pi(S) \subset \mathcal{D}_y(\varphi)$. At each stage β_q, S consists of approximately $\beta_q h(\mu)$ intervals of diameter approximately $\exp(-\beta_q \int \psi d\mu)$. Moreover, for all $\omega \in S$, $\beta_q|\omega|\gamma_q \in \mathcal{R}_q$. The total diameter of cylinders corresponding to words from $\mathcal{R}_q$ is about $r_q \approx \exp(-\beta_q \int \varphi d\mu)$, and so at stage γ_q S consists of approximately $\beta_q h(\mu)$ intervals of diameter roughly $\exp(-\beta_q \int (\psi + \varphi) d\mu)$, giving an optimal covering exponent of $t > s$. The fact that $\beta_q \geq \alpha_q$ will be shown to imply that we cannot obtain a cover which is more efficient, and as such $\dim_{\mathcal{H}} \pi(S) \geq t$.

Proof of Proposition 5.1. Choose $s < s(\varphi)$ so that $P(-s(\phi+\varphi)) > 0$. Without loss of generality we may assume that $s > 0$. Now take $\epsilon \in (0, P(-s(\phi + \varphi)))$. Since $\lim_{n\to\infty} \rho_n = 0$, it follows from the definition of pressure that for all sufficiently large n we have,

$$\sum_{\omega \in \mathcal{A}^n} \exp(S_n(-s(\psi + \varphi))(\omega)) > e^{\epsilon n + 2ns\rho_n}. \tag{5.4}$$

Consequently, for all sufficiently large n we have,

$$\sum_{\tau \in \mathcal{A}^n} e^{-s(S_n(\psi)(\tau) + S_n(\phi)(\tau))} > e^{\epsilon n}. \tag{5.5}$$

By choosing some large k we obtain,

$$\sum_{\tau \in \mathcal{A}^k} e^{-s(S_k(\psi)(\tau) + S_k(\phi)(\tau))} > 6. \tag{5.6}$$

Thus, there exists some finite subset $\mathcal{F} \subseteq \mathcal{A}^k$ with

$$\sum_{\tau \in \mathcal{F}} e^{-s(S_k(\psi)(\tau) + S_k(\phi)(\tau))} > 6. \tag{5.7}$$

Note that $s > 0$ and for each $\tau \in \mathcal{F}$, $S_k(\psi)(\tau) > 0$ and $S_k(\varphi)(\tau) > 0$, so $e^{-s(S_k(\psi)(\tau) + S_k(\phi)(\tau))} \in (0, 1)$ for every $\tau \in \mathcal{F}$.

The finite set $\mathcal{F}$ inherits an order $<_*$ from the order on $[0, 1]$ in a natural way by $\tau_1 <_* \tau_2$ if and only if $\sup \phi_{\tau_1}([0, 1]) \leq \inf \phi_{\tau_2}([0, 1])$. Partition $\mathcal{F}$ into two disjoint sets $\mathcal{F}_1$ and $\mathcal{F}_2$ so that if $\tau \in \mathcal{F}_1$ then its succesor under

$<_*$ is in $\mathcal{F}_2$ and if $\tau \in \mathcal{F}_2$ then its succesor under $<_*$ is in $\mathcal{F}_1$. Clearly we may choose one $m \in \{1,2\}$ so that

$$\sum_{\tau\in\mathcal{F}_m} e^{-s(S_k(\psi)(\tau)+S_k(\phi)(\tau))} \geq \frac{1}{2}\sum_{\tau\in\mathcal{F}} e^{-s(S_k(\psi)(\tau)+S_k(\phi)(\tau))} > 3. \tag{5.8}$$

Since $s>0$, $S_k(\psi)(\tau)>0$ and $S_k(\varphi)(\tau)\geq 0$, $e^{-s(S_k(\psi)(\tau)+S_k(\phi)(\tau))}<1$ for every $\tau\in\mathcal{F}$. Thus we may remove both the smallest and the largest element from $\mathcal{F}_m$, under the order $<_*$, to obtain a set $\mathcal{B}\subset\mathcal{F}_m$ satisfying

$$\sum_{\tau\in\mathcal{B}} e^{-s(S_k(\psi)(\tau)+S_k(\phi)(\tau))} > 1. \tag{5.9}$$

Let $c := \max\{S_k(\psi)(\tau)+S_k(\varphi)(\tau) : \tau\in\mathcal{F}\} > 0$. Given any $\omega_1,\omega_2\in\mathcal{A}^n$ and $\tau_1,\tau_2\in\mathcal{B}$ with either $\omega_1\neq\omega_2$ or $\tau_1\neq\tau_2$, or both, we have,

$$|x-y| \geq \max\left\{e^{-S_n(\psi)(\omega_1)-c}, e^{-S_n(\psi)(\omega_2)-c}\right\} \tag{5.10}$$

for all $x\in(\phi_{\omega_1}\circ\phi_{\tau_1})([0,1])$ and $y\in(\phi_{\omega_1}\circ\phi_{\tau_1})([0,1])$. When $\omega_1\neq\omega_2$ this follows from the fact that $\mathcal{B}$ contains neither the maximal nor the minimal element of $\mathcal{F}$ under $<_*$. When $\omega_1=\omega_2$ but $\tau_1\neq\tau_2$ this follows from the fact that since $\tau_1,\tau_2\in\mathcal{B}\subset\mathcal{F}_m$, τ_1 cannot be the successor of τ_2 and τ_2 cannot be the successor of τ_1.

Since $\mathcal{B}$ is finite and for each $\omega\in\Sigma$ $S_k(\psi)(\omega)\geq k\log\xi$ and $S_k(\psi)(\omega)\geq 0$, we may take $t\in(s,1)$ satisfying

$$\sum_{\tau\in\mathcal{B}} e^{-t(S_k(\psi)(\tau)+S_k(\phi)(\tau))} = 1. \tag{5.11}$$

We define a k-th level Bernoulli measure μ on $\mathcal{B}^{\mathbb{N}}$ by defining $p(\tau)$ for $\tau\in\mathcal{A}^k$ by $p(\tau) := e^{-t(S_k(\psi)(\tau)+S_k(\phi)(\tau))}$ and setting $\mu([\tau_1,\cdots,\tau_n]) = p_{\tau_1}\cdots p_{\tau_n}$ for each $(\tau_1,\cdots,\tau_n)\in\mathcal{B}^n$. We define,

$$\begin{aligned}\mathbb{E}(S_k(\psi)) &:= \sum_{\tau\in\mathcal{B}} p(\tau)S_k(\psi)(\tau)\\ \mathbb{E}(S_k(\varphi)) &:= \sum_{\tau\in\mathcal{B}} p(\tau)S_k(\varphi)(\tau).\end{aligned}$$

Choose a decreasing sequence $\{\delta_q\}_{q\in\mathbb{N}}\subset\mathbb{R}_{>0}$ so that $\prod_{q\in\mathbb{N}}(1-\delta_q)>0$. Take $q\in\mathbb{N}$. By Kolmogorov's strong law of large numbers combined with Egorov's theorem there exists set $S_q\subseteq\mathcal{B}^{\mathbb{N}}$ with $\mu(S_q)>1-\delta_q$ and $N(q)\in\mathbb{N}$ such that for all $\omega=(\omega_\nu)_{\nu\in\mathbb{N}}\in S_q$ with $\omega_\nu\in\mathcal{B}$ for each $\nu\in\mathbb{N}$ and all

