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LOWER BOUNDS FOR FINITENESS OF GENERALIZED LOCAL COHOMOLOGY MODULES AND THEIR ASSOCIATED PRIMES

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ABSTRACT. Let R be a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ an ideal of R, M a finite R–module and X an arbitrary R–module. In this paper, we study relations between finiteness of local cohomology and generalized local cohomology modules in several cases. We characterize the membership of generalized local cohomology modules in a certain Serre class from lower bounds and we found the least integer such that these modules belong to that Serre class. Let n be a non-negative integer, we prove that $\bigcup_{i<n} \operatorname{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M,X)) = \bigcup_{i<n} \operatorname{Supp}_R(\mathrm{H}^i_{\mathfrak{a}+\operatorname{Ann}_R M}(X)) = \bigcup_{i<n} \operatorname{Supp}_R(\operatorname{Ext}^i_R(M/\mathfrak{a}M,X))$ and if $\mathrm{H}^i_{\mathfrak{a}}(M,X)=0$ for all $i<n$ then $\operatorname{Ass}_R(\mathrm{H}^n_{\mathfrak{a}}(M,X)) = \operatorname{Ass}_R(\operatorname{Ext}^n_R(M/\mathfrak{a}M,X))$, these imply that if $\operatorname{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M,X))$ is finite for all $i<n$, then the finiteness of $\operatorname{Ass}_R(\mathrm{H}^n_{\mathfrak{a}}(M,X))$ is equivalent to the finiteness of $\operatorname{Ass}_R(\operatorname{Ext}^n_R(M/\mathfrak{a}M,X))$.

1. Introduction

Throughout R is a commutative noetherian ring. $\mathfrak{a}$ an ideal of R, M a finite (i.e., finitely generated) R–module and X an arbitrary R–module. The generalized local cohomology modules

$$\mathrm{H}^i_{\mathfrak{a}}(M,X) \cong \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, X).$$

was introduced by J. Herzog [12]. This concept was studied in the articles [20], [4], [12] and [21]. Note that this is in fact a generalization of the usual local cohomology, because if $M=R$, then $\mathrm{H}^i_{\mathfrak{a}}(R,N) = \mathrm{H}^i_{\mathfrak{a}}(N)$. Important problems concerning local cohomology are vanishing, finiteness, artinianness and finiteness of associated primes results (see [14]).

Recall that a subclass of the class of all modules is called Serre class, if it is closed under taking submodules, quotients and extensions. Examples are given by the class of finite modules, Artinian modules and weakly Laskerian modules. For unexplained terminology we refer to [5] and [6].

In Section 2, it is shown that for a Serre subcategory $\mathcal{S}$, if $\operatorname{Ext}^{n-r}_R(M,\mathrm{H}^r_{\mathfrak{a}}(X))$ is in $\mathcal{S}$ for all r, $0 \leqslant r \leqslant n$. Then $\mathrm{H}^n_{\mathfrak{a}}(M,X)$ is in $\mathcal{S}$ that provides the following equivalent conditions:

(i) $\mathrm{H}^i_{\mathfrak{a}}(X)$ is in $\mathcal{S}$ for all i, $0 \leqslant i \leqslant n$.

(ii) $\mathrm{H}^i_{\mathfrak{a}}(M,X)$ is in $\mathcal{S}$ for all i, $0 \leqslant i \leqslant n$, and for any finite R–module M.

We also discuss the relation between finiteness of $\mathrm{H}^{s+t}_{\mathfrak{a}}(M,X)$ and $\operatorname{Ext}^s_R(M,\mathrm{H}^t_{\mathfrak{a}}(X))$ under some condition on s and t. Some applications of it are indicated.

In the third section, In theorem 3.1 and corollary 3.3 we are interested in finding least integer such that generalized local cohomology modules belong to a certain Serre class that we introduced it in [2, Difinition 2.1] and in consequence for a positive integer n, we prove that $\bigcup_{i<n} \operatorname{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M,X)) = \bigcup_{i<n} \operatorname{Supp}_R(\mathrm{H}^i_{\mathfrak{a}+\operatorname{Ann}_R M}(X)) = \bigcup_{i<n} \operatorname{Supp}_R(\operatorname{Ext}^i_R(M/\mathfrak{a}M,X))$ and if $\mathrm{H}^i_{\mathfrak{a}}(M,X)=0$ for all $i<n$ then

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$\mathrm{Ass}_R(\mathrm{H}^n_{\mathfrak{a}}(M,X)) = \mathrm{Ass}_R(\mathrm{Ext}^n_R(M/\mathfrak{a}M,X))$, which implies that if $\mathrm{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M,X))$ is finite for all $i<n$, then the finiteness of $\mathrm{Ass}_R(\mathrm{H}^n_{\mathfrak{a}}(M,X))$ is equivalent to the finiteness of $\mathrm{Ass}_R(\mathrm{Ext}^n_R(M/\mathfrak{a}M,X))$.

We are however avoiding the use of spectral sequences completely in this work, even if we can show some of our results with this technique. So we provide a more elementary treatment.

2. Finiteness results

Lemma 2.1. *Let M and X be respectively finite and arbitrary R–modules and $\mathfrak{a}$ be an ideal of R. Then*

(a) *if* $\mathrm{Supp}_R(M)\cap\mathrm{Supp}_R(X)\subseteq \mathrm{V}(\mathfrak{a})$*. Then for all $i\geq 0$, we have*

$$\mathrm{H}^i_{\mathfrak{a}}(M,X)\cong \mathrm{Ext}^i_R(M,X).$$

(b) *If $f: R\longrightarrow S$ is a flat ring homomorphism, then*

$$\mathrm{H}^i_{\mathfrak{a}}(M,X)\otimes_R S\cong \mathrm{H}^i_{\mathfrak{a}S}(M\otimes_R S, X\otimes_R S).$$

Proof. (i) There is a minimal injective resolution $E^\bullet$ of X such that $\mathrm{Supp}_R(E^i)\subseteq \mathrm{Supp}_R(X)$ for all $i\geq 0$. Since $\mathrm{Supp}_R(\mathrm{Hom}_R(M,E^i))\subseteq \mathrm{Supp}_R(M)\cap\mathrm{Supp}_R(X)\subseteq \mathrm{V}(\mathfrak{a})$, $\mathrm{Hom}_R(M,E^i)$ is $\mathfrak{a}$–torsion. Therefore, for all $i\geq 0$,

$$\begin{aligned}\mathrm{H}^i_{\mathfrak{a}}(M,X) &= \mathrm{H}^i(\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(M,E^\bullet)))\\ &= \mathrm{H}^i(\mathrm{Hom}_R(M,E^\bullet))\\ &= \mathrm{Ext}^i_R(M,X),\end{aligned}$$

as we desired.

