FACTORIZATION OF LINEAR AND NONLINEAR DIFFERENTIAL OPERATORS: NECESSARY AND SUFFICIENT CONDITIONS

MAHOUTON NORBERT HOUNKONNOU*
International Chair in Mathematical Physics and Applications
(ICMPA-UNESCO Chair)
University of Abomey-Calavi
072 B.P. 50 Cotonou, Republic of Benin

PASCAL DKENGNE SIELENOU†
International Chair in Mathematical Physics and Applications
(ICMPA-UNESCO Chair)
University of Abomey-Calavi
072 B.P. 50 Cotonou, Republic of Benin

Abstract

An algebraic approach for factorizing nonlinear partial differential equations (PDEs) and systems of PDEs is provided. In the particular case of second order linear and nonlinear PDEs and systems of PDEs, necessary and sufficient conditions of factorization are given.

AMS Subject Classification: 34-01, 34A30, 34A34, 35-01, 47E05, 47F05.

Keywords: Linear differential equations; nonlinear differential equations; factorization.

1 Introduction

The search for exact solutions of differential equations is very challenging in mathematics, but their usefulness in the proper understanding of qualitative features of phenomena and processes in various areas of natural science merits to get down to such an investigation. Indeed, exact solutions can be used to verify the consistency and estimate errors of various numerical, asymptotic and approximate analytical methods. Unfortunately, there does not

*E-mail address: norbert.hounkonnou@cipma.uac.bj
†E-mail address: sielenou_alain@yahoo.fr

always exist a method adapted for the resolution of any type of differential equations. Very often, one tries to reduce the equation in order to make easier its resolution. But this reduction requires the knowledge of suitable transformations or changes of variables. The latters usually give rise to another problem the issue of which is not always favourable.

A simple approach for the reduction of a differential equation consists in seeking a factorization, if there exists, of the differential operator associated with it. Note that for the particular case of second order linear ordinary differential equations of Schrödinger or Sturm-Liouville type, the factorization of the associated differential operators also allows to obtain partially or completely their spectrum, under certain assumptions of integrability [1, 2, 3, 4]. In recent years, there has been much interest devoted to the problem of factorization of differential equations, especially based on linear ordinary [5, 6, 8] and nonlinear differential operators [7, 8, 9]. Although effective, the used methods are rather restrictive in their applications.

Recently, a purely algebraic method of factorization of the second order linear ordinary differential equations has been presented by the authors in [10, 11, 12, 13]. The same procedure of factorization has been exploited in [14] and extended to second order nonlinear ordinary differential equations (NLODES) and systems of NLODEs. This work generalizes previous works by applying the above mentioned algebraic method of factorization to linear and nonlinear systems of partial differential equations (PDEs). Necessary and sufficient conditions of factorization are derived in the case of second order equations.

First of all, some useful notations are required. Consider X, an n-dimensional independent variable space, and U, an m-dimensional dependent variable space. Let $x = \left(x^1, \cdots, x^n\right) \in X$ and $u = \left(u^1, \cdots, u^m\right) \in U$. We define the space $U^{(s)}$, $s \in \mathbb{N}$ as:

$$U^{(s)} := \left\{ u^{(s)} \,:\, u^{(s)} = \bigotimes_{j=1}^{m} \left(\bigotimes_{k=0}^{s} u^j_{(k)} \right) \right\}, \tag{1.1}$$

where $u^j_{(k)}$ is the

$$p_k = n^k \tag{1.2}$$

of all k-th order partial derivatives of u^j. The $u^j_{(k)}$ vector components are recursively obtained as follows:

i) $u^j_{(0)} = u^j$ and $u^j_{(1)} = \left(u^j_{x^1}, u^j_{x^2}, \cdots, u^j_{x^n}\right)$;

ii) Assume that $u^j_{(k)}$ is known. Then,

- Form the tuples $\widetilde{u}^j_{(k+1)}(l)$ as follows:

$$\widetilde{u}^j_{(k+1)}(l) = \left(\frac{\partial}{\partial x^1} u^j_{(k)}[l], \frac{\partial}{\partial x^2} u^j_{(k)}[l], \cdots, \frac{\partial}{\partial x^n} u^j_{(k)}[l] \right), \quad l = 1, 2, \cdots, p_k,$$

  where $u^j_{(k)}[l]$ is the l-th component of the vector $u^j_{(k)}$;

- Finally, form the vector

$$u^j_{(k+1)} = \left(\widetilde{u}^j_{(k+1)}(1), \widetilde{u}^j_{(k+1)}(2), \cdots, \widetilde{u}^j_{(k+1)}(p_k) \right).$$

An element $u^{(s)}$, in the space $U^{(s)}$, is the

$$q_s = m(1+p_1+p_2+\cdots+p_s)\text{-tuple} \tag{1.3}$$

defined by

$$u^{(s)} = \left(u^1_{(0)}, u^1_{(1)}, \cdots, u^1_{(s)}, u^2_{(0)}, u^2_{(1)}, \cdots, u^2_{(s)}, \cdots, u^m_{(0)}, u^m_{(1)}, \cdots, u^m_{(s)}\right). \tag{1.4}$$

The coordinates in the space $X \times U^{(s)}$ are denoted by $\left(x, u^{(s)}\right)$.
In the sequel, the q_s-uple $u^{(s)}$ will be referred to (1.4), whereas the integers p_k and q_s are defined by (1.2) and (1.3), respectively. Define differential operators $\mathrm{D}_{k,h}$ whose action on a regular function u is

$$\mathrm{D}_{k,h}\, u = u_{(k)}[h] \tag{1.5}$$

These operators $\mathrm{D}_{k,h}$ satisfy the following properties:

(i) $\mathrm{D}_{0,1}\, u = u$ (identity),

(ii) $\mathrm{D}_{1,h}\, \mathrm{D}_{k',h'}\, u = \mathrm{D}_{k'+1,n(h'-1)+h}\, u$ (composition rule),

(iii) $\mathrm{D}_{k,h}\, u = \mathrm{D}_{1,h}\, \mathrm{D}_{k-1,1}\, u, \quad k \geq 1$ (decomposition rule).

Remark 1.1. Operators $\mathrm{D}_{k,h}$ allow the simplification of the writing of certain differential operators. For example, the operator

$$\mathcal{T} = \sum_{l_1+l_2+\cdots+l_n=0}^{s} \frac{\partial^{l_1+l_2+\cdots+l_n}}{(\partial x^1)^{l_1}(\partial x^2)^{l_2}\cdots(\partial x^n)^{l_n}}$$

can be shortly expressed as

$$\mathcal{T} = \sum_{k=0}^{s}\sum_{h=1}^{p_k} D_{k,h}.$$

2 Linear differential operators

In this section, we develop an algebraic method of factorization applicable to linear differential operators (LDOs) and to systems of LDOs.

2.1 Factorizations of linear differential equations

The general setting of the factorization problem for LDOs is developed. Necessary and sufficient conditions are derived for the factorization of second order linear ordinary and partial differential operators with two independent variables.

2.1.1 General setting

Let $s \geq 2$ be a positive integer and Λ be an open subset of $\mathbb{R}^n$. Let

$$\mathcal{P}(s) = \sum_{k=0}^{s} \sum_{h=1}^{p_k} g_{k,h}(x) \mathrm{D}_{k,h} \tag{2.1}$$

a linear differential operator of order s, where $g_{k,h} \in \mathcal{C}(\Lambda, \mathbb{R})$. The operator $\mathcal{P}(s)$ acts on a function $u \in \mathcal{C}^s(\Lambda, \mathbb{R})$ as follows

$$\mathcal{P}(s)\, u = \sum_{k=0}^{s} \sum_{h=1}^{p_k} g_{k,h}(x) \mathrm{D}_{k,h}\, u. \tag{2.2}$$

The method of factorization consists in seeking a decomposition of the differential operator (2.1) in the following form

$$\mathcal{P}(s) = \prod_{i=1}^{l} \mathcal{Q}_i(s_i) \tag{2.3}$$

with $\sum_{i=1}^{l} s_i = s$ and

$$\mathcal{Q}_i(s_i) = \sum_{k=0}^{s_i} \sum_{h=1}^{p_k} b_{i,k,h}(x) \mathrm{D}_{k,h}, \tag{2.4}$$

where $b_{1,k,h} \in \mathcal{C}(\Lambda, \mathbb{R})$ and $b_{i,k,h} \in \mathcal{C}^{\sum_{j=1}^{i-1} s_j}(\Lambda, \mathbb{R})$, $i = 2, 3, \cdots, l$.

Proposition 2.1. *Let $\mathcal{P}(s)$ be an operator which can be decomposed into the form (2.3). If the function u_0 satisfies*

$$\mathcal{Q}_l(s_l) u_0 = 0, \tag{2.5}$$

and $u_1, \ldots, u_{l-1}$ are solutions of the system

$$\prod_{k=l-j+1}^{l} \mathcal{Q}_k(s_k) u_j \quad = \quad v_j,\ j = 1, 2, \ldots, l-1, \tag{2.6}$$

where $v_j,\ j = 1, 2, \ldots, l-1$, are solutions of

$$\prod_{i=1}^{l-j} \mathcal{Q}_i(s_i) v_j \quad = \quad 0, \tag{2.7}$$

then $u_0, u_1, \ldots, u_{l-1}$ are l particular solutions of the equation $\mathcal{P}(s)\, u = 0$.

Proof. Let u_0 and $u_j,\ j = 1, 2, \ldots, l-1$ be solutions of (2.5) and (2.6), respectively. Then

$$\mathcal{P}(s) u_0 = \left(\prod_{i=1}^{l-1} \mathcal{Q}_i(s_i) \right) \mathcal{Q}_l(s_l) u_0 = 0,$$

and for $j = 1, 2, \ldots, l-1$,

$$\begin{aligned} \mathcal{P}(s) u_j \quad &= \quad \left(\prod_{i=1}^{l-j} \mathcal{Q}_i(s_i) \right) \left(\prod_{k=l-j+1}^{l} \mathcal{Q}_k(s_k) \right) u_j \\ &= \quad \prod_{i=1}^{l-j} \mathcal{Q}_i(s_i) v_j = 0, \end{aligned}$$

where the use of (2.6) and (2.7) has been made. □

Expanding (2.3) leads to the relations between unknown functions $b_{i,k,h}$ of the differential operators $\mathcal{Q}_i(s_i)$ and the known functions $g_{k,h}$ of the original differential operator $\mathcal{P}(s)$. Without loss of generality and as matter of clarity, this study will be concentrated to second order equations, the generalization being straightforward.

