A survey of quantum Teichmüller space and Kashaev algebra

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Abstract.
In this chapter, we survey the algebraic aspects of quantum Teichmüller space, generalized Kashaev algebra and a natural relationship between the two algebras.

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Contents

1 Introduction

A quantization of the Teichmüller space $\mathcal{T}(S)$ of a punctured surface S was developed by L. O. Chekhov and V. V. Fock [9, 10, 11] and, independently, by R. Kashaev [18, 19, 20, 21]. This is a deformation of the C*–algebra of functions on Teichmüller space $\mathcal{T}(S)$. The quantization was expressed in terms of self-adjoint operators on Hilbert spaces and the quantum dilogarithm function. Although these two approaches of quantization use the same ingredients, the relationship between them is still mysterious. Chekhov and Fock worked with shear coordinates of Teichmüller space while Kashaev worked with a new coordinate.

The pure algebraic foundation of Chekhov-Fock's quantization was established in the work of F. Bonahon, H. Bai and X. Liu [3, 6, 22]. The algebraic aspect of Kashaev's quantization is investigated and generalized in [13]. And a natural relationship between quantum Teichmüller space and generalized Kashaev algebra is established in [13]. In this chapter we make a survey of the ideas and results mentioned.

Recently, I. B. Frenkel and H. K. Kim [12] derived the quantum Teichmüller space from tensor products of a single canonical representation of the modular double of the quantum plane and showed that the quantum universal Teichmüller space is realized in the infinite tensor power of the canonical representation naturally indexed by rational numbers including the infinity.

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2 The quantum Teichmüller space: foundation

In this section we review the finite-dimensional Chekhov-Fock's quantization of the Teichmüller space following [22] closely.

2.1 Ideal triangulations

Let S be an oriented surface with genus g and with $p \geq 1$ punctures, obtained by removing p points $\{v_1, \dots, v_p\}$ from a closed oriented surface $\bar{S}$ of genus g. An *ideal triangulation* of S is a triangulation of the closed surface $\bar{S}$ whose vertex set is exactly $\{v_1, \dots, v_p\}$. If the Euler characteristic of S is negative, i.e., $m := 2g - 2 + p > 0$, S has an ideal triangulation. Any ideal triangulation of S has $2m$ ideal triangles and $3m$ edges. The edges of an ideal triangulation λ of S are numerated as $\{\lambda_1, ..., \lambda_{3m}\}$.

Let $\Lambda(S)$ denote the set of isotopy classes of ideal triangulations of S. The set $\Lambda(S)$ admits a natural action of the symmetric group on the set $\{1, 2, ..., 3m\}$, $\mathfrak{S}_{3m}$, acting by permuting the indices of the edges of λ. Namely $\lambda' = \alpha(\lambda)$ for $\alpha \in \mathfrak{S}_{3m}$ if $\lambda_i = \lambda'_{\alpha(i)}$.

Another important transformation of $\Lambda(S)$ is provided by the *i–th diagonal exchange map* $\Delta_i : \Lambda(S) \to \Lambda(S)$ defined as follows. Suppose that the i–th edge λ_i of an ideal triangulation $\lambda \in \Lambda(S)$ is adjacent to two triangles. Then $\Delta_i(\lambda)$ is obtained from λ by replacing the edge λ_i by the other diagonal λ'_i of the square formed by the two triangles, as illustrated in Figure 1.

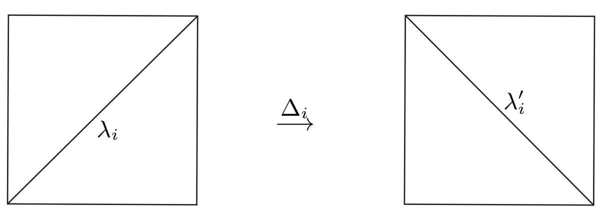

Figure 1.

Lemma 2.1. *The reindexings and diagonal exchanges satisfy the following relations:*

(1) $(\alpha\beta)(\lambda) = \alpha(\beta(\lambda))$ *for every* α, $\beta \in \mathfrak{S}_{3m}$*;*

(2) $(\Delta_i)^2 = \mathrm{Id}$*;*

(3) $\alpha \circ \Delta_i = \Delta_{\alpha(i)} \circ \alpha$ *for every* $\alpha \in \mathfrak{S}_{3m}$*;*

(4) *If* λ_i *and* λ_j *do not belong to the same triangle of* $\lambda \in \Lambda(S)$*, then* $\Delta_i \circ \Delta_j(\lambda) = \Delta_j \circ \Delta_i(\lambda)$*;*

(5) *If three triangles of an ideal triangulation* $\lambda \in \Lambda(S)$ *form a pentagon with diagonals* λ_i*,* λ_j *as in Figure 2, then*

$$\Delta_i \circ \Delta_j \circ \Delta_i \circ \Delta_j \circ \Delta_i(\lambda) = \alpha_{i\leftrightarrow j}(\lambda),$$

where $\alpha_{i\leftrightarrow j} \in \mathfrak{S}_{3m}$ *denotes the transposition exchanging* i *and* j*.*

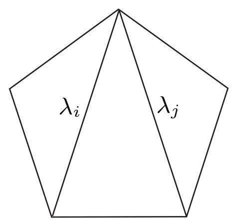

Figure 2.

To construct the quantum Teichmüller space, we need the following two results of R. C. Penner [24] (see also J. L. Harer [14]).

Theorem 2.2. *Given two ideal triangulations* $\lambda, \lambda' \in \Lambda(S)$*, there exists a finite sequence of ideal triangulations* $\lambda = \lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(n)} = \lambda'$ *such that each* $\lambda_{(k+1)}$ *is obtained from* $\lambda_{(k)}$ *by a diagonal exchange or by a reindexing of its edges.*

Theorem 2.3. *Given two ideal triangulations* $\lambda, \lambda' \in \Lambda(S)$ *and given two sequences* $\lambda = \lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(n)} = \lambda'$ *and* $\lambda = \overline{\lambda}_{(0)}, \overline{\lambda}_{(1)}, \ldots, \overline{\lambda}_{(\overline{n})} = \lambda'$ *of diagonal exchanges and reindexings connecting them as in Theorem 2.2, these two sequences can be related to each other by successive applications of the following moves and of their inverses. These moves correspond to the relations in Lemma 2.1.*

(1) *Replace* $\ldots, \lambda_{(k)}, \beta(\lambda_{(k)}), \alpha(\beta(\lambda_{(k)})), \ldots$
by $\ldots, \lambda_{(k)}, (\alpha\beta)(\lambda_{(k)}), \ldots$ *where* $\alpha, \beta \in \mathfrak{S}_{3m}$*.*

(2) *Replace* $\ldots, \lambda_{(k)}, \Delta_i(\lambda_{(k)}), \lambda_{(k)}, \ldots$
by $\ldots, \lambda_{(k)}, \ldots$ *.*

(3) *Replace* $\ldots, \lambda_{(k)}, \Delta_i(\lambda_{(k)}), \alpha \circ \Delta_i(\lambda_{(k)}), \ldots$
by $\ldots, \lambda_{(k)}, \alpha(\lambda_{(k)}), \Delta_{\alpha(i)} \circ \alpha(\lambda_{(k)}), \ldots$ *where* $\alpha \in \mathfrak{S}_{3m}$*.*

(4) *Replace* $\ldots, \lambda_{(k)}, \Delta_i(\lambda_{(k)}), \Delta_j \circ \Delta_i(\lambda_{(k)}), \ldots$
by $\ldots, \lambda_{(k)}, \Delta_j(\lambda_{(k)}), \Delta_i \circ \Delta_j(\lambda_{(k)}), \ldots$ *where* λ_i, λ_j *are two edges which do not belong to a same triangle of* $\lambda_{(k)}$*.*

(5) *Replace* $\ldots, \lambda_{(k)}, \Delta_i(\lambda_{(k)}), \Delta_j \circ \Delta_i(\lambda_{(k)}), \Delta_i \circ \Delta_j \circ \Delta_i(\lambda_{(k)}), \Delta_j \circ \Delta_i \circ \Delta_j \circ \Delta_i(\lambda_{(k)}), \Delta_i \circ \Delta_j \circ \Delta_i \circ \Delta_j \circ \Delta_i(\lambda_{(k)}), \ldots$

by $\ldots, \lambda_{(k)}, \alpha_{i\leftrightarrow j}(\lambda_{(k)}), \ldots$ *where* λ_i, λ_j *are two diagonals of a pentagon of* $\lambda_{(k)}$ *as in Figure 2.*

2.2 Shear coordinates for the Teichmüller space

If the Euler characteristic of S is negative, i.e., $m := 2g - 2 + p > 0$, S admits complete hyperbolic metrics. The *Teichmüller space* $\mathcal{T}(S)$ of S consists of all isotopy classes of complete hyperbolic metrics on S. W. Thurston [27] associated to each ideal triangulation a global coordinate system which is called *shear coordinate* for the Teichmüller space $\mathcal{T}(S)$ (see also [4, 11]).

An end of a surface S with a complete hyperbolic metric $d \in \mathcal{T}(S)$ can be of two types: a *cusp* with finite area bounded on one side by a horocycle; and a *funnel* with infinite area bounded on one side by a simple closed geodesic. The *convex core* $\mathrm{Conv}(S, d)$ of (S, d) is the smallest non-empty closed convex subset of (S, d), and is bounded in S by a family of disjoint simple closed geodesics. Each cusp end of (S, d) is also a cusp end of $\mathrm{Conv}(S, d)$, while each funnel end of S faces a boundary component of $\mathrm{Conv}(S, d)$.

The *enhanced Teichmüller space* $\widetilde{\mathcal{T}}(S)$ consists of all isotopy classes of complete hyperbolic metrics $d \in \mathcal{T}(S)$ enhanced with an orientation of each boundary component of $\mathrm{Conv}(S, d)$.