$n \geq N(q)$ we have,

$$\begin{aligned}
(5.12)\quad \frac{1}{n}\sum_{\nu=1}^{n} S_k(\psi)(\omega_\nu) &< \mathbb{E}(S_k(\psi)) + \frac{1}{q} \\
(5.13)\quad \frac{1}{n}\sum_{\nu=1}^{n} S_k(\varphi)(\omega_\nu) &< \mathbb{E}(S_k(\varphi)) + \frac{1}{q} \\
(5.14)\quad \frac{1}{n}\sum_{\nu=1}^{n} \log p_{\omega_\nu} &< \sum_{\tau\in\mathcal{B}} p(\tau)\log p(\tau) + \frac{1}{q} \\
&= -t\left(\mathbb{E}(S_k(\psi)) + \mathbb{E}(S_k(\varphi))\right) + \frac{1}{q} \\
&< -t\left(\frac{1}{n}\sum_{\nu=1}^{n} S_k(\psi)(\omega_\nu) + \mathbb{E}(S_k(\varphi))\right) + \frac{2}{q} \\
&\leq -t\left(\frac{1}{n} S_{nk}(\psi)(\omega_\nu)_{\nu=1}^{n} + \mathbb{E}(S_k(\varphi))\right) + \frac{2}{q}.
\end{aligned}$$

Clearly we may assume that $(N(q))_{q\in\mathbb{N}}$ is increasing and $N(1) \geq 2$.
Now fix

$$\begin{aligned}
\zeta &\in \left(0, \limsup_{r\to 0} r^{-1} \sum_{A\in C(y,\lambda_r,r)} \operatorname{diam}(A)\right), \\
d &\in \left(\limsup_{r\to 0} \lambda_r^{-1}\log r, 0\right).
\end{aligned}$$

We shall now give an inductive construction consisting of a quadruple of rapidly increasing sequences of natural numbers $(\alpha_q)_{q\in\mathbb{N}\cup\{0\}}$, $(\beta_q)_{q\in\mathbb{N}\cup\{0\}}$, $(\gamma_q)_{q\in\mathbb{N}\cup\{0\}}$, $(\lambda_q)_{q\in\mathbb{N}\cup\{0\}}$, a sequence of positive real numbers $(r_q)_{q\in\mathbb{N}\cup\{0\}}$ and a pair of sequences of finite sets of words $(\mathcal{R}_q)_{q\in\mathbb{N}\cup\{0\}}$ and $(\Gamma_q)_{q\in\mathbb{N}\cup\{0\}}$. First set $\alpha_0 = \beta_0 = \gamma_0 = 0$, $\lambda_0 = 1$ and $\Lambda_0 = \Gamma_0 = \emptyset$. For each $q \in \mathbb{N}$ we define

$$\alpha_q := 10kq^2\gamma_{q-1}N(q)N(q+1)\left\lceil \log \zeta^{-1} c(3+2\rho_{\lambda_{q-1}}) \max\left\{S_{\gamma_{q-1}}(\psi)(\tau) + S_{\gamma_{q-1}}(\varphi)(\tau) : \tau \in \Gamma_{q-1}\right\}\right\rceil.$$

Note that since Γ_{q-1} is finite α_q is well defined.
We then choose $r_q > 0$ so that,

$$-\log r_q > k^{-1}(\alpha_q - \gamma_{q-1})\left(\mathbb{E}(S_k(\varphi)) + \frac{1}{q}\right) + \gamma_{q-1}c + q, \tag{5.15}$$

and also

$$\sum_{A\in C(y,\lambda_{r_q},r_q)} \operatorname{diam}(A) > \zeta r_q$$

and $\lambda_r^{-1}\log r < d$.

Let $\lambda_q := \lambda_{r_q}$. We may choose $\mathcal{R}_q$ to be a finite set of words $\tau \in \mathcal{A}^{\lambda_q}$ so that for each $\tau \in \mathcal{R}_q$ $\phi_\tau([0,1]) \subset B(y, r_q)$ and

$$\sum_{\tau \in \mathcal{R}_q} \operatorname{diam}(\phi_\tau([0,1])) > \zeta r_q.$$

Let β_q be the largest integer satisfying $k|(\beta_q - \gamma_{q-1})$ and

$$-\log r_q > k^{-1}(\beta_q - \gamma_{q-1})\left(\mathbb{E}(S_k(\varphi)) + \frac{1}{q}\right) + \gamma_{q-1}c + q. \tag{5.16}$$

We let $\gamma_q := \beta_q + \lambda_q$. We define Γ_q by,

$$\Gamma_q := \left\{\omega \in \mathcal{A}^{\gamma_q} : \omega|\gamma_{q-1} \in \Gamma_{q-1}, \gamma_{q-1}|\omega|\beta_q \in \mathcal{B}^{k^{-1}(\beta_q - \gamma_{q-1})}, \beta_q|\omega|\gamma_q \in \mathcal{R}_q\right\}.$$

Note that since $\mathcal{B}$, Γ_{q-1} and $\mathcal{R}_q$ are finite, so is Γ_q.

We inductively define a sequence of measures $\mathcal{W}_q$ supported on Γ_q.

For each $\omega \in \mathcal{A}^n$ and $\tau \in \mathcal{R}_q$ we let

$$q(\omega, \tau) := \frac{\operatorname{diam}(\phi_\omega \circ \phi_\tau([0,1]))}{\sum_{\tau \in \mathcal{R}_q} \operatorname{diam}(\phi_\omega \circ \phi_\tau([0,1]))}.$$

Now by the definition of Γ_q, each $\omega^q \in \Gamma_q$ is of the form $\omega^q = (\omega^{q-1}, \kappa_1^q, \cdots, \kappa_{k^{-1}(\beta_q - \gamma_{q-1})}, \tau_q)$ where $\omega^{q-1} \in \Gamma_{q-1}$, $\kappa_\nu^q \in \mathcal{B}$ for $\nu = 1, \cdots, k^{-1}(\beta_q - \gamma_{q-1})$ and $\tau_q \in \mathcal{R}_q$. We set,

$$\mathcal{W}_q(\omega^q) = \mathcal{W}_{q-1}([\omega^{q-1}])\left(\prod_{\nu=1}^{k^{-1}(\beta_q - \gamma_{q-1})} p(\kappa_\nu)\right) q\left((\omega^{q-1}, \kappa_1^q, \cdots, \kappa_{k^{-1}(\beta_q - \gamma_{q-1})}), \tau_q\right)$$

Define $\Gamma := \{\omega \in \Sigma : \omega|\gamma_q \in \Gamma_q \text{ for all } q \in \mathbb{N}\}$ and extend the sequence $(\mathcal{W}_q)_{q \in \mathbb{N}}$ to a measure $\mathcal{W}$ on Γ in the natural way.