2.1.2 Necessary and sufficient conditions for the factorization of second order linear ODEs

Let Λ and Λ_0 be two open subsets of $\mathbb{R}$ such that $\Lambda_0 \subset \Lambda$. Consider the second order linear ordinary differential operator

$$\begin{aligned} \mathcal{P}(2) &= \sum_{k=0}^{2}\sum_{h=1}^{p_k} g_{k,h}(x)\mathrm{D}_{k,h} \\ &= g_{0,1}(x)\mathrm{D}_{0,1} + g_{1,1}(x)\mathrm{D}_{1,1} + g_{2,1}(x)\mathrm{D}_{2,1}, \end{aligned} \tag{2.8}$$

where $g_{k,h} \in \mathcal{C}(\Lambda,\mathbb{R})$ and $x = x^1$. Write $\mathcal{P}(2)$ in the form

$$\begin{aligned} \mathcal{P}(2) &= \mathcal{Q}_1(1)\cdot\mathcal{Q}_2(1) \\ &= \left[\sum_{k=0}^{1}\sum_{h=1}^{p_k} b_{1,k,h}(x)\mathrm{D}_{k,h}\right]\left[\sum_{k=0}^{1}\sum_{h=1}^{p_k} b_{2,k,h}(x)\mathrm{D}_{k,h}\right] \\ &= \left[b_{1,0,1}(x)\mathrm{D}_{0,1} + b_{1,1,1}(x)\mathrm{D}_{1,1}\right]\left[b_{2,0,1}(x)\mathrm{D}_{0,1} + b_{2,1,1}(x)\mathrm{D}_{1,1}\right], \end{aligned} \tag{2.9}$$

where $b_{1,k,h} \in \mathcal{C}(\Lambda,\mathbb{R})$ and $b_{2,k,h} \in \mathcal{C}^1(\Lambda,\mathbb{R})$. Let $u \in \mathcal{C}^2(\Lambda_0,\mathbb{R})$. Then we have

$$\mathcal{P}(2)\,u = g_{0,1}(x)\,u + g_{1,1}(x)\,u_x + g_{2,1}(x)\,u_{2x} \tag{2.10}$$

and after expansion

$$\begin{aligned} \mathcal{P}(2)\,u &= \left[b_{1,0,1}(x)\mathrm{D}_{0,1} + b_{1,1,1}(x)\mathrm{D}_{1,1}\right]\left[b_{2,0,1}(x)\mathrm{D}_{0,1} + b_{2,1,1}(x)\mathrm{D}_{1,1}\right]u \\ &= b_{1,1,1}b_{2,1,1}\,u_{2x} + \left[b_{1,0,1}b_{2,1,1} + b_{1,1,1}b_{2,0,1} + b_{1,1,1}\mathrm{D}_{1,1}(b_{2,1,1})\right]u_x \\ &+ \left[b_{1,0,1}b_{2,0,1} + b_{1,1,1}\mathrm{D}_{1,1}(b_{2,0,1})\right]u. \end{aligned} \tag{2.11}$$

Identifying (2.10) with (2.11) yields

Proposition 2.2. *A necessary and sufficient condition for the differential operator $\mathcal{P}(2)$ defined by (2.8) be decomposed into the form (2.9) is:*

$$g_{2,1} = b_{1,1,1}b_{2,1,1}, \tag{2.12}$$

$$g_{1,1} = b_{1,0,1}b_{2,1,1} + b_{1,1,1}b_{2,0,1} + b_{1,1,1}D_{1,1}(b_{2,1,1}), \tag{2.13}$$

$$g_{0,1} = b_{1,0,1}b_{2,0,1} + b_{1,1,1}D_{1,1}(b_{2,0,1}). \tag{2.14}$$

Propose an approach to solve system (2.12)-(2.14). Assume that $g_{2,1}$ does not vanish on Λ. Thus, it is always possible to find two nonzero functions on Λ, namely $b_{1,1,1}$ and $b_{2,1,1}$, which satisfy (2.12). Substituting $X = b_{1,0,1}$ and $Y = b_{2,0,1}$ in (2.13) gives

$$X = \frac{1}{b_{2,1,1}}\left[g_{1,1} - b_{1,1,1}\mathrm{D}_{1,1}(b_{2,1,1}) - b_{1,1,1}Y\right]. \tag{2.15}$$

The substitution of (2.15) into (2.14) implies that the decomposition (2.9) is strongly related to the existence of a solution to the following Riccati equation in Y

$$\mathrm{D}_{1,1}(Y)-\frac{b_{1,1,1}}{g_{2,1}}Y^2+\frac{g_{1,1}-b_{1,1,1}\mathrm{D}_{1,1}(b_{2,1,1})}{g_{2,1}}Y-\frac{g_{0,1}}{b_{1,1,1}}=0. \tag{2.16}$$

2.1.3 Necessary and sufficient conditions for the factorization of second order linear PDEs with two independent variables

Let Λ and Λ_0 be two open subsets of $\mathbb{R}^2$ such that $\Lambda_0\subset\Lambda$. Consider the second order linear partial differential operator

$$\begin{aligned}\mathcal{P}(2) &= \sum_{k=0}^{2}\sum_{h=1}^{p_k} g_{k,h}(x)\mathrm{D}_{k,h}\\ &= g_{0,1}(x)\mathrm{D}_{0,1}+g_{1,1}(x)\mathrm{D}_{1,1}+g_{1,2}(x)\mathrm{D}_{1,2}\\ &+ g_{2,1}(x)\mathrm{D}_{2,1}+g_{2,2}(x)\mathrm{D}_{2,2}+g_{2,3}(x)\mathrm{D}_{2,3}+g_{2,4}(x)\mathrm{D}_{2,4},\end{aligned} \tag{2.17}$$

where $g_{k,h}\in\mathcal{C}(\Lambda,\mathbb{R})$ and $x=\left(x^1,x^2\right)$. Write $\mathcal{P}(2)$ in the form

$$\begin{aligned}\mathcal{P}(2) &= \mathcal{Q}_1(1)\cdot\mathcal{Q}_2(1)\\ &= \left[\sum_{k=0}^{1}\sum_{h=1}^{p_k} b_{1,k,h}(x)\mathrm{D}_{k,h}\right]\left[\sum_{k=0}^{1}\sum_{h=1}^{p_k} b_{2,k,h}(x)\mathrm{D}_{k,h}\right]\\ &= \left[b_{1,0,1}(x)\mathrm{D}_{0,1}+b_{1,1,1}(x)\mathrm{D}_{1,1}+b_{1,1,2}(x)\mathrm{D}_{1,2}\right]\\ &\times \left[b_{2,0,1}(x)\mathrm{D}_{0,1}+b_{2,1,1}(x)\mathrm{D}_{1,1}+b_{2,1,2}(x)\mathrm{D}_{1,2}\right],\end{aligned} \tag{2.18}$$

where $b_{1,k,h}\in\mathcal{C}(\Lambda,\mathbb{R})$ and $b_{2,k,h}\in\mathcal{C}^1(\Lambda,\mathbb{R})$. Let $u\in\mathcal{C}^2(\Lambda_0,\mathbb{R})$. Then we have

$$\mathcal{P}(2)u = g_{0,1}u+g_{1,1}u_{x^1}+g_{1,2}u_{x^2}+g_{2,1}u_{2x^1}+(g_{2,2}+g_{2,3})u_{x^1x^2}+g_{2,4}u_{2x^2} \tag{2.19}$$

and after expansion

$$\begin{aligned}\mathcal{P}(2)u &= \left[b_{1,0,1}(x)\mathrm{D}_{0,1}+b_{1,1,1}(x)\mathrm{D}_{1,1}+b_{1,1,2}(x)\mathrm{D}_{1,2}\right]\\ &\times \left[b_{2,0,1}(x)\mathrm{D}_{0,1}+b_{2,1,1}(x)\mathrm{D}_{1,1}+b_{2,1,2}(x)\mathrm{D}_{1,2}\right]u\\ &= \left[b_{1,0,1}b_{2,0,1}+b_{1,1,1}\mathrm{D}_{1,1}(b_{2,0,1})+b_{1,1,2}\mathrm{D}_{1,2}(b_{2,0,1})\right]u\\ &+ \left[b_{1,0,1}b_{2,1,1}+b_{1,1,1}b_{2,0,1}+b_{1,1,1}\mathrm{D}_{1,1}(b_{2,1,1})+b_{1,1,2}\mathrm{D}_{1,2}(b_{2,1,1})\right]u_{x^1}\\ &+ \left[b_{1,0,1}b_{2,1,2}+b_{1,1,2}b_{2,0,1}+b_{1,1,1}\mathrm{D}_{1,1}(b_{2,1,2})+b_{1,1,2}\mathrm{D}_{1,2}(b_{2,1,2})\right]u_{x^2}\\ &+ b_{1,1,1}b_{2,1,1}u_{2x^1}+\left[b_{1,1,2}b_{2,1,1}+b_{1,1,1}b_{2,1,2}\right]u_{x^1x^2}+b_{1,1,2}b_{2,1,2}u_{2x^2}.\end{aligned} \tag{2.20}$$

Identifying (2.19) with (2.20) leads to the following

Proposition 2.3. *A necessary and sufficient condition for the differential operator $\mathcal{P}(2)$ defined by (2.17) be decomposed into the form (2.18) is:*

$$g_{2,1} = b_{1,1,1}b_{2,1,1}, \tag{2.21}$$

$$g_{2,2}+g_{2,3} = b_{1,1,2}b_{2,1,1}+b_{1,1,1}b_{2,1,2}, \tag{2.22}$$

$$g_{2,4} = b_{1,1,2}b_{2,1,2}, \tag{2.23}$$

$$g_{1,1} = b_{1,0,1}b_{2,1,1}+b_{1,1,1}b_{2,0,1}+\mathcal{L}(b_{2,1,1}), \tag{2.24}$$

$$g_{1,2} = b_{1,0,1}b_{2,1,2}+b_{1,1,2}b_{2,0,1}+\mathcal{L}(b_{2,1,2}), \tag{2.25}$$

$$g_{0,1} = b_{1,0,1}b_{2,0,1}+\mathcal{L}(b_{2,0,2}), \tag{2.26}$$

where $\mathcal{L} = b_{1,1,1}D_{1,1} + b_{1,1,2}D_{1,2}$.