Under an enhanced hyperbolic metric $d \in \widetilde{\mathcal{T}}(S)$, each edge λ_i of an ideal triangulation λ is realized by a unique d–geodesic g_i such that each end of g_i, either converges towards a cusp end of S, or spirals around a boundary component of $\mathrm{Conv}(S, d)$ in the orientation specified by $d \in \widetilde{\mathcal{T}}(S)$.

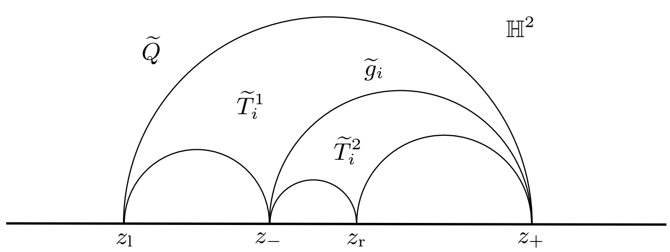

Figure 3.

The enhanced hyperbolic metric $d \in \widetilde{\mathcal{T}}(S)$ associates to the edge λ_i of λ a positive number x_i defined as follows. The geodesic g_i separates two triangle components T_i^1 and T_i^2 of $\mathrm{Conv}(S, d) - \{g_i\}$. The hyperbolic plane $\mathbb{H}^2$ is the universal covering of S endowed with the metric d. Lift g_i, T_i^1 and T_i^2 to a geodesic $\widetilde{g}_i$ and two triangles $\widetilde{T}_i^1$ and $\widetilde{T}_i^2$ in $\mathbb{H}^2$ so that the union $\widetilde{g}_i \cup \widetilde{T}_i^1 \cup \widetilde{T}_i^2$

forms a square $\widetilde{Q}$ in $\mathbb{H}^2$. See Figure 3. In the upper half-space model for $\mathbb{H}^2$, let z_-, z_+, $z_{\rm r}$, $z_{\rm l}$ be the vertices of $\widetilde{Q}$ in such a way that $\widetilde{g}_i$ goes from z_- to z_+ and, for this orientation of $\widetilde{g}_i$, $z_{\rm r}$, $z_{\rm l}$ are respectively to the right and to the left of $\widetilde{g}_i$ for the orientation of $\widetilde{Q}$ given by the orientation of S. Then,

$$x_i := -\,\text{cross-ratio}\,(z_{\rm r}, z_{\rm l}, z_-, z_+) = -\frac{(z_{\rm r}-z_-)(z_{\rm l}-z_+)}{(z_{\rm r}-z_+)(z_{\rm l}-z_-)}.$$

The real numbers $\{x_i\}$ are the *exponential shear coordinates* of the enhanced hyperbolic metric $d \in \widetilde{\mathfrak{T}}(S)$. The shear coordinates are $\ln x_i$.

It turns out that $\{x_i\}$ defines a homeomorphism $\phi_\lambda : \widetilde{\mathfrak{T}}(S) \to \mathbb{R}_+^{3m}$.

Therefore the exponential shear coordinates associates a parametrization $\phi_\lambda : \widetilde{\mathfrak{T}}(S) \to \mathbb{R}_+^{3m}$ to each ideal triangulation $\lambda \in \Lambda(S)$ (endowed with an indexing of its edges). We now investigate the coordinate changes $\phi_{\lambda'} \circ \phi_\lambda^{-1}$ associated to two ideal triangulations.

If $\lambda' = \alpha(\lambda)$ is obtained by reindexing the edges of λ by $\alpha \in \mathfrak{S}_{3m}$, then $\phi_{\lambda'} \circ \phi_\lambda^{-1}$ is the permutation of the coordinates by α. For a diagonal exchange, we have the following result.

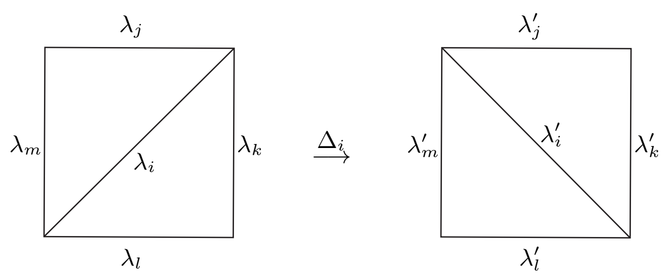

Figure 4.

Proposition 2.4 (Liu [22]). *Suppose that the ideal triangulations λ, λ' are obtained from each other by a diagonal exchange, namely that $\lambda' = \Delta_i(\lambda)$. Label the edges of λ involved in this diagonal exchange as λ_i, λ_j, λ_k, λ_l, λ_m as in Figure 4. If $(x_1, x_2, \ldots, x_{3m})$ and $(x'_1, x'_2, \ldots, x'_{3m})$ are the exponential shear coordinates associated to λ and λ' of the same enhanced hyperbolic metric, then $x'_h = x_h$ for every $h \notin \{i,j,k,l,m\}$, $x'_i = x_i^{-1}$ and:*

Case 1 *if the edges λ_j, λ_k, λ_l, λ_m are distinct, then*

$$x'_j = (1+x_i)x_j \quad x'_k = (1+x_i^{-1})^{-1}x_k \quad x'_l = (1+x_i)x_l \quad x'_m = (1+x_i^{-1})^{-1}x_m;$$

Case 2 *if λ_j is identified with λ_k, and λ_l is distinct from λ_m, then*

$$x'_j = x_i x_j \quad x'_l = (1+x_i)x_l \quad x'_m = (1+x_i^{-1})^{-1}x_m;$$

Case 3 *(the inverse of Case 2) if λ_j is identified with λ_m, and λ_k is distinct from λ_l, then*

$$x'_j = x_i x_j \quad x'_k = (1 + x_i^{-1})^{-1} x_k \quad x'_l = (1 + x_i) x_l;$$

Case 4 *if λ_j is identified with λ_l, and λ_k is distinct from λ_m, then*

$$x'_j = (1 + x_i)^2 x_j \quad x'_k = (1 + x_i^{-1})^{-1} x_k \quad x'_m = (1 + x_i^{-1})^{-1} x_m$$

Case 5 *(the inverse of Case 4) if λ_k is identified with λ_m, and λ_j is distinct from λ_l, then*

$$x'_j = (1 + x_i) x_j \quad x'_k = (1 + x_i^{-1})^{-2} x_k \quad x'_l = (1 + x_i) x_l;$$

Case 6 *if λ_j is identified with λ_k, and λ_l is identified with λ_m (in which case S is a 3–times punctured sphere), then*

$$x'_j = x_i x_j \quad x'_l = x_i x_l;$$

Case 7 *(the inverse of Case 6) if λ_j is identified with λ_m, and λ_k is identified with λ_l (in which case S is a 3–times punctured sphere), then*

$$x'_j = x_i x_j \quad x'_k = x_i x_k;$$

Case 8 *if λ_j is identified with λ_l, and λ_k is identified with λ_m (in which case S is a once punctured torus), then*

$$x'_j = (1 + x_i)^2 x_j \quad x'_k = (1 + x_i^{-1})^{-2} x_k.$$

2.3 The Chekhov-Fock algebra

Fix an ideal triangulation $\lambda \in \Lambda(S)$. The complement $S - \lambda$ has $6m$ spikes converging towards the punctures, and each spike is delimited by one λ_i on one side and one λ_j on the other side, with possibly $i = j$. For $i, j \in \{1, \ldots, 3m\}$, let a_{ij}^λ denote the number of spikes of $S - \lambda$ which are delimited on the left by λ_i and on the right by λ_j, and set

$$\sigma_{ij}^\lambda = a_{ij}^\lambda - a_{ji}^\lambda.$$

Note that $\sigma_{ij}^\lambda \in \{-2, -1, 0, 1, 2\}$, and that $\sigma_{ji}^\lambda = -\sigma_{ij}^\lambda$.

Let q be an arbitrary complex number. The *Chekhov-Fock algebra* associated to the ideal triangulation λ is the algebra $\mathcal{T}_\lambda^q$ defined by generators X_1, X_1^{-1}, X_2, X_2^{-1}, …, X_{3m}, X_{3m}^{-1}, with each pair $X_i^{\pm 1}$ associated to an edge λ_i of λ, and by the relations

$$X_i X_j = q^{2\sigma_{ij}^\lambda} X_j X_i.$$

This algebra has a well-defined fraction division algebra $\widehat{\mathcal{T}}^q_\lambda$ which consists of all formal fractions PQ^{-1} with $P,\, Q \in \mathcal{T}^q_\lambda$ and $Q \neq 0$, and two such fractions $P_1Q_1^{-1}$ and $P_2Q_2^{-1}$ are identified if there exists $S_1,\, S_2 \in \mathcal{T}^q_\lambda - \{0\}$ such that $P_1S_1 = P_2S_2$ and $Q_1S_1 = Q_2S_2$.

The algebras $\mathcal{T}^q_\lambda$ and $\widehat{\mathcal{T}}^q_\lambda$ strongly depend on the ideal triangulation λ. As one moves from one ideal triangulation λ to another λ', Chekhov and Fock [11, 9, 10] (see also [22]) introduce *coordinate change isomorphisms* $\Phi^q_{\lambda\lambda'}$: $\widehat{\mathcal{T}}^q_{\lambda'} \to \widehat{\mathcal{T}}^q_\lambda$. We denote by $X'_1,\, X'_2,\, \ldots,\, X'_n$ the generators of $\widehat{\mathcal{T}}^q_{\lambda'}$ associated to the edges $\lambda'_1,\, \lambda'_2,\, \ldots,\, \lambda'_n$ of λ', and by $X_1,\, X_2,\, \ldots,\, X_n$ the generators of $\widehat{\mathcal{T}}^q_\lambda$ associated to the edges $\lambda_1,\, \lambda_2,\, \ldots,\, \lambda_n$ of λ.

