Nonlinear self-adjointness and conservation laws

Nail H. Ibragimov
Department of Mathematics and Science, Blekinge Institute of Technology, 371 79 Karlskrona, Sweden

Abstract. The general concept of nonlinear self-adjointness of differential equations is introduced. It includes the linear self-adjointness as a particular case. Moreover, it embraces the strict self-adjointness and quasi self-adjointness introduced earlier by the author. It is shown that the equations possessing the nonlinear self-adjointness can be written equivalently in a strictly self-adjoint form by using appropriate multipliers. All linear equations possess the property of nonlinear self-adjointness, and hence can be rewritten in a nonlinear strictly self-adjoint. For example, the heat equation $u_t - \Delta u = 0$ becomes strictly self-adjoint after multiplying by u^{-1}. Conservation laws associated with symmetries can be constructed for all differential equations and systems having the property of nonlinear self-adjointness.

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1 Preliminaries

The concept of self-adjointness of nonlinear equations was introduced [1] for constructing conservation laws associated with symmetries of differential equations. To extend the possibilities of the new method for constructing conservation laws [2] the notion of quasi self-adjointness was used in [3]. I suggest here the general concept of *nonlinear self-adjointness*. It embraces the strict self-adjointness and quasi self-adjointness introduced earlier as well as the usual linear self-adjointness. Moreover, it will be shown that *all linear equations* possess the property of nonlinear self-adjointness.

It is demonstrated that the equations possessing the nonlinear self-adjointness can be written equivalently in a strictly self-adjoint form by using appropriate multipliers. Consequently, any linear equation can be rewritten in an equivalent nonlinear form which is strictly self-adjoint. For example, the heat equation $u_t - \Delta u = 0$ becomes strictly self-adjoint if we rewrite it in the form $u^{-1}(u_t - \Delta u) = 0$.

The construction of conservation laws demonstrates a practical significance of the nonlinear self-adjointness. Namely, *conservation laws can be associated with symmetries for all linear and nonlinear self-adjoint differential equations.*

1.1 Notation

Let $x = (x^1, \ldots, x^n)$ be independent variables. We consider two sets of dependent variables, $u = (u^1, \ldots, u^m)$ and $v = (v^1, \ldots, v^m)$ with respective partial derivatives

$$u_{(1)} = \{u_i^\alpha\}, \quad u_{(2)} = \{u_{ij}^\alpha\}, \ \ldots, \ u_{(s)} = \{u_{i_1 \cdots i_s}^\alpha\}$$

and

$$v_{(1)} = \{v_i^\alpha\}, \quad v_{(2)} = \{v_{ij}^\alpha\}, \ \ldots, \ v_{(s)} = \{v_{i_1 \cdots i_s}^\alpha\},$$

where

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_i D_j(u^\alpha), \quad u_{i_1 \cdots i_s}^\alpha = D_{i_1} \cdots D_{i_s}(u^\alpha),$$

$$v_i^\alpha = D_i(v^\alpha), \quad v_{ij}^\alpha = D_i D_j(v^\alpha), \quad v_{i_1 \cdots i_s}^\alpha = D_{i_1} \cdots D_{i_s}(v^\alpha).$$

Here and in what follows D_i denotes the operator of total differentiation:

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + v_i^\alpha \frac{\partial}{\partial v^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + v_{ij}^\alpha \frac{\partial}{\partial v_j^\alpha} + u_{ijk}^\alpha \frac{\partial}{\partial u_{jk}^\alpha} + v_{ijk}^\alpha \frac{\partial}{\partial v_{jk}^\alpha} + \cdots .$$

1.2 Adjoint equations

We will consider systems of m differential equations (linear or non-linear)

$$F_\alpha\big(x, u, u_{(1)}, \ldots, u_{(s)}\big) = 0, \quad \alpha = 1, \ldots, m, \tag{1}$$

with m dependent variables. The *adjoint equations* to equations (1) are written [1]

$$F_\alpha^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) = 0, \quad \alpha = 1, \ldots, m, \tag{2}$$

with the adjoint operator F_α^* defined by

$$F_\alpha^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) = \frac{\delta \mathcal{L}}{\delta u^\alpha}\,, \tag{3}$$

where $\mathcal{L}$ the *formal Lagrangian* for equations (1) given by

$$\mathcal{L} = \sum_{\beta=1}^{m} v^\beta F_\beta(x, u, u_{(1)}, \ldots, u_{(s)}) \tag{4}$$

and $\delta/\delta u^\alpha$ is the variational derivative

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^{\infty}(-1)^s D_{i_1}\cdots D_{i_s}\,\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}\,\cdot$$

For a linear equation $L[u]=0$ the adjoint operator defined by (3) is identical with the classical adjoint operator $L^*[v]$ determined by the equation $vL[u]-uL^*[v]=D_i(p^i)$.