Propose an approach to solve system (2.21)-(2.26). Assume that at least one of the functions $g_{2,1}$ and $g_{2,4}$ does not vanish on Λ, says $g_{2,1}$. It is always possible to find two nonzero functions on Λ, namely $b_{1,1,1}$ and $b_{2,1,1}$ which satisfy (2.21). Substituting $X_1 = b_{1,1,2}$ and $X_2 = b_{2,1,2}$ into (2.22) yields

$$X_1 = \frac{1}{b_{2,1,1}}\left(g_{2,2} + g_{2,3} - b_{1,1,1}X_2\right). \tag{2.27}$$

The substitution of (2.27) into (2.23) shows that $b_{2,1,2}$ is a solution of the second degree algebraic equation

$$\frac{b_{1,1,1}}{g_{2,1}}X_2^2 - \frac{g_{2,2} + g_{2,3}}{g_{2,1}}X_2 + \frac{g_{2,4}}{b_{1,1,1}} = 0. \tag{2.28}$$

The discriminant of equation (2.28) is

$$\Delta = (g_{2,2} + g_{2,3})^2 - 4g_{2,1}g_{2,4} = (b_{1,1,2}b_{2,1,1} - b_{1,1,1}b_{2,1,2})^2 \geq 0. \tag{2.29}$$

If $\Delta > 0$, then the substitution of $Y = b_{1,0,1}$ and $Z = b_{2,0,1}$ into (2.24) and (2.25) implies that the decomposition (2.18) is possible if the unique solution to the following algebraic system in Y and Z

$$\begin{aligned} g_{1,1} - \mathcal{L}(b_{2,1,1}) &= b_{2,1,1}Y + b_{1,1,1}Z \\ g_{1,2} - \mathcal{L}(b_{2,1,2}) &= b_{2,1,2}Y + b_{1,1,2}Z \end{aligned} \tag{2.30}$$

satisfies (2.26). Indeed, the determinant of the system (2.30) is

$$b_{1,1,2}b_{2,1,1} - b_{1,1,1}b_{2,1,2} = \pm\sqrt{\Delta} \neq 0.$$

If $\Delta = 0$, then the substitution of $Y = b_{1,0,1}$ and $Z = b_{2,0,1}$ into (2.24) yields

$$Y = \frac{1}{b_{2,1,1}}\left[g_{1,1} - \mathcal{L}(b_{2,1,1}) - b_{1,1,1}Z\right]. \tag{2.31}$$

Then, the substitution of (2.31) into (2.26) implies that the decomposition (2.18) is strongly related to the existence of a solution to the following first order quasi-linear partial differential equation in Z

$$\mathcal{L}(Z) - \frac{b_{1,1,1}}{b_{2,1,1}}Z^2 + \frac{g_{1,1} - \mathcal{L}(b_{2,1,1})}{b_{2,1,1}}Z - g_{0,1} = 0 \tag{2.32}$$

which satisfies (2.25).

2.2 Factorizations of systems of linear differential equations

The previous analysis is now made for systems of linear differential equations.

2.2.1 General considerations

Let Λ be an open subset of $\mathbb{R}^n$. Examine now the factorization process for systems of s-th order, $(s \geq 2)$, linear differential equations with n independent variables $x = \left(x^1, \cdots, x^n\right)$ and $m \geq 2$ dependent variables $u = {}^t\left(u^1, \cdots, u^m\right)$, $u = u(x)$ whose associated matrix operator, $\mathcal{M}\,(s)$, is of the form

$$\mathcal{M}\,(s) = \left[\mathcal{R}_{p,q}\,(s_{p,q})\right]_{1\leq p,q\leq m}; \tag{2.33}$$

the $\mathcal{R}_{p,q}\,(s_{p,q})$ are $s_{p,q}$-th order linear differential operators

$$\mathcal{R}_{p,q}\,(s_{p,q}) = \sum_{k=0}^{s_{p,q}} \sum_{h=1}^{p_k} f_{p,q,k,h}(x) \mathrm{D}_{k,h}, \tag{2.34}$$

where $f_{p,q,k,h} \in \mathcal{C}\,(\Lambda, \mathbb{R})$, $s_{p,q} = s - 1 + \delta_{p,q}$, $\delta_{p,p} = 1$ and $\delta_{p,q} = 0$ if $p \neq q$.
Let Λ and Λ_0 be two open subsets of $\mathbb{R}^n$ such that $\Lambda_0 \subset \Lambda$. The matrix operator $\mathcal{M}\,(s)$ acts on a vector valued function $u = {}^t\left(u^1, \cdots, u^m\right) \in \mathcal{C}^s(\Lambda_0, \mathbb{R}^m)$ as follows

$$\mathcal{M}\,(s)\,u = \left[\mathcal{R}_{p,q}\,(s_{p,q})\right]_{1\leq p,q\leq m} u = \left[\sum_{q=1}^{m} \mathcal{R}_{p,q}\,(s_{p,q})\; u^q\right]_{1\leq p\leq m}.$$

The method of factorization consists in seeking a decomposition of the matrix $\mathcal{M}\,(s)$ under the following form

$$\mathcal{M}\,(s) = \prod_{i=1}^{l} \mathcal{N}_i(s_i) \tag{2.35}$$

where

$$\mathcal{N}_i(s_i) = \left[\mathcal{T}_{i,p,q}\,(s_{i,p,q})\right]_{1\leq p,q\leq m} \tag{2.36}$$

and

$$\mathcal{T}_{i,p,q}\,(s_{i,p,q}) = \sum_{k=0}^{s_{i,p,q}} \sum_{h=1}^{p_k} a_{i,p,q,k,h}(x) \mathrm{D}_{k,h}, \tag{2.37}$$

with $\sum_{i=1}^{l} s_i = s$, $s_{i,p,q} = s_i - 1 + \delta_{p,q}$, $a_{1,p,q,k,h} \in \mathcal{C}\,(\Lambda, \mathbb{R})$ and $a_{i,p,q,k,h} \in \mathcal{C}^{\sum_{j=1}^{i-1} s_{i,p,q}}(\Lambda, \mathbb{R})$, $i = 2, 3, \cdots, l$.

Proposition 2.4. *Let $\mathcal{M}\,(s)$ be a matrix of differential operators defined by (2.33) which can be decomposed into the form (2.35). If the function $u_0 = {}^t\left(u_0^1, \cdots, u_0^m\right)$ satisfies*

$$\mathcal{N}_l(s_l) u_0 = 0, \tag{2.38}$$

and $u_j = {}^t\left(u_j^1, \cdots, u_j^m\right)$, $j = 1, 2, \cdots, l-1$ are solutions of the system

$$\prod_{k=l-j+1}^{l} \mathcal{N}_k(s_k) u_j \;\; = \;\; v_j,\; j = 1, 2, \ldots, l-1, \tag{2.39}$$

where $v_j = {}^t\left(v_j^1, \cdots, v_j^m\right)$, $j = 1, 2, \ldots, l-1$, are solutions of

$$\prod_{i=1}^{l-j} \mathcal{N}_i(s_i) v_j \;\; = \;\; 0, \tag{2.40}$$

then $u_0, u_1, \ldots, u_{l-1}$ are l particular solutions of the equation $\mathcal{M}\,(s)\,u = 0$.

Proof. The proof is similar to that of the Proposition 2.1. □

Expanding (2.35) leads to the relations between the unknown functions $a_{i,p,q,k,h}$ of $\mathcal{N}_i(s_i)$ and the known functions $f_{p,q,k,h}$ of $\mathcal{M}(s)$.

As matter of clarity, in the sequel we explicitly derive necessary and sufficient conditions for the factorization of systems of second order linear ordinary and partial differential operators with two independent variables.

2.2.2 Necessary and sufficient conditions for the factorization of systems of second order linear ODEs

Let Λ and Λ_0 be two open subsets of $\mathbb{R}$ such that $\Lambda_0 \subset \Lambda$. Consider the matrix operator

$$\mathcal{M}(2) = [\mathcal{R}_{p,q}]_{1\leq p,q\leq m}, \tag{2.41}$$

where

$$\mathcal{R}_{p,p} = \sum_{k=0}^{2}\sum_{h=1}^{p_k} f_{p,q,k,h}(x)\mathrm{D}_{k,h} = f_{p,p,0,1}\mathrm{D}_{0,1} + f_{p,p,1,1}\mathrm{D}_{1,1} + f_{p,p,2,1}\mathrm{D}_{2,1} \tag{2.42}$$

and for $p \neq q$

$$\mathcal{R}_{p,q} = \sum_{k=0}^{1}\sum_{h=1}^{p_k} f_{p,q,k,h}(x)\mathrm{D}_{k,h} = f_{p,q,0,1}\mathrm{D}_{0,1} + f_{p,q,1,1}\mathrm{D}_{1,1} \tag{2.43}$$

with $f_{p,q,k,h} \in \mathcal{C}(\Lambda,\mathbb{R})$, $x = x^1$. Write $\mathcal{M}(2)$ in the form

$$\mathcal{M}(2) = \mathcal{N}_1(1)\cdot\mathcal{N}_2(1), \tag{2.44}$$

where

$$\mathcal{N}_i(1) = [\mathcal{T}_{i,p,q}]_{1\leq p,q\leq m} \tag{2.45}$$

with

$$\mathcal{T}_{i,p,p} = \sum_{k=0}^{1}\sum_{h=1}^{p_k} a_{i,p,p,k,h}(x)\mathrm{D}_{k,h} = a_{i,p,p,0,1}\mathrm{D}_{0,1} + a_{i,p,p,1,1}\mathrm{D}_{1,1} \tag{2.46}$$

and for $p \neq q$

$$\mathcal{T}_{i,p,q} = a_{i,p,q,0,1}\mathrm{D}_{0,1}, \tag{2.47}$$

$a_{1,p,q,k,h} \in \mathcal{C}(\Lambda,\mathbb{R})$ and $a_{2,p,q,k,h} \in \mathcal{C}^1(\Lambda,\mathbb{R})$. Let $u = {}^t\left(u^1,\cdots,u^m\right) \in \mathcal{C}^2(\Lambda_0,\mathbb{R})$. Then we have

$$\mathcal{M}(2)\,u = [\mathcal{R}_{p,q}]_{1\leq p,q\leq m}\,u = \left[\sum_{q=1}^{m}\mathcal{R}_{p,q}\,u^q\right]_{1\leq p\leq m} \tag{2.48}$$