Definition 2.5. Suppose that the ideal triangulations $\lambda,\, \lambda' \in \Lambda(S)$ are obtained from each other by an edge reindexing, namely that $\lambda'_i = \lambda_{\alpha(i)}$ for some permutation $\alpha \in \mathfrak{S}_{3m}$. Then we define a map $\widehat{\alpha}$ from the set of the generators of the algebra $\widehat{\mathcal{T}}^q_{\lambda'}$ to $\widehat{\mathcal{T}}^q_\lambda$ by

$$\widehat{\alpha}(X'_i) = X_{\alpha(i)}, \quad \text{for any} \quad i = 1, ..., 3m.$$

Suppose that the ideal triangulations $\lambda,\, \lambda'$ are obtained from each other by a diagonal exchange, namely that $\lambda' = \Delta_i(\lambda)$. Label the edges of λ involved in this diagonal exchange as $\lambda_i,\, \lambda_j,\, \lambda_k,\, \lambda_l,\, \lambda_m$ as in Figure 4. Then we define a map $\widehat{\Delta}_i$ on the set of the generators of the algebra $\widehat{\mathcal{T}}^q_{\lambda'}$ to $\widehat{\mathcal{T}}^q_\lambda$ such that $X'_h \mapsto X_h$ for every $h \notin \{i, j, k, l, m\}$, $X'_i \mapsto X_i^{-1}$ and:

Case 1 if the edges $\lambda_j,\, \lambda_k,\, \lambda_l,\, \lambda_m$ are distinct, then

$$\begin{aligned} X'_j &\mapsto (1 + qX_i)X_j & X'_k &\mapsto (1 + qX_i^{-1})^{-1}X_k \\ X'_l &\mapsto (1 + qX_i)X_l & X'_m &\mapsto (1 + qX_i^{-1})^{-1}X_m; \end{aligned}$$

Case 2 if λ_j is identified with λ_k, and λ_l is distinct from λ_m, then

$$X'_j \mapsto X_iX_j \quad X'_l \mapsto (1 + qX_i)X_l \quad X'_m \mapsto (1 + qX_i^{-1})^{-1}X_m$$

Case 3 (the inverse of Case 2) if λ_j is identified with λ_m, and λ_k is distinct from λ_l, then

$$X'_j \mapsto X_iX_j \quad X'_k \mapsto (1 + qX_i^{-1})^{-1}X_k \quad X'_l \mapsto (1 + qX_i)X_l$$

Case 4 if λ_j is identified with λ_l, and λ_k is distinct from λ_m, then

$$X'_j \mapsto (1 + qX_i)(1 + q^3X_i)X_j$$
$$X'_k \mapsto (1 + qX_i^{-1})^{-1}X_k \quad X'_m \mapsto (1 + qX_i^{-1})^{-1}X_m$$

Case 5 (the inverse of Case 4) if λ_k is identified with λ_m, and λ_j is distinct from λ_l, then

$$X'_j \mapsto (1+qX_i)X_j \quad X'_l \mapsto (1+qX_i)X_l$$
$$X'_k \mapsto (1+qX_i^{-1})^{-1}(1+q^3X_i^{-1})^{-1}X_k$$

Case 6 if λ_j is identified with λ_k, and λ_l is identified with λ_m (in which case S is a 3–times punctured sphere), then

$$X'_j \mapsto X_iX_j \quad X'_l \mapsto X_iX_l;$$

Case 7 (the inverse of Case 6) if λ_j is identified with λ_m, and λ_k is identified with λ_l (in which case S is a 3–times punctured sphere), then

$$X'_j \mapsto X_iX_j \quad X'_k \mapsto X_iX_k;$$

Case 8 if λ_j is identified with λ_l, and λ_k is identified with λ_m (in which case S is a once punctured torus), then

$$X'_j \mapsto (1+qX_i)(1+q^3X_i)X_j$$
$$X'_k \mapsto (1+qX_i^{-1})^{-1}(1+q^3X_i^{-1})^{-1}X_k$$

It turns out that the maps $\widehat{\alpha}$ and $\widehat{\Delta}_i$ can be extended to the whole algebra $\widehat{\mathfrak{T}}^q_{\lambda'}$ as algebra homomorphisms from $\widehat{\mathfrak{T}}^q_{\lambda'}$ to $\widehat{\mathfrak{T}}^q_{\lambda}$.

The motivation of the definition of $\widehat{\alpha}$ and $\widehat{\Delta}_i$ is that they are reduced to the corresponding shear coordinate changes (Proposition 2.4) when $q=1$.

Proposition 2.6 (Liu [22]). *If an ideal triangulation λ' is obtained from another one λ by an operation π, where $\pi=\alpha$ for some $\alpha\in\mathfrak{S}_{3m}$, or $\pi=\Delta_i$ for some i, then $\widehat{\pi}:\widehat{\mathfrak{T}}^q_{\lambda'}\to\widehat{\mathfrak{T}}^q_{\lambda}$ as in Definition 2.5 is an isomorphism between the two algebras.*

Proposition 2.7 (Liu [22]). *The map $\widehat{\alpha}$ and $\widehat{\Delta}_i$ satisfy the following relations which correspond to the relations in Lemma 2.1:*

(1) $\widehat{\alpha\beta}=\widehat{\alpha}\circ\widehat{\beta}$ *for every α, $\beta\in\mathfrak{S}_{3m}$;*

(2) $\widehat{\Delta}_i\circ\widehat{\Delta}_i=\mathrm{Id}$*;*

(3) $\widehat{\alpha}\circ\widehat{\Delta}_i=\widehat{\Delta}_{\alpha(i)}\circ\widehat{\alpha}$ *for every $\alpha\in\mathfrak{S}_{3m}$;*

(4) *If λ_i and λ_j do not belong to the same triangle of $\lambda\in\Lambda(S)$, then $\widehat{\Delta}_i\circ\widehat{\Delta}_j=\widehat{\Delta}_j\circ\widehat{\Delta}_i$;*

(5) *If three triangles of an ideal triangulation $\lambda\in\Lambda(S)$ form a pentagon with diagonals λ_i, λ_j as in Figure 2, then*

$$\widehat{\Delta}_i\circ\widehat{\Delta}_j\circ\widehat{\Delta}_i\circ\widehat{\Delta}_j\circ\widehat{\Delta}_i=\widehat{\alpha}_{i\leftrightarrow j}. \tag{2.1}$$

2.4 The quantum Teichmüller space

Theorem 2.8 (Liu [22]). *There is a family of algebra isomorphisms*

$$\Phi^q_{\lambda\lambda'} : \widehat{\mathfrak{T}}^q_{\lambda'} \to \widehat{\mathfrak{T}}^q_{\lambda}$$

defined as λ, $\lambda' \in \Lambda(S)$ ranges over all pairs of ideal triangulations, such that:

(1) $\Phi^q_{\lambda\lambda''} = \Phi^q_{\lambda\lambda'} \circ \Phi^q_{\lambda'\lambda''}$ *for every λ, λ', $\lambda'' \in \Lambda(S)$;*

(2) $\Phi^q_{\lambda\lambda'}$ *is the isomorphism defined in Definition 2.5 when λ' is obtained from λ by a reindexing or a diagonal exchange.*

(3) $\Phi^q_{\lambda\lambda'}$ *depends only on λ and λ'.*

The *quantum (enhanced) Teichmüller space* of S can now be defined as the algebra

$$\widehat{\mathfrak{T}}^q_S = \Big(\bigsqcup_{\lambda \in \Lambda(S)} \widehat{\mathfrak{T}}^q_\lambda \Big) / \sim$$

where the relation $\sim$ is defined by the property that, for $X \in \widehat{\mathfrak{T}}^q_\lambda$ and $X' \in \widehat{\mathfrak{T}}^q_{\lambda'}$,

$$X \sim X' \Leftrightarrow X = \Phi^q_{\lambda,\lambda'}(X').$$

The quantum Teichmüller space $\widehat{\mathfrak{T}}^q_S$ is a noncommutative deformation of the algebra of rational functions on the enhanced Teichmüller space $\widetilde{\mathfrak{T}}(S)$.

3 The quantum Teichmüller space: properties

In this section we survey some interesting properties and applications of the quantum Teichmüller space. The uniqueness of the construction of the quantum Teichmüller space is established by H. Bai [1]. In [3, 5, 6, 23], it is shown that the quantum Teichmüller space $\widehat{\mathfrak{T}}^q_S$ has a rich representation theory which also produces an invariant of hyperbolic 3-manifolds.

We would like to mention the following related important works without providing more details.

H. Bai [2] shows that Kashaev's $6j$-symbols [16, 17] are intertwining operators of local representations of quantum Teichmller spaces introduced in [3]. Note that appearance of Kashaev's $6j$-symbols in quantum Teichmller theory at roots of unity is already explicit in [18](see the operator $T_{h,x,y}$ in Proposition 10).

C. Hiatt [15] proves that for the torus with one hole and $p \geq 1$ punctures and the sphere with four holes there is a family of quantum trace functions in the quantum Teichmüller space, analog to the non-quantum trace functions in

Teichmüller space, satisfying the properties proposed by Chekhov and Fock in [10].

For a punctured surface S, a point of its Teichmüller space $\mathcal{T}(S)$ determines an irreducible representation of its quantization $\mathcal{T}_S^q$. J. Roger [25] analyzes the behavior of these representations as one goes to infinity in $\mathcal{T}_S$. He shows that an irreducible representation of $\mathcal{T}_S^q$ limits to a direct sum of representations of $\mathcal{T}_{S_\gamma}^q$, where S_γ is obtained from S by pinching a multicurve γ to a set of nodes. The result is analogous to the factorization rule found in conformal field theory.