(ii) It is easy and we leave it to the reader. □

Theorem 2.2. *Let M be a finite R–module and X be an arbitrary R–module such that $\mathrm{Ext}^{n-r}_R(M,\mathrm{H}^r_{\mathfrak{a}}(X))$ is in $\mathcal{S}$ for all r, $0\leqslant r\leqslant n$. Then $\mathrm{H}^n_{\mathfrak{a}}(M,X)$ is in $\mathcal{S}$.*

Proof. We prove by using induction on n. Let $n=0$, $\overline{X}=X/\Gamma_{\mathfrak{a}}(X)$ and consider the exact sequence $0\to\Gamma_{\mathfrak{a}}(X)\to X\to\overline{X}\to 0$. Since $\Gamma_{\mathfrak{a}}(\overline{X})=0$, $\Gamma_{\mathfrak{a}}(M,\Gamma_{\mathfrak{a}}(X))\cong\Gamma_{\mathfrak{a}}(M,X)$ from [13, Lemma 2.1]. Now, by the above lemma, we get $\mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(X))\cong\Gamma_{\mathfrak{a}}(M,X)$.

Suppose that $n>0$ and that $n-1$ is settled. Let $\overline{X}=X/\Gamma_{\mathfrak{a}}(X)$ and $L=E(\overline{X})/\overline{X}$ where $E(\overline{X})$ is an injective hull of $\overline{X}$. Since $\Gamma_{\mathfrak{a}}(\overline{X})=0=\Gamma_{\mathfrak{a}}(E(\overline{X}))$, $\Gamma_{\mathfrak{a}}(M,\overline{X})=0=\Gamma_{\mathfrak{a}}(M,E(\overline{X}))$ by [13, Lemma 2.1]. Applying the derived functors of $\Gamma_{\mathfrak{a}}(-)$ and $\Gamma_{\mathfrak{a}}(M,-)$ to the exact sequence $0\to\overline{X}\to E(\overline{X})\to L\to 0$, we obtain, for all $i>0$, the isomorphisms:

$$\mathrm{H}^{i-1}_{\mathfrak{a}}(L)\cong\mathrm{H}^i_{\mathfrak{a}}(\overline{X}) \quad\text{and}\quad \mathrm{H}^{i-1}_{\mathfrak{a}}(M,L)\cong\mathrm{H}^i_{\mathfrak{a}}(M,\overline{X}).$$

Note that, for all r, $0\leqslant r\leqslant n-1$, we have

$$\begin{aligned}\mathrm{Ext}^{(n-1)-r}_R(M,\mathrm{H}^r_{\mathfrak{a}}(L)) &\cong \mathrm{Ext}^{(n-1)-r}_R(M,\mathrm{H}^{r+1}_{\mathfrak{a}}(\overline{X}))\\ &\cong \mathrm{Ext}^{n-(r+1)}_R(M,\mathrm{H}^{r+1}_{\mathfrak{a}}(X))\end{aligned}$$

from the above isomorphisms. Thus $\mathrm{H}^{n-1}_{\mathfrak{a}}(M,L)$ is in $\mathcal{S}$ by the induction hypothesis on L. Therefore $\mathrm{H}^n_{\mathfrak{a}}(M,\overline{X})$ belongs to $\mathcal{S}$. From the exact sequence $0\to\Gamma_{\mathfrak{a}}(X)\to X\to\overline{X}\to 0$, we get the exact sequence

$$\ldots\longrightarrow\mathrm{Ext}^n_R(M,\Gamma_{\mathfrak{a}}(X))\longrightarrow\mathrm{H}^n_{\mathfrak{a}}(M,X)\longrightarrow\mathrm{H}^n_{\mathfrak{a}}(M,\overline{X})\longrightarrow\ldots$$

which shows that $\mathrm{H}^n_{\mathfrak{a}}(M,X)$ is in $\mathcal{S}$. □

Corollary 2.3. (*cf.* [16, Theorem 3.3]) *Let M be a finite R–module and X be an arbitrary R–module such that $\mathrm{Ext}_R^{n-r}(M, \mathrm{H}_{\mathfrak{a}}^r(X))$ is weakly Laskerian for all r, $0 \leqslant r \leqslant n$. Then $\mathrm{H}_{\mathfrak{a}}^n(M,X)$ is weakly Laskerian and so for any submodule T of $\mathrm{H}_{\mathfrak{a}}^n(M,X))$ the set $\mathrm{Ass}(\mathrm{H}_{\mathfrak{a}}^n(M,X)/T)$ is finite.*

Definition 2.4. (*see* [1, Definition 2.1] *and* [2, Definition 3.1]) *Let $\mathcal{M}$ be a Serre subcategory of the category of R–modules. We say that $\mathcal{M}$ is a Melkersson subcategory with respect to the ideal $\mathfrak{a}$ if for any $\mathfrak{a}$–torsion R–module X, $0 :_X \mathfrak{a}$ is in $\mathcal{M}$ implies that X is in $\mathcal{M}$. $\mathcal{M}$ is called Melkersson subcategory when it is a Melkersson subcategory with respect to all ideals of R.*

To see some examples of Melkersson subcategories, we refer the reader to [1, Examples 2.4 and 2.5].

The first author and Melkersson in [1, Theorem 2.9 (i)⇔(vi)] proved the following corollary for Melkersson subcategories but we prove it for any Serre subcategories, also Mafi in [16, Lemma 3.1] proved it in special case for weakly Laskerian modules using spectral sequences argument, while it is a simple conclusion of theorem 2.2.