where

$$\mathcal{R}_{p,p}\,u^p = f_{p,p,0,1}\,u^p + f_{p,p,1,1}\,u^p_x + f_{p,p,2,1}\,u^p_{2x}$$

and for $p \neq q$

$$\mathcal{R}_{p,q}\,u^q = f_{p,q,0,1}\,u^q + f_{p,q,1,1}\,u^q_x.$$

On the other hand, after expansion of (2.44), we have

$$\mathcal{M}\,(2)\,u=\left[\widetilde{\mathcal{R}}_{p,q}\right]_{1\le p,q\le m}u=\left[\sum_{q=1}^{m}\widetilde{\mathcal{R}}_{p,q}\,u^{q}\right]_{1\le p\le m}, \tag{2.49}$$

where

$$\begin{aligned}\widetilde{\mathcal{R}}_{p,p}\,u^{p} &= \left[\sum_{l=1}^{m}a_{1,p,l,0,1}a_{2,l,p,0,1}+a_{1,p,p,1,1}\mathrm{D}_{1,1}(a_{2,p,p,0,1})\right]u^{p}+a_{1,p,p,1,1}a_{2,p,p,1,1}\,u^{p}_{2x}\\ &+ \left[a_{1,p,p,0,1}a_{2,p,p,1,1}+a_{1,p,p,1,1}a_{2,p,p,0,1}+a_{1,p,p,1,1}\mathrm{D}_{1,1}(a_{2,p,p,1,1})\right]u^{p}_{x}\end{aligned}$$

and for $p\neq q$

$$\begin{aligned}\widetilde{\mathcal{R}}_{p,q}\,u^{q} &= \left[a_{1,p,p,1,1}a_{2,p,q,0,1}+a_{1,p,q,0,1}a_{2,q,q,1,1}\right]u^{q}_{x}\\ &+ \left[\sum_{l=1}^{m}a_{1,p,l,0,1}a_{2,l,q,0,1}+a_{1,p,p,1,1}\mathrm{D}_{1,1}(a_{2,p,q,0,1})\right]u^{q}.\end{aligned}$$

Identifying (2.48) with (2.49) yields

Proposition 2.5. *A necessary and sufficient condition for the differential operator $\mathcal{M}\,(2)$ defined by (2.41) be decomposed into the form (2.44) is:*

$$f_{p,p,0,1} = \sum_{l=1}^{m}a_{1,p,l,0,1}a_{2,l,p,0,1}+a_{1,p,p,1,1}D_{1,1}(a_{2,p,p,0,1}), \tag{2.50}$$

$$f_{p,p,1,1} = a_{1,p,p,0,1}a_{2,p,p,1,1}+a_{1,p,p,1,1}a_{2,p,p,0,1}+a_{1,p,p,1,1}D_{1,1}(a_{2,p,p,1,1}), \tag{2.51}$$

$$f_{p,p,2,1} = a_{1,p,p,1,1}a_{2,p,p,1,1} \tag{2.52}$$

and for $p\neq q$

$$f_{p,q,0,1} = \sum_{l=1}^{m}a_{1,p,l,0,1}a_{2,l,q,0,1}+a_{1,p,p,1,1}D_{1,1}(a_{2,p,q,0,1}), \tag{2.53}$$

$$f_{p,q,1,1} = a_{1,p,p,1,1}a_{2,p,q,0,1}+a_{1,p,q,0,1}a_{2,q,q,1,1}. \tag{2.54}$$

2.2.3 Necessary and sufficient conditions for the factorization of systems of second order linear PDEs with two independent variables

Let Λ and Λ_0 be two open subsets of $\mathbb{R}^2$ such that $\Lambda_0\subset\Lambda$. Consider the matrix operator

$$\mathcal{M}\,(2)=\left[\mathcal{R}_{p,q}\right]_{1\le p,q\le m}, \tag{2.55}$$

where

$$\begin{aligned}\mathcal{R}_{p,p} &= \sum_{k=0}^{2}\sum_{h=1}^{p_k}f_{p,q,k,h}(x)\mathrm{D}_{k,h}\\ &= f_{p,p,0,1}\mathrm{D}_{0,1}+f_{p,p,1,1}\mathrm{D}_{1,1}+f_{p,p,1,2}\mathrm{D}_{1,2}\\ &+ f_{p,p,2,1}\mathrm{D}_{2,1}+f_{p,p,2,2}\mathrm{D}_{2,2}+f_{p,p,2,3}\mathrm{D}_{2,3}+f_{p,p,2,4}\mathrm{D}_{2,4}\end{aligned} \tag{2.56}$$

and for $p \neq q$

$$\mathcal{R}_{p,q} = \sum_{k=0}^{1} \sum_{h=1}^{p_k} f_{p,q,k,h}(x) \mathrm{D}_{k,h} = f_{p,q,0,1}\mathrm{D}_{0,1} + f_{p,q,1,1}\mathrm{D}_{1,1} + f_{p,q,1,2}\mathrm{D}_{1,2} \tag{2.57}$$

with $f_{p,q,k,h} \in \mathcal{C}(\Lambda, \mathbb{R})$, $x = \left(x^1, x^2\right)$. Write $\mathcal{M}(2)$ in the form

$$\mathcal{M}(2) = \mathcal{N}_1(1) \cdot \mathcal{N}_2(1), \tag{2.58}$$

where

$$\mathcal{N}_i(1) = [\mathcal{T}_{i,p,q}]_{1 \leq p,q \leq m} \tag{2.59}$$

with

$$\mathcal{T}_{i,p,p} = \sum_{k=0}^{1} \sum_{h=1}^{p_k} a_{i,p,p,k,h}(x) \mathrm{D}_{k,h} = a_{i,p,p,0,1}\mathrm{D}_{0,1} + a_{i,p,p,1,1}\mathrm{D}_{1,1} + a_{i,p,p,1,2}\mathrm{D}_{1,2} \tag{2.60}$$

and for $p \neq q$

$$\mathcal{T}_{i,p,q} = a_{i,p,q,0,1}\mathrm{D}_{0,1}, \tag{2.61}$$

$a_{1,p,q,k,h} \in \mathcal{C}(\Lambda, \mathbb{R})$ and $a_{2,p,q,k,h} \in \mathcal{C}^1(\Lambda, \mathbb{R})$. Let $u = {}^t\left(u^1, \cdots, u^m\right) \in \mathcal{C}^2(\Lambda_0, \mathbb{R})$. Then we have

$$\mathcal{M}(2)\, u = [\mathcal{R}_{p,q}]_{1 \leq p,q \leq m}\, u = \left[\sum_{q=1}^{m} \mathcal{R}_{p,q}\, u^q\right]_{1 \leq p \leq m}, \tag{2.62}$$

where

$$\begin{aligned} \mathcal{R}_{p,p}\, u^p &= f_{p,p,0,1}\, u^p + f_{p,p,1,1}\, u^p_{x^1} + f_{p,p,1,2}\, u^p_{x^2} \\ &+ f_{p,p,2,1}\, u^p_{2x^1} + (f_{p,p,2,2} + f_{p,p,2,3})\, u^p_{x^1x^2} + f_{p,p,2,4}\, u^p_{2x^2} \end{aligned}$$

and for $p \neq q$

$$\mathcal{R}_{p,q}\, u^q = f_{p,q,0,1}\, u^q + f_{p,q,1,1}\, u^q_{x^1} + f_{p,q,1,2}\, u^q_{x^2}.$$

On the other hand, after expansion of (2.58), we have

$$\mathcal{M}(2)\, u = \left[\widetilde{\mathcal{R}}_{p,q}\right]_{1 \leq p,q \leq m} u = \left[\sum_{q=1}^{m} \widetilde{\mathcal{R}}_{p,q}\, u^q\right]_{1 \leq p \leq m}, \tag{2.63}$$

where

$$\begin{aligned} \widetilde{\mathcal{R}}_{p,p}\, u^p &= a_{1,p,p,1,1}a_{2,p,p,1,1}\, u^p_{2x^1} + (a_{1,p,p,1,2}a_{2,p,p,1,1} + a_{1,p,p,1,1}a_{2,p,p,1,2})\, u^p_{x^1x^2} \\ &+ [a_{1,p,p,0,1}a_{2,p,p,1,1} + a_{1,p,p,1,1}a_{2,p,p,0,1} + \mathcal{L}_p(a_{2,p,p,1,1})]\, u^p_{x^1} \\ &+ [a_{1,p,p,0,1}a_{2,p,p,1,2} + a_{1,p,p,1,2}a_{2,p,p,0,1} + \mathcal{L}_p(a_{2,p,p,1,2})]\, u^p_{x^2} \\ &+ \left[\sum_{l=1}^{m} a_{1,p,l,0,1}a_{2,l,p,0,1} + \mathcal{L}_p(a_{2,p,p,0,1})\right] u^p + a_{1,p,p,1,2}a_{2,p,p,1,2}\, u^p_{2x^2} \end{aligned}$$

and for $p \neq q$

$$\begin{aligned}\widetilde{\mathcal{R}}_{p,q} u^q &= \left[a_{1,p,p,1,1} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,1}\right] u^q_{x^1} \\ &+ \left[a_{1,p,p,1,2} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,2}\right] u^q_{x^2} \\ &+ \left[\sum_{l=1}^{m} a_{1,p,l,0,1} a_{2,l,q,0,1} + \mathcal{L}_p(a_{2,p,q,0,1})\right] u^q,\end{aligned}$$

where $\mathcal{L}_p = a_{1,p,p,1,1}\mathrm{D}_{1,1} + a_{1,p,p,1,2}\mathrm{D}_{1,2}$. From the Identification of (2.62) with (2.63) results

Proposition 2.6. *A necessary and sufficient condition for the differential operator $\mathcal{M}(2)$ defined by (2.55) be decomposed into the form (2.58) is:*

$$f_{p,p,0,1} = \sum_{l=1}^{m} a_{1,p,l,0,1} a_{2,l,p,0,1} + \mathcal{L}_p(a_{2,p,p,0,1}), \tag{2.64}$$

$$f_{p,p,1,1} = a_{1,p,p,0,1} a_{2,p,p,1,1} + a_{1,p,p,1,1} a_{2,p,p,0,1} + \mathcal{L}_p(a_{2,p,p,1,1}), \tag{2.65}$$

$$f_{p,p,1,2} = a_{1,p,p,0,1} a_{2,p,p,1,2} + a_{1,p,p,1,2} a_{2,p,p,0,1} + \mathcal{L}_p(a_{2,p,p,1,2}), \tag{2.66}$$

$$f_{p,p,2,1} = a_{1,p,p,1,1} a_{2,p,p,1,1}, \tag{2.67}$$

$$f_{p,p,2,2} + f_{p,p,2,3} = a_{1,p,p,1,2} a_{2,p,p,1,1} + a_{1,p,p,1,1} a_{2,p,p,1,2}, \tag{2.68}$$

$$f_{p,p,2,4} = a_{1,p,p,1,2} a_{2,p,p,1,2} \tag{2.69}$$

and for $p \neq q$

$$f_{p,q,0,1} = \sum_{l=1}^{m} a_{1,p,l,0,1} a_{2,l,q,0,1} + \mathcal{L}_p(a_{2,p,q,0,1}), \tag{2.70}$$

$$f_{p,q,1,1} = a_{1,p,p,1,1} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,1}, \tag{2.71}$$

$$f_{p,q,1,2} = a_{1,p,p,1,2} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,2}. \tag{2.72}$$

3 Nonlinear differential operators

In this section, we investigate the factorization of nonlinear differential operators.