The skein algebra and the quantum Teichmüller space are considered as two different quantizations of the character variety consisting of all representations of surface groups in $\mathrm{PSL}_2(\mathbb{C})$. F. Bonahon and H. Wong [7, 8] construct a homomorphism from the skein algebra to the quantum Teichmüller space which, when restricted the classical case, corresponds to the equivalence between these two algebras through trace functions.

3.1 Uniqueness

The original definition of the quantum Teichmüller space was motivated by geometry. However, H. Bai [1] shows that it is intrinsically tied to the combinatorics of the set $\Lambda(S)$. Indeed, H. Bai proves that the coordinate change isomorphisms considered in Definition 2.5 are the only ones which satisfy a certain number of natural conditions.

The *discrepancy span* $D(\lambda, \lambda')$ of two ideal triangulations λ, λ' is the closure of the union of those connected components of $S - \lambda$ which are not isotopic to a component of $S - \lambda'$.

The coordinate change isomorphisms $\Phi_{\lambda\lambda'}^q$ are said to satisfy the *Locality Condition* if the following holds. Let λ and λ' be two ideal triangulations indexed in such a way that $\lambda_i \subset D(\lambda, \lambda')$ when $i \leq k$, and $\lambda'_i = \lambda_i$ when $i > k$. Then

(1) $\Phi_{\lambda\lambda'}^q(X'_i) = X_i$ for every $i > k$;

(2) $\Phi_{\lambda\lambda'}^q(X'_i) = f_i(X_1, X_2, \cdots, X_k)$ for every $i \leq k$, where f_i is a multivariable rational function depending only on the combinatorics of λ and λ' in $D(\lambda, \lambda')$ in the following sense: For any two pairs of ideal triangulation (λ, λ'), (λ'', λ''') for which there exists a diffeomorphism $\psi : D(\lambda, \lambda') \to D(\lambda'', \lambda''')$ sending λ_i to λ''_j and λ'_j to λ'''_j for every $1 \leq j \leq k$, then

$$\Phi_{\lambda\lambda'}^q(X'_i) = f_i(X_1, X_2, \cdots, X_k) \text{ and}$$
$$\Phi_{\lambda''\lambda'''}^q(X'''_i) = f_i(X''_1, X''_2, \cdots, X''_k)$$

for the same rational function f_i.

Proposition 3.1 (Bai [1]). *The algebra isomorphisms $\Phi^q_{\lambda\lambda'} : \widehat{\mathcal{T}}^q_{\lambda'} \to \widehat{\mathcal{T}}^q_{\lambda}$ in Theorem 2.8 satisfies the Locality Condition.*

Theorem 3.2 (Bai [1]). *Assume that the surface S satisfies $\chi(S) < -2$. Then the family of coordinate change isomorphisms $\Phi^q_{\lambda\lambda'}$ in Theorem 2.8 is unique up to a uniform rescaling and/or inversion of the X_i.*

Namely, if

$$\Psi^q_{\lambda\lambda'} : \mathbb{C}(X'_1, X'_2, \dots, X'_n)^q_{\lambda'} \to \mathbb{C}(X_1, X_2, \dots, X_n)^q_{\lambda}$$

is another family of isomorphisms satisfying the conditions of Theorem 2.8 and the Locality Condition, then there exists a non-zero constant $\xi \in \mathbb{C}(q)$ and a sign $\varepsilon = \pm 1$ such that $\Psi^q_{\lambda\lambda'} = \Theta_\lambda \circ \Phi^q_{\lambda\lambda'} \circ \Theta^{-1}_{\lambda'}$ for any pair of ideal triangulations λ, λ', where $\Theta_\lambda : \mathbb{C}(X_1, X_2, \dots, X_n)^q_\lambda \to \mathbb{C}(X_1, X_2, \dots, X_n)^q_\lambda$ is the isomorphism defined by $\Theta_\lambda(X_i) = \xi X_i^\varepsilon$ for every i.

Theorem 3.2 is false when S is the once-punctured torus or the 3–times punctured sphere. The uniqueness property for the twice-punctured torus and the 4–times punctured sphere has not been established.

3.2 Representations

In this subsecton, our exposition follows [6] closely.

A standard method to move from abstract algebraic constructions to more concrete applications is to consider finite-dimensional representations. In the case of algebras, this means algebra homomorphisms valued in the algebra $\mathrm{End}(V)$ of endomorphisms of a finite-dimensional vector space V over $\mathbb{C}$. Elementary considerations show that these can exist only when q is a root of unity.

Theorem 3.3 (Bonahon-Liu [6]). *Suppose that q^2 is a primitive N–th root of unity. For any ideal triangulation λ of a surface S, every irreducible finite-dimensional representation of the Chekhov-Fock algebra $\mathcal{T}^q_\lambda$ has dimension N^{3g+p-3} if N is odd, and $N^{3g+p-3}/2^g$ if N is even. Up to isomorphism, such a representation is classified by:*

(1) *a non-zero complex number $x_i \in \mathbb{C}^*$ associated to each edge of λ;*

(2) *a choice of an N–th root for each of p explicit monomials in the numbers x_i;*

(3) *when N is even, a choice of square root for each of $2g$ explicit monomials in the numbers x_i.*

Conversely, any such data can be realized by an irreducible finite-dimensional representation of $\mathcal{T}^q_\lambda$.

Theorem 3.3 shows that the Chekhov-Fock algebra has a rich representation theory. Unfortunately, for dimension reasons, its fraction algebra $\widehat{\mathcal{T}}^q_\lambda$ and, consequently, the quantum Teichmüller space $\widehat{\mathcal{T}}^q_S$ cannot have any finite-dimensional representation. This leads us to introduce the *polynomial core* $\mathcal{T}^q_S$ of the quantum Teichmüller space $\widehat{\mathcal{T}}^q_S$, defined as the family $\{\mathcal{T}^q_\lambda\}_{\lambda\in\Lambda(S)}$ of all Chekhov-Fock algebras $\mathcal{T}^q_\lambda$, considered as subalgebras of $\widehat{\mathcal{T}}^q_S$, as λ ranges over the set $\Lambda(S)$ of all isotopy classes of ideal triangulations of the surface S.

Theorem 3.3 says that, up to a finite number of choices, an irreducible representation of $\mathcal{T}^q_\lambda$ is classified by certain numbers $x_i \in \mathbb{C}^*$ associated to the edges of the ideal triangulation λ of S. There is a classical geometric object which is also associated to λ with the same edge weights x_i. Namely, we can consider in the hyperbolic 3–space $\mathbb{H}^3$ the pleated surface that has pleating locus λ, that has shear parameter along the i–th edge of λ equal to the real part of $\ln x_i$, and that has bending angle along this edge equal to the imaginary part of $\ln x_i$. In turn, this pleated surface has a *monodromy representation*, namely a group homomorphism from the fundamental group $\pi_1(S)$ to the group $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$ of orientation-preserving isometries of $\mathbb{H}^3$. This construction associates to a representation of the Chekhov-Fock algebra $\mathcal{T}^q_\lambda$ a group homomorphism $r : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$, well-defined up to conjugation by an element of $\mathrm{PSL}_2(\mathbb{C})$.

Theorem 3.4 (Bonahon-Liu [6]). *Let q be a primitive N–th root of $(-1)^{N+1}$, for instance $q = -\mathrm{e}^{2\pi \mathrm{i}/N}$. If $\rho = \{\rho_\lambda : \mathcal{T}^q_\lambda \to \mathrm{End}(V)\}_{\lambda\in\Lambda(S)}$ is a finite-dimensional irreducible representation of the polynomial core $\mathcal{T}^q_S$ of the quantum Teichmüller space $\widehat{\mathcal{T}}^q_S$, the representations ρ_λ induce the same monodromy homomorphism $r_\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$.*

Theorem 3.4 is essentially equivalent to the property that, for the choice of q indicated, the pleated surfaces respectively associated to the representations $\rho_\lambda : \mathcal{T}^q_\lambda \to \mathrm{End}(V)$ and $\rho_\lambda \circ \Phi^q_{\lambda\lambda'} : \mathcal{T}^q_{\lambda'} \to \mathrm{End}(V)$ have (different pleating loci but) the same monodromy representation $r_\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$.

The homomorphism r_ρ is the *hyperbolic shadow* of the representation ρ. Not every homomorphism $r : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ is the hyperbolic shadow of a representation of the polynomial core, but many of them are:

Theorem 3.5 (Bonahon-Liu [6]). *An injective homomorphism $r : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ is the hyperbolic shadow of a finite number of irreducible finite-dimensional representations of the polynomial core $\mathcal{T}^q_S$, up to isomorphism. More precisely, this number of representations is equal to $2^l N^p$ if N is odd, and $2^{2g+l} N^p$ if N is even, where l is the number of ends of S whose image under r is loxodromic.*

Let φ be a diffeomorphism of the surface S. Suppose in addition that φ is homotopically aperiodic (also called homotopically pseudo-Anosov), so that its (3–dimensional) mapping torus M_φ admits a complete hyperbolic metric. The hyperbolic metric of M_φ gives an injective homomorphism $r_\varphi : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ such that $r_\varphi \circ \varphi^*$ is conjugate to r_φ, where φ^* is the isomorphism of $\pi_1(S)$ induced by φ.

The diffeomorphism φ also acts on the quantum Teichmüller space and on its polynomial core $\mathcal{T}^q_S$. In particular, it acts on the set of representations of $\mathcal{T}^q_S$ and, because $r_\varphi \circ \varphi^*$ is conjugate to r_φ, it sends a representation with hyperbolic shadow r_φ to another representation with shadow r_φ. Actually, when N is odd, there is a preferred representation ρ_φ of $\mathcal{T}^q_S$ which is fixed by the action of φ, up to isomorphism. This statement means that, for every ideal triangulation λ, we have a representation $\rho_\lambda : \mathcal{T}^q_\lambda \to \mathrm{End}(V)$ of dimension N^{3g+p-3} and an isomorphism L^q_φ of V such that

$$\rho_{\varphi(\lambda)} \circ \Phi_{\varphi(\lambda)\lambda}(X) = L^q_\varphi \cdot \rho_\lambda(X) \cdot (L^q_\varphi)^{-1}$$

in $\mathrm{End}(V)$ for every $X \in \mathcal{T}^q_\lambda$, for a suitable interpretation of the left hand side of the equation.