Corollary 2.5. *Let X be an R–module and n be a non-negative integer. Then the following statements are equivalent.*

(i) $\mathrm{H}_{\mathfrak{a}}^i(X)$ *is in $\mathcal{S}$ for all i, $0 \leqslant i \leqslant n$.*
(ii) $\mathrm{H}_{\mathfrak{a}}^i(M,X)$ *is in $\mathcal{S}$ for all i, $0 \leqslant i \leqslant n$, and for any finite R–module M.*

Theorem 2.6. *Let M be a finite R–module, X be an arbitrary R–module and s,t be non-negative integers. Assume also that:*

(i) $\mathrm{H}_{\mathfrak{a}}^{s+t}(M,X)$ *is in $\mathcal{S}$,*
(ii) $\mathrm{Ext}_R^{s+r+1}(M, \mathrm{H}_{\mathfrak{a}}^{t-r}(X))$ *is in $\mathcal{S}$ for all r, $1 \leq r \leq t$, and*
(iii) $\mathrm{Ext}_R^{s-r-1}(M, \mathrm{H}_{\mathfrak{a}}^{t+r}(X))$ *is in $\mathcal{S}$ for all r, $1 \leq r \leq s-1$.*

Then $\mathrm{Ext}_R^s(M, \mathrm{H}_{\mathfrak{a}}^t(X))$ is in $\mathcal{S}$.

Proof. We prove by induction on t. Let $t = 0$ and $\overline{X} = X/\Gamma_{\mathfrak{a}}(X)$. By Theorem 2.2, $\mathrm{H}_{\mathfrak{a}}^{s-1}(M, \overline{X})$ belongs to $\mathcal{S}$ since $\mathrm{Ext}_R^{s-1-r}(M, \mathrm{H}_{\mathfrak{a}}^r(\overline{X}))$ is in $\mathcal{S}$ for all r, $0 \leqslant r \leqslant s-1$. Applying the derived functor of $\Gamma_{\mathfrak{a}}(M,-)$ to the short exact sequence $0 \to \Gamma_{\mathfrak{a}}(X) \to X \to \overline{X} \to 0$, we obtain the long exact sequence

$$\cdots \longrightarrow \mathrm{H}_{\mathfrak{a}}^{s-1}(M, \overline{X}) \longrightarrow \mathrm{H}_{\mathfrak{a}}^s(M, \Gamma_{\mathfrak{a}}(X)) \longrightarrow \mathrm{H}_{\mathfrak{a}}^s(M,X) \longrightarrow \cdots$$

which shows that $\mathrm{H}_{\mathfrak{a}}^s(M, \Gamma_{\mathfrak{a}}(X))$ is in $\mathcal{S}$. Thus $\mathrm{Ext}_R^s(M, \Gamma_{\mathfrak{a}}(X))$ belongs to $\mathcal{S}$ by Lemma 2.1(a).

Now, Suppose that $t > 0$ and that $t-1$ is settled. Let $\overline{X} = X/\Gamma_{\mathfrak{a}}(X)$ and $L = E(\overline{X})/\overline{X}$ where $E(\overline{X})$ is an injective hull of $\overline{X}$. Since $\Gamma_{\mathfrak{a}}(\overline{X}) = 0 = \Gamma_{\mathfrak{a}}(E(\overline{X}))$, $\Gamma_{\mathfrak{a}}(M, \overline{X}) = 0 = \Gamma_{\mathfrak{a}}(M, E(\overline{X}))$ by [13, Lemma 2.1]. Applying the derived functors of $\Gamma_{\mathfrak{a}}(-)$ and $\Gamma_{\mathfrak{a}}(M,-)$ to the exact sequence $0 \to \overline{X} \to E(\overline{X}) \to L \to 0$, we obtain, for all $i > 0$, the isomorphisms:

$$\mathrm{H}_{\mathfrak{a}}^{i-1}(L) \cong \mathrm{H}_{\mathfrak{a}}^i(\overline{X}) \cong \mathrm{H}_{\mathfrak{a}}^i(X) \quad \text{and} \quad \mathrm{H}_{\mathfrak{a}}^{i-1}(M,L) \cong \mathrm{H}_{\mathfrak{a}}^i(M, \overline{X}).$$

By the above isomorphisms, we have

$$\mathrm{Ext}_R^{s+r+1}(M, \mathrm{H}_{\mathfrak{a}}^{(t-1)-r}(L)) \cong \mathrm{Ext}_R^{s+r+1}(M, \mathrm{H}_{\mathfrak{a}}^{t-r}(X))$$

for r, $1 \leqslant r \leqslant t-1$, and

$$\mathrm{Ext}_R^{s-r-1}(M, \mathrm{H}_{\mathfrak{a}}^{(t-1)+r}(L)) \cong \mathrm{Ext}_R^{s-r-1}(M, \mathrm{H}_{\mathfrak{a}}^{t+r}(X))$$

for r, $1 \leqslant r \leqslant s-1$ which are in $\mathcal{S}$ by assumptions (ii) and (iii).

On the other hand, by the exact sequence $0 \to \Gamma_{\mathfrak{a}}(X) \to X \to \overline{X} \to 0$ and Lemma 2.1(a), we get the long exact sequence

$$\cdots \longrightarrow \mathrm{H}_{\mathfrak{a}}^{s+t}(M,X) \longrightarrow \mathrm{H}_{\mathfrak{a}}^{s+t}(M,\overline{X}) \longrightarrow \mathrm{Ext}_R^{s+t+1}(M,\Gamma_{\mathfrak{a}}(X)) \longrightarrow \cdots$$

which shows that $\mathrm{H}_{\mathfrak{a}}^{s+t}(M,\overline{X})$ is in $\mathcal{S}$ by assumptions (i) and (ii). That is $\mathrm{H}_{\mathfrak{a}}^{s+(t-1)}(M,L)$ is in $\mathcal{S}$. Now, by the induction hypothesis on L, $\mathrm{Ext}_R^s(M,\mathrm{H}_{\mathfrak{a}}^{t-1}(L))$ belongs to $\mathcal{S}$. Therefore $\mathrm{Ext}_R^s(M,\mathrm{H}_{\mathfrak{a}}^{t}(X))$ is in $\mathcal{S}$ which terminates the induction argument. The proof is completed. □

Corollary 2.7. *Let M be a finite R–module and X be an arbitrary R–module. Assume that n is a non-negative integer such that $\mathrm{Ext}_R^{j-i}(M,\mathrm{H}_{\mathfrak{a}}^{i}(X))$ is in $\mathcal{S}$ for all i, j with $0 \le i \le n-1$ and $j = n, n+1$. Then $\mathrm{H}_{\mathfrak{a}}^{n}(M,X)$ is in $\mathcal{S}$ if and only if $\mathrm{Hom}_R(M,\mathrm{H}_{\mathfrak{a}}^{n}(X))$ is in $\mathcal{S}$.*

Proof. ($\Rightarrow$) Apply Theorem 2.6 with $s = 0$ and $t = n$.