3.1 Factorizations of nonlinear differential equations

We start with general considerations and then deduce the main results on conditions of factorization.

3.1.1 General setting and results

Let $s \geq 2$ be a positive integer, Λ be an open subset of $\mathbb{R}^n$ and Ω an open subset of $\mathbb{R}$. Let

$$\mathcal{P}(s) = \sum_{k=0}^{s} \sum_{h=1}^{p_k} g_{k,h}(x,\cdot)\mathrm{D}_{k,h} \tag{3.1}$$

be a nonlinear differential operator of order s, where $g_{k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$. The operator $\mathcal{P}(s)$ acts on a function $u \in \mathcal{C}^s(\Lambda, \Omega)$ as follows

$$\mathcal{P}(s)\,u = \sum_{k=0}^{s} \sum_{h=1}^{p_k} g_{k,h}(x,u)\mathrm{D}_{k,h}\,u. \tag{3.2}$$

The method of factorization consists in seeking a decomposition of the differential operator (3.1) in the following form

$$\mathcal{P}(s) = \prod_{i=1}^{l} \mathcal{Q}_i(s_i) \tag{3.3}$$

with $\sum_{i=1}^{l} s_i = s$ and

$$\mathcal{Q}_i(s_i) = \sum_{k=0}^{s_i} \sum_{h=1}^{p_k} b_{i,k,h}(x,\cdot)\mathrm{D}_{k,h}, \tag{3.4}$$

where $b_{1,k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$ and $b_{i,k,h} \in \mathcal{C}^{\sum_{j=1}^{i-1} s_j}(\Lambda \times \Omega, \mathbb{R})$, $i = 2,3,\cdots,l$.
Expanding (3.3) leads to the relations between unknown functions $b_{i,k,h}$ of the differential operators $\mathcal{Q}_i(s_i)$ and the known functions $g_{k,h}$ of the original differential operator $\mathcal{P}(s)$.

3.1.2 Necessary and sufficient conditions for the factorization of second order nonlinear ODEs

Let Ω, Λ and Λ_0 be three open subsets of $\mathbb{R}$ such that $\Lambda_0 \subset \Lambda$. Consider the second order nonlinear ordinary differential operator

$$\begin{aligned} \mathcal{P}(2) &= \sum_{k=0}^{2} \sum_{h=1}^{p_k} g_{k,h}(x,\cdot)\mathrm{D}_{k,h} \\ &= g_{0,1}(x,\cdot)\mathrm{D}_{0,1} + g_{1,1}(x,\cdot)\mathrm{D}_{1,1} + g_{2,1}(x,\cdot)\mathrm{D}_{2,1}, \end{aligned} \tag{3.5}$$

where $g_{k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$ and $x = x^1$. Write $\mathcal{P}(2)$ in the form

$$\begin{aligned} \mathcal{P}(2) &= \mathcal{Q}_1(1) \cdot \mathcal{Q}_2(1) \\ &= \left[\sum_{k=0}^{1} \sum_{h=1}^{p_k} b_{1,k,h}(x,\cdot)\mathrm{D}_{k,h}\right] \left[\sum_{k=0}^{1} \sum_{h=1}^{p_k} b_{2,k,h}(x,\cdot)\mathrm{D}_{k,h}\right] \\ &= \left[b_{1,0,1}(x,\cdot)\mathrm{D}_{0,1} + b_{1,1,1}(x,\cdot)\mathrm{D}_{1,1}\right]\left[b_{2,0,1}(x,\cdot)\mathrm{D}_{0,1} + b_{2,1,1}(x,\cdot)\mathrm{D}_{1,1}\right], \end{aligned} \tag{3.6}$$

where $b_{1,k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$ and $b_{2,k,h} \in \mathcal{C}^1(\Lambda \times \Omega, \mathbb{R})$. Let $u \in \mathcal{C}^2(\Lambda_0, \Omega)$. Then we have

$$\mathcal{P}(2)\,u = g_{0,1}(x,u)\,u + g_{1,1}(x,u)\,u_x + g_{2,1}(x,u)\,u_{2x} \tag{3.7}$$

and after expansion

$$\begin{aligned} \mathcal{P}(2)\,u &= \left[b_{1,0,1}(x,\cdot)\mathrm{D}_{0,1} + b_{1,1,1}(x,\cdot)\mathrm{D}_{1,1}\right]\left[b_{2,0,1}(x,\cdot)\mathrm{D}_{0,1} + b_{2,1,1}(x,\cdot)\mathrm{D}_{1,1}\right]u \\ &= \left[b_{1,0,1}b_{2,1,1} + b_{1,1,1}b_{2,0,1} + b_{1,1,1}\mathrm{D}_{1,1}(b_{2,1,1}) + b_{1,1,1}\mathrm{D}_{1,2}(b_{2,0,1})u\right]u_x \\ &+ b_{1,1,1}\mathrm{D}_{1,2}(b_{2,1,1})u_x^2 + \left[b_{1,0,1}b_{2,0,1} + b_{1,1,1}\mathrm{D}_{1,1}(b_{2,0,1})\right]u + b_{1,1,1}b_{2,1,1}\,u_{2x}. \end{aligned} \tag{3.8}$$

Identifying (3.7) with (3.8) furnishes

Proposition 3.1. *A necessary and sufficient condition for the differential operator* $\mathcal{P}(2)$ *defined by (3.5) be decomposed into the form (3.6) is:*

$$\begin{aligned} g_{2,1} &= b_{1,1,1}b_{2,1,1}, && (3.9)\\ g_{1,1} &= b_{1,0,1}b_{2,1,1}+b_{1,1,1}b_{2,0,1}+b_{1,1,1}D_{1,1}(b_{2,1,1})+b_{1,1,1}D_{1,2}(b_{2,0,1})u, && (3.10)\\ 0 &= b_{1,1,1}D_{1,2}(b_{2,1,1}), && (3.11)\\ g_{0,1} &= b_{1,0,1}b_{2,0,1}+b_{1,1,1}D_{1,1}(b_{2,0,1}). && (3.12) \end{aligned}$$

3.1.3 Necessary and sufficient conditions for the factorization of second order nonlinear PDEs with two independent variables

Let Λ and Λ_0 be two open subsets of $\mathbb{R}^2$ such that $\Lambda_0 \subset \Lambda$. Let Ω be an open subset of $\mathbb{R}$. Consider the second order nonlinear partial differential operator

$$\begin{aligned} \mathcal{P}(2) &= \sum_{k=0}^{2}\sum_{h=1}^{p_k} g_{k,h}(x,\cdot)\mathrm{D}_{k,h}\\ &= g_{0,1}(x,\cdot)\mathrm{D}_{0,1}+g_{1,1}(x,\cdot)\mathrm{D}_{1,1}+g_{1,2}(x,\cdot)\mathrm{D}_{1,2}+g_{2,1}(x,\cdot)\mathrm{D}_{2,1}\\ &+ g_{2,2}(x,\cdot)\mathrm{D}_{2,2}+g_{2,3}(x,\cdot)\mathrm{D}_{2,3}+g_{2,4}(x,\cdot)\mathrm{D}_{2,4}, \end{aligned} \tag{3.13}$$

where $g_{k,h} \in \mathcal{C}(\Lambda\times\Omega,\mathbb{R})$ and $x=\left(x^1,x^2\right)$. Write $\mathcal{P}(2)$ in the form

$$\begin{aligned} \mathcal{P}(2) &= \mathcal{Q}_1(1)\cdot\mathcal{Q}_2(1)\\ &= \left[\sum_{k=0}^{1}\sum_{h=1}^{p_k} b_{1,k,h}(x,\cdot)\mathrm{D}_{k,h}\right]\left[\sum_{k=0}^{1}\sum_{h=1}^{p_k} b_{2,k,h}(x,\cdot)\mathrm{D}_{k,h}\right]\\ &= [b_{1,0,1}(x,\cdot)\mathrm{D}_{0,1}+b_{1,1,1}(x,\cdot)\mathrm{D}_{1,1}+b_{1,1,2}(x,\cdot)\mathrm{D}_{1,2}]\\ &\times [b_{2,0,1}(x,\cdot)\mathrm{D}_{0,1}+b_{2,1,1}(x,\cdot)\mathrm{D}_{1,1}+b_{2,1,2}(x,\cdot)\mathrm{D}_{1,2}], \end{aligned} \tag{3.14}$$

where $b_{1,k,h} \in \mathcal{C}(\Lambda\times\Omega,\mathbb{R})$ and $b_{2,k,h} \in \mathcal{C}^1(\Lambda\times\Omega,\mathbb{R})$. Let $u \in \mathcal{C}^2(\Lambda_0,\Omega)$. Then we have

$$\begin{aligned} \mathcal{P}(2)u &= g_{0,1}(x,u)\,u+g_{1,1}(x,u)\,u_{x^1}+g_{1,2}(x,u)\,u_{x^2}+g_{2,1}(x,u)\,u_{2x^1}\\ &+ \left(g_{2,2}(x,u)+g_{2,3}(x,u)\right)u_{x^1x^2}+g_{2,4}(x,u)\,u_{2x^2} \end{aligned} \tag{3.15}$$