Theorem 3.6 (Bonahon-Liu [6]). *Let N be odd. Up to conjugation and up to multiplication by a constant, the isomorphism L^q_φ depends only on the homotopically aperiodic diffeomorphism $\varphi : S \to S$ and on the primitive N–th root q of* 1.

Explicit computations of these invariants for the once-punctured torus or the 4–times punctured sphere are provided in X. Liu [23].

H. Bai, F. Bonahon and X. Liu [3] investigate another type of representations of the quantum Teichmüller space, called local representations, which are somewhat simpler to analyze and more closely connected to the combinatorics of ideal triangulations.

4 Kashaev algebra

In this section, we establish the theory of Kashaev algebra which is parallel to the theory of the quantum Teichmüller space. The exposition follows [13] closely.

4.1 Decorated ideal triangulations

Let S be an oriented surface of genus g with $p \geq 1$ punctures and negative Euler characteristic, i.e., $m = 2g - 2 + p > 0$.

A decorated ideal triangulation τ of S introduced by Kashaev [18] is an ideal triangulation such that the ideal triangles are numerated as $\{\tau_1, \tau_2, ..., \tau_{2m}\}$ and there is a mark (a star symbol) at a corner of each ideal triangle. Denote by $\triangle(S)$ the set of isotopy classes of decorated ideal triangulations of the surface S.

The set $\triangle(S)$ admits a natural action of the symmetric group on the set $\{1, 2, ..., 2m\}$, $\mathfrak{S}_{2m}$, acting by permuting the indexes of the ideal triangles of τ. Namely $\tau' = \alpha(\tau)$ for $\alpha \in \mathfrak{S}_{2m}$ if $\tau_i = \tau'_{\alpha(i)}$.

Another important transformation of $\triangle(S)$ is provided by the *diagonal exchange* $\varphi_{ij} : \triangle(S) \to \triangle(S)$ defined as follows. Suppose that two ideal triangles τ_i, τ_j share an edge e such that the marked corners are opposite to the edge e. Then $\varphi_{ij}(\tau)$ is obtained by rotating the interior of the union $\tau_i \cup \tau_j$ $90°$ in the clockwise order, as illustrated in Figure 5(2).

The last transformation of $\triangle(S)$ is the *mark rotation* $\rho_i : \triangle(S) \to \triangle(S)$. $\rho_i(\tau)$ is obtained by relocating the mark of the ideal triangle τ_i from one corner to the next corner in the counterclockwise order, as illustrated in Figure 5(1).

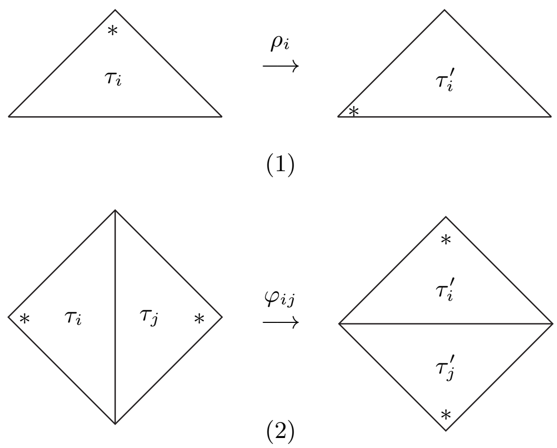

Figure 5.

Lemma 4.1. *The reindexings, diagonal exchanges and mark rotations satisfy the following relations:*

(1) $(\alpha\beta)(\tau) = \alpha(\beta(\tau))$ *for every* α, $\beta \in \mathfrak{S}_{2m}$*;*

(2) $\varphi_{ij} \circ \varphi_{ij} = \alpha_{i\leftrightarrow j}$, *where* $\alpha_{i\leftrightarrow j}$ *denotes the transposition exchanging* i *and* j*;*

(3) $\alpha \circ \varphi_{ij} = \varphi_{\alpha(i)\alpha(j)} \circ \alpha$ *for every* $\alpha \in \mathfrak{S}_{2m}$*;*

(4) $\varphi_{ij} \circ \varphi_{kl}(\tau) = \varphi_{kl} \circ \varphi_{ij}(\tau)$, *for* $\{i, j\} \neq \{k, l\}$*;*

(5) *If three triangles* τ_i, τ_j, τ_k *of an ideal triangulation* $\tau \in \triangle(S)$ *form a pentagon and their marked corners are located as in Figure 6, then the*

Pentagon Relation holds:

$$\omega_{jk} \circ \omega_{ik} \circ \omega_{ij}(\tau) = \omega_{ij} \circ \omega_{jk}(\tau),$$

where $\omega_{\mu\nu} = \rho_\mu \circ \varphi_{\mu\nu} \circ \rho_\nu$*;*

(6) $\rho_i \circ \rho_i \circ \rho_i = \mathrm{Id}$*;*

(7) $\rho_i \circ \rho_j = \rho_j \circ \rho_i$*;*

(8) $\alpha \circ \rho_i = \rho_{\alpha(i)} \circ \alpha$ *for for every* $\alpha \in \mathfrak{S}_{2m}$*.*

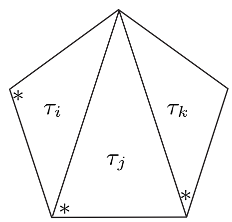

Figure 6.

Lemma 4.1 is essentially contained in Kashaev [19] where ω_{ij} is used as the diagonal exchange.

The following two results about decorated ideal triangulations can be easily proved using Penner's result about ideal triangulations [24].

Theorem 4.2. *Given two decorated ideal triangulations* $\tau, \tau' \in \triangle(S)$*, there exists a finite sequence of decorated ideal triangulations* $\tau = \tau_{(0)}, \tau_{(1)}, \dots, \tau_{(n)} = \tau'$ *such that each* $\tau_{(k+1)}$ *is obtained from* $\tau_{(k)}$ *by a diagonal exchange or by a mark rotation or by a reindexing of its ideal triangles.*

Theorem 4.3. *Given two decorated ideal triangulations* $\tau, \tau' \in \triangle(S)$ *and given two sequences* $\tau = \tau_{(0)}, \tau_{(1)}, \dots, \tau_{(n)} = \tau'$ *and* $\tau = \overline{\tau}_{(0)}, \overline{\tau}_{(1)}, \dots, \overline{\tau}_{(\overline{n})} = \tau'$ *of diagonal exchanges, mark rotations and reindexings connecting them as in Theorem 4.2, these two sequences can be related to each other by successive applications of the following moves and of their inverses. These moves correspond to the relations in Lemma 4.1.*

(1) *Replace* $\dots, \tau_{(k)}, \beta(\tau_{(k)}), \alpha \circ \beta(\tau_{(k)}), \dots$
by $\dots, \tau_{(k)}, (\alpha\beta)(\tau_{(k)}), \dots$ *where* $\alpha, \beta \in \mathfrak{S}_n$*.*

(2) *Replace* $\dots, \tau_{(k)}, \varphi_{ij}(\tau_{(k)}), \varphi_{ij} \circ \varphi_{ij}(\tau_{(k)}) \dots$
by $\dots, \tau_{(k)}, \alpha_{i\leftrightarrow j}(\tau_{(k)}), \dots.$

(3) *Replace* $\dots, \tau_{(k)}, \varphi_{ij}(\tau_{(k)}), \alpha \circ \varphi_{ij}(\tau_{(k)}), \dots$
by $\dots, \tau_{(k)}, \alpha(\tau_{(k)}), \varphi_{\alpha(i)\alpha(j)} \circ \alpha(\tau_{(k)}), \dots$ *where* $\alpha \in \mathfrak{S}_n$*.*

(4) *Replace* $\ldots, \tau_{(k)}, \varphi_{kl}(\tau_{(k)}), \varphi_{ij} \circ \varphi_{kl}(\tau_{(k)}), \ldots$
by $\ldots, \tau_{(k)}, \varphi_{ij}(\tau_{(k)}), \varphi_{kl} \circ \varphi_{ij}(\tau_{(k)}), \ldots$ *where* $\{i, j\} \neq \{k, l\}$.

(5) *Replace* $\ldots, \tau_{(k)}, \omega_{ij}(\tau_{(k)}), \omega_{ik} \circ \omega_{ij}(\tau_{(k)}), \omega_{jk} \circ \omega_{ik} \circ \omega_{ij}(\tau_{(k)}), \ldots,$
by $\ldots, \tau_{(k)}, \omega_{jk}(\tau_{(k)}), \omega_{ij} \circ \omega_{jk}(\tau_{(k)}), \ldots$ *where* $\omega_{\mu\nu} = \rho_\mu \circ \varphi_{\mu\nu} \circ \rho_\nu$.