We let $S \subseteq \Gamma$ denote the subset,

$$S := \{\omega \in \Gamma : [\gamma_{q-1}|\omega|\beta_q] \cap S_q \neq \emptyset \text{ for all } q \in \mathbb{N}\}. \tag{5.17}$$

Lemma 5.3. *For all $\omega \in S$ and $n \in \mathbb{N}$ we have $\pi(\omega) \in \phi_{\omega|n}((0,1))$.*

Proof. Suppose for a contradiction that $\omega \in S$ and for some $N \in \mathbb{N}$ $\pi(\omega) \notin \phi_{\omega|N}((0,1))$. Then for all $n \geq N$ we have $\pi(\omega) \in \phi_{\omega|n}(\{0,1\}) = \partial\phi_{\omega|n}([0,1])$. However, given $N \in \mathbb{N}$ we may choose q with $\gamma_q > N$. Then $\omega_{\gamma_q+1} \in \mathcal{B}$ by the construction of S. Consequently $\phi_{\gamma_q+1}([0,1])$ is in neither the left most, nor the right most interval amongst,

$$\left\{\phi_{\omega|\kappa(l)} \circ \phi_\tau([0,1]) : \tau \in \mathcal{F}\right\}.$$

Hence, $\pi(\omega) \notin \partial\phi_{\omega|\gamma_q}([0,1])$. □

Lemma 5.4. $\pi(S) \subseteq \mathcal{D}_y(\varphi)$.

Proof. Take $\omega \in S$. By Lemma 5.3 we have $\pi(\omega) \in \phi_{\omega|n}((0,1)) \subseteq V_{\omega|n}$ and hence $S_n(\varphi)(\omega) \leq S_n(\varphi)(\omega|n)$ for all $n \in \mathbb{N}$ and in particular for each $q \in \mathbb{N}$,

$$\begin{aligned}
S_{\beta_q}(\varphi)(\omega) &\leq S_{\beta_q}(\varphi)(\omega|\beta_q) \\
&\leq S_{\beta_q-\gamma_{q-1}}(\varphi)(\gamma_{q-1}|\omega|\beta_q) + c\gamma_{q-1} \\
&\leq \sum_{\nu=1}^{k^{-1}(\beta_q-\gamma_{q-1})} S_k(\varphi)(\gamma_{q-1}+(\nu-1)k|\omega|\gamma_{q-1}+\nu k) + c\gamma_{q-1}.
\end{aligned}$$

By (5.13) combined with the fact that $[\gamma_{q-1}|\omega|\beta_q] \cap S_q \neq \emptyset$,

$$S_{\beta_q}(\varphi)(\omega) \leq k^{-1}(\beta_q - \gamma_{q-1})\left(\mathbb{E}(S_k(\varphi)) + \frac{1}{q}\right) + c\gamma_{q-1}.$$

Thus, by the definition of r_q we have, $r_q < e^{-S_{\beta_q}(\varphi)(\omega)}$.

$$T^{\beta_q}(\pi(\omega)) = \pi(\sigma^{\beta_q}(\omega)) \in \phi_{\beta_q|\omega|\gamma_q}([0,1])$$

Since $\omega \in S \subseteq \Gamma$, $\beta_q|\omega|\gamma_q \in \mathcal{R}_q$ and hence

$$T^{\beta_q}(\pi(\omega)) \in \phi_{\beta_q|\omega|\gamma_q}([0,1]) \subseteq B(y, r_q) \subseteq B(y, e^{-S_{\beta_q}(\varphi)(\omega)}).$$

Since this holds for all $q \in \mathbb{N}$, $\pi(\omega) \in \mathcal{F}_y(\varphi)$. □

Lemma 5.5. *Suppose $\omega \in S$. Given $q \in \mathbb{N}$ and $\gamma_{q-1} < n \leq \beta_q$ we have,*

$$\begin{aligned}
-\log \mathcal{W}_q([\omega|n]) \geq t\left(S_n(\psi)(\omega|n) + k^{-1}(n-\gamma_{q-1})\mathbb{E}(S_k(\varphi))\right) \\
-\frac{3\gamma_{q-1}}{q-1} - 2\gamma_{q-1}\rho_{\lambda_{q-1}} - \frac{2n}{q} - N(q)c,
\end{aligned}$$

$$-\log \mathcal{W}_q([\omega|\gamma_q]) \geq tS_{\gamma_q}(\psi)(\omega|\gamma_q) - \frac{3\gamma_q}{q} - 2\gamma_q\rho_{\lambda_q}.$$

Proof. We prove the lemma by induction. The lemma is trivial for $q = 0$. Now suppose that

$$-\log \mathcal{W}_{q-1}([\omega|\gamma_q]) \geq tS_{\gamma_{q-1}}(\psi)(\omega|\gamma_{q-1}) - \frac{3\gamma_{q-1}}{q-1} - 2\gamma_{q-1}\rho_{\lambda_{q-1}}.$$

Take $\gamma_{q-1} < n \leq \beta_q$ consider $\ell(n) := \lfloor k^{-1}(n-\gamma_{q-1})\rfloor$. If $\ell(n) < N(q)$ then clearly

$$\begin{aligned}
S_n(\psi)(\omega|n) &\leq S_{\gamma_{q-1}}(\psi)(\omega|\gamma_{q-1}) + S_{n-\gamma_q}(\psi)(\gamma_{q-1}|\omega|n) \\
&\leq S_{\gamma_q}(\psi)(\omega|\gamma_{q-1}) + N(q)c,
\end{aligned}$$

$$k^{-1}(n-\gamma_{q-1})\mathbb{E}(S_k(\varphi)) \leq N(q)c$$

Since $t < 1$ and $N(q-1) \leq N(q)$ it follows from the inductive hypothesis together with the definition of $\mathcal{W}_q$ that,

$$\begin{aligned}
-\log \mathcal{W}_q([\omega|n]) &\geq -\log \mathcal{W}_{q-1}([\omega|\gamma_{q-1}]) \\
&\geq t\left(S_n(\psi)(\omega|n) + k^{-1}(n-\gamma_{q-1})\mathbb{E}(S_k(\varphi))\right) \\
&\quad -\frac{3\gamma_{q-1}}{q-1} - 2\gamma_{q-1}\rho_{\lambda_{q-1}} - 2N(q)c.
\end{aligned}$$

On the other hand, if $\ell(n) \geq N(q)$ then by equation (5.14) together with $[\gamma_{q-1}|\omega|\beta_q] \cap S_q \neq \emptyset$ we have

$$\begin{aligned}\sum_{\nu=k^{-1}\gamma_{q-1}}^{k^{-1}\gamma_q+\ell(n)-1} \log p(\omega_{k\nu+1},\cdots,\omega_{k\nu+k}) &< -t\left(S_{k\ell(n)}(\psi)(\gamma_{q-1}|\omega|\gamma_{q-1}+k\ell(n))+\ell(n)\mathbb{E}(S_k(\varphi))\right)+\frac{2n}{q} \\ &< -t\left(S_{n-\gamma_{q-1}}(\psi)(\omega|n-\gamma_{q-1})+k^{-1}(n-\gamma_{q-1})\mathbb{E}(S_k(\varphi))\right) \\ &\quad +2c+\frac{2n}{q}.\end{aligned}$$