and after expansion

$$\begin{aligned} \mathcal{P}(2)u &= [b_{1,0,1}(x,\cdot)\mathrm{D}_{0,1}+b_{1,1,1}(x,\cdot)\mathrm{D}_{1,1}+b_{1,1,2}(x,\cdot)\mathrm{D}_{1,2}]\\ &\times [b_{2,0,1}(x,\cdot)\mathrm{D}_{0,1}+b_{2,1,1}(x,\cdot)\mathrm{D}_{1,1}+b_{2,1,2}(x,\cdot)\mathrm{D}_{1,2}]\,u\\ &= [b_{1,0,1}b_{2,0,1}+b_{1,1,1}\mathrm{D}_{1,1}(b_{2,0,1})+b_{1,1,2}\mathrm{D}_{1,2}(b_{2,0,1})]\,u\\ &+ [b_{1,0,1}b_{2,1,1}+b_{1,1,1}b_{2,0,1}+b_{1,1,1}\mathrm{D}_{1,1}(b_{2,1,1})+b_{1,1,2}\mathrm{D}_{1,2}(b_{2,1,1})\\ &+ b_{1,1,1}\mathrm{D}_{1,3}(b_{2,0,1})\,u]\,u_{x^1}+[b_{1,0,1}b_{2,1,2}+b_{1,1,2}b_{2,0,1}+b_{1,1,1}\mathrm{D}_{1,1}(b_{2,1,2})\\ &+ b_{1,1,2}\mathrm{D}_{1,2}(b_{2,1,2})+b_{1,1,2}\mathrm{D}_{1,3}(b_{2,0,1})\,u]\,u_{x^2}+b_{1,1,1}\mathrm{D}_{1,3}(b_{2,1,1})u_{x^1}^2\\ &+ [b_{1,1,1}\mathrm{D}_{1,3}(b_{2,1,2})+b_{1,1,2}\mathrm{D}_{1,3}(b_{2,1,1})]\,u_{x^1}u_{x^2}+b_{1,1,2}\mathrm{D}_{1,3}(b_{2,1,2})u_{x^2}^2\\ &+ b_{1,1,1}b_{2,1,1}\,u_{2x^1}+[b_{1,1,2}b_{2,1,1}+b_{1,1,1}b_{2,1,2}]\,u_{x^1x^2}+b_{1,1,2}b_{2,1,2}\,u_{2x^2}. \end{aligned} \tag{3.16}$$

Identifying (3.15) with (3.16) yields

Proposition 3.2. *A necessary and sufficient condition for the differential operator* $\mathcal{P}(2)$ *defined by (3.13) be decomposed into the form (3.14) is:*

$$\begin{aligned}
g_{2,1} &= b_{1,1,1}b_{2,1,1}, && (3.17)\\
g_{2,2}+g_{2,3} &= b_{1,1,2}b_{2,1,1}+b_{1,1,1}b_{2,1,2}, && (3.18)\\
g_{2,4} &= b_{1,1,2}b_{2,1,2}, && (3.19)\\
g_{1,1} &= b_{1,0,1}b_{2,1,1}+b_{1,1,1}b_{2,0,1}+\mathcal{L}(b_{2,1,1})+b_{1,1,1}D_{1,3}(b_{2,0,1})\,u, && (3.20)\\
g_{1,2} &= b_{1,0,1}b_{2,1,2}+b_{1,1,2}b_{2,0,1}+\mathcal{L}(b_{2,1,2})+b_{1,1,2}D_{1,3}(b_{2,0,1})\,u, && (3.21)\\
g_{0,1} &= b_{1,0,1}b_{2,0,1}+\mathcal{L}(b_{2,0,2}), && (3.22)\\
0 &= b_{1,1,2}D_{1,3}(b_{2,1,2}), && (3.23)\\
0 &= b_{1,1,1}D_{1,3}(b_{2,1,1}), && (3.24)\\
0 &= b_{1,1,1}D_{1,3}(b_{2,1,2})+b_{1,1,2}D_{1,3}(b_{2,1,1}), && (3.25)
\end{aligned}$$

where $\mathcal{L}=b_{1,1,1}D_{1,1}+b_{1,1,2}D_{1,2}$.

3.2 Factorizations of systems of nonlinear differential equations

3.2.1 Theoretical considerations and principles

Let Λ be an open subset of $\mathbb{R}^n$ and Ω, an open subset of $\mathbb{R}^m$. Examine now the factorization process for systems of s-th order, $(s\geq 2)$, nonlinear differential equations with n independent variables $x=\left(x^1,\cdots,x^n\right)$ and $m\geq 2$ dependent variables $u={}^t\left(u^1,\cdots,u^m\right)$, $u=u(x)$ whose associated matrix operator, $\mathcal{M}(s)$, is of the form

$$\mathcal{M}(s)=\left[\mathcal{R}_{p,q}(s_{p,q})\right]_{1\leq p,q\leq m}; \tag{3.26}$$

the $\mathcal{R}_{p,q}(s_{p,q})$ are $s_{p,q}$-th order linear differential operators

$$\mathcal{R}_{p,q}(s_{p,q})=\sum_{k=0}^{s_{p,q}}\sum_{h=1}^{p_k} f_{p,q,k,h}(x,\underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{k,h}, \tag{3.27}$$

where $f_{p,q,k,h}\in\mathcal{C}(\Lambda\times\Omega,\mathbb{R})$, $s_{p,q}=s-1+\delta_{p,q}$, $\delta_{p,p}=1$ and $\delta_{p,q}=0$ if $p\neq q$.
Let Λ and Λ_0 be two open subsets of $\mathbb{R}^n$ such that $\Lambda_0\subset\Lambda$. The matrix operator $\mathcal{M}(s)$ acts on a vector valued function $u={}^t\left(u^1,\cdots,u^m\right)\in\mathcal{C}^s(\Lambda_0,\Omega)$ as follows

$$\mathcal{M}(s)\,u=\left[\mathcal{R}_{p,q}(s_{p,q})\right]_{1\leq p,q\leq m}u=\left[\sum_{q=1}^{m}\mathcal{R}_{p,q}(s_{p,q})\,u^q\right]_{1\leq p\leq m},$$

with

$$\mathcal{R}_{p,q}(s_{p,q})\,u^q=\sum_{k=0}^{s_{p,q}}\sum_{h=1}^{p_k} f_{p,q,k,h}\left(x,u^1,\ldots,u^m\right)\mathrm{D}_{k,h}\,u^q. \tag{3.28}$$

The method of factorization consists in seeking a decomposition of the matrix $\mathcal{M}(s)$ under the following form

$$\mathcal{M}(s)=\prod_{i=1}^{l}\mathcal{N}_i(s_i) \tag{3.29}$$

where

$$\mathcal{N}_i(s_i) = [\mathcal{T}_{i,p,q}(s_{i,p,q})]_{1\leq p,q\leq m} \tag{3.30}$$

and

$$\mathcal{T}_{i,p,q}(s_{i,p,q}) = \sum_{k=0}^{s_{i,p,q}} \sum_{h=1}^{p_k} a_{i,p,q,k,h}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{k,h}, \tag{3.31}$$

with $\sum_{i=1}^{l} s_i = s$, $s_{i,p,q} = s_i - 1 + \delta_{p,q}$, $a_{1,p,q,k,h} \in \mathcal{C}(\Lambda\times\Omega,\mathbb{R})$ and $a_{i,p,q,k,h} \in \mathcal{C}^{\sum_{j=1}^{i-1} s_{i,p,q}}(\Lambda\times\Omega,\mathbb{R})$, $i = 2,3,\cdots,l$.
Expanding (3.29) leads to the relations between the unknown functions $a_{i,p,q,k,h}$ of $\mathcal{N}_i(s_i)$ and the known functions $f_{p,q,k,h}$ of $\mathcal{M}(s)$.

3.2.2 Necessary and sufficient conditions for the factorization of systems of second order nonlinear ODEs

Let Λ, Λ_0 be two open subsets of $\mathbb{R}$ such that $\Lambda_0 \subset \Lambda$, and Ω an open subset of $\mathbb{R}^m$. Consider the matrix operator

$$\mathcal{M}(2) = [\mathcal{R}_{p,q}]_{1\leq p,q\leq m}, \tag{3.32}$$

where

$$\begin{aligned}\mathcal{R}_{p,p} &= \sum_{k=0}^{2}\sum_{h=1}^{p_k} f_{p,q,k,h}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{k,h} \\ &= f_{p,p,0,1}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{0,1} + f_{p,p,1,1}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{1,1} + f_{p,p,2,1}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{2,1}\end{aligned} \tag{3.33}$$

and for $p \neq q$

$$\begin{aligned}\mathcal{R}_{p,q} &= \sum_{k=0}^{1}\sum_{h=1}^{p_k} f_{p,q,k,h}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{k,h} \\ &= f_{p,q,0,1}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{0,1} + f_{p,q,1,1}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{1,1}\end{aligned} \tag{3.34}$$

with $f_{p,q,k,h} \in \mathcal{C}(\Lambda\times\Omega,\mathbb{R})$, $x = x^1$. Write $\mathcal{M}(2)$ in the form

$$\mathcal{M}(2) = \mathcal{N}_1(1)\cdot\mathcal{N}_2(1), \tag{3.35}$$

where

$$\mathcal{N}_i(1) = [\mathcal{T}_{i,p,q}]_{1\leq p,q\leq m} \tag{3.36}$$

with

$$\begin{aligned}\mathcal{T}_{i,p,p} &= \sum_{k=0}^{1}\sum_{h=1}^{p_k} a_{i,p,p,k,h}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{k,h} \\ &= a_{i,p,p,0,1}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{0,1} + a_{i,p,p,1,1}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{1,1}\end{aligned} \tag{3.37}$$

and for $p \neq q$

$$\mathcal{T}_{i,p,q} = a_{i,p,q,0,1}(x, \underbrace{\cdot, \ldots, \cdot}_{m\text{-entries}})\mathrm{D}_{0,1}, \tag{3.38}$$