(6) *Replace* $\ldots, \tau_{(k)}, \rho_i(\tau_{(k)}), \rho_i \circ \rho_i(\tau_{(k)}), \tau_{(k)} \ldots$
by $\ldots, \tau_{(k)}, \ldots.$

(7) *Replace* $\ldots, \tau_{(k)}, \rho_i(\tau_{(k)}), \rho_j \circ \rho_i(\tau_{(k)}), \ldots$
by $\ldots, \tau_{(k)}, \rho_j(\tau_{(k)}), \rho_i \circ \rho_j(\tau_{(k)}), \ldots.$

(8) *Replace* $\ldots, \tau_{(k)}, \rho_i(\tau_{(k)}), \alpha \circ \rho_i(\tau_{(k)}), \ldots$
by $\ldots, \tau_{(k)}, \alpha(\tau_{(k)}), \rho_{\alpha(i)} \circ \alpha(\tau_{(k)}), \ldots.$

4.2 Kashaev coordinates

For a decorated ideal triangulation τ of a punctured surface S, Kashaev [18] associated to each ideal triangle τ_i two numbers $\ln y_i, \ln z_i$. A Kashaev coordinate is a vector $(\ln y_1, \ln z_1, ..., \ln y_{2m}, \ln z_{2m}) \in \mathbb{R}^{4m}$.

Denote by $(y_1, z_1, ..., y_{2m}, z_{2m})$ the exponential Kashaev coordinate for the decorated ideal triangulation τ. Denote by $(y'_1, z'_1, ..., y'_{2m}, z'_{2m})$ the exponential Kashaev coordinate for the decorated ideal triangulation τ'. Kashaev [18] introduces the change of coordinates as follows.

Definition 4.4 (Kashaev [18]). Suppose that a decorated ideal triangulation τ' is obtained from another one τ by reindexing the ideal triangles, i.e., $\tau' = \alpha(\tau)$ for some $\alpha \in \mathfrak{S}_{2m}$, then we define $(y'_i, z'_i) = (y_{\alpha(i)}, z_{\alpha(i)})$ for any $i = 1, ..., 2m$.

Suppose that a decorated ideal triangulation τ' is obtained from another one τ by a mark rotation, i.e., $\tau' = \rho_i(\tau)$ for some i, then we define $(y'_j, z'_j) = (y_j, z_j)$ for any $j \neq i$ while

$$(y'_i, z'_i) = (\frac{z_i}{y_i}, \frac{1}{y_i}).$$

Suppose a decorated ideal triangulation τ' is obtained from another one τ by a diagonal exchange, i.e., $\tau' = \varphi_{ij}(\tau)$ for some i, j, then we define $(y'_k, z'_k) = (y_k, z_k)$ for any $k \notin \{i, j\}$ while

$$(y'_i, z'_i, y'_j, z'_j) = (\frac{z_j}{y_i y_j + z_i z_j}, \frac{y_i}{y_i y_j + z_i z_j}, \frac{z_i}{y_i y_j + z_i z_j}, \frac{y_j}{y_i y_j + z_i z_j}).$$

Kashaev [18] considered ω_{ij} instead of φ_{ij}.

There is a natural relationship between Kashaev coordinates and Penner coordinates for the decorated Teichmüller space which is established in [18]. For an exposition, see also Teschner [26].

4.3 Generalized Kashaev algebra: triangulation-dependent

For a decorated ideal triangulation τ of a punctured surface S, Kashaev [18] introduced an algebra $\mathcal{K}^q_\tau$ on $\mathbb{C}$ generated by $Y_1^{\pm}, Z_1^{\pm}, Y_2^{\pm}, Z_2^{\pm}, ..., Y_{2m}^{\pm}, Z_{2m}^{\pm}$, with $Y_i^{\pm}, Z_i^{\pm}$ associated to an ideal triangle τ_i, and by the relations:

$$\begin{aligned} Y_iY_j &= Y_jY_i, \\ Z_iZ_j &= Z_jZ_i, \\ Y_iZ_j &= Z_jY_i \quad \text{if} \quad i \neq j, \\ Z_iY_i &= q^2Y_iZ_i. \end{aligned}$$

Kashaev's original definition is $Y_iZ_i = q^2Z_iY_i$. We adopt our convention to make it compatible with the quantum Teichmüller space [22]. Kashaev's parameter q corresponds to our q^{-1}.

The algebra $\widehat{\mathcal{K}}^q_\tau$ is the fraction division algebra of $\mathcal{K}^q_\tau$.

In particular, when $q = 1$, $\mathcal{K}^q_\tau$ and $\widehat{\mathcal{K}}^q_\tau$ respectively coincide with the Laurent polynomial algebra $\mathbb{C}[Y_1^{\pm}, Z_1^{\pm}, ..., Y_{2m}^{\pm}, Z_{2m}^{\pm}]$ and the rational fraction algebra $\mathbb{C}(Y_1, Z_1, ..., Y_{2m}, Z_{2m})$. The general $\mathcal{K}^q_\tau$ and $\widehat{\mathcal{K}}^q_\tau$ can be considered as deformations of $\mathcal{K}^1_\tau$ and $\widehat{\mathcal{K}}^1_\tau$.

The algebra $\widehat{\mathcal{K}}^q_\tau$ depends on the decorated ideal triangulation τ. We introduce algebra isomorphisms in the following.

Definition 4.5. Let a, b be two arbitrary nonzero complex numbers.

Suppose that a decorated ideal triangulation τ' is obtained from another one τ by reindexing the ideal triangles, i.e., $\tau' = \alpha(\tau)$ for some $\alpha \in \mathfrak{S}_{2m}$, then we define a map $\widehat{\alpha}$ on the set of generators of $\widehat{\mathcal{K}}^q_{\tau'}$ to $\widehat{\mathcal{K}}^q_\tau$ by

$$\begin{aligned} \widehat{\alpha}(Y_i') &= Y_{\alpha(i)}, \quad \text{for any} \quad i = 1, ..., 2m, \\ \widehat{\alpha}(Z_i') &= Z_{\alpha(i)}, \quad \text{for any} \quad i = 1, ..., 2m. \end{aligned}$$

Suppose that a decorated ideal triangulation τ' is obtained from another one τ by a mark rotation, i.e., $\tau' = \rho_i(\tau)$ for some i, then we define a map $\widehat{\rho}_i$ on the set of generators of $\widehat{\mathcal{K}}^q_{\tau'}$ to $\widehat{\mathcal{K}}^q_\tau$ by

$$\begin{aligned} \widehat{\rho}_i(Y_j') &= Y_j, \quad \text{if} \quad j \neq i, \\ \widehat{\rho}_i(Z_j') &= Z_j, \quad \text{if} \quad j \neq i, \\ \widehat{\rho}_i(Y_i') &= aY_i^{-1}Z_i, \\ \widehat{\rho}_i(Z_i') &= Y_i^{-1}. \end{aligned}$$

Suppose a decorated ideal triangulation τ' is obtained from another one τ by a diagonal exchange, i.e., $\tau' = \varphi_{ij}(\tau)$ for some i, j, then we define a map

$\widehat{\varphi}_{ij}$ on the set of generators of $\widehat{\mathcal{K}}^q_{\tau'}$ to $\widehat{\mathcal{K}}^q_{\tau}$ by

$$\begin{aligned}\widehat{\varphi}_{ij}(Y_i') &= \ (bY_iY_j + Z_iZ_j)^{-1}Z_j,\\ \widehat{\varphi}_{ij}(Z_i') &= b(bY_iY_j + Z_iZ_j)^{-1}Y_i,\\ \widehat{\varphi}_{ij}(Y_j') &= \ (bY_iY_j + Z_iZ_j)^{-1}Z_i,\\ \widehat{\varphi}_{ij}(Z_j') &= b(bY_iY_j + Z_iZ_j)^{-1}Y_j.\end{aligned}$$

It turns out that the maps $\widehat{\alpha}$, $\widehat{\rho}_i$ and $\widehat{\varphi}_{ij}$ can be extended to the whole algebra $\widehat{\mathcal{K}}^q_{\tau'}$ as algebra homomorphisms between from $\widehat{\mathcal{K}}^q_{\tau'}$ to $\widehat{\mathcal{K}}^q_{\tau}$.

Kashaev [18] considered a special case of these maps when $a = q^{-1}, b = q$.

From the definition, when $q = 1$, we get the coordinate change formula in Definition 4.4.

Proposition 4.6 (Guo-Liu [13]). *If a decorated ideal triangulation τ' is obtained from another one τ by an operation π, where $\pi = \alpha$ for some $\alpha \in \mathfrak{S}_{2m}$, or $\pi = \rho_i$ for some i, or $\pi = \varphi_{ij}$ for some i, j, then $\widehat{\pi} : \widehat{\mathcal{K}}^q_{\tau'} \to \widehat{\mathcal{K}}^q_{\tau}$ as in Definition 4.5 is an isomorphism between the two algebras.*

Proposition 4.7 (Guo-Liu [13]). *The maps $\widehat{\alpha}, \widehat{\rho}_i$ and $\widehat{\varphi}_{ij}$ satisfy the following relations which correspond to the relations in Lemma 4.1:*

(1) $\widehat{\alpha\beta} = \widehat{\alpha} \circ \widehat{\beta}$ *for every* α, $\beta \in \mathfrak{S}_{2m}$*;*

(2) $\widehat{\varphi}_{ij} \circ \widehat{\varphi}_{ij} = \widehat{\alpha}_{i\leftrightarrow j}$*;*

(3) $\widehat{\alpha} \circ \widehat{\varphi}_{ij} = \widehat{\varphi}_{\alpha(i)\alpha(j)} \circ \widehat{\alpha}$ *for every* $\alpha \in \mathfrak{S}_{2m}$*;*

(4) $\widehat{\varphi}_{ij} \circ \widehat{\varphi}_{kl} = \widehat{\varphi}_{kl} \circ \widehat{\varphi}_{ij}$ *for* $\{i, j\} \neq \{k, l\}$*;*

(5) *If three triangles τ_i, τ_j, τ_k of an ideal triangulation $\tau \in \triangle(S)$ form a pentagon and their marked corners are located as in Figure 7, then the Pentagon Relation holds:*

$$\widehat{\omega}_{jk} \circ \widehat{\omega}_{ik} \circ \widehat{\omega}_{ij} = \widehat{\omega}_{ij} \circ \widehat{\omega}_{jk},$$

where $\widehat{\omega}_{\mu\nu} = \widehat{\rho}_\mu \circ \widehat{\varphi}_{\mu\nu} \circ \widehat{\rho}_\nu$*;*

