COHEN-MACAULAYNESS OF GENERICALLY COMPLETE INTERSECTION MONOMIAL IDEALS

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Abstract. In this paper we try to understand which generically complete intersection monomial ideals with fixed radical are Cohen-Macaulay. We are able to give a complete characterization for a special class of simplicial complexes, namely the Cohen-Macaulay complexes without cycles in codimension 1. Moreover, we give sufficient conditions when the square-free monomial ideal has minimal multiplicity.

1. Introduction

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and Δ be a simplicial complex on $V = \{v_1, \ldots, v_n\}$. The Stanley-Reisner ideal of Δ is:

$$I_\Delta = \bigcap_{F \in \Im(\Delta)} (x_i : \ v_i \notin F),$$

where $\Im(\Delta)$ is the set of facets of Δ. Given an ideal $J \subset R$ such that $\sqrt{J} = I_\Delta$, it turns out that R/I_Δ is Cohen-Macaulay whenever R/J is Cohen-Macaulay. Of course the converse is not true, so in this paper we are going to study the following problem: *How to discribe a family of ideals J such that R/J is Cohen-Macaulay and $\sqrt{J} = I_\Delta$?*

We restrict our attention on monomial ideals J. This problem has been already considered, for instance see the paper of Miller, Sturmfels and Yanagawa [MSY]. Also, independently and with different proofs, Minh and Trung in [MT] and the second author of this paper in [Va], characterized the simplicial complexes Δ for which all the symbolic powers of I_Δ are Cohen-Macaulay. However we consider a different type of family of monomial ideals with a fixed radical, namely the

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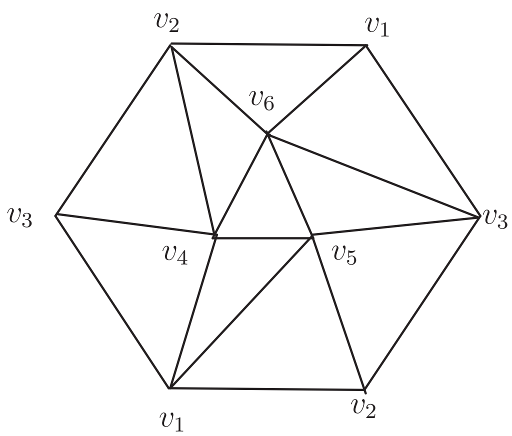
generically complete intersection monomial ideals:

$$I_{\Delta(\alpha)} = \bigcap_{F \in \Im(\Delta)} (x_i^{\alpha_i(F)} : v_i \notin F),$$

where $\alpha_i(F)$ are positive integers. In [HTT], Herzog, Takayama and Terai characterized those simplicial complexes for which $R/I_{\Delta(\alpha)}$ is Cohen-Macaulay for any choice of α. It turns out that such complexes are very rare.

The purpose of this paper is to give conditions, depending on Δ, on the values $\alpha_i(F)$ in such a way that $R/I_{\Delta(\alpha)}$ is Cohen-Macaulay. It is easy to see that if $\alpha_i(F)$ is constant for any i, then the depth of $R/I_{\Delta(\alpha)}$ is equal to the depth of R/I_Δ. However, even if R/I_Δ is Cohen-Macaulay, $R/I_{\Delta(\alpha)}$ might not be Cohen-Macaulay for "simple" functions α. For instance consider the triangulation of the projective plane in the picture below (all the visible triangles are actually faces):



Simplicial complex Δ

With the help of CoCoA [CT] we can check that, for any vertex i_0 and any facet F_0 not containing i_0, we have $R/I_{\Delta(\alpha)}$ is not Cohen-Macaulay for the following α:

$$\alpha_i(F) = \begin{cases} 2 & \text{if } i = i_0, F = F_0, \\ 1 & \text{otherwise} \end{cases}$$

In this paper we are going to face the above problem for a special kind of simplicial complexes, namely the *Cohen-Macaulay complexes without cycles in codimension 1*, which we are going to introduce in Definition 2.3. In this case we give necessary and sufficient conditions on α for $R/I_{\Delta(\alpha)}$ being Cohen-Macaulay. Without entering into the details, every α_i has to be weakly decreasing along particular shellings (Theorem 3.5).

By similar tools, in the last section we give sufficient conditions on α for $R/I_{\Delta(\alpha)}$ to be Cohen-Macaulay when R/I_{Δ} has minimal multiplicity (Theorem 4.8). We will also notice that such conditions are, in general, not necessary.

Some results in this paper have been conjectured and confirmed by using the computer algebra package CoCoA [CT]. We wish to thank Aldo Conca for suggesting the problem. We want also to thank Satoshi Murai for introducing us to Example 4.9.

2. Cohen-Macaulay complex without cycles in codimension 1

For general facts about commutative algebra and combinatorics see the books of Bruns and Herzog [BH], Björner [B], Stanley [St2] or Miller and Sturmfels [MS].