$a_{1,p,q,k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$ and $a_{2,p,q,k,h} \in \mathcal{C}^1(\Lambda \times \Omega, \mathbb{R})$. Let $u = {}^t\left(u^1, \cdots, u^m\right) \in \mathcal{C}^2(\Lambda_0, \Omega)$. Then we have

$$\mathcal{M}(2)\, u = [\mathcal{R}_{p,q}]_{1\leq p,q\leq m}\, u = \left[\sum_{q=1}^{m} \mathcal{R}_{p,q}\, u^q\right]_{1\leq p\leq m} \tag{3.39}$$

where

$$\mathcal{R}_{p,p}\, u^p = f_{p,p,0,1}(x,u)\, u^p + f_{p,p,1,1}(x,u)\, u_x^p + f_{p,p,2,1}(x,u)\, u_{2x}^p$$

and for $p \neq q$

$$\mathcal{R}_{p,q}\, u^q = f_{p,q,0,1}(x,u)\, u^q + f_{p,q,1,1}(x,u)\, u_x^q.$$

On the other hand, after expansion of (3.35), we have

$$\mathcal{M}(2)\, u = \left[\widetilde{\mathcal{R}}_{p,q}\right]_{1\leq p,q\leq m} u = \left[\sum_{q=1}^{m} \widetilde{\mathcal{R}}_{p,q}\, u^q\right]_{1\leq p\leq m}, \tag{3.40}$$

where

$$\begin{aligned}
\widetilde{\mathcal{R}}_{p,p}\, u^p \;=\;& a_{1,p,p,1,1}a_{2,p,p,1,1}\, u_{2x}^p + a_{1,p,p,1,1}\sum_{\widetilde{h}=1}^{m} \mathrm{D}_{1,\widetilde{h}+1}(a_{2,p,p,1,1})\, u_x^{\widetilde{h}}\, u_x^p \\
&+ a_{1,p,p,1,1}\sum_{\substack{\widetilde{h}=1\\ \widetilde{h}\neq p}}^{m} \mathrm{D}_{1,\widetilde{h}+1}(a_{2,p,p,0,1})\, u_x^{\widetilde{h}}\, u^p + [a_{1,p,p,0,1}a_{2,p,p,1,1} + a_{1,p,p,1,1}a_{2,p,p,0,1} \\
&+ a_{1,p,p,1,1}\mathrm{D}_{1,1}(a_{2,p,p,1,1}) + a_{1,p,p,1,1}\mathrm{D}_{1,p+1}(a_{2,p,p,0,1})\, u^p]\, u_x^p \\
&+ \left[\sum_{l=1}^{m} a_{1,p,l,0,1}a_{2,l,p,0,1} + a_{1,p,p,1,1}\mathrm{D}_{1,1}(a_{2,p,p,0,1})\right] u^p
\end{aligned}$$

and for $p \neq q$

$$\begin{aligned}
\widetilde{\mathcal{R}}_{p,q}\, u^q \;=\;& [a_{1,p,p,1,1}a_{2,p,q,0,1} + a_{1,p,q,0,1}a_{2,q,q,1,1} + a_{1,p,p,1,1}\mathrm{D}_{1,q+1}(a_{2,p,q,0,1})\, u^q]\, u_x^q \\
&+ a_{1,p,p,1,1}\sum_{\substack{\widetilde{h}=1\\ \widetilde{h}\neq q}}^{m} \mathrm{D}_{1,\widetilde{h}+1}(a_{2,p,q,0,1})\, u_x^{\widetilde{h}}\, u^q \\
&+ \left[\sum_{l=1}^{m} a_{1,p,l,0,1}a_{2,l,q,0,1} + a_{1,p,p,1,1}\mathrm{D}_{1,1}(a_{2,p,q,0,1})\right] u^q.
\end{aligned}$$

Identifying (3.39) with (3.40) yields

Proposition 3.3. *A necessary and sufficient condition for the differential operator* $\mathcal{M}(2)$

defined by (3.32) be decomposed into the form (3.35) is:

$$\begin{aligned} f_{p,p,0,1} &= \sum_{l=1}^{m} a_{1,p,l,0,1} a_{2,l,p,0,1} + a_{1,p,p,1,1} D_{1,1}(a_{2,p,p,0,1}), & (3.41)\\ f_{p,p,1,1} &= a_{1,p,p,0,1} a_{2,p,p,1,1} + a_{1,p,p,1,1} a_{2,p,p,0,1} \\ &+ a_{1,p,p,1,1} D_{1,1}(a_{2,p,p,1,1}) + a_{1,p,p,1,1} D_{1,p+1}(a_{2,p,p,0,1})\, u^{p}, & (3.42)\\ f_{p,p,2,1} &= a_{1,p,p,1,1} a_{2,p,p,1,1}, & (3.43)\\ 0 &= D_{1,\widetilde{h}+1}(a_{2,p,p,0,1}), \quad \widetilde{h} \in \{1,2,\cdots,m\} \setminus \{p\}, & (3.44)\\ 0 &= D_{1,\widetilde{h}+1}(a_{2,p,p,1,1}), \quad \widetilde{h} = 1,2,\cdots,m & (3.45) \end{aligned}$$

and for $p \neq q$

$$\begin{aligned} f_{p,q,0,1} &= \sum_{l=1}^{m} a_{1,p,l,0,1} a_{2,l,q,0,1} + a_{1,p,p,1,1} D_{1,1}(a_{2,p,q,0,1}), & (3.46)\\ f_{p,q,1,1} &= a_{1,p,p,1,1} a_{2,p,q,0,1} + a_{1,p,q,0,1} a_{2,q,q,1,1} + a_{1,p,p,1,1} D_{1,q+1}(a_{2,p,q,0,1})\, u^{q}, & (3.47)\\ 0 &= D_{1,\widetilde{h}+1}(a_{2,p,q,0,1}), \quad \widetilde{h} \in \{1,2,\cdots,m\} \setminus \{q\}. & (3.48) \end{aligned}$$

3.2.3 Necessary and sufficient conditions for the factorization of systems of second order nonlinear PDEs with two independent variables

Let Λ, Λ_0 be two open subsets of $\mathbb{R}^2$ such that $\Lambda_0 \subset \Lambda$, and Ω an open subset of $\mathbb{R}^m$. Consider the matrix operator

$$\mathcal{M}(2) = [\mathcal{R}_{p,q}]_{1 \leq p,q \leq m}, \tag{3.49}$$

where

$$\begin{aligned} \mathcal{R}_{p,p} &= \sum_{k=0}^{2} \sum_{h=1}^{p_k} f_{p,q,k,h}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}}) \mathrm{D}_{k,h} & (3.50)\\ &= f_{p,p,0,1}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}}) \mathrm{D}_{0,1} + f_{p,p,1,1}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}}) \mathrm{D}_{1,1} + f_{p,p,1,2}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}}) \mathrm{D}_{1,2} \\ &+ f_{p,p,2,1}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}}) \mathrm{D}_{2,1} + f_{p,p,2,2}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}}) \mathrm{D}_{2,2} \\ &+ f_{p,p,2,3}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}}) \mathrm{D}_{2,3} + f_{p,p,2,4}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}}) \mathrm{D}_{2,4} \end{aligned}$$

and for $p \neq q$

$$\begin{aligned} \mathcal{R}_{p,q} &= \sum_{k=0}^{1} \sum_{h=1}^{p_k} f_{p,q,k,h}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}}) \mathrm{D}_{k,h} & (3.51)\\ &= f_{p,q,0,1}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}}) \mathrm{D}_{0,1} + f_{p,q,1,1}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}}) \mathrm{D}_{1,1} + f_{p,q,1,2}(x, \underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}}) \mathrm{D}_{1,2} \end{aligned}$$

with $f_{p,q,k,h} \in \mathcal{C}(\Lambda \times \Omega, \mathbb{R})$, $x = \left(x^1, x^2\right)$. Write $\mathcal{M}(2)$ in the form

$$\mathcal{M}(2) = \mathcal{N}_1(1) \cdot \mathcal{N}_2(1), \tag{3.52}$$

where

$$\mathcal{N}_i(1) = [\mathcal{T}_{i,p,q}]_{1\leq p,q\leq m} \tag{3.53}$$

with

$$\begin{aligned} \mathcal{T}_{i,p,p} &= \sum_{k=0}^{1}\sum_{h=1}^{p_k} a_{i,p,p,k,h}(x\underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{k,h} \\ &= a_{i,p,p,0,1}(x\underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{0,1} + a_{i,p,p,1,1}(x\underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{1,1} + a_{i,p,p,1,2}(x\underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{1,2} \end{aligned} \tag{3.54}$$

and for $p \neq q$

$$\mathcal{T}_{i,p,q} = a_{i,p,q,0,1}(x\underbrace{\cdot,\ldots,\cdot}_{m\text{-entries}})\mathrm{D}_{0,1}, \tag{3.55}$$

$a_{1,p,q,k,h} \in \mathcal{C}(\Lambda\times\Omega,\mathbb{R})$ and $a_{2,p,q,k,h} \in \mathcal{C}^1(\Lambda\times\Omega,\mathbb{R})$. Let $u = {}^t\left(u^1,\cdots,u^m\right) \in \mathcal{C}^2(\Lambda_0,\Omega)$. Then we have

$$\mathcal{M}\,(2)\,u = [\mathcal{R}_{p,q}]_{1\leq p,q\leq m}\,u = \left[\sum_{q=1}^{m}\mathcal{R}_{p,q}\,u^q\right]_{1\leq p\leq m}, \tag{3.56}$$

where

$$\begin{aligned} \mathcal{R}_{p,p}\,u^p &= f_{p,p,0,1}\,u^p + f_{p,p,1,1}\,u^p_{x^1} + f_{p,p,1,2}\,u^p_{x^2} \\ &+ f_{p,p,2,1}\,u^p_{2x^1} + (f_{p,p,2,2}+f_{p,p,2,3})\,u^p_{x^1x^2} + f_{p,p,2,4}\,u^p_{2x^2} \end{aligned} \tag{3.57}$$

and for $p \neq q$

$$\mathcal{R}_{p,q}\,u^q = f_{p,q,0,1}\,u^q + f_{p,q,1,1}\,u^q_{x^1} + f_{p,q,1,2}\,u^q_{x^2}.$$