(6) $\widehat{\rho}_i \circ \widehat{\rho}_i \circ \widehat{\rho}_i = \mathrm{Id}$*;*

(7) $\widehat{\rho}_i \circ \widehat{\rho}_j = \widehat{\rho}_j \circ \widehat{\rho}_i$*;*

(8) $\widehat{\alpha} \circ \widehat{\rho}_i = \widehat{\rho}_{\alpha(i)} \circ \widehat{\alpha}$ *for every* $\alpha \in \mathfrak{S}_{2m}$*.*

4.4 Generalized Kashaev algebra: triangulation-independent

Theorem 4.8 (Guo-Liu [13]). *For two arbitrary complex numbers a, b, there is a family of algebra isomorphisms*

$$\Psi^q_{\tau\tau'}(a, b) : \widehat{\mathcal{K}}^q_{\tau'} \to \widehat{\mathcal{K}}^q_{\tau}$$

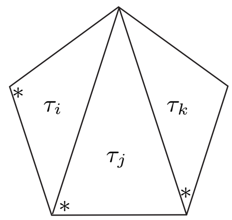

Figure 7. (Same as Figure 6)

defined as τ, $\tau' \in \triangle(S)$ ranges over all pairs of decorated ideal triangulations, such that:

(1) $\Psi^q_{\tau\tau''}(a,b) = \Psi^q_{\tau\tau'}(a,b) \circ \Psi^q_{\tau'\tau''}(a,b)$ *for every* τ, τ', $\tau'' \in \triangle(S)$;

(2) $\Psi^q_{\tau\tau'}(a,b)$ *is the isomorphism of Definition 4.5 when* τ' *is obtained from* τ *by a reindexing or a mark rotation or a diagonal exchange.*

(3) $\Psi^q_{\tau\tau'}(a,b)$ *depends only on* τ *and* τ'.

The *generalized Kashaev algebra* $\widehat{\mathcal{K}}^q_S(a,b)$ associated to a surface S is defined as the algebra

$$\widehat{\mathcal{K}}^q_S(a,b) = \Big(\bigsqcup_{\tau \in \triangle(S)} \widehat{\mathcal{K}}^q_\tau(a,b) \Big) / \sim$$

where the relation $\sim$ is defined by the property that, for $X \in \widehat{\mathcal{K}}^q_\tau(a,b)$ and $X' \in \widehat{\mathcal{K}}^q_{\tau'}(a,b)$,

$$X \sim X' \Leftrightarrow X = \Psi^q_{\tau,\tau'}(a,b)(X').$$

5 Kashaev coordinates and shear coordinates

To understand the quantization using shear coordinates and the quantization using Kashaev coordinates, we first need to understand the relationship between these two coordinates.

5.1 Decorated ideal triangulations

Given a decorated ideal triangulation τ, by forgetting the mark at each corner, we obtain an ideal triangulation λ. We call λ the *underlying ideal triangulation* of τ. let $\lambda_1, \lambda_2, ..., \lambda_{3m}$ be the components of ideal triangulation λ. Denote by $\tau_1, .., \tau_{2m}$ the ideal triangles.

For the ideal triangulation λ, we may consider its dual graph. Each ideal triangle τ_μ corresponds to a vertex τ_μ^* of the dual graph. Denote by $\lambda_1^*, \lambda_2^*, ..., \lambda_{3m}^*$ the dual edges. If an edge λ_i bounds one side of the ideal triangles τ_μ and one side of τ_ν, then the dual edge λ_i^* connects the vertexes τ_μ^* and τ_ν^*.

In a decorated ideal triangulation τ, each ideal triangle τ_μ (embedded or not) has three sides which correspond to the three half-edges incident to the vertex τ_μ^* of the dual graph. The three sides are numerated by $0, 1, 2$ in the counterclockwise order such that the $0-$side is opposite to the marked corner.

5.2 Space of Kashaev coordinates

Let's recall that a Kashaev coordinate associated to a decorated ideal triangulation τ is a vector $(\ln y_1, \ln z_1, ..., \ln y_{2m}, \ln z_{2m}) \in \mathbb{R}^{4m}$, where $\ln y_\mu$ and $\ln z_\mu$ are associated to the ideal triangle τ_μ. Denote by $\mathcal{K}_\tau$ the space of Kahsaev coordinates associated to τ. We see that $\mathcal{K}_\tau = \mathbb{R}^{4m}$.

Given a vector $(\ln y_1, \ln z_1, ..., \ln y_{2m}, \ln z_{2m}) \in \mathcal{K}_\tau$, we associate a number to each side of each ideal triangle as follows. For the ideal triangle τ_μ, we associate

- $\ln h_\mu^0 := \ln y_\mu - \ln z_\mu$ to the 0-side;
- $\ln h_\mu^1 := \ln z_\mu$ to the 1-side;
- $\ln h_\mu^2 := -\ln y_\mu$ to the 2-side.

Therefore $\ln h_\mu^0 + \ln h_\mu^1 + \ln h_\mu^2 = 0$. We can identify the space $\mathcal{K}_\tau = \mathbb{R}^{4m}$ with a subspace of $\mathbb{R}^{6m} = \{(..., \ln h_\mu^0, \ln h_\mu^1, \ln h_\mu^2, ...)\}$ satisfying $\ln h_\mu^0 + \ln h_\mu^1 + \ln h_\mu^2 = 0$ for each ideal triangle τ_μ.

5.3 Exact sequence

The enhanced Teichmüller space parametrized by shear coordinates is $\widetilde{\mathcal{T}}_\lambda = \mathbb{R}^{3m} = \{(\ln x_1, \ln x_2, ..., \ln x_{3m})\}$, where $\ln x_i$ is the shear coordinate at edge λ_i. We define a map $f_1 : \widetilde{\mathcal{T}}_\lambda \to \mathbb{R}$ by sending $(\ln x_1, \ln x_2, ..., \ln x_{3m})$ to the sum of entries $\sum_{i=1}^{3m} \ln x_i$.

Suppose λ is the underlying ideal triangulation of the decorated ideal triangulation τ. We define a map $f_2 : \mathcal{K}_\tau \to \widetilde{\mathcal{T}}_\lambda$ as a linear function by setting

$$\ln x_i = \ln h_\mu^s + \ln h_\nu^t$$

whenever λ_i bounds the $s-$side of τ_μ and the $t-$side of τ_ν (μ may equal ν), where $s, t \in \{0, 1, 2\}$.

Another map $f_3 : H_1(S, \mathbb{R}) \to \mathcal{K}_\tau$ is defined as follows. A homology class in $H_1(S, \mathbb{R})$ is represented by a linear combination of oriented dual edges: $\sum_{i=1}^{3m} c_i \lambda_i^*$. If the orientation of λ_i^* is from the $s-$side of τ_μ to the

$t-$side of τ_ν, by setting $\ln h^s_\mu = -c_i$ and $\ln h^t_\nu = c_i$, we obtain a vector $(..., \ln h^0_\mu, \ln h^1_\mu, \ln h^2_\mu, ...) \in \mathbb{R}^{6m}$. The boundary map of chain complexes sends $\sum_{i=1}^{3m} c_i \lambda^*_i$ to a linear combination of vertexes. In this combination, the term involving the vertex τ^*_μ is $(c_i\epsilon_i + c_j\epsilon_j + c_k\epsilon_k)\tau^*_\mu$ where $\lambda_i, \lambda_j, \lambda_k$ (two of them may coincide) bound three sides of τ_μ and $\epsilon_t = -1$ if λ^*_t starts at the side of τ_μ bounded by λ_t while $\epsilon_t = 1$ if λ^*_t ends at the side of τ_μ bounded by λ_t. Therefore

$$(c_i\epsilon_i + c_j\epsilon_j + c_k\epsilon_k)\tau^*_\mu = (\ln h^0_\mu + \ln h^1_\mu + \ln h^2_\mu)\tau^*_\mu.$$

Since the chain $\sum_{i=1}^{3m} c_i \lambda^*_i$ is a cycle, we must have $\ln h^0_\mu + \ln h^1_\mu + \ln h^2_\mu = 0$. Therefore this vector $(..., \ln h^0_\mu, \ln h^1_\mu, \ln h^2_\mu, ...)$ is in the subspace $\mathcal{K}_\tau$.

Combining the three maps, we obtain

Theorem 5.1 (Guo-Liu [13]). *The following sequence is exact:*

$$0 \to H_1(S,\mathbb{R}) \xrightarrow{f_3} \mathcal{K}_\tau \xrightarrow{f_2} \widetilde{\mathcal{T}}_\lambda \xrightarrow{f_1} \mathbb{R} \to 0.$$

From the theorem above, we see that $\mathcal{K}_\tau$ is a fiber bundle over the space $\mathrm{Ker}(f_1)$ whose fiber is an affine space modeled on $H_1(S,\mathbb{R})$. To be precise, given a vector $s \in \mathrm{Ker}(f_1)$, let $v \in f_2^{-1}(s)$. Then $f_2^{-1}(s) = v + H_1(S,\mathbb{R})$.