Let $V = \{v_1, \dots, v_n\}$ be a finite set. A *simplicial complex* Δ on V is a collection of subsets of V such that $F \in \Delta$ whenever $F \subset G$ for some $G \in \Delta$, and such that $\{v_i\} \in \Delta$ for $i = 1, \dots, n$. Given finite sets $F_1, \dots, F_m$ the simplicial complex on $V = \cup_{i=1}^{m} F_i$ generated by them, i.e. consisting in all the subsets of any F_i, is denoted by $< F_1, \dots, F_m >$. The elements of a simplicial complex Δ are its *faces*. Maximal faces under inclusion are called *facets*. The set of facets is denoted by $\Im(\Delta)$. The *dimension of a face* F, $\dim F$, is the number $|F| - 1$. The *dimension of* Δ is:

$$\dim\Delta = \max\{\dim F : F \in \Delta\}.$$

A simplicial complex is *pure* if all its facets are of the same dimension. It is called *strongly connected* if each pair $F, G \in \Im(\Delta)$ can be connected by a *strongly connected sequence*, i.e. a sequence of facets $F = F_0, F_1, \dots, F_k = G$ such that $|F_i \cap F_{i+1}| = d - 1$ for all $i = 0, \dots, k-1$, where $\dim \Delta = d - 1$. We will say that Δ is *shellable* if it is pure and it can be given a linear order $F_1, \dots, F_m$ to the facets of Δ in a way that $< F_i > \cap < F_1, \dots, F_{i-1} >$ is generated by a non-empty set of maximal proper faces of $< F_i >$ for all $i = 2, \dots, m$. Such a linear order is called a *shelling* of Δ. The *link* of a face F of Δ is the simplicial complex $\mathrm{lk}_{\Delta}(F) = \{G : F \cup G \in \Delta, F \cap G = \emptyset\}$.

The relations between commutative algebra and combinatorics come from the *Stanley-Reisner ideal* of Δ, denoted by I_{Δ}: it is the ideal generated by all monomials $x_{i_1} \dots x_{i_s}$ such that $\{v_{i_1}, \dots, v_{i_s}\} \notin \Delta$. If the *Stanley-Reisner ring* $k[\Delta] = k[x_1, \dots, x_n]/I_{\Delta}$ is a Cohen-Macaulay ring, then Δ is called a *Cohen-Macaulay complex*.

The following are well known facts:

- Δ is shellable $\Rightarrow$ Δ is Cohen-Macaulay $\Rightarrow$ Δ is pure.
- If Δ is Cohen-Macaulay, then Δ and $\mathrm{lk}_{\Delta}(F)$ are strongly connected for all faces F of Δ.

Lemma 2.1. *Let Δ be a $(d-1)$-dimensional Cohen-Macaulay complex and $F, G \in \Im(\Delta)$ with $|F \cap G| < d-1$. Then, there exists a facet $H \in \Im(\Delta)$ such that $(F \cap G) \subset (H \cap G)$ and $|H \cap G| = d-1$.*

Proof. From what said above $\mathrm{lk}_\Delta(F \cap G)$ is strongly connected. Set $G' = G \setminus (F \cap G)$ and $F' = F \setminus (G \cap F)$. There exists a strongly connected sequence $F' = F'_0, F'_1, \ldots, F'_k = G'$ of facets of $\mathrm{lk}_\Delta(F \cap G)$. Then it is enough to set $H = F'_{k-1} \cup (F \cap G)$. The lemma is proved. □

Let F be a face of Δ. Denote by B_F the ideal $(x_i : v_i \notin F)$. Lemma 2.1 yields the useful corollary below.

Corollary 2.2. *Let Δ be a $(d-1)$-dimensional Cohen-Macaulay complex with $\Im(\Delta) = \{F_1, \ldots, F_m\}$. Then, for all $i = 1, \ldots, m$,*

$$\bigcap_{j \neq i} B_{F_j} + B_{F_i} = \bigcap_{\substack{j \neq i \\ |F_j \cap F_i| = d-1}} B_{F_j \cap F_i}.$$

Proof. For $i = 1, \ldots, m$ we have

$$\bigcap_{j \neq i} B_{F_j} + B_{F_i} = \bigcap_{j \neq i} (B_{F_j} + B_{F_i}) = \bigcap_{j \neq i} (B_{F_j \cap F_i}).$$

Using Lemma 2.1, we have the corollary. □

Definition 2.3. Let Δ be a $(d-1)$-dimensional pure simplicial complex. We recall that the facet graph of Δ (see White [Wh]), denoted by $G(\Delta)$, is defined as follow:

- The set of vertices is $V(G(\Delta)) = \Im(\Delta)$,
- The set of egdes is

$$E(G(\Delta)) = \{\{F, G\} : F, G \in \Im(\Delta) \text{ and } |F \cap G| = d-1\}.$$

Remark 2.4. *Notice that a pure simplicial complex Δ is strongly connected if and only if $G(\Delta)$ is connected.*

We say that Δ is a *Cohen-Macaulay complex without cycles in codimension 1* if Δ is Cohen-Macaulay and $G(\Delta)$ is a tree.

Lemma 2.5. *Let Δ be a $(d-1)$-dimensional Cohen-Macaulay complex without cycles in codimension 1 and $F_1, \ldots, F_k$ be a strongly connected sequence with $k \geq 2$. Then we have $(F_k \cap F_1) \subset (F_2 \cap F_1)$.*

Proof. We can assume $F_1 = \{v_1, \ldots, v_d\}, F_2 = \{v_2, \ldots, v_{d+1}\}$ and $k > 2$. Because $G(\Delta)$ is a tree, $|F_1 \cap F_k| < d-1$. If $(F_k \cap F_1) \not\subset (F_2 \cap F_1)$, then $v_1 \in F_k$. Moreover, we have $\mathrm{lk}_\Delta\{v_1\}$ is strongly connected. Set $F'_1 = F_1 \setminus \{v_1\}$ and $F'_k = F_k \setminus \{v_1\}$. There exists a sequence of facets of $\mathrm{lk}_\Delta\{v_1\}$, namely $F'_1, F'_{t_1}, \ldots, F'_{t_h}, F'_k$, such that $|F'_1 \cap F'_{t_1}| = |F'_{t_1} \cap F'_{t_2}| = \cdots = |F'_{t_h} \cap F'_k| = d-2$. So we have the strongly connected sequence $F_1, F_{t_1}, \ldots, F_{t_h}, F_k$, with $F_{t_j} = \{v_1\} \cup F'_{t_j}$ for all $j = 1, \ldots, h$. On the other hand, since $G(\Delta)$ is a tree, then the sequence $F_1, F_{t_1}, \ldots, F_{t_h}, F_k$

coincides with the sequence $F_1, F_2, \ldots, F_k$. So $F_2 = \{v_1\} \cup F'_{t_1}$. This is a contradiction. $\square$