Moreover, by the defintion of $\mathcal{W}_q$ we have,

$$\begin{aligned}-\log\mathcal{W}_q\left([\omega|n]\right) &\geq -\log\mathcal{W}_{q-1}\left([\omega|\gamma_{q-1}]\right)-\sum_{\nu=0}^{\ell(n)-1}\log p(\omega_{k\nu+1},\cdots,\omega_{k\nu+k}) \\ &\geq t\left(S_{\gamma_{q-1}}(\psi)(\omega|\gamma_{q-1})+S_{n-\gamma_{q-1}}(\psi)(\omega|n-\gamma_{q-1})+k^{-1}(n-\gamma_{q-1})\mathbb{E}(S_k(\varphi))\right) \\ &\quad --\frac{3\gamma_{q-1}}{q-1}-2\gamma_{q-1}\rho_{\lambda_{q-1}}-2c-\frac{2n}{q} \\ &\geq t\left(S_n(\psi)(\omega|n)+k^{-1}(n-\gamma_{q-1})\mathbb{E}(S_k(\varphi))\right) \\ &\quad -\frac{3\gamma_{q-1}}{q-1}-2\gamma_{q-1}\rho_{\lambda_{q-1}}-N(q)c-\frac{2n}{q}.\end{aligned}$$

In particular we have

$$\begin{aligned}-\log\mathcal{W}_q\left([\omega|\beta_q]\right) &\geq t\left(S_{\beta_q}(\psi)(\omega|\beta_q)+k^{-1}(\beta_q-\gamma_{q-1})\mathbb{E}(S_k(\varphi))\right) \\ &\quad -\frac{3\gamma_{q-1}}{q-1}-2\gamma_{q-1}\rho_{\lambda_{q-1}}-N(q)c-\frac{2\beta_q}{q}.\end{aligned}$$

Note that,

$$\begin{aligned}-\log\mathcal{W}_q\left([\omega|\gamma_q]\right) &= -\log\mathcal{W}_q\left([\omega|\beta_q]\right)-\log q(\omega|\beta_q,\beta_q|\omega|\gamma_q) \\ &= -\log\mathcal{W}_q\left([\omega|\beta_q]\right)-\log\left(\frac{\operatorname{diam}\left(\phi_{\omega|\gamma_q}([0,1])\right)}{\sum_{\tau\in\mathcal{R}_q}\operatorname{diam}\left(\phi_{\omega|\beta_q}\circ\phi_\tau([0,1])\right)}\right) \\ &\geq -\log\mathcal{W}_q\left([\omega|\beta_q]\right)-t\log\left(\frac{\operatorname{diam}\left(\phi_{\omega|\gamma_q}([0,1])\right)}{\sum_{\tau\in\mathcal{R}_q}\operatorname{diam}\left(\phi_{\omega|\beta_q}\circ\phi_\tau([0,1])\right)}\right).\end{aligned}$$

Clearly,

$$-\log\operatorname{diam}\left(\phi_{\omega|\gamma_q}([0,1])\right)\geq S_{\gamma_q}(\psi)(\omega|\gamma_q)-\gamma_q\rho_{\gamma_q}$$

Moreover,

$$\begin{aligned}\sum_{\tau\in\mathcal{R}_q}\text{diam}\left(\phi_{\omega|\beta_q\circ\tau}([0,1])\right) &\geq \sum_{\tau\in\mathcal{R}_q}\exp\left(-S_{\gamma_q}(\psi)(\omega|\beta_q,\tau)\right)\\ &\geq e^{-S_{\beta_q}(\psi)(\omega|\beta_q)}\sum_{\tau\in\mathcal{R}_q}e^{-S_{\lambda_q}(\psi)(\tau)}\\ &\geq e^{-S_{\beta_q}(\psi)(\omega|\beta_q)-\lambda_q\rho_{\lambda_q}}\sum_{\tau\in\mathcal{R}_q}\text{diam}\left(\phi_\tau([0,1])\right)\\ &\geq e^{-S_{\beta_q}(\psi)(\omega|\beta_q)-\lambda_q\rho_{\lambda_q}}\zeta r_q.\end{aligned}$$

Note that from the definition of β_q and c we have,

$$-\log r_q \leq k^{-1}(\beta_q-\gamma_{q-1})\mathbb{E}(S_k(\varphi))+c(\gamma_{q-1}+1)+q$$

Combining these inequalities we see that,

$$\begin{aligned}-\log\mathcal{W}_q\left([\omega|\gamma_q]\right) &\geq tS_{\gamma_q}(\psi)(\omega|\gamma_q)-\gamma_q\rho_{\gamma_q}-N(q)c-\frac{2\beta_q}{q}\\ &\quad -\frac{3\gamma_{q-1}}{q-1}-2\gamma_{q-1}\rho_{\lambda_{q-1}}-\lambda_q\rho_{\lambda_q}-c(\gamma_{q-1}+1)-q+\log\zeta\\ &\geq tS_{\gamma_q}(\psi)(\omega|\gamma_q)-\frac{3\gamma_q}{q}-2\gamma_q\rho_{\lambda_q},\end{aligned}$$

since $\gamma_q\geq\beta_q\geq\alpha_q$ and by the definition of α_q,

$$\alpha_q > q\left(\frac{3\gamma_{q-1}}{q-1}+2\gamma_{q-1}\rho_{\lambda_{q-1}}+c(\gamma_{q-1}+1)+q-\log\zeta\right).$$

□

We define a Borel measure μ by $\mu(A):=\mathcal{W}(S\cap\pi^{-1}(A))$ for Borel sets $A\subseteq[0,1]$.

Lemma 5.6. *$\mu([0,1])>0$.*

Proof. This follows immediately from the fact that

$$\mathcal{W}(S)\geq\prod_{q\in\mathbb{N}}(1-\delta_q)>0.$$

□

Lemma 5.7. *For all $\omega\in S$ we have*

$$\liminf_{r\to 0}\frac{\log\mu(B(\pi(\omega),r))}{\log r}\geq t.$$

Proof. For the proof of Lemma 5.7 we shall require some additional notation. Given a pair of functions f and g, depending on $q\in\mathbb{N}$ and $r\in(0,1)$, we shall write,

$$f(q,r)\geq g(q,r)-\eta(q,r), \tag{5.18}$$

to denote that for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ and a $\delta > 0$ such that given any $(q,r) \in \mathbb{N} \times (0,1)$ with $q > N$ and $r < \delta$ we have

$$f(q,r) \geq g(q,r) - \epsilon. \tag{5.19}$$

Note that by (5.15) $r_q < e^{-q}$ for all $q \in \mathbb{N}$ and by Definition 5.1 this implies that $\lim_{q\to\infty} \lambda_q = \lim_{q\to\infty} \lambda_{r_q} = \infty$ and hence $\lim_{q\to\infty} \rho_{\lambda_q} = 0$. Thus for any function $g : \mathbb{N} \times (0,1) \to \mathbb{R}$,

$$g(q,r) - \rho_{\lambda_q} \geq g(q,r) - \eta(q,r).$$

Similarly, it follows from the definition of β_q that

$$g(q,r) - cN(q)N(q+1)\beta_q^{-1} \geq g(q,r) - \eta(q,r).$$

Firstly we show that for any $x = \pi(\omega)$ with $\omega \in S$ $B(x,r)$ and $r > 0$ for which there exists $q \in \mathbb{N}$ and $l \in \mathbb{N}$ with $\gamma_{q-1} \leq l < \beta_q$ such that

$$B(x,r) \cap \pi(S) \subseteq \phi_{\omega|l}([0,1]) \text{ but } B(x,r) \cap \pi(S) \quad \nsubseteq \quad \phi_{\omega|l+1}([0,1])$$