($\Leftarrow$) Apply Theorem 2.2. □

The following remark is another proof for corollary 2.7 using [3, Proposition 3.1]. Also it is the study of finiteness of kernel and the cokernel of the natural homomorphism $f : \mathrm{H}_{\mathfrak{a}}^{n}(M,X) \longrightarrow \mathrm{Hom}_R(M,\mathrm{H}_{\mathfrak{a}}^{n}(X))$.

Remark 2.8. *Let M be a finite R–module and X be an arbitrary R–module. Let $\mathfrak{a}$ be an ideal of R, n be a non-negative integer and $\mathcal{S}$ be a Serre subcategory of the category of R–modules. Consider the natural homomorphism*

$$f : \mathrm{H}_{\mathfrak{a}}^{n}(M,X) \longrightarrow \mathrm{Hom}_R(M,\mathrm{H}_{\mathfrak{a}}^{n}(X)).$$

(a) *If $\mathrm{Ext}_R^{n-j}(M,\mathrm{H}_{\mathfrak{a}}^{j}(X))$ belongs to $\mathcal{S}$ for all $j < n$, then $\mathrm{Ker}\, f$ belongs to $\mathcal{S}$.*

(b) *If $\mathrm{Ext}_R^{n+1-j}(M,\mathrm{H}_{\mathfrak{a}}^{j}(X))$ belongs to $\mathcal{S}$ for all $j < n$, then $\mathrm{Coker}\, f$ belongs to $\mathcal{S}$.*

(c) *If $\mathrm{Ext}_R^{t-j}(M,\mathrm{H}_{\mathfrak{a}}^{j}(X))$ belongs to $\mathcal{S}$ for $t = n, n+1$ and for all $j < n$, then $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ both belong to $\mathcal{S}$. Thus $\mathrm{H}_{\mathfrak{a}}^{n}(M,X)$ belongs to $\mathcal{S}$ if and only if $\mathrm{Hom}_R(M,\mathrm{H}_{\mathfrak{a}}^{n}(M))$ belongs to $\mathcal{S}$.*

Proof. We apply [3, Proposition 3.1] with $FX = \mathrm{Hom}_R(M,X)$, and $GX = \Gamma_{\mathfrak{a}}(X)$. Observe that $FG(-) = \Gamma_{\mathfrak{a}}(M,-)$. □

It is clear from the above remark that, if $\mathrm{Ext}_R^{n-j}(M,\mathrm{H}_{\mathfrak{a}}^{j}(X)) = 0$ for all $j < n$, then f is injective and if $\mathrm{Ext}_R^{n+1-j}(M,\mathrm{H}_{\mathfrak{a}}^{j}(X)) = 0$ for all $j < n$, then f is surjective. If both of this condition fulfilled together, then f if an isomorphism. See also 2.11.

Corollary 2.9. *Let X be an R–module and s, t be non-negative integers such that $\mathrm{H}_{\mathfrak{a}}^{i}(X)$ is in $\mathcal{S}$ for all i, $0 \leqslant i \leqslant t-1$ or $t+1 \leqslant r \leqslant s+t$. Then $\mathrm{H}_{\mathfrak{a}}^{s+t}(M,X)$ is in $\mathcal{S}$ if and only if $\mathrm{Ext}_R^{s}(M,\mathrm{H}_{\mathfrak{a}}^{t}(X))$ is in $\mathcal{S}$.*

Proof. ($\Rightarrow$) This follows from Theorem 2.6.

($\Leftarrow$) Apply Theorem 2.2 with $n = s + t$. □

Theorem 2.10. *Let M be a finite R–module, X be an arbitrary R–module and s, t be non-negative integers. Assume also that:*

(i) *$\mathrm{Ext}_R^{s+t-r}(M,\mathrm{H}_{\mathfrak{a}}^{r}(X)) = 0$ for all r, $1 \le r \le t-1$ or $t+1 \le r \le s+t$,*

(ii) *$\mathrm{Ext}_R^{s+r+1}(M,\mathrm{H}_{\mathfrak{a}}^{t-r}(X)) = 0$ for all r, $1 \le r \le t$, and*

(iii) *$\mathrm{Ext}_R^{s-r-1}(M,\mathrm{H}_{\mathfrak{a}}^{t+r}(X)) = 0$ for all r, $1 \le r \le s-1$.*

Then we have $\mathrm{H}_{\mathfrak{a}}^{s+t}(M,X) \cong \mathrm{Ext}_R^{s}(M,\mathrm{H}_{\mathfrak{a}}^{t}(X))$.

Proof. We prove by using induction on t. Let $t = 0$. We have $\mathrm{H}_{\mathfrak{a}}^{s-1}(M, X/\Gamma_{\mathfrak{a}}(X)) = 0 = \mathrm{H}_{\mathfrak{a}}^{s}(M, X/\Gamma_{\mathfrak{a}}(X))$ from hypothesis (iii) and (i), and Theorem 2.2 with $\mathcal{S} = 0$. Now the assertion follows by the long exact sequence

$$\cdots \longrightarrow \mathrm{H}_{\mathfrak{a}}^{s-1}(M, X/\Gamma_{\mathfrak{a}}(X)) \longrightarrow \mathrm{Ext}_R^s(M, \Gamma_{\mathfrak{a}}(X)) \longrightarrow \mathrm{H}_{\mathfrak{a}}^s(M, X) \longrightarrow \mathrm{H}_{\mathfrak{a}}^s(M, X/\Gamma_{\mathfrak{a}}(X)) \longrightarrow \cdots$$

obtained from the short exact sequence $0 \longrightarrow \Gamma_{\mathfrak{a}}(X) \longrightarrow X \longrightarrow X/\Gamma_{\mathfrak{a}}(X) \longrightarrow 0$.