Corollary 2.6. *A Cohen-Macaulay complex without cycles in codimension 1 is shellable.*

Proof. Let Δ be a Cohen-Macaulay complex without cycles in codimension 1. Because $G(\Delta)$ is a tree, we can choose a linear order $F_1, \ldots, F_m$ over $\Im(\Delta)$ such that F_j is a free vertex of $G(\Delta)_{|\{F_1,\ldots,F_j\}}$, i.e., there exists only one edge of $G(\Delta)_{|\{F_1,\ldots,F_j\}}$ which contains F_j. By using Lemma 2.5 and induction on m, it is easy to show that $F_1, \ldots, F_m$ is a shelling of Δ. Hence, Δ is shellable. $\square$

Lemma 2.7. *Let $F_1, \ldots, F_m$ be a shelling of a Cohen-Macaulay complex Δ without cycles in codimension 1. Then F_m is a free vertex of $G(\Delta)$.*

Proof. If F_m is not a free vertex of $G(\Delta)$, then there exist distinct numbers $h, k < m$ such that $|F_h \cap F_m| = |F_k \cap F_m| = d - 1$, where $\dim(\Delta) = d - 1$. But $< F_1, \ldots, F_{m-1} >$ is shellable too. In particular, it is strongly connected. Then there exists a strongly connected sequence $F_h, F_{t_1}, \ldots, F_{t_s}, F_k$, with each $t_i < m$. Therefore we have a cycle $F_h, F_{t_1}, \ldots, F_{t_s}, F_k, F_m, F_h$ in $G(\Delta)$, a contradiction. $\square$

Definition 2.8. Let Δ be a $(d-1)$-dimensional pure simplicial complex. For any $i = 1, \ldots, n$ we define the graph $G^i(\Delta)$ as follow:

- The set of vertices is $V(G^i(\Delta)) = \{V_i\} \cup \{F \in \Im(\Delta) : v_i \notin F\}$, where V_i is a new vertex.
- The set of egdes is

$$E(G^i(\Delta)) = \{\{F, G\} : |F \cap G| = d - 1\} \cup$$

$$\{\{V_i, F\} : \text{ there exists a facet } G \ni v_i \text{ and } |G \cap F| = d - 1\}.$$

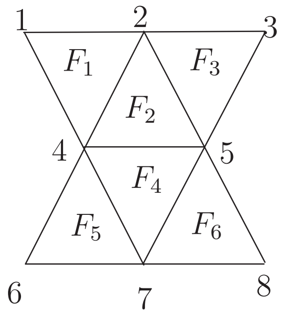
The graph $G^i(\Delta)$ is called the v_i-*graph* of Δ.

Remark 2.9. *If Δ is a Cohen-Macaulay complex, $G(\Delta)$ and $G^i(\Delta)$ are connected for $i = 1, \ldots, n$.*

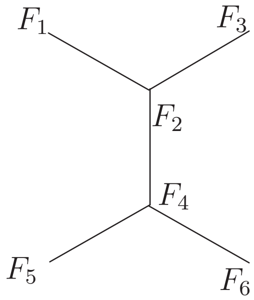
Lemma 2.10. *Let Δ be a Cohen-Macaulay complex without cycles in codimension 1. Then $G^i(\Delta)$ is a tree for all $i = 1, \ldots, n$.*

Proof. Because $G(\Delta)$ is a tree, $G^i(\Delta)$ is not a tree if and only if there exists a strongly connected sequence of facets $F_1, \ldots, F_k$ such that $v_i \in F_1, F_k$ and $v_i \notin F_j$ for $j = 2, \ldots, k - 1$. But by Lemma 2.5 we have $(F_k \cap F_1) \subset (F_2 \cap F_1)$. The proof is completed. $\square$

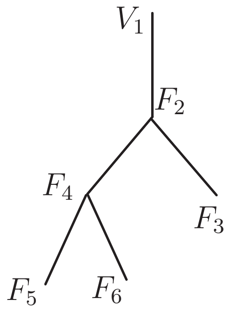
Example 2.11. Consider the following simplicial complex Δ:



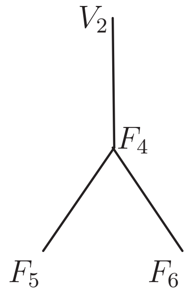
Simpilicial complex Δ



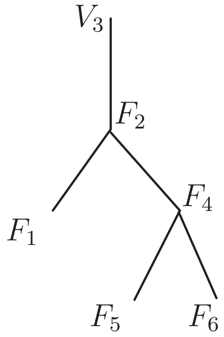
Graph $G(\Delta)$



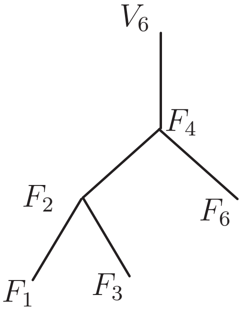
Graph $G^1(\Delta)$



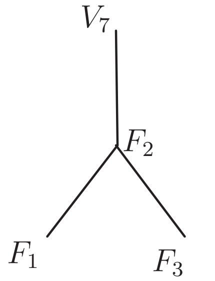
Graph $G^2(\Delta)$



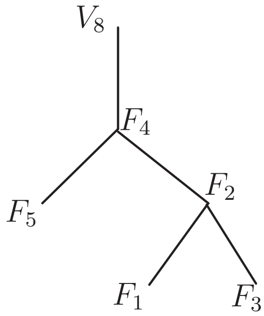
Graph $G^3(\Delta)$



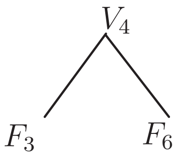
Graph $G^6(\Delta)$



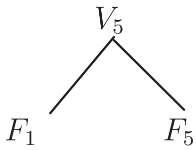
Graph $G^7(\Delta)$



Graph $G^8(\Delta)$



Graph $G^4(\Delta)$



Graph $G^5(\Delta)$

3. The Cohen-Macaulayness for a simplicial complex without cycles in codimension 1

Throughout this section, Δ will be a $(d-1)$-dimensional Cohen-Macaulay complex without cycles in codimension 1. Moreover the set of its facets will be $\Im(\Delta) = \{F_1, \ldots, F_m\}$. The Stanley-Reisner ideal of Δ is:

$$I_\Delta = \bigcap_{j=1}^{m} (x_i : \ v_i \notin F_j).$$

For $i = 1, \ldots, n$, let $\alpha_i = (\alpha_i(j) : j \in \{1, \ldots, m\}$ and $v_i \notin F_j)$ be positive integer vectors. Set $Q_j = (x_i^{\alpha_i(j)} : v_i \notin F_j)$ for all $j = 1, \ldots, m$ and define the following ideal:

$$I_{\Delta(\alpha)} = \bigcap_{j=1}^{m} Q_j.$$

Obviously, Q_j is the B_{F_j}-primary component of $I_{\Delta(\alpha)}$ and $\sqrt{I_{\Delta(\alpha)}} = I_\Delta$.

For any vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ denote by $\Delta(\alpha)_\mathbf{a}$ the subcomplex of Δ with the set of facets

$$\Im(\Delta(\alpha)_\mathbf{a}) = \{F_j \in \Im(\Delta) | a_i < \alpha_i(j) \text{ for all } i \text{ such that } v_i \notin F_j\}.$$

By [MT, Theorem 1.6], we have:

Theorem 3.1. *$I_{\Delta(\alpha)}$ is Cohen-Macaulay if and only if $\Delta(\alpha)_\mathbf{a}$ is a Cohen-Macaulay complex for all $\mathbf{a} \in \mathbb{N}^n$.*

Albeit Theorem 3.1 gives necessary and sufficient conditions for $I_{\Delta(\alpha)}$ to be Cohen-Macaulay, we would like to give a simpler characterization on the numbers $\alpha_i(j)$. By some experiments with CoCoA [CT] on some concrete examples, we came to the followings:

Definition 3.2. Let G be a tree. For any vertex v of G, we consider the directed graph (G, v) as follow:

- The set of vertices is $V((G, v)) = V(G)$.
- The pair $(u_2, u_1) \in E((G, v))$ iff there is a path $v, u_k, \ldots, u_2, u_1$ in G. We will call it *a directed edge* of (G, v).

By Lemma 2.10, $G^i(\Delta)$ is a tree for all $i = 1, \ldots, n$. We have the following definition:

Definition 3.3. A vector $\alpha_i = (\alpha_i(j) : v_i \notin F_j)$ is called $G^i(\Delta)$-*satisfying* if $\alpha_i(h) \geq \alpha_i(k)$ for all directed edges (F_h, F_k) of $(G^i(\Delta), V_i)$. Moreover, $\alpha = (\alpha_i(j))$ is called Δ-*satisfying* if α_i is $G^i(\Delta)$-satisfying for all $i = 1, \ldots, n$.

Lemma 3.4. *Let $F_1, \ldots, F_m$ be a shelling of Δ. If α is Δ-satisfying, then there exists $i \in \{1, \ldots, n\}$ and a positive integer s such that*

$$\bigcap_{j=1}^{m-1} Q_j + Q_m = (x_i^s) + Q_m.$$

Proof. By Lemma 2.7, F_m is a free vertex of $G(\Delta)$. We can assume $F_m = \{v_1, \ldots, v_d\}$ and there exists a facet $F_h = \{v_2, \ldots, v_{d+1}\}$ with $F_j \cap F_m \subsetneq F_h \cap F_m$ for all $j \neq h, m$, see Lemma 2.5. So $F_j \cap F_m$ is a proper subset of $\{v_2,\dots,v_d\}$ for all $j \neq h,m$. Notice that for each $i > d+1$, the pair (F_h, F_m) is a directed edge of $(G^i(\Delta), V_i)$. Then, because α is Δ-satisfying, we have $Q_h+Q_m = (x_1^{\alpha_1(h)})+Q_m$. Moreover, $\alpha_1(h) \geq \alpha_1(j)$ for all $j \neq h,m$, since α_1 is $G^1(\Delta)$-satisfying. Hence $(x_1^{\alpha_1(h)}) \subset Q_j$ for all $j \neq h,m$. So $Q_j+Q_m \supset Q_h+Q_m$ for all $j \neq h,m$. We have:

$$\bigcap_{j=1}^{m-1} Q_j+Q_m = \bigcap_{j=1}^{m-1}(Q_j+Q_m) \supseteq (Q_h \cap Q_m) = (x_1^{\alpha_1(h)})+Q_m \supseteq \bigcap_{j=1}^{m-1} Q_j+Q_m.$$

So the Lemma is proved. □

Theorem 3.5. *Let Δ be a Cohen-Macaulay complex without cycles in codimension 1. Then $I_{\Delta(\alpha)}$ is Cohen-Macaulay if and only if α is Δ-satisfying.*