satisfies

$$\frac{\log \mu(B(x,r))}{\log r} \quad \geq \quad t - \eta(q,r). \tag{5.20}$$

Indeed, as $B(x,r) \cap \pi(S) \subseteq \phi_{\omega|l}([0,1])$ it follows from Lemma 5.5 that,

$$\begin{aligned}
-\log \mu(B(x,r)) \quad &\geq \quad -\log \mathcal{W}([\omega|l]) \\
&= \quad -\log \mathcal{W}_q([\omega|l]) \\
&\geq \quad tS_l(\psi)(\omega|l) - \frac{3\gamma_{q-1}}{q-1} - 2\gamma_{q-1}\rho_{\lambda_{q-1}} - \frac{2l}{q} - N(q)c \\
&= \quad -\log \mathcal{W}_q([\omega|l]) \\
&\geq \quad tS_l(\psi)(\omega|l) - \frac{6l}{q-1} - 2l\rho_{\lambda_{q-1}},
\end{aligned}$$

since $l \geq \gamma_{q-1} > qN(q)c$. Since $S_l(\psi)(\omega|l) \geq l \log \xi$ this implies

$$\frac{\log \mu(B(x,r))}{S_l(\psi)(\omega|l)} \quad \geq \quad t - \log \xi^{-1}\left(\frac{6}{q-1} + 2\rho_{\lambda_{q-1}}\right).$$

However, $B(x,r) \cap \pi(S) \nsubseteq \phi_{\omega|l+1}([0,1])$ and hence $B(x,r) \cap \pi(S) \nsubseteq \phi_{\omega|\kappa(l)}([0,1])$ where $\kappa(l) := k\lceil k^{-1}(l+1)\rceil$. It follows that $B(x,r) \cap \pi(S)$ intersects $\phi_{\tau|\kappa(l)}([0,1])$, for some $\tau \in S$, as well as $\phi_{\omega|\kappa(l)}([0,1])$. Since $\kappa(l) \leq \beta_q$ and $\omega, \tau \in S$, $(\kappa(l)-k)|\omega|\kappa(l), (\kappa(l)-k)|\tau|\kappa(l) \in \mathcal{B}$. Thus, by (5.10),

$$\begin{aligned}
r \quad &\geq \quad \frac{1}{2} e^{-S_n(\psi)(\omega|\kappa(l)-k)-c} \\
&\geq \quad e^{-S_n(\psi)(\omega|l)-c-\log 2}.
\end{aligned}$$

Thus,

$$\frac{\log \mu(B(x,r))}{\log r} \quad \geq \quad \left(1 + \frac{c+\log 2}{\log r}\right)\left(t - \log \xi^{-1}\left(\frac{6}{q-1} + 2\rho_{\lambda_{q-1}}\right)\right)$$

which implies the first claim (5.20).

Secondly, we show that given $\omega \in S$, $x \in [0,1]$ and $r > 0$ for which $B(x,r) \cap \pi(S) \subseteq \phi_{\omega|\beta_q}([0,1])$ and yet $B(x,r) \cap \pi(S) \nsubseteq \phi_{\omega|\beta_q} \circ \phi_\tau([0,1])$ for any $\tau \in \mathcal{R}_q$ we have,

$$\frac{\log \mu\left(B(x,r)\right)}{\log r} \geq t - \eta(q,r). \tag{5.21}$$

From the proof of Lemma 5.5 we have,

$$\begin{aligned}
-\log \mathcal{W}_q\left([\omega|\beta_q]\right) &\geq t\left(S_{\beta_q}(\psi)(\omega|\beta_q) + k^{-1}(\beta_q - \gamma_{q-1})\mathbb{E}(S_k(\varphi))\right) \\
&\quad -\frac{3\gamma_{q-1}}{q-1} - 2\gamma_{q-1}\rho_{\lambda_{q-1}} - N(q)c - \frac{2\beta_q}{q} \\
-\log r_q &\leq k^{-1}(\beta_q - \gamma_{q-1})\mathbb{E}(S_k(\varphi)) + c(\gamma_{q-1}+1) + q \\
\sum_{\tau \in \mathcal{R}_q} \operatorname{diam}\left(\phi_{\omega|\beta_q \circ \tau}([0,1])\right) &\geq e^{-S_{\beta_q}(\psi)(\omega|\beta_q) - \lambda_q \rho_{\lambda_q}} \zeta r_q.
\end{aligned}$$

Suppose $r > r_q$. Then by the first two inequalities together with the fact that $B(x,r) \subseteq \phi_{\omega|\beta_q}([0,1])$ we have

$$\begin{aligned}
-\log \mu(B(x,r)) &\geq -\log \mathcal{W}_q\left([\omega|\beta_q]\right) \\
&\geq -t\log r - \left(\frac{3\gamma_{q-1}}{q-1} + 2\gamma_{q-1}\rho_{\lambda_{q-1}} + N(q)c + \frac{2\beta_q}{q} + c(\gamma_{q-1}+1) + q\right).
\end{aligned}$$

Note also that $B(x,r) \subseteq \phi_{\omega|\beta_q}([0,1])$ implies $-\log r > \beta_q \log \xi > \gamma_{q-1} \log \xi$ and hence,

$$\begin{aligned}
\frac{\log \mu(B(x,r))}{\log r} &\geq t - \log \xi^{-1}\left(\frac{3}{q-1} + 2\rho_{\lambda_{q-1}} + \frac{N(q)c + c(\gamma_{q-1}+1) + q}{\beta_q} + \frac{2}{q}\right) \\
&\geq t - \eta(q,r).
\end{aligned}$$

Now suppose that $r \leq r_q$ and let $\mathcal{T}$ denote the following collection,

$$\mathcal{T} := \left\{\tau \in \mathcal{R}_q : \frac{\operatorname{diam}\left(\phi_{\omega|\beta_q} \circ \phi_\tau([0,1]) \cap B(x,r)\right)}{\operatorname{diam}\left(\phi_{\omega|\beta_q} \circ \phi_\tau([0,1])\right)} > \frac{1}{2}\right\}.$$

We also define $B_{\mathcal{T}}(x,r) \subseteq B(x,r)$ by,

$$B_{\mathcal{T}}(x,r) := \bigcup_{\tau \in \mathcal{T}} \phi_{\omega|\beta_q} \circ \phi_\tau([0,1])$$

From the definition of μ and $\mathcal{W}$ we see that for each $\tau \in \mathcal{R}_q$ we have,

$$\begin{aligned}
\mu(\phi_{\omega|\beta_q} \circ \phi_\tau([0,1])) &\leq \mathcal{W}_q\left([\omega|\beta_q, \tau]\right) \\
&\leq \mathcal{W}_q\left([\omega|\beta_q]\right) \cdot \frac{\operatorname{diam}\left(\phi_{\omega|\beta_q} \circ \phi_\tau([0,1])\right)}{\sum_{\tau \in \mathcal{R}_q} \operatorname{diam}\left(\phi_{\omega|\beta_q} \circ \phi_\tau([0,1])\right)}.
\end{aligned}$$