Assume that $t > 0$ and that $t - 1$ is settled. This is sufficiently similar to that of Theorem 2.6 to be omitted. We leave the proof to the reader. □

Yassemi in [21, Example 3.6] have given an example to show that the R–modules $\mathrm{H}_{\mathfrak{a}}^n(M, X)$ and $\mathrm{Hom}_R(M, \mathrm{H}_{\mathfrak{a}}^n(X))$ are not always equal. We show that with some condition they are isomorph.

Corollary 2.11. (cf. [13, Proposition 2.3(ii)]) *Let M be a finite R–module, X be an arbitrary R–module and n be a non-negative integer such that $\mathrm{Ext}_R^{j-i}(M, \mathrm{H}_{\mathfrak{a}}^i(X)) = 0$ for all i, j with $0 \leqslant i \leqslant n-1$ and $j = n, n+1$. Then we have $\mathrm{H}_{\mathfrak{a}}^n(M, X) \cong \mathrm{Hom}_R(M, \mathrm{H}_{\mathfrak{a}}^n(X))$.*

Proof. Apply Theorem 2.10 with $s = 0$ and $t = n$. □

Corollary 2.12. *Suppose that M is a finite R–module, X is an R–module, and n, m are non-negative integers such that $n \leq m$. Assume also that $\mathrm{H}_{\mathfrak{a}}^i(X) = 0$ for all i, $i \neq n$ (resp. $0 \leq i \leq n-1$ or $n+1 \leq i \leq m$). Then we have $\mathrm{H}_{\mathfrak{a}}^{i+n}(M, X) \cong \mathrm{Ext}_R^i(M, H_{\mathfrak{a}}^n(X))$ for all i, $i \geq 0$ (resp. $0 \leq i \leq m-n$).*

Proof. For all i, $i \geq 0$ (resp. $0 \leq i \leq m-n$), apply Theorem 2.10 with $s = i$ and $t = n$. □

Corollary 2.13. *Let M be a finite R–module with $\mathrm{pd}_R(M) < \infty$ and X be an arbitrary R–module. Then the following statements hold true.*

(a) $\mathrm{H}_{\mathfrak{a}}^i(M, X) = 0$ *for all i, $i > \mathrm{pd}_R(M) + \mathrm{cd}(\mathfrak{a}, X)$.*
(b) $\mathrm{H}_{\mathfrak{a}}^{\mathrm{pd}_R(M)+\mathrm{cd}(\mathfrak{a},X)}(M, X) \cong \mathrm{Ext}_R^{\mathrm{pd}_R(M)}(M, \mathrm{H}_{\mathfrak{a}}^{\mathrm{cd}(\mathfrak{a},X)}(X))$.

Proof. (i) Let n and r be non-negative integers such that $n > \mathrm{pd}_R(M) + \mathrm{cd}(\mathfrak{a}, X)$ and $0 \leq r \leq n$. If $r > \mathrm{cd}(\mathfrak{a}, X)$, then $\mathrm{Ext}_R^{n-r}(M, \mathrm{H}_{\mathfrak{a}}^r(X)) = 0$. Otherwise $n - r > \mathrm{pd}_R(M)$ and so $\mathrm{Ext}_R^{n-r}(M, \mathrm{H}_{\mathfrak{a}}^r(X)) = 0$. Thus $\mathrm{H}_{\mathfrak{a}}^n(M, X) = 0$ by Theorem 2.2.

(ii) By apply Theorem 2.10 with $s = \mathrm{pd}(M)$ and $t = \mathrm{cd}(\mathfrak{a}, M)$, the assertion follows. □

Corollary 2.14. *Let M and X be two finite R–modules. Assume that $d = \dim_R(X)$ and $p = \mathrm{pd}_R(X)$ are finite. Then $\mathrm{H}_{\mathfrak{a}}^{p+d}(M, X)$ is an $\mathfrak{a}$–cofinite artinian R–module.*

Proof. This is immediate by [19, Proposition 5.1] and corollary 2.13 (b). □

3. Lower bounds for finiteness of generalzed local cohomology modules

The first author and Melkersson in [1, Difinition 2.6 and Example 2.8] introduced the concept of regular sequences on a module with respect to Serre classes that recovered Poor sequences, filter regular sequences, generalized regular sequences and sequences in dimension> s on a module where s is a non-negative integer. They also found the relation of these notion on a finite module and the membership of the local cohomology modules in Melkersson subcategories . See [1, Theorem 2.9 (i)↔(vii)]. Here for given R–modules M and X we prove a similar characterization for generalized local cohomolgy module $\mathrm{H}_{\mathfrak{a}}^i(M, X)$ in Melkersson subcategories. Coung and Hoang in [10, Theorem 3.1] proved Part [(i)↔(v)] of the following theorem in the special case of Artinianness when R is a local ring.

Theorem 3.1. *Let M be a finite R–module and X be an arbitrary R–module. Assume also that n is a non-negative integer and $\mathcal{M}$ is a Melkersson subcategory with respect to $\mathfrak{a}$ of the category of R–modules. Then the following statements are equivalent.*

(i) $\mathrm{H}^i_{\mathfrak{a}}(M, X)$ *is in* $\mathcal{M}$ *for all* i, $0 \leqslant i < n$ (for all i) .

(ii) $\mathrm{H}^i_{\mathfrak{a}+\mathrm{Ann}(M)}(X)$ is in $\mathcal{M}$ for all i, $0 \leqslant i < n$ (for all i).

(iii) $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is in $\mathcal{M}$ for all i, $0 \leqslant i < n$ (for all i).

(iv) $\mathrm{H}^i(x_1, \ldots, x_r; X)$ is in $\mathcal{M}$ for all i, $0 \leqslant i < n$ (for all i), where $\mathfrak{a} + \mathrm{Ann}(M) = (x_1, \ldots, x_r)$.