Proof. We choose a shelling $F_1,\dots,F_m$ of Δ. We denote by Δ_j the simplicial complex with the set of facets $\Im(\Delta_j) = \{F_1,\dots,F_j\}$ and $I_{\Delta_j(\alpha)}$ the ideal $\bigcap_{t=1}^{j} Q_t$. We will prove the theorem by induction on m. This is obvious for $m=1$. We assume that the assertion is true for $j=1,\dots,m-1$. By Lemma 2.7 we have F_m is a free vertex of $G(\Delta)$. So F_m is a free vertex of $G^i(\Delta)$ for all $i=1,\dots,n$ whenever F_m is a vertex of $G^i(\Delta)$. If α_i is $G^i(\Delta)$-satisfying for all $i=1,\dots,n$, then $(\alpha_i)_{|\Delta_{m-1}}$ is $G^i(\Delta_{m-1})$-satisfying for all $i=1,\dots,n$. By induction, we have $R/I_{\Delta_k(\alpha)}$ are d-dimensional Cohen-Macaulay rings for all $k=1,\dots,m-1$. We have the following exact sequence:

(3.1)

$$0 \to R/I_{\Delta_m(\alpha)} \xrightarrow{f} R/I_{\Delta_{m-1}(\alpha)} \oplus R/Q_m \xrightarrow{g} R/(I_{\Delta_{m-1}(\alpha)}+Q_m) \to 0.$$

By using Lemma 3.4 we have $R/(I_{\Delta_{m-1}(\alpha)}+Q_m)$ is a $(d-1)$-dimensional Cohen-Macaulay ring. Because $R/I_{\Delta_{m-1}(\alpha)}$ and R/Q_m are Cohen-Macaulay rings of dimension d, we have that $R/I_{\Delta_m(\alpha)}$ is d-dimensional Cohen-Macaulay ring by [BH, Proposition 1.2.9].

Conversely, if there exists an index i such that α_i is not $G^i(\Delta)$-satisfying, then there exists a directed edge (F_h, F_k) in $G^i(\Delta)$ such that $\alpha_i(k) > \alpha_i(h)$. We choose the vector $\mathbf{a}=(a_1,\dots,a_n)$ with

$$a_t = \begin{cases} \alpha_i(h) & \text{if } t=i, \\ 0 & \text{otherwise.} \end{cases}$$

It turns out that if a facet F of Δ contains the vertex v_i, then $F \in \Im(\Delta(\alpha)_{\mathbf{a}})$. Moreover, $F_k \in \Im(\Delta(\alpha)_{\mathbf{a}})$ and $F_h \notin \Im(\Delta(\alpha)_{\mathbf{a}})$. So, $\Delta(\alpha)_{\mathbf{a}}$ is not strongly connected. Hence, $\Delta(\alpha)_{\mathbf{a}}$ is not Cohen-Macaulay. This is a contradiction with Theorem 3.1. □

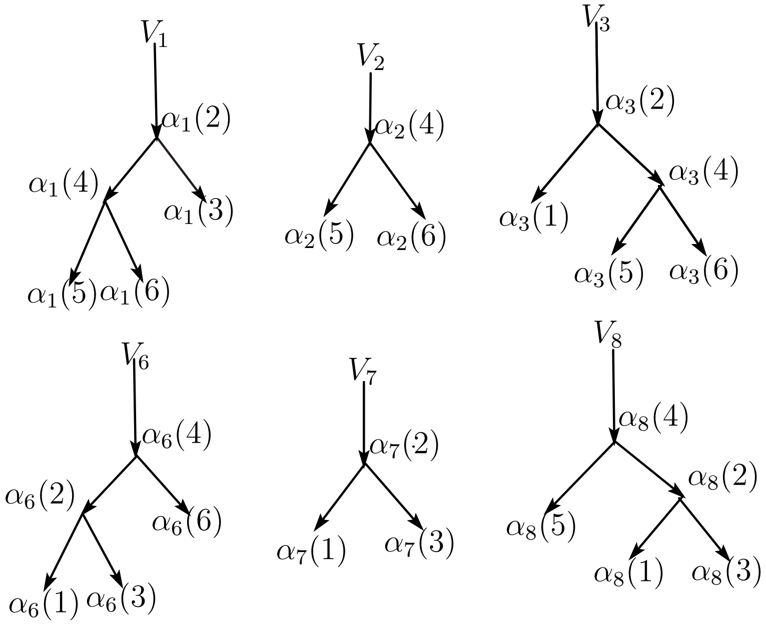
Example 3.6. Let Δ be the simplicial complex of Example 2.11.

$$I_\Delta = (x_3,x_5,x_6,x_7,x_8) \cap (x_1,x_3,x_6,x_7,x_8) \cap (x_1,x_4,x_6,x_7,x_8)$$
$$\cap (x_1,x_2,x_3,x_6,x_8) \cap (x_1,x_2,x_3,x_5,x_8) \cap (x_1,x_2,x_3,x_4,x_6).$$

The ideal $I_{\Delta(\alpha)}$ is:

$$(x_3^{\alpha_3(1)}, x_5^{\alpha_5(1)}, x_6^{\alpha_6(1)}, x_7^{\alpha_7(1)}, x_8^{\alpha_8(1)}) \cap (x_1^{\alpha_1(2)}, x_3^{\alpha_3(2)}, x_6^{\alpha_6(2)}, x_7^{\alpha_7(2)}, x_8^{\alpha_8(2)})$$
$$\cap (x_1^{\alpha_1(3)}, x_4^{\alpha_4(3)}, x_6^{\alpha_6(3)}, x_7^{\alpha_7(3)}, x_8^{\alpha_8(3)}) \cap (x_1^{\alpha_1(4)}, x_2^{\alpha_2(4)}, x_3^{\alpha_3(4)}, x_6^{\alpha_6(4)}, x_8^{\alpha_8(4)})$$
$$\cap (x_1^{\alpha_1(5)}, x_2^{\alpha_2(5)}, x_3^{\alpha_3(5)}, x_5^{\alpha_5(5)}, x_8^{\alpha_8(5)}) \cap (x_1^{\alpha_1(6)}, x_2^{\alpha_2(6)}, x_3^{\alpha_3(6)}, x_4^{\alpha_4(6)}, x_6^{\alpha_6(6)}).$$