Hence, as $t < 1$,

$$\begin{aligned}\mu(B_{\mathcal{T}}(x,r)) &\leq \sum_{\tau\in\mathcal{T}} \mu(\phi_{\omega|\beta_q}\circ\phi_\tau([0,1]))\\ &\leq \mathcal{W}_q\left([\omega|\beta_q]\right)\cdot\frac{\sum_{\tau\in\mathcal{T}}\operatorname{diam}\left(\phi_{\omega|\beta_q}\circ\phi_\tau([0,1])\right)}{\sum_{\tau\in\mathcal{R}_q}\operatorname{diam}\left(\phi_{\omega|\beta_q}\circ\phi_\tau([0,1])\right)}\\ &\leq \mathcal{W}_q\left([\omega|\beta_q]\right)\left(\frac{\sum_{\tau\in\mathcal{T}}\operatorname{diam}\left(\phi_{\omega|\beta_q}\circ\phi_\tau([0,1])\right)}{\sum_{\tau\in\mathcal{R}_q}\operatorname{diam}\left(\phi_{\omega|\beta_q}\circ\phi_\tau([0,1])\right)}\right)^t\\ &\leq 2\mathcal{W}_q\left([\omega|\beta_q]\right)\left(\sum_{\tau\in\mathcal{R}_q}\operatorname{diam}\left(\phi_{\omega|\beta_q}\circ\phi_\tau([0,1])\right)\right)^{-t} r^t.\end{aligned}$$

Piecing the previous inequalities together with the observations from the proof of Lemma 5.5 we obtain

$$-\log\mu\left(B_{\mathcal{T}}(x,r)\right)$$

$$\geq -t\log r - \left(\frac{3\gamma_{q-1}}{q-1} + 2\gamma_{q-1}\rho_{\lambda_{q-1}} + N(q)c + \frac{2\beta_q}{q} + c(\gamma_{q-1}+1) + q + \lambda_q\rho_{\lambda_q} - \log\zeta - \log 2\right).$$

Now $\lambda_q < d\log r_q \leq d\log r$, where $d < 0$ is the constant as appears in the positive upper cylinder density condition. Hence,

$$\frac{\log\mu\left(B_{\mathcal{T}}(x,r)\right)}{\log r} \tag{5.22}$$

$$\begin{aligned}&\geq t - \log\xi^{-1}\left(\frac{3}{q-1} + 2\rho_{\lambda_{q-1}} + \frac{N(q)c + c(\gamma_{q-1}+1) + q - \log\zeta + \log 2}{\beta_q} + \frac{2}{q}\right) + d\rho_{\lambda_q}\\ &\geq t - \eta(q,r).\end{aligned}$$

Consider the set $\mathcal{C} := \left\{\tau\in\mathcal{R}_q : \phi_{\omega|\beta_q}\circ\phi_\tau([0,1])\cap B(x,r)\neq\emptyset, \tau\notin\mathcal{T}\right\}$. It is clear that $\mathcal{C}$ contains at most two elements, with $\phi_{\omega|\beta_q}\circ\phi_\tau([0,1])$ containing either $\inf B(x,r)$ or $\sup B(x,r)$. We shall show that for $\tau\in\mathcal{C}$ we have,

$$\frac{\log\mu\left((\phi_{\omega|\beta_q}\circ\phi_\tau)([0,1])\cap B(x,r)\right)}{\log r} \geq t - \eta(q,r). \tag{5.23}$$

Take $\tau\in\mathcal{C}$ and assume that $\sup B(x,r)\in\phi_{\omega|\beta_q}\circ\phi_\tau([0,1])$ ie. $\phi_{\omega|\beta_q}\circ\phi_\tau([0,1])$ intersects the right hand boundary of $B(x,r)$. Since $\tau\notin\mathcal{T}$ we have $\operatorname{diam}\left(\phi_{\omega|\beta_q}\circ\phi_\tau([0,1])\cap B(x,r)\right) < \frac{1}{2}\operatorname{diam}\left(\phi_{\omega|\beta_q}\circ\phi_\tau([0,1])\right)$. Choose $\tilde\omega\in S$ such that $\pi(\tilde\omega)$ is on the right hand side of $\phi_{\omega|\beta_q}\circ\phi_\tau([0,1])\cap B(x,r)\cap\pi(S)$. Define $\tilde r := |\pi(\tilde\omega) - \inf(\phi_{\omega|\beta_q}\circ\phi_\tau)([0,1])|$, and consider $B(\pi(\tilde\omega),\tilde r)$. Since $\pi(\tilde\omega)$ is on the right hand side of $(\phi_{\omega|\beta_q}\circ\phi_\tau)([0,1])\cap B(x,r)\cap\pi(S)$ and

$$\operatorname{diam}\left(\phi_{\omega|\beta_q}\circ\phi_\tau([0,1])\cap B(x,r)\right) < \frac{1}{2}\operatorname{diam}\left(\phi_{\omega|\beta_q}\circ\phi_\tau([0,1])\right),$$

we have

$$(\phi_{\omega|\beta_q} \circ \phi_\tau)([0,1]) \cap B(x,r) \cap \pi(S) \subseteq B(\pi(\tilde{\omega}), \tilde{r}) \subseteq (\phi_{\omega|\beta_q} \circ \phi_\tau)([0,1])$$

and $\tilde{\omega}|\gamma_q = (\omega|\beta_q, \tau)$.

We consider two cases. First suppose that $B(\pi(\tilde{\omega}), \tilde{r}) \subseteq \phi_{\tilde{\omega}|\beta_{q+1}}([0,1])$. It follows from Lemma 5.5 that,

$$\begin{aligned} -\log \mu\left(B(\pi(\tilde{\omega}), \tilde{r})\right) &\geq -\log \mathcal{W}_{q+1}\left([\tilde{\omega}|\beta_{q+1}]\right) \\ &\geq t\left(S_{\beta_{q+1}}(\psi)(\omega|\beta_{q+1}) + k^{-1}(\beta_{q+1} - \gamma_{q-1})\exp(S_k(\varphi))\right) \\ &\quad -\frac{3\gamma_q}{q} - 2\gamma_q\rho_{\lambda_q} - \frac{2\beta_{q+1}}{q+1} - N(q+1)c \\ &\geq t\beta_{q+1}\log\xi - \left(k\log\xi + cN(q+1) + \frac{5\beta_{q+1}}{q} + 2\beta_{q+1}\rho_{\lambda_q}\right). \end{aligned}$$

Hence,

$$\frac{-\log \mu\left((\phi_{\omega|\beta_q} \circ \phi_\tau)([0,1]) \cap B(x,r)\right)}{\beta_{q+1}\log\xi} \geq t - \log\xi^{-1}\left(\frac{k\log\xi + cN(q+1)}{\beta_{q+1}} + \frac{5}{q} + 2\rho_{\lambda_q}\right).$$

Since $B(x,r) \cap \pi(S) \nsubseteq (\phi_{\omega|\beta_q} \circ \phi_{\tau'})([0,1])$ for any $\tau' \in \mathcal{R}_q$, it follows from (5.10) that

$$\begin{aligned} -\log r &\leq -\max\left\{S_{\gamma_q}(\psi)(\tau') : \tau' \in \Gamma_q\right\} - c \\ &\leq \alpha_{q+1}\log\xi < \beta_{q+1}\log\xi. \end{aligned} \tag{5.24}$$