When X is finite these conditions are also equivalent to:

(v) *There is a sequence of length n in $\mathfrak{a} + \mathrm{Ann}(M)$ that is $\mathcal{M}$–regular on X(the same thing for all n).*

Proof. (ii) ⇒ (i). Since $\mathrm{H}^i_{\mathfrak{a}}(M, X) \cong \mathrm{H}^i_{\mathfrak{a}+\mathrm{Ann}(M)}(M, X)$ for all i, the assertion holds from Corollary 2.5.

(i) ⇒ (ii). We use induction on n. Let $n = 0$. Since $\Gamma_{\mathfrak{a}}(M, X) \cong \Gamma_{\mathfrak{a}}(M, \Gamma_{\mathfrak{a}}(X)) \cong \mathrm{Hom}_R(M, \Gamma_{\mathfrak{a}}(X))$, $\Gamma_{\mathrm{Ann}(M)}(\Gamma_{\mathfrak{a}}(X))$ is in $\mathcal{M}$ (see [1, Theorem 2.9]). Thus $\Gamma_{\mathfrak{a}+\mathrm{Ann}(M)}(X)$ is in $\mathcal{M}$.

Assume that $n > 0$ and that $n - 1$ is settled. By the induction hypothesis, $\mathrm{H}^i_{\mathfrak{a}+\mathrm{Ann}(M)}(X)$ is in $\mathcal{M}$ for all i, $0 \leqslant i < n - 1$. Apply Theorem 2.6 with $s = 0$ and $t = n$ for the ideal $\mathfrak{a} + \mathrm{Ann}(M)$ to see that $\mathrm{Hom}_R(M, \mathrm{H}^n_{\mathfrak{a}+\mathrm{Ann}(M)}(X))$ is in $\mathcal{M}$. Thus $\Gamma_{\mathrm{Ann}(M)}(\mathrm{H}^n_{\mathfrak{a}+\mathrm{Ann}(M)}(X))$ is in $\mathcal{M}$ and so $\mathrm{H}^n_{\mathfrak{a}+\mathrm{Ann}(M)}(X)$ belongs to $\mathcal{M}$. This completes the induction argument.

(ii) ⇔ (iii) ⇔ (iv) ⇔ (v). Follows from [1, Theorem 2.9 (i) ⇔ (iii) ⇔ (v) ⇔ (vii)]. □

Chu and Tang in [8, Proposition 2.4] proved the following corollary in the local case while it is an immediate result of theorem 3.1[(i)⇔(ii)] even in general case. See also [10, Corrollary 3.2], [11, Theorem 2.2] and [18, Corrollary 2.6].

Corollary 3.2. *Let $t \geqslant 1$ and M, X be two finite R–modules,* $\mathrm{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M, X)) \subseteq \mathrm{Max}(R)$ *hold for all $i < t$ if and only if $\mathrm{H}^i_{\mathfrak{a}}(M, X)$ is Artinian for all $i < t$. In particular, If $\dim(R/\mathfrak{a}) = 0$ then $\mathrm{H}^i_{\mathfrak{a}}(M, X)$ is Artinian for all i.*

Proof. Use Theorem 3.1[(i)⇔(ii)] when $\mathcal{M}$ is the category of zero dimensional R–module and note that a zero dimensional Noetherian R–module is Artinian. □

Let X be a finite R–module, $\mathfrak{a}$ be an ideal of R and $\mathcal{M}$ be a Melkersson subcategory with respect to $\mathfrak{a}$ such that $X/\mathfrak{a}X$ is not in $\mathcal{M}$. In [1, Lemma 2.14, Difinition 2.15], the authors proved that every sequence in $\mathfrak{a}$ which is $\mathcal{M}$–regular on X can be extend to maximal one and all maximal $\mathcal{M}$–regular on X in $\mathfrak{a}$ have the same length. They denote this common length by $\mathcal{M}$–$\mathrm{depth}_{\mathfrak{a}}(M)$. They also proved that it is the least integer such that $\mathrm{H}^i_{\mathfrak{a}}(X)$, $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ or Koszol cohomology with respect to $\mathfrak{a}$ are not in $\mathcal{M}$ (see [1, Theorem 2.18]). Using the Melkersson subcategories of [1, Example 2.4] this notion gives ordinary depth, filter-depth, generalized depth and s-depth where s is a non-negative integer. See [1, Example 2.16] and [7, Difinition 3.1]. In the following we prove that $\mathcal{M} - \mathrm{depth}(\mathfrak{a} + \mathrm{Ann}(M), X)$ is the least integer such that $\mathrm{H}^i_{\mathfrak{a}}(M, X)$, $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ or Koszol cohomology with respect to $\mathfrak{a} + \mathrm{Ann}(M)$ are not in $\mathcal{M}$. This generalize the result of Bijan-Zadeh [4, Proposition 5.5] when we consider $\mathcal{M} = \{0\}$. In the case of Artinianness and finiteness of support it recover [8, Theorem 2.2], [10, Theorem 3.1], [9, Theorem 4.1] and [17, Theorem 2.8]. Note that all of these Theorems are in local case while our corollary is in general case.

Corollary 3.3. *Let M, X be finite R–modules, and $\mathcal{M}$ be a Melkersson subcategory with respect to $\mathfrak{a}$ of $\mathcal{C}(R)$ such that $X/(\mathfrak{a}+\operatorname{Ann}(M))X$ is not in $\mathcal{M}$. Then*

(a) $\mathcal{M}-\operatorname{depth}(\mathfrak{a}+\operatorname{Ann}(M), X)=\inf\{i : \mathrm{H}^i_{\mathfrak{a}}(M, X)\notin\mathcal{M}\}$.

(b) $\mathcal{M}-\operatorname{depth}(\mathfrak{a}+\operatorname{Ann}(M), X)=\inf\{i : \operatorname{Ext}^i_R(M/\mathfrak{a}M, X)\notin\mathcal{M}\}$.

(c) $\mathcal{M}-\operatorname{depth}(\mathfrak{a}+\operatorname{Ann}(M), X)=\inf\{i : \mathrm{H}^i(x_1,\ldots,x_r; X)\notin\mathcal{M}$ *where* $\mathfrak{a}+\operatorname{Ann}(M)=(x_1,\ldots,x_r)\}$.