Theorem 3.5 tells us that $I_{\Delta(\alpha)}$ is Cohen-Macaulay if and only if $\alpha_4(3)$, $\alpha_4(6)$, $\alpha_5(1)$ and $\alpha_5(5)$ are arbitrary positive integers and $\alpha_i(j)$ are positive integers which satisfy the order as in the following figure:



Of course, we can define $I_{\Delta(\alpha+\mathbf{1})}$ for any vector $\alpha \in (\mathbb{N}^n)^m$ in the obvious way. For such an α, we say that it is Δ-satisfying if the collection of numbers $((\alpha_i)_j + 1)$ where $i = 1, \dots, n$ and $v_i \notin F_j$ is Δ-satisfying.

Corollary 3.7. *Let Δ be a Cohen-Macaulay complex without cycles in codimension 1 and α, β be vectors in $(\mathbb{N}^n)^m$ such that $I_{\Delta(\alpha+\mathbf{1})}, I_{\Delta(\beta+\mathbf{1})}$ are Cohen-Macaulay, then $I_{\Delta(\alpha+\beta+\mathbf{1})}$ is Cohen-Macaulay.*

Proof. Because $I_{\Delta(\alpha+\mathbf{1})}$ and $I_{\Delta(\beta+\mathbf{1})}$ are Cohen-Macaulay, then α and β are Δ-satisfying. Thus, $\alpha + \beta$ is Δ-satisfying. So $I_{\Delta(\alpha+\beta+\mathbf{1})}$ is Cohen-Macaulay. □

Corollary 3.7 says that, if Δ is a Cohen-Macaulay complex without cycles in codimension 1, the set

$$S = \{\alpha \in (\mathbb{N}^n)^m \ : \ I_{\Delta(\alpha+\mathbf{1})} \text{ is Cohen-Macaulay}\}$$

is an affine semigroup. It is possible to describe a finite system of generators of S. Fixed $i \in \{1, \dots, n\}$, the idea is to pick the vectors $\alpha_H = ((\alpha_p)_q)$, for any poset ideal H of $(G^i(\Delta), v_i)$, such that the nonzero entries of α are just in α_i and

$$(\alpha_i)_j = \begin{cases} 1 & \text{if } F_j \in G^i(\Delta) \setminus H, \\ 0 & \text{otherwise} \end{cases}.$$

Remark 3.8. *The conclusion of Corollary 3.7 is not true for general complexes. For instance, consider the square*

$$< \{1,2\}, \{2,3\}, \{3,4\}, \{4,1\} > .$$

Corollary 3.9. *Let Δ be a Cohen-Macaulay complex without cycles in codimension 1 and*

$$\alpha_i(j) = \begin{cases} a_i & \text{if } i \in H, j \in K, \\ 1 & \text{otherwise,} \end{cases}$$

where H is a subset of $[n]$, K is a subset of $[m]$ and a_i are integer numbers bigger than 1 for all $i \in H$. Then $I_{\Delta(\alpha)}$ is Cohen-Macaulay if and only if $G^i(\Delta)_{|\{V_i\} \cup \{F_j | j \in K\}}$ are trees for all $i \in H$.

Proof. If $G^i(\Delta)_{|\{V_i\} \cup \{F_j | j \in K\}}$ are trees for all $i \in H$, we have α_i is $G^i(\Delta)$-satisfying for all $i = 1, \ldots, n$. It implies that $I_{\Delta(\alpha)}$ is Cohen-Macaulay. Conversely, if $G^i(\Delta)_{|\{V_i\} \cup \{F_j | j \in K\}}$ is not a tree for some i, then α_i is not $G^i(\Delta)$-satisfying. Therefore we conclude by Theorem 3.5. $\square$

4. The Cohen-Macaulayness for a strongly connected quasi-tree

Let Δ be a $(d-1)$-dimensional simplicial complex. Denote by f_i the number of i-dimensional faces of Δ. The vector $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ is called *f-vector* of Δ. The Hilbert series of the Stanley-Reisner ring is:

$$H_{k[\Delta]}(t) = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^d},$$

where $s \leq d$. The finite sequence of integers $h(\Delta) = (h_0, h_1, \ldots, h_s)$ is called the *h-vector* of Δ. The multiplicity of the Stanley-Reisner ring is $e(k[\Delta]) = \sum_{i=0}^{s} h_i$. The h-vector and the f-vector of a simplicial complex are related by a formula. In particular, we have:

$$h_0 = 1, h_1 = f_0 - d \text{ and } \sum_{i=0}^{s} h_i = f_{d-1},$$

for instance see [BH, Corollary 5.1.9]. So, $e(k[\Delta]) \geq 1 + (n-d)$ for all Cohen-Macaulay simplicial complexes Δ. A Cohen-Macaulay simplicial complex has *minimal multiplicity* if $e(k[\Delta]) = 1 + (n-d)$.