On the other hand, after expansion of (3.52), we have

$$\mathcal{M}\,(2)\,u = \left[\widetilde{\mathcal{R}}_{p,q}\right]_{1\leq p,q\leq m} u = \left[\sum_{q=1}^{m}\widetilde{\mathcal{R}}_{p,q}\,u^q\right]_{1\leq p\leq m}, \tag{3.58}$$

where

$$\begin{aligned} \widetilde{\mathcal{R}}_{p,p}\,u^p &= a_{1,p,p,1,1}a_{2,p,p,1,1}\,u^p_{2x^1} + (a_{1,p,p,1,2}a_{2,p,p,1,1} + a_{1,p,p,1,1}a_{2,p,p,1,2})\,u^p_{x^1x^2} \\ &+ a_{1,p,p,1,2}a_{2,p,p,1,2}\,u^p_{2x^2}\,[a_{1,p,p,0,1}a_{2,p,p,1,1} + a_{1,p,p,1,1}a_{2,p,p,0,1} \\ &+ \mathcal{L}_p(a_{2,p,p,1,1}) + a_{1,p,p,1,1}\mathrm{D}_{1,p+2}(a_{2,p,p,0,1})\,u^p]\,u^p_{x^1} + [a_{1,p,p,0,1}a_{2,p,p,1,2} \\ &+ a_{1,p,p,1,2}a_{2,p,p,0,1} + \mathcal{L}_p(a_{2,p,p,1,2}) + a_{1,p,p,1,2}\mathrm{D}_{1,p+2}(a_{2,p,p,0,1})\,u^p]\,u^p_{x^2} \\ &+ a_{1,p,p,1,1}\sum_{\widetilde{h}=1}^{m}\mathrm{D}_{1,\widetilde{h}+2}(a_{2,p,p,1,1})\,u^{\widetilde{h}}_{x^1}\,u^p_{x^1} + a_{1,p,p,1,1}\sum_{\widetilde{h}=1}^{m}\mathrm{D}_{1,\widetilde{h}+2}(a_{2,p,p,1,2})\,u^{\widetilde{h}}_{x^1}\,u^p_{x^2} \\ &+ a_{1,p,p,1,2}\sum_{\widetilde{h}=1}^{m}\mathrm{D}_{1,\widetilde{h}+2}(a_{2,p,p,1,1})\,u^{\widetilde{h}}_{x^2}\,u^p_{x^1} + a_{1,p,p,1,2}\sum_{\widetilde{h}=1}^{m}\mathrm{D}_{1,\widetilde{h}+2}(a_{2,p,p,1,2})\,u^{\widetilde{h}}_{x^2}\,u^p_{x^2} \\ &+ a_{1,p,p,1,1}\sum_{\substack{\widetilde{h}=1\\ \widetilde{h}\neq p}}^{m}\mathrm{D}_{1,\widetilde{h}+2}(a_{2,p,p,0,1})\,u^{\widetilde{h}}_{x^1}\,u^p + a_{1,p,p,1,2}\sum_{\substack{\widetilde{h}=1\\ \widetilde{h}\neq p}}^{m}\mathrm{D}_{1,\widetilde{h}+2}(a_{2,p,p,0,1})\,u^{\widetilde{h}}_{x^2}\,u^p \\ &+ \left[\sum_{l=1}^{m}a_{1,p,l,0,1}a_{2,l,p,0,1} + \mathcal{L}_p(a_{2,p,p,0,1})\right]u^p \end{aligned}$$

and for $p \neq q$

$$\begin{aligned}
\widetilde{\mathcal{R}}_{p,q}\, u^q &= [a_{1,p,p,1,1}a_{2,p,q,0,1} + a_{1,p,q,0,1}a_{2,q,q,1,1} + a_{1,p,p,1,1}\mathrm{D}_{1,q+2}(a_{2,p,q,0,1})\, u^q]\, u^q_{x^1} \\
&+ [a_{1,p,p,1,2}a_{2,p,q,0,1} + a_{1,p,q,0,1}a_{2,q,q,1,2} + a_{1,p,p,1,2}\mathrm{D}_{1,q+2}(a_{2,p,q,0,1})\, u^q]\, u^q_{x^2} \\
&+ a_{1,p,p,1,1}\sum_{\substack{\widetilde{h}=1\\ \widetilde{h}\neq q}}^{m}\mathrm{D}_{1,\widetilde{h}+2}(a_{2,p,q,0,1})\, u^{\widetilde{h}}_{x^1}\, u^q + a_{1,p,p,1,2}\sum_{\substack{\widetilde{h}=1\\ \widetilde{h}\neq q}}^{m}\mathrm{D}_{1,\widetilde{h}+2}(a_{2,p,q,0,1})\, u^{\widetilde{h}}_{x^2}\, u^q \\
&+ \left[\sum_{l=1}^{m} a_{1,p,l,0,1}a_{2,l,q,0,1} + \mathcal{L}_p(a_{2,p,q,0,1})\right] u^q,
\end{aligned}$$

where $\mathcal{L}_p = a_{1,p,p,1,1}\mathrm{D}_{1,1} + a_{1,p,p,1,2}\mathrm{D}_{1,2}$. Identifying (3.56) with (3.58) yields

Proposition 3.4. *A necessary and sufficient condition for the differential operator $\mathcal{M}(2)$ defined by (3.49) be decomposed into the form (3.52) is:*

$$\begin{aligned}
f_{p,p,0,1} &= \sum_{l=1}^{m} a_{1,p,l,0,1}a_{2,l,p,0,1} + \mathcal{L}_p(a_{2,p,p,0,1}), && (3.59)\\
f_{p,p,1,1} &= a_{1,p,p,0,1}a_{2,p,p,1,1} + a_{1,p,p,1,1}a_{2,p,p,0,1} \\
&+ \mathcal{L}_p(a_{2,p,p,1,1}) + a_{1,p,p,1,1}D_{1,p+2}(a_{2,p,p,0,1})\, u^p, && (3.60)\\
f_{p,p,1,2} &= a_{1,p,p,0,1}a_{2,p,p,1,2} + a_{1,p,p,1,2}a_{2,p,p,0,1} \\
&+ \mathcal{L}_p(a_{2,p,p,1,2}) + a_{1,p,p,1,2}D_{1,p+2}(a_{2,p,p,0,1})\, u^p, && (3.61)\\
f_{p,p,2,1} &= a_{1,p,p,1,1}a_{2,p,p,1,1}, && (3.62)\\
f_{p,p,2,2} + f_{p,p,2,3} &= a_{1,p,p,1,2}a_{2,p,p,1,1} + a_{1,p,p,1,1}a_{2,p,p,1,2}, && (3.63)\\
f_{p,p,2,4} &= a_{1,p,p,1,2}a_{2,p,p,1,2}, && (3.64)\\
0 &= D_{1,\widetilde{h}+2}(a_{2,p,p,0,1}), \quad \widetilde{h} \in \{1,2,\cdots,m\}\setminus\{p\}, && (3.65)\\
0 &= D_{1,\widetilde{h}+2}(a_{2,p,p,1,1}), \quad \widetilde{h} = 1,2,\cdots,m, && (3.66)\\
0 &= D_{1,\widetilde{h}+2}(a_{2,p,p,1,2}), \quad \widetilde{h} = 1,2,\cdots,m && (3.67)
\end{aligned}$$

and for $p \neq q$

$$\begin{aligned}
f_{p,q,0,1} &= \sum_{l=1}^{m} a_{1,p,l,0,1}a_{2,l,q,0,1} + \mathcal{L}_p(a_{2,p,q,0,1}), && (3.68)\\
f_{p,q,1,1} &= a_{1,p,p,1,1}a_{2,p,q,0,1} + a_{1,p,q,0,1}a_{2,q,q,1,1} + a_{1,p,p,1,1}D_{1,q+2}(a_{2,p,q,0,1})\, u^q, && (3.69)\\
f_{p,q,1,2} &= a_{1,p,p,1,2}a_{2,p,q,0,1} + a_{1,p,q,0,1}a_{2,q,q,1,2} + a_{1,p,p,1,2}D_{1,q+2}(a_{2,p,q,0,1})\, u^q, && (3.70)\\
0 &= D_{1,\widetilde{h}+2}(a_{2,p,q,0,1}), \quad \widetilde{h} \in \{1,2,\cdots,m\}\setminus\{q\}. && (3.71)
\end{aligned}$$

Acknowledgments

This work is partially supported by the ICTP through the OEA-ICMPA-Prj-15. The ICMPA is in partnership with the Daniel Iagolnitzer Foundation (DIF), France.

References

[1] Fred Cooper, Avinash Khare, Uday Sukhatme, Los Alamos, *Supersymmetry in quantum mechanics*, World scientific publishing Co. pte. Ltd. (2001).

[2] M. N. Hounkonnou, K. Sodoga, and E. C. Azatassou, Factorization of Sturm Liouville operators: solvable potentials and underlying algebraic structure. *J. Phys. A: Math. Gen.*, vol. **38** (2005), 371-390.

[3] G. Junker, and P. Roy Conditionally exactly solvable potentials: A supersymmetric construction method. *Annals of Physics* **270** 1 (1998), 155-177.

[4] V. M. Tkachuk, Supersymmetric method for constructing quasi-exactly and conditionally-exactly solvable potentials. *J. Phys. A: Math. Gen.* **32** (1999) 1291.

[5] E. Beke, Die Irreducibilität der homogenen linearen Differentialgleichungen. *Mathematische Annalen* 45 (1894), 185-195.

[6] M. Van Hoeij, Factorization of differential operators with rational functions coefficients, *J. Symbolic Computation* **24** (1997), 537-561.

[7] L. M. Berkovich, Factorization of some classes of nonlinear ordinary differential equations: methods and algorithms, Poster Session Abstracts of the 1998 International Symposium on Symbolic and Algebraic Computation ISSAC'1998, Rostock, Germany, Aug. 13-15, 1998, George Labahn, Mohamed Omar rayes, Editors, 1-15.

[8] L. M. Berkovich, method of factorization of ordinary differential operators and some of its applications, *Applicable Analysis and Discrete Mathematics*, *1* (2007), 122-149.

[9] L. M. Berkovich, Factorization and transformations of ordinary differential equations, *Saratov University Publ.*, 1989.

[10] M. N. Hounkonnou, A. Ronveaux and K. Sodoga, Factorization of some confluent Heun's differential equations, *Appl. Math. Comput.* vol. **189** (2007), 816-820.

[11] A. Ronveaux, Factorization of the Heun's differential equation, *Appl. Math. Comput.* **141** (2003), 177-184.

[12] M. N. Hounkonnou and A. Ronveaux, Factorization of the Lamé differential operator, *Commun. Math. Anal.* Volume **11**, Number 1 (2011), 121-136.

[13] M. N. Hounkonnou and P. A. Dkengne Sielenou, On factorizable classes of second order linear differential equations with rational functions coefficients, *SUT Journal of Mathematics* Vol. 46, No. 2 (2010), 205-229.

[14] M. N. Hounkonnou and P. A. Dkengne Sielenou, An algebraic method of factorization of ordinary differential equations, *XXVIII WORKSHOP ON GEOMETRICAL METHODS IN PHYSICS. AIP Conference Proceedings*, Volume 1191 (2009), 104-109, DOI: 10.1063/1.3275580.