Proof. Follows from Theorem 3.1. □

Remark 3.4. *Theorem 3.1 and corollary 3.3 can be applied to each Melkersson subcategory mentioned in* [1, Example 2.4 and 2.5] *resulting in each case in a number of equivalent conditions*

Corollary 3.5. $\inf\{i : \mathrm{H}^i_{\mathfrak{a}}(X)\notin\mathcal{M}\}\leqslant\inf\{i : \mathrm{H}^i_{\mathfrak{a}}(M, X)\notin\mathcal{M}\}$ *and the equality hold whenever* $\operatorname{Ann}(M)\subseteq\mathfrak{a}$*, e.g., M is faithful.*

Proof. It is easy to see that $\mathcal{M}-\operatorname{depth}(\mathfrak{a}, X)\leqslant\mathcal{M}-\operatorname{depth}(\mathfrak{a}+\operatorname{Ann}(M), X)$ so by [1, Example 2.18 (a)]and 3.3, the result follows. □

Part (a) of the following corollary in local case has been proved in [9, Lemma 2.8] by Cuong and Hoang when X is finite R–module but in our proof X is an arbitrary R–module and R is non-local ring.

Corollary 3.6. *Let M be a finite R–module then*

(a) $\bigcup\limits_{i<n}\operatorname{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M, X))=\bigcup\limits_{i<n}\operatorname{Supp}_R(\mathrm{H}^i_{\mathfrak{a}+\operatorname{Ann}_R M}(X))=\bigcup\limits_{i<n}\operatorname{Supp}_R(\operatorname{Ext}^i_R(M/\mathfrak{a}M, X))$ *for every R–module X and every positive integer n.*

(b) *If X is a finite R–module, then* $\bigcup\limits_{i<n}\operatorname{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M, X))$ *is a closed set for each positive integer n. In particular* $\bigcup\limits_{i}\operatorname{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M, X))$ *is a closed set.*

Proof. (a) Respectively by Lemma 2.1(b), Theorem 3.1 and [1, Theorem 2.9 (i)↔(ii)↔(iii)] we have

$$\begin{aligned}
\mathfrak{p}\notin\bigcup_{i<n}\operatorname{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M, X)) &\Leftrightarrow \mathrm{H}^i_{\mathfrak{a}}(M, X)_{\mathfrak{p}}=0 \quad \forall i<n\\
&\Leftrightarrow \mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, X_{\mathfrak{p}})=0 \quad \forall i<n\\
&\Leftrightarrow \mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}+\operatorname{Ann}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}}(X_{\mathfrak{p}})=0 \quad \forall i<n\\
&\Leftrightarrow \mathrm{H}^i_{\mathfrak{a}+\operatorname{Ann}_R M}(X)_{\mathfrak{p}}=0 \quad \forall i<n\\
&\Leftrightarrow \mathfrak{p}\notin\bigcup_{i<n}\operatorname{Supp}_R(\mathrm{H}^i_{\mathfrak{a}+\operatorname{Ann}_R M}(X))
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{p}\notin\bigcup_{i<n}\operatorname{Supp}_R(\mathrm{H}^i_{\mathfrak{a}+\operatorname{Ann}_R M}(X)) &\Leftrightarrow \mathrm{H}^i_{\mathfrak{a}+\operatorname{Ann}_R M}(X)_{\mathfrak{p}}=0 \quad \forall i<n\\
&\Leftrightarrow \mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}+\operatorname{Ann}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}}(X_{\mathfrak{p}})=0 \quad \forall i<n\\
&\Leftrightarrow \operatorname{Ext}^i_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}}+\operatorname{Ann}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}, X_{\mathfrak{p}})=0 \quad \forall i<n\\
&\Leftrightarrow \operatorname{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}}M_{\mathfrak{p}}, X_{\mathfrak{p}})=0 \quad \forall i<n\\
&\Leftrightarrow \operatorname{Ext}^i_R(M/\mathfrak{a}M, X)_{\mathfrak{p}}=0 \quad \forall i<n\\
&\Leftrightarrow \mathfrak{p}\notin\bigcup_{i<n}\operatorname{Supp}_R(\operatorname{Ext}^i_R(M/\mathfrak{a}M, X))
\end{aligned}$$

as we desired.

(b) Use this fact that the support of a finite R–module is a closed set, then use [5, Theorem 3.3.1] and part (a). □

The vanishing of generalized local cohomology modules from upper bounds needs special condition and in all of them M must has finite projective dimension for example see [21, Theorem 2.5 and 3.7] , [8, Theorem 3.1] or corollary 2.13 (a). However, the following result together with corollary 3.6 (a) shows that there is a union of finitely many supports of generalized local cohomology modules so that the other supports can be viewed as its subset even if M has infinite projective dimension.

Corollary 3.7. *Let M be a finite R–module and $n = \mathrm{ara}(\mathfrak{a} + \mathrm{Ann}_R M)$ then*

$$\mathrm{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M, X)) \subseteq \bigcup_{i \leqslant n} \mathrm{Supp}_R(\mathrm{H}^i_{\mathfrak{a}+\mathrm{Ann}_R M}(X))$$

for all $i \geqslant 0$.

Cuong and Hoang in [10, Theorem 2.4] proved the following theorem when X is a finite module and $n = \mathrm{grade}(\mathfrak{a} + \mathrm{Ann}_R M, X)$. Our proof is more general and its method is completely different.