We recall the following definition. The facet F of Δ is called a *leaf* of Δ if there exists a facet G such that $(H \cap F) \subseteq (G \cap F)$ for all $H \in \Im(\Delta)$. The facet G is called a *branch* of F. A simplicial complex Δ is called a *quasi-forest* if there exists a total order $\Im(\Delta) = \{F_1, \ldots, F_m\}$ such that F_i is a leaf of $< F_1, \ldots, F_i >$ for all $i = 1, \ldots, m$. This order is called a *leaf order* of the quasi-forest. A connected quasi-forest is called a *quasi-tree*. For properties about quasi-tree see the paper of the first author with Constantinescu [CN]. Maybe the following statement is already known. However we did not find it anywhere, so we prefer to include a proof here.

Proposition 4.1. *Let Δ be a simplicial complex. The following conditions are equivalent:*
(i) Δ is a strongly connected complex with minimal multiplicity;
(ii) Δ is a Cohen-Macaulay complex with minimal multiplicity;
(iii) Δ is a shellable complex with minimal multiplicity;
(iv) Δ is a strongly connected quasi-tree.

Proof. We assume Δ is a $(d-1)$-dimensional simplicial complex with n vertices and m facets.

If Δ is strongly connected, we build the facets order by choosing the facet F_i such that $< F_1, \ldots, F_i >$ is strongly connected for all $i = 1, \ldots, m$. We have

$$|F_i \setminus \bigcup_{j=1}^{i-1} F_j| \leq 1,$$

for all $i = 1, \ldots, m$. However, $e(k[\Delta]) = 1 + (n-d) = m$. So, $n = d + (m-1)$. This implies $|F_i \setminus \bigcup_{j=1}^{i-1} F_j| = 1$ for all $i = 2, \ldots, m$. By this fact, (i),(ii),(iii) and (iv) are easily seen to be equivalent. □

Notice that by Proposition 4.1 one can easily deduce that the notion of "strongly connected quasi-tree" coincides with the one of "tree" introduced in the paper of Jarrah and Laubenbacher [JL, Definition 4.4]. However, we do not call them trees because such a term is also used by other authors with a different meaning (for instance see the paper of Faridi [Fa, Definition 9]). An interesting consequence of Proposition 4.1 and [JL, Theorem 4.10] is that strongly connected quasi-trees are exactly the clique complexes of a chordal graph.

Remark 4.2. *(i) Δ is a Cohen-Macaulay complex without cycles in codimension 1 $\Rightarrow$ Δ is a strongly connected quasi-tree.*
(ii) The converse is not true. For example, $\Delta = < \{1,2\}, \{1,3\}, \{1,4\} >$.

Definition 4.3. Let Δ be a strongly connected quasi-tree with the leaf order $F_1, \ldots, F_m$. We define *a relation tree of* Δ, denoted by $T(\Delta)$, in the following way:

- The vertices of $T(\Delta)$ are the facets of Δ.
- The edges are obtained recursively as follows:
  - Take the leaf F_m of Δ and choose a branch G of F_m.
  - Set $\{F_m, G\}$ to be an edge of $T(\Delta)$.
  - Remove F_m from Δ and proceed with the remaining complex as before to determine the other edges of $T(\Delta)$.

Remark 4.4. *(i)The graph $T(\Delta)$ depends on the leaf order and the choice of the branch for each leaf. However it is always a tree.*
(ii) The tree $T(\Delta)$ is a spanning tree of $G(\Delta)$.
(iii) If Δ is a Cohen-Macaulay complex without cycles in codimension 1, then the relation tree of Δ is $G(\Delta)$.

Lemma 4.5. *Let Δ be a strongly connected quasi-tree with the relation tree $T(\Delta)$ and $F_1, F_2, \dots, F_k$ adjacent vertices in $T(\Delta)$. If the vertex $v \in F_1 \cap F_k$, then $v \in F_i$ for all $i = 1, \dots, k$.*

Proof. Let $G_1, \dots, G_m$ be the leaf order corresponding with the relation tree $T(\Delta)$ and $F_i = G_{t_i}$ for all $i = 1, \dots, k$. Because $F_1, F_2, \dots, F_k$ are adjacent vertices in $T(\Delta)$, for all $i = 1, \dots, k-1$ we have:

- If $t_i < t_{i+1}$, then F_i is a branch of F_{i+1}.
- If $t_i > t_{i+1}$, then F_{i+1} is a branch of F_i.

We have two following cases:
case 1: $t_1 < t_2 < \dots < t_k$. So, F_i is a branch of F_{i+1} for all $i = 1, \dots, k-1$. This implies $(F_1 \cap F_k) \subseteq (F_{k-1} \cap F_k)$, $(F_1 \cap F_{k-1}) \subseteq (F_{k-2} \cap F_{k-1})$,..., $(F_1 \cap F_3) \subseteq (F_2 \cap F_3)$. Hence, $v \in F_i$ for all $i = 1, \dots, k$.
case 2: $t_1 > t_2 > \dots > t_h < t_{h+1} < \dots < t_k$. We can assume $t_1 < t_k$, then t_k is the biggest number in $\{t_1, \dots, t_k\}$. So, $v \in F_1 \cap F_k \subseteq F_{k-1} \cap F_k$. This implies $v \in F_1 \cap F_{k-1}$. We continue with the pair (t_1, t_{k-1}), so on. Hence, $v \in F_i$ for all $i = 1, \dots, k$. □

For all $i = 1, \dots, n$, we define the graph $T^i(\Delta)$ with the set of vertices $V(T^i(\Delta)) = V(G^i(\Delta))$ and the set of edge $E(T^i(\Delta)) = E(G^i(\Delta)) \cap E(T(\Delta))$. By Lemma 4.5, we have:

Corollary 4.6. *With the above assumptions, $T^i(\Delta)$ are trees for all $i = 1, \dots, n$.*

We consider the directed trees $(T^i(\Delta), V_i)$.