The adjointness of *linear* operators L is a *symmetric relation*, namely $(L^*)^* = L$. Nonlinear equations do not possess this property so that, in general, $(F^*)^* \neq F$.

1.3 Self-adjointness

A linear operator L is said to be self-adjoint if $L^* = L$. Then we also say that the equation $L[u]=0$ is self-adjoint. Thus, the self-adjointness of a linear equation $L[u]=0$ means that the adjoint equation $L^*[v]=0$ coincides with $L[u]=0$ upon the substitution

$$v = u. \tag{5}$$

This property has been extended to nonlinear equations [1] by the following definition.

Definition 1. Equation (1) is *self-adjoint* if the adjoint equation (2) becomes equivalent to the original equation (1) upon the substitution (5).

For example, the Korteweg-de Vries (KdV) equation $u_t = u_{xxx} + uu_x$ is self-adjoint. Indeed, its adjoint equation (2) has the form $v_t = v_{xxx} + uv_x$ and coincides with the KdV equation upon setting $v=u$.

The concept of quasi self-adjointness introduced in [3] generalizes Definition 1 by replacing (5) with the substitution of the form

$$v = \varphi(u), \quad \varphi'(u) \neq 0. \tag{6}$$

Thus, equation (1) is *quasi self-adjoint* if the adjoint equation (2) becomes equivalent to equation (1) upon the substitution (6). Let us consider as an example the equation

$$u_t - u^2 u_{xx} = 0 \tag{7}$$

describing the nonlinear heat conduction in solid hydrogen [1]. Its adjoint equation (2) is

$$v_t + 4uvu_{xx} + u^2 v_{xx} + 4uu_x v_x + 2vu_x^2 = 0.$$

[1]It is recalled that (7) is related to the classical 1+1-dimensional heat equation by a differential substitution [4] or a reciprocal transformation [5]. This connection, together with its extensions, allows the analytic solution of certain moving boundary problems in nonlinear heat conduction [6].

It becomes equivalent to equation (7) not after the substitution (5) but after the following substitution of the form (6):

$$v = u^{-2}\,. \tag{8}$$

1.4 Theorem on conservation laws

We will use the following statement proved in [2].

Theorem 1. Any infinitesimal symmetry (Lie point, Lie-Bäcklund, nonlocal)

$$X = \xi^i(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial u^\alpha}$$

of equations (1) leads to a conservation law $D_i(C^i) = 0$ constructed by the formula

$$C^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\Big(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\Big) + D_j D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\Big) - \ldots\right] \tag{9}$$

$$+ D_j\big(W^\alpha\big)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k\Big(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\Big) + \ldots\right] + D_j D_k\big(W^\alpha\big)\left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \ldots\right],$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$ and $\mathcal{L}$ is the formal Lagrangian (4). In applying the formula (9) the formal Lagrangian $\mathcal{L}$ should be written in the symmetric form with respect to all mixed derivatives $u_{ij}^\alpha,\ u_{ijk}^\alpha, \ldots$.

2 Definition and main properties of nonlinear self-adjointness

2.1 Heuristic discussion

Definition (4) of the formal Lagrangian $\mathcal{L}$ shows that the vector (9) involves the 'non-physical' variable v. Therefore the validity of the conservation equation $D_i(C^i) = 0$ requires that we should take into account not only equations (1) but also the adjoint equations (2). But if the system (1) is quasi self-adjoint (in particular self-adjoint), one can eliminate v via the substitution (6) and obtain a conservation law for equations (1).