Theorem 3.8. *Let M be a finite R–module and $\mathfrak{a}$ be an ideal of R. Let X be an arbitrary R–module such that $\mathrm{H}^i_{\mathfrak{a}}(M, X) = 0$ for all $i < n$ (in particular, when X is finite and $n = \mathrm{grade}(\mathfrak{a} + \mathrm{Ann}_R M, X)$) then*

$$\mathrm{Ass}(\mathrm{H}^n_{\mathfrak{a}}(M, X)) = \mathrm{Ass}(\mathrm{Ext}^n_R(M/\mathfrak{a}M, X))$$

Proof. We first prove by using induction on n that

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^n_{\mathfrak{a}}(M, X)) \cong \mathrm{Ext}^n_R(M/\mathfrak{a}M, X).$$

Let $n = 0$, since $\Gamma_{\mathfrak{a}}(M, X) \cong \mathrm{Hom}_R(M, \Gamma_{\mathfrak{a}}(X))$ so we have

$$\begin{aligned} \mathrm{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M, X)) &\cong \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{Hom}_R(M, \Gamma_{\mathfrak{a}}(X)) \\ &\cong \mathrm{Hom}_R(R/\mathfrak{a} \otimes_R M, \Gamma_{\mathfrak{a}}(X)) \\ &\cong \mathrm{Hom}_R(M/\mathfrak{a}M, \Gamma_{\mathfrak{a}}(X)) \\ &\cong \mathrm{Hom}_R(M/\mathfrak{a}M, X). \end{aligned}$$

Suppose that $n > 0$ and that $n - 1$ is settled. Let E be an injective hull of X and $L = E/X$. By hypothesis $\Gamma_{\mathfrak{a}}(M, X) = 0$, it follows by 3.1[(i)⇔(ii)] that $\Gamma_{\mathfrak{a}+Ann(M)}(X) = 0$. Thus $\Gamma_{\mathfrak{a}+Ann(M)}(E) = 0$. Therefore $\mathrm{Hom}_R(R/(\mathfrak{a} + Ann(M)), E) = 0$ and again by using 3.1[(i)⇔(ii)] we have $\Gamma_{\mathfrak{a}}(M, E) = 0$. So by [1, Theorem 2.9 (ii)⇔(iii)] it follows that $\mathrm{Hom}_R(M/\mathfrak{a}M, E) = 0$. Applying the derived functors of $\Gamma_{\mathfrak{a}}(M, -)$ and $\mathrm{Hom}_R(M/\mathfrak{a}M, -)$ to the exact sequence $0 \to X \to E \to L \to 0$, we obtain, for all $i > 0$, the isomorphisms:

$$\mathrm{H}^{i-1}_{\mathfrak{a}}(M, L) \cong \mathrm{H}^i_{\mathfrak{a}}(M, X) \quad \text{and} \quad \mathrm{Ext}^{i-1}_R(M/\mathfrak{a}M, L) \cong \mathrm{Ext}^i_R(M/\mathfrak{a}M, X).$$

Note that, from the above isomorphisms we get $\mathrm{H}^{i-1}_{\mathfrak{a}}(M, L) = 0$ for all $i < n-1$. Therefore by induction hypothesis

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{n-1}_{\mathfrak{a}}(M, L)) \cong \mathrm{Ext}^{n-1}_R(M/\mathfrak{a}M, L).$$

Now using the above isomorphisms we have

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^n_{\mathfrak{a}}(M, X)) \cong \mathrm{Ext}^n_R(M/\mathfrak{a}M, X).$$

Since $\mathrm{H}^n_{\mathfrak{a}}(M, X)$ is $\mathfrak{a}$–torsion it follows

$$\mathrm{Ass}(\mathrm{H}^n_{\mathfrak{a}}(M, X)) = \mathrm{Ass}(\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^n_{\mathfrak{a}}(M, X))) = \mathrm{Ass}(\mathrm{Ext}^n_R(M/\mathfrak{a}M, X))$$

which terminates the induction argument. The proof is completed.

□

In [9, Theorem 4.5], part (a) of the following corollary has been proved when X is a finite module and R is local ring. We generalize the statement by removing all conditions on X and R. Moreover the finiteness of $\mathrm{Ass}_R(\mathrm{H}^n_{\mathfrak{a}}(M,X))$ under the assumption in (c) has been proved in [9, Theorem 4.5](in local case) and [17, Theorem 2.4] when X is a finite module. See also [15, Theorem 3.1]. Here we prove, for an arbitrary module X under the assumption in (c), that the finiteness of $\mathrm{Ass}_R(\mathrm{H}^n_{\mathfrak{a}}(M,X))$ is equivalent to the finiteness of $\mathrm{Ass}_R(\mathrm{Ext}^n_R(M/\mathfrak{a}M,X))$.

Corollary 3.9. *Let X be an R–module and let n be a positive integer.*
Put $P_n = \bigcup_{i=0}^{n-1} \mathrm{Supp}_R(\mathrm{Ext}^i_R(M/\mathfrak{a}M,X))$. Then

(a) $\mathrm{Ass}_R(\mathrm{Ext}^n_R(M/\mathfrak{a}M,X)) \cup P_n = \mathrm{Ass}_R(\mathrm{H}^n_{\mathfrak{a}}(M,X)) \cup P_n$.
(b) $\mathrm{Ass}_R(\mathrm{H}^n_{\mathfrak{a}}(M,X)) \subset \mathrm{Ass}_R(\mathrm{Ext}^n_R(M/\mathfrak{a}M,X)) \cup P_n$.
(c) *If $\mathrm{H}^i_{\mathfrak{a}}(M,X)$ has finite support for all $i < n$, then $\mathrm{Ass}_R(\mathrm{H}^n_{\mathfrak{a}}(M,X))$ is a finite set if and only if $\mathrm{Ass}_R(\mathrm{Ext}^n_R(M/\mathfrak{a}M,X))$ is a finite set.*

Proof. (a) If $\mathfrak{p} \notin \bigcup_{i<n} \mathrm{Supp}_R(\mathrm{H}^i_{\mathfrak{a}}(M,X))$, then by 3.8

$$\mathrm{Ass}_{R_\mathfrak{p}}(\mathrm{Ext}^n_{R_\mathfrak{p}}(M_\mathfrak{p}/\mathfrak{a}M_\mathfrak{p},X_\mathfrak{p})) = \mathrm{Ass}_{R_\mathfrak{p}}(\mathrm{H}^n_{\mathfrak{a}R_\mathfrak{p}}(M_\mathfrak{p},X_\mathfrak{p})).$$

Now it is clear that, $\mathfrak{p}$ is not in the left side if and only if it is not in the right side by corollary 3.6 (a).

(b) Follows from (a).

(c) Use part (a) and corollary 3.6 (a). □

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