Definition 4.7. Let Δ be a strongly connected quasi-tree and $\alpha = (\alpha_i(j))$ for $i = 1, \dots, n$ and j such that $v_i \notin F_j$. The collection α is called Δ-*satisfying* if there exists a relation tree $T(\Delta)$ such that if the directed edge $(F_h, F_k) \in E((T^i(\Delta), V_i))$ then $\alpha_i(h) \geq \alpha_i(k)$.

The proof of Lemma 3.4 works also if Δ is a strongly connected quasi-tree. So, arguing as in the proof of Theorem 3.5, we have:

Theorem 4.8. *Let Δ be a strongly connected quasi-tree and α be Δ-satisfying. Then, $I_{\Delta(\alpha)}$ is Cohen-Macaulay.*

The converse is not true. For example, let Δ be the strongly connected quasi-tree with the set of facets:

$$\Im(\Delta) = \{\{1,6\}, \{2,6\}, \{3,6\}, \{4,6\}, \{5,6\}\}.$$

The graph $G(\Delta)$ is the complete graph on $\{F_1, \dots, F_5\}$. The Stanley-Reisner ideal I_Δ is:

$$(x_2, x_3, x_4, x_5) \cap (x_1, x_3, x_4, x_5) \cap (x_1, x_2, x_4, x_5) \cap (x_1, x_2, x_3, x_5) \cap (x_1, x_2, x_3, x_4).$$

Consider $I_{\Delta(\alpha)}$:

$(x_2^2, x_3, x_4, x_5) \cap (x_1, x_3^2, x_4, x_5) \cap (x_1, x_2, x_4^2, x_5) \cap (x_1, x_2, x_3, x_5^2) \cap (x_1^2, x_2, x_3, x_4)$.

It is easy to check that $I_{\Delta(\alpha)}$ is Cohen-Macaulay but α is not Δ-satisfying.

We end the paper by observing that we do not see how to extend the obtained results to more general simplicial complexes.

Given a shellable simplicial complex Δ with $\Im(\Delta) = \{F_1, \dots, F_m\}$, we could define a collection of positive integers $\alpha = (\alpha_i(j))$, for $i = 1, \dots, n$ and j such that $v_i \notin F_j$, to be Δ-*satisfying* if: *For any* $i = 1, \dots, n$ *there exists a shelling* $F_{i_1}, \dots, F_{i_m}$ *such that:*

(1) *There exists* $p = 1, \dots, m$ *for which* $v_i \in \bigcap_{h=1}^{p} F_{i_h}$ *and* $v_i \notin \bigcup_{h=p+1}^{m} F_{i_h}$.
(2) *If* $\alpha_i(i_t) > \alpha_i(i_s)$, *then* $t < s$.

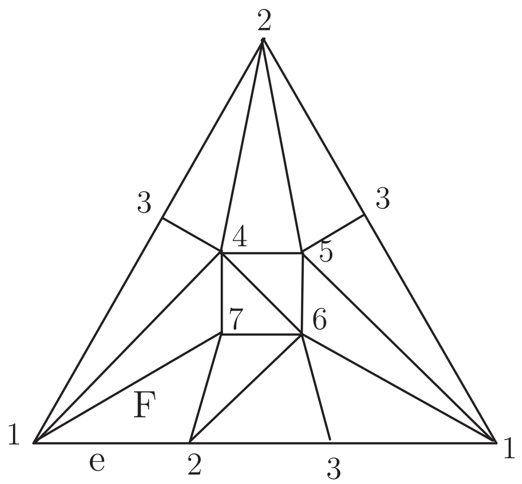
It is easy to see that Definitions 3.3 and 4.7 are included in the one above. However the analog of Theorem 4.8 does not hold in the general setting. For instance consider Δ to be the square and the collection α corresponding to the following ideal:

$$I_{\Delta(\alpha)} = (x_1, x_2^2) \cap (x_1, x_3^3) \cap (x_2^3, x_4) \cap (x_3^2, x_4).$$

Albeit α is Δ-satisfying, $I_{\Delta(\alpha)}$ is not Cohen-Macaulay.

We can prove that $I_{\Delta(\alpha)}$ is Cohen-Macaulay whenever α is Δ-satisfying and there is an index $i = 1, \dots, n$ such that α_j is constant for any $j \neq i$. But this is not so nice, since in general, given a vertex of a shellable simplicial complex, we cannot find any shelling for which the first condition of the general definition of "Δ-satisfying" holds.

Example 4.9. The following example, due to Hachimori ([Ha]), is a modification of the dunce hat. Consider the 2-dimensional simplicial complex Δ:



The above simplicial complex is easily seen to be shellable. However for any shelling $F_1, \dots, F_{13}$ we must have $F_{13} = F$. In fact e is the only boundary of Δ, so if $F_{13} \neq F$ then $\Delta_{12} \cap < F_{13} >= \partial F_{13}$, where Δ_{12} denotes the simplicial complex $< F_1, \ldots, F_{12} >$. The Mayer-Vietories sequence yields the below long exact sequence of singular homology groups:

$$\ldots \to H_2(\Delta) \to H_1(\Delta_{12} \cap < F_{13} >) \to H_1(\Delta_{12}) \oplus H_1(< F_{13} >) \to \ldots .$$

Because Δ_{12} is a 2-dimensional shellable simplicial complex, Reisner's theorem (see [BH, Corollary 5.3.9]) implies $H_1(\Delta_{12}) = H_1(< F_{13} >) = 0$. On the other hand $H_1(\Delta_{12} \cap < F_{13} >) = H_1(\partial F_{13}) \neq 0$. Thus the above exact sequence yields $H_2(\Delta) \neq 0$. But this is a contradiction, since, as it is easy to show, Δ is collapsible, in particular contractible.

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