5.4 Relation to bivecotrs

Consider the linear isomorphism

$$\begin{aligned} M : \mathcal{K}_\tau &\longrightarrow \mathcal{K}_\tau \\ (\ln y_1, \ln z_1, ..., \ln y_{2m}, \ln z_{2m}) &\longmapsto (..., \ln h^0_\mu, \ln h^1_\mu, \ln h^2_\mu, ...). \end{aligned} \tag{5.1}$$

Proposition 5.2 (Guo-Liu [13]).
If $(\ln x_1, \ln x_2, ..., \ln x_{3m}) = f_2 \circ M(\ln y_1, \ln z_1, ..., \ln y_{2m}, \ln z_{2m})$, *then*

$$\sum_{i,j=1}^{3m} \sigma^\lambda_{ij} \frac{\partial}{\partial \ln x_i} \wedge \frac{\partial}{\partial \ln x_j} = (f_2)_* \circ M_*(\sum_{\mu=1}^{2m} \frac{\partial}{\partial \ln y_\mu} \wedge \frac{\partial}{\partial \ln z_\mu}),$$

where $\sigma^\lambda_{ij} = a^\lambda_{ij} - a^\lambda_{ji}$ *and* a^λ_{ij} *is the number of corners of the ideal triangulation* λ *which is delimited in the left by* λ_i *and on the right by* λ_j.

The left hand side of the equality is the Weil-Petersson Poisson structure on the enhanced Teichmüller space [9].

5.5 Compatibility of coordinate changes

Proposition 5.3 (Guo-Liu [13]). *Suppose that the decorated ideal triangulations* τ *and* τ' *have underlying ideal triangulations* λ *and* λ' *respectively. The* *following diagram is commutative:*

$$\begin{array}{ccc} \widetilde{\mathcal{T}}_{\lambda} & \xleftarrow{f_2} & \mathcal{K}_{\tau} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{T}}_{\lambda'} & \xleftarrow{f_2} & \mathcal{K}_{\tau'} \end{array}$$

where the two vertical maps are corresponding coordinate changes. The coordinate changes of Kashaev coordinates are given in Definition 4.4. The coordinate changes of shear coordinates are given in Proposition 2.4.

6 Relationship between quantum Teichmüller space and Kashaev algebra

In this section, we establish a natural relationship between the quantum Teichmüller space $\widehat{\mathcal{T}}_S^q$ and the generalized Kashaev algebra $\widehat{\mathcal{K}}_S^q(a,b)$.

6.1 Homomorphism

For a ideal triangle τ_μ, we associate three elements in $\mathcal{K}_\tau^q$ to the three sides of τ_μ as follows:

- $H_\mu^0 := Y_\mu Z_\mu^{-1}$ to the $0-$side;
- $H_\mu^1 := Z_\mu$ to the $1-$side;
- $H_\mu^2 := Y_\mu^{-1}$ to the $2-$side.

Lemma 6.1. *For any $s,t \in \{0,1,2\}$ and $\mu \in 1,2,...,3m$,*

$$H_\mu^s H_\mu^t = q^{2\sigma_{st}} H_\mu^t H_\mu^s,$$

where $\sigma_{st} + \sigma_{ts} = 0$ and $\sigma_{10} = \sigma_{02} = \sigma_{21} = 1$.

Suppose λ is the underlying ideal triangulation of τ, the Chekhov-Fock algebra $\mathcal{T}_\lambda^q$ is the algebra over $\mathbb{C}$ defined by generators $X_1^{\pm 1}$, $X_2^{\pm 1}$, $\ldots$, $X_n^{\pm 1}$ associated to the components of λ and by relations $X_i X_j = q^{2\sigma_{ij}^{\lambda}} X_j X_i$.

We define a map F_τ from the set of the generators of $\mathcal{T}_\lambda^q$ to $\mathcal{K}_\tau^q$. Suppose that the edge λ_i bounds the $s-$side of τ_μ and the $t-$side of τ_ν. We define

$$F_\tau(X_i) = q^{\delta_{\mu\nu}\sigma_{ts}} H_\mu^s H_\nu^t \in \mathcal{K}_\tau^q, \tag{6.1}$$

where σ_{ts} is defined in Lemma 6.1 and $\delta_{\mu\nu}$ is the Kronecker delta, i.e., $\delta_{\mu\mu} = 1$ and $\delta_{\mu\nu} = 0$ if $\mu \neq \nu$. When $\mu = \nu$, X_i is well-defined, since

$$q^{\sigma_{ts}} H_\mu^s H_\mu^t = q^{\sigma_{st}} H_\mu^t H_\mu^s$$

due to Lemma 6.1.

This definition is natural since when $q = 1$ we get the relationship between Kashaev coordinates and shear coordinates which is given by the map M and f_2. In fact when $q = 1$ the generators Y_μ, Z_μ are commutative. They reduce to the geometric quantities y_μ, z_μ associate to τ_μ. H_μ^s and X_i are reduced to h_μ^s and x_i.

Lemma 6.2. *$F_\tau(X_i)F_\tau(X_j) = q^{2\sigma_{ij}^\lambda} F_\tau(X_j)F_\tau(X_i)$ for any generators X_i and X_j.*

It turns out that F_τ can be extended to the whole algebra $\mathcal{T}_\lambda^q$ as an algebra homomorphism from $\mathcal{T}_\lambda^q$ to $\mathcal{K}_\tau^q$.

6.2 Compatibility

Recall that $\widehat{\mathcal{K}}_\tau^q$ is the fraction division algebra of $\mathcal{K}_\tau^q$. The algebraic isomorphism between $\widehat{\mathcal{K}}_\tau^q$ and $\widehat{\mathcal{K}}_{\tau'}^q$ is defined in Definition 4.5.

Lemma 6.3. *Suppose that a decorated ideal triangulation τ' is obtained from τ by a mark rotation ρ_μ for some $\mu \in \{1, 2, ..., 2m\}$. Let λ be the common underlying ideal triangulation of τ and τ'. The following diagram is commutative if and only if $a = q^{-2}$.*

$$\begin{array}{ccc} \widehat{\mathcal{T}}_\lambda^q & \xrightarrow{F_\tau} & \widehat{\mathcal{K}}_\tau^q \\ {\scriptstyle \mathrm{Id}}\uparrow & & \uparrow{\scriptstyle \widehat{\rho}_\mu} \\ \widehat{\mathcal{T}}_\lambda^q & \xrightarrow{F_{\tau'}} & \widehat{\mathcal{K}}_{\tau'}^q \end{array}$$

Lemma 6.4. *Suppose that a decorated ideal triangulation τ' is obtained from τ by a diagonal exchange $\varphi_{\mu\nu}$. Let λ and λ' be the underlying ideal triangulation of τ and τ' respectively. Then λ' is obtained from λ by a diagonal exchange with respect to the edge λ_i which is the common edge of τ_μ and τ_ν. The following diagram is commutative if and only if $b = q^3$.*

$$\begin{array}{ccc} \widehat{\mathcal{T}}_\lambda^q & \xrightarrow{F_\tau} & \widehat{\mathcal{K}}_\tau^q \\ {\scriptstyle \widehat{\Delta}_i}\uparrow & & \uparrow{\scriptstyle \widehat{\varphi}_{\mu\nu}} \\ \widehat{\mathcal{T}}_{\lambda'}^q & \xrightarrow{F_{\tau'}} & \widehat{\mathcal{K}}_{\tau'}^q \end{array}$$

Theorem 6.5 (Guo-Liu [13]). *Suppose the decorated ideal triangulations τ and τ' have underlying ideal triangulations λ and λ' respectively. The following diagram is commutative if and only if $a = q^{-2}, b = q^3$.*

$$\begin{array}{ccc} \widehat{\mathcal{T}}^q_\lambda & \xrightarrow{F_\tau} & \widehat{\mathcal{K}}^q_\tau \\ \Phi^q_{\lambda,\lambda'} \uparrow & & \uparrow \Psi^q_{\tau,\tau'}(a,b) \\ \widehat{\mathcal{T}}^q_{\lambda'} & \xrightarrow{F_{\tau'}} & \widehat{\mathcal{K}}^q_{\tau'} \end{array}$$

Recall that the quantum Teichmüller space of S is defined as the algebra

$$\widehat{\mathcal{T}}^q_S = \left(\bigsqcup_{\lambda \in \Lambda(S)} \widehat{\mathcal{T}}^q_\lambda \right) / \sim$$

where the relation $\sim$ is defined by the property that, for $X \in \widehat{\mathcal{T}}^q_\lambda$ and $X' \in \widehat{\mathcal{T}}^q_{\lambda'}$,

$$X \sim X' \Leftrightarrow X = \Phi^q_{\lambda,\lambda'}(X').$$

The generalized Kashaev algebra $\widehat{\mathcal{K}}^q_S(a,b)$ associated to a surface S is defined as the algebra

$$\widehat{\mathcal{K}}^q_S(a,b) = \left(\bigsqcup_{\tau \in \triangle(S)} \widehat{\mathcal{K}}^q_\tau(a,b) \right) / \sim$$

where the relation $\sim$ is defined by the property that, for $X \in \widehat{\mathcal{K}}^q_\tau(a,b)$ and $X' \in \widehat{\mathcal{K}}^q_{\tau'}(a,b)$,

$$X \sim X' \Leftrightarrow X = \Psi^q_{\tau,\tau'}(a,b)(X').$$

Corollary 6.6. *The homomorphism F_τ induces a homomorphism*

$$\widehat{\mathcal{T}}^q_S \to \widehat{\mathcal{K}}^q_S(a,b)$$

if and only if $a = q^{-2}, b = q^3$.

6.3 Quotient algebra

Furthermore, consider the element

$$H = q^{-\sum_{i<j} \sigma^\lambda_{ij}} X_1 X_2 ... X_{3m} \in \mathcal{T}^q_\lambda.$$

It is proved in [22](Proposition 14) that H is independent of the ideal triangulation λ. Therefore H is a well-defined element of the quantum Teichmüller space $\widehat{\mathcal{T}}^q_S$.

Theorem 6.7 (Guo-Liu [13]). *The homomorphism F_τ induces a homomorphism*

$$\widehat{\mathfrak{T}}^q_S/(q^{-2m}H-1) \to \widehat{\mathcal{K}}^q_S(q^{-2},q^3)$$

where $(q^{-2m}H-1)$ is the ideal generated by $q^{-2m}H-1$.

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