Thus,

$$\begin{aligned} \frac{\log \mu\left((\phi_{\omega|\beta_q} \circ \phi_\tau)([0,1]) \cap B(x,r)\right)}{\log r} &\geq t - \log\xi^{-1}\left(\frac{k\log\xi + cN(q+1)}{\beta_{q+1}} + \frac{5}{q} + 2\rho_{\lambda_q}\right) \\ &\geq t - \eta(q,r). \end{aligned}$$

Now suppose that $B(\pi(\tilde{\omega}), \tilde{r}) \nsubseteq \phi_{\tilde{\omega}|\beta_{q+1}}([0,1])$. Then we may apply (5.20) to obtain

$$\frac{\log \mu(B(\pi(\tilde{\omega}, \tilde{r}))}{\log \tilde{r}} \geq t - \eta(q+1, \tilde{r}). \tag{5.25}$$

Clearly $\tilde{r} < 2r$ and so $\lim_{r\to\infty} \frac{\log \tilde{r}}{\log r} \geq 1$ and hence,

$$\frac{\log \mu\left((\phi_{\omega|\beta_q} \circ \phi_\tau)([0,1]) \cap B(x,r)\right)}{\log r} \geq t - \eta(q,r).$$

By symmetry the same holds if $\phi_{\omega|\beta_q} \circ \phi_\tau([0,1])$ intersects the left hand boundary of $B(x,r)$. This proves the claim (5.23).

Recall that,

$$B(x,r) \cap \pi(S) \subseteq B_{\mathcal{T}}(x,r) \cup \left(\bigcup_{\tau\in\mathcal{C}} (\phi_{\omega|\beta_q} \circ \phi_\tau)([0,1]) \cap B(x,r)\right).$$

Noting that $\#\mathcal{C} \leq 2$ we obtain,

$$\begin{aligned}\mu\left(B(x,r)\right) &\leq \mu\left(B_{\mathcal{T}}(x,r)\right) + \sum_{\tau\in\mathcal{C}} \mu\left((\phi_{\omega|\beta_q} \circ \phi_\tau)([0,1]) \cap B(x,r)\right) \\ &\leq 3\max\left\{\mu\left(B_{\mathcal{T}}(x,r)\right)\right\} \cup \left\{\mu\left((\phi_{\omega|\beta_q} \circ \phi_\tau)([0,1]) \cap B(x,r)\right) : \tau \in \mathcal{C}\right\}.\end{aligned}$$

By combining with (5.22) and (5.23),

$$\frac{\log \mu\left(B(x,r)\right) - \log 3}{\log r} \geq t - \eta(q,r),$$

which implies (5.21).

To complete the proof of the Lemma we fix $\omega \in S$, let $x = \pi(\omega)$ and consider a ball $B(\pi(\omega), r)$ of radius $r > 0$. Now choose $q(r) \in \mathbb{N}$ so that

$$B(x,r) \cap \pi(S) \subseteq \phi_{\omega|\gamma_{q(r)-1}}([0,1]) \text{ but } B(x,r) \cap \pi(S) \quad \not\subseteq \quad \phi_{\omega|\gamma_{q(r)}}([0,1]).$$

Now either $B(x,r) \cap \pi(S) \not\subseteq \phi_{\omega|\beta_{q(r)}}([0,1])$, in which case we apply (5.20) or $B(x,r) \cap \pi(S) \not\subseteq \phi_{\omega|\beta_{q(r)}}([0,1])$ in which case we apply (5.21). In both cases we obtain,

$$\frac{\log \mu(B(x,r))}{\log r} \geq t - \eta(q(r), r). \tag{5.26}$$

By (5.24) whenver $q(r) \leq Q$ we have

$$r \geq \exp\left(-\max\left\{S_{\gamma_Q}(\psi)(\tau') : \tau' \in \Gamma_Q\right\} - c\right) > 0.$$

Hence, $\lim_{r\to 0} q(r) = \infty$. Therefore, by (5.26) we have

$$\liminf_{r\to 0} \frac{\log \mu(B(\pi(\omega), r))}{\log r} \geq t. \tag{5.27}$$

□

To complete the proof of Proposition 5.1 we recall the following standard Lemma.

Lemma 5.8. *Let ν be a finite Borel measure on some metric space X. Suppose we have $J \subseteq X$ with $\nu(J) > 0$ such that for all $x \in J$*

$$\liminf_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \geq d.$$

Then $\dim_{\mathcal{H}} J \geq d$.

Proof. See [F2, Proposition 2.2] . □

Thus by Lemmas 5.7 and 5.6 we have

$$\dim_{\mathcal{H}} \pi(S) \geq t > s.$$

Hence, by Lemma 5.4 the Hausdorff dimension of $\mathcal{D}_y(\varphi)$ is at least s. Since this for all $s < s(\varphi)$, we have

$$\dim_{\mathcal{H}} \mathcal{D}_y(\varphi) \geq s(\varphi).$$

□

6. Proof of Theorem 5

Proof of Theorem 5. We begin by defining a sequence $(r_n)_{n\in\mathbb{N}}$ by

$$r_n := \min\left\{\left(2+\sum_{q\in\mathbb{N}} e^{-q/n}\right)^{-n^2}\cdot e^{-2n^2}, \frac{1}{2}\left(\Phi(n)-\Phi(n+1)\right)\right\}. \tag{6.1}$$

Note that since Φ is strictly decreasing each $r_n > 0$. Now take $n_0 > 2$ so that $\Phi(n_0) < \left(1-2^{1-\beta^{-1}}\right)$ and $\sum_{n\geq n_0} e^{-\beta n} < 1$. For each $n \geq n_0$ we choose some closed interval $V_n \subset (\Phi_{n+1}, \Phi_n)$ of length r_n, which is always possible, since $r_n < \Phi(n) - \Phi(n+1)$. Note that since each $r_n < e^{-n}$ we have $\sum_{n\geq n_0} r_n^\beta \leq \sum_{n\geq n_0} e^{-\beta n} < 1$. Hence, $r_1 = r_2 := 2^{-\beta^{-1}}\left(1-\sum_{n\geq n_0} r_n^\beta\right)^{\beta^{-1}} > 0$. Note also that $1-\Phi(n_0) > 2^{1-\beta^{-1}} > 2r_1$. Thus, we may choose two disjoint closed intervals V_1, V_2 of width $r_1 = r_2$ contained within $(\Phi(n_0), 1)$.

We now let $\mathcal{A} := \{n \in \mathbb{N} : n \geq n_0\} \cup \{1,2\}$. Define $T : \bigcup_{n\in\mathcal{A}} V_n \to [0,1]$ to be the unique expanding Markov map which maps each of the intervals $\{V_n\}_{n\in\mathcal{A}}$ onto $[0,1]$ in an affine and orientation preserving way. First note that,

$$\sum_{n\in\mathcal{A}} \operatorname{diam}(V_n)^\beta = r_1^\beta + r_2^\beta + \sum_{n\geq n_0} r_n^\beta = 1. \tag{6.2}$$

Thus, $\dim_{\mathcal{H}}\Lambda = \beta$ by Moran's formula.

Take $n \geq n_0$ and consider $\mathcal{S}_0^{(n)}(\Phi) := \{x \in \Lambda : |T^n(x)| < \Phi(n)\}$. Since T is orientation preserving it follows from the construction of T that we can cover $S_n(\Phi)$ with sets of the form $V_\omega = \cap_{j=0}^n T^{-j} V_{\omega_j}$ where $\omega \in \mathcal{C}_n := \left\{\omega \in \mathcal{A}^{n+1} : \omega_{n+1} \geq n\right\}$. Since T is piecewise linear we have $\operatorname{diam} V_\omega = \prod_{j=1}^{n+1} r_{\omega_j}$ for each $\omega \in \mathcal{A}^{n+1}$. It follows that for any $m > n_0$ we may cover $\mathcal{S}_0(\Phi)$ with the family $\bigcup_{n\geq m}\{V_\omega : \omega \in \mathcal{C}_n\}$.

Now take $\epsilon > 0$. For all $n > \epsilon^{-1}$ we have,

$$\begin{aligned}
\sum_{\omega \in \mathcal{C}_n} (\mathrm{diam} V_\omega)^\epsilon &\leq \sum_{\omega \in \mathcal{C}_n} (r_{\omega_1} \cdots r_{\omega_n})^\epsilon \\
&= \left(\sum_{n \in \mathcal{A}} r_n^\epsilon \right)^n \cdot \sum_{q \geq n} r_n^\epsilon \\
&\leq \left(2 + \sum_{q \in \mathbb{N}} e^{-\epsilon q} \right)^n \cdot \sum_{k \geq n} \left(\left(2 + \sum_{q \in \mathbb{N}} e^{-q/k} \right)^{-k^2} \cdot e^{-2k^2} \right)^\epsilon \\
&\leq \left(2 + \sum_{q \in \mathbb{N}} e^{-\epsilon q} \right)^n \cdot \left(2 + \sum_{q \in \mathbb{N}} e^{-q/n} \right)^{-n^2 \epsilon} \cdot \sum_{k \geq n} e^{-2kn\epsilon} \\
&\leq \left(2 + \sum_{q \in \mathbb{N}} e^{-\epsilon q} \right)^n \cdot \left(2 + \sum_{q \in \mathbb{N}} e^{-q/n} \right)^{-n} \cdot e^{-n} \sum_{k \geq n} e^{-k} \\
&\leq e^{-n} \sum_{k \in \mathbb{N}} e^{-k}.
\end{aligned}$$

Thus, for all $m > \epsilon^{-1}$ we have,

$$\sum_{n \geq m} \sum_{\omega \in \mathcal{C}_n} (\mathrm{diam} V_\omega)^\epsilon \leq \sum_{n \geq m} e^{-n} \sum_{k \in \mathbb{N}} e^{-k} \leq \left(\sum_{k \in \mathbb{N}} e^{-k} \right)^2 < \infty.$$

Since $\lim_{m \to \infty} \sup \{\mathrm{diam} V_\omega : \omega \in \mathcal{C}_n\} = 0$ it follows that $\dim_{\mathcal{H}} S_0(\Phi) < \epsilon$. As this holds for all $\epsilon > 0$ we have $\dim_{\mathcal{H}} S_0(\Phi) = 0$. □

We note that by Corollary 2 $s(\alpha) > 0$ for all $\alpha \in \mathbb{R}_{>0}$.

7. Remarks

Both Theorems 3 and 4 may be extended in a number of ways with some minor alterations of the proof.

Given $\Phi : \mathbb{N} \times \Lambda \to (0, 1)$ we define

$$\mathcal{S}_y(\Phi) := \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \{x \in \Lambda : |T^n(x) - y| < \Phi(n, x)\}.$$

Theorems 3 and 4 both deal with the case where Φ is multiplicative, ie. $\Phi(n + m, x) = \Phi(n, T^m(x)) \cdot \Phi(m, x)$, for all $n, m \in \mathbb{N} \cup \{0\}$ and $x \in \Lambda$. Indeed, when Φ is multiplicative, we may take $\varphi : x \mapsto -\log \Phi(0, x)$ so that $\Phi(n, x) = \exp(-S_n(\varphi)(x))$ and $\mathcal{S}_y(\Phi) = \mathcal{D}_y(\varphi)$.

We say that Φ is almost multiplicative if there exists some constant $C > 1$ such that,

$$C^{-1} < \frac{\Phi(n, T^m(x)) \cdot \Phi(m, x)}{\Phi(n + m, x)} < C,$$

for all $n, m \in \mathbb{N}$ and $x \in \Lambda$. Examples include the norms of certain matrix products (see [FL, IY]). Given $\omega \in \mathcal{A}^n$ we let $\Phi(\omega) := \sup\{\Phi(n,x) : x \in V_\omega\}$. Following Feng and Lau [FL] one may define a pressure function, $P(s,\Phi) \to \mathbb{R}$ by

$$P(s,\Phi) := \lim_{n\to\infty} \frac{1}{n} \sum_{\omega\in\mathcal{A}^n} \left(\Phi(\omega)\cdot ||\psi'_\omega||_\infty\right)^s,$$

and let $s(\Phi) := \inf\{s : P(s,\Phi) \leq 0\}$. Technical modifications to the proof of Theorems 3 and 4 show that whenever T is a countable Markov map and Φ is almost multiplicative, $\dim_{\mathcal{H}} \mathcal{S}_y(\Phi) = s(\Phi)$ for all $y \in \Lambda$, and if $\overline{\Lambda} = [0,1]$ then $\dim_{\mathcal{H}} \mathcal{S}_y(\Phi) = s(\Phi)$ for all $y \in \overline{\Lambda}$.

Instead of considering the sets $\mathcal{D}_y(\varphi)$ we can consider sets of the form,

$$\mathcal{L}_y(\varphi) := \left\{x \in \Lambda : \limsup_{n\to\infty} \frac{\log d(T^n(x),y)}{S_n(\varphi)(x)} = -1\right\}.$$

When T is a countable Markov map we have $\dim_{\mathcal{H}} \mathcal{L}_y(\varphi) = \dim_{\mathcal{H}} \mathcal{D}_y(\varphi) = s(\varphi)$ for all $y \in \Lambda$ and when T is a countable Markov map satisfying $\overline{\Lambda} = [0,1]$ we have $\dim_{\mathcal{H}} \mathcal{L}_y(\varphi) = \dim_{\mathcal{H}} \mathcal{D}_y(\varphi) = s(\varphi)$ for all $y \in [0,1]$. To prove the upper bound we note that $\mathcal{L}_y(\varphi) \subset \dim_{\mathcal{H}} \mathcal{D}_y((1-\delta)\varphi)$ for all $\delta \in (0,1)$ and $\lim_{\delta\to 0} \dim_{\mathcal{H}} \mathcal{D}_y((1-\delta)\varphi) = \lim_{\delta\to 0} s((1-\delta)\varphi) = s(\varphi)$. To prove the lower bound requires a technical adaptation of the proof of Proposition 5.1, removing those points x for which $T^n(x)$ moves too close to y.

One can also consider what happens when we replace assumption (1) in Definition 2.1 with the weaker assumption that T is modelled by a subshift of finite type. If the corresponding matrix is finitely primitive (see [MU2, Section 2.1]) then one may adapt the proofs of Theorems 3 and 4 with only mino modifications. However, to determine the dimension of $\mathcal{D}_y(\varphi)$ for an arbitrary countable subshift of finite type would require further innovation.

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