However, the quasi self-adjointness is not the only case when the variables v can eliminated from the conserved vector (9). Let us note first of all that we can relax the condition $\varphi'(u) \neq 0$ in (6) since it is used only to guarantee the *equivalence* of equation (2) to equation (1) after eliminating v by setting $v = \varphi(u)$. In constructing conservation laws, it is important only that v does not vanish identically, because otherwise $\mathcal{L} = 0$ and (9) gives the trivial vector $C^i = 0$. Therefore we can replace condition $\varphi'(u) \neq 0$ (6) with the weaker condition $\varphi(u) \neq 0$. Secondly, the substitution (6) can be replaced with a more general substitution where φ involves not only the variable u but also its derivatives as well as the independent variables x. This will be a differential substitution

$$v^\alpha = \varphi^\alpha(x, u, u_{(1)}, \ldots), \quad \alpha = 1, \ldots, m. \tag{10}$$

The only requirement is that not all φ^α vanish. Moreover, φ may involve nonlocal variables, e.g. $D_i^{-1}(u^\alpha)$. Then it is more convenient to determine v implicitly by

$$\Phi_\alpha(x, u, u_{(1)}, \ldots; v, v_{(1)}, \ldots) = 0, \quad \alpha = 1, \ldots, m. \tag{11}$$

2.2 Definition

In this paper we will consider the substitutions (10) that do not involve the derivatives and use the following definition of *nonlinear self-adjointness*.

Definition 2. The system (1) is said to be *self-adjoint* if the adjoint system (2) is satisfied for all solutions u of equations (1) upon a substitution

$$v^\alpha = \varphi^\alpha(x, u), \quad \alpha = 1, \ldots, m, \tag{12}$$

such that

$$\varphi(x, u) \neq 0. \tag{13}$$

In other words, the following equations hold:

$$F_\alpha^*\big(x, u, \varphi(x, u), \ldots, u_{(s)}, \varphi_{(s)}\big) = \lambda_\alpha^\beta\, F_\beta\big(x, u, \ldots, u_{(s)}\big), \quad \alpha = 1, \ldots, m, \tag{14}$$

where λ_α^β are undetermined coefficients. Here φ is the m-dimensional vector $\varphi = (\varphi^1, \ldots, \varphi^m)$ and $\varphi_{(\sigma)}$ are its derivatives,

$$\varphi_{(\sigma)} = \big\{D_{i_1} \cdots D_{i_\sigma}\big(\varphi^\alpha(x, u)\big)\big\}, \quad \sigma = 1, \ldots, s.$$

Eq. (13) means that not all components $\varphi^\alpha(x, u)$ of the vector φ vanish simultaneously.

2.3 Properties

Proposition 1. The system (1) is self-adjoint in the sense of Definition 2 if and only of there exist functions v^α given by (12) and satisfying the condition (13) that solve the adjoint system (2) *for all solutions* $u(x)$ of equations (1).

Proposition 2. Any linear equation is self-adjoint.
Proof. This is a consequence of the fact that the adjoint equation $L^*[v] = 0$ to the linear equation $L[u] = 0$ does not involve the variable u. Therefore any non-vanishing solution $v = \varphi(x)$ of the adjoint equation gives a vector function (12) which is independent of u and hence satisfies the requirement of Definition 2.

In the case of one dependent variable, i.e. $m = 1$, we can easily prove the following.
Proposition 3. Equation (1) is self-adjoint in the sense of Definition 2 if and only if it becomes self-adjoint in the sense of Definition 1 upon rewriting in the equivalent form

$$\mu(x,u)F\big(x,u,u_{(1)},\ldots,u_{(s)}\big) = 0, \quad \mu(x,u) \neq 0, \tag{15}$$

with an appropriate multiplier $\mu(x,u)$. In particular, any linear equation can be made self-adjoint in the sense of Definition 1.
Proof. The computation reveals the following relation between the multiplier (15) and the substitution (12):

$$\varphi(x,u) = u\mu(x,u). \tag{16}$$

Namely, if equation (1) is self-adjoint in the sense of Definition 2 with the substitution (12), then equation (15) whose multiplier μ is determined by (16) is self-adjoint in the restricted sense of Definition 1, and visa versa.

2.4 Example: the Kompaneets equation

Let us write the Kompaneets equation [7] in the form

$$u_t = \frac{1}{x^2}\, D_x\left[x^4(u_x + u + u^2)\right]. \tag{17}$$

The reckoning shows that the adjoint equation to equation (17),

$$v_t + x^2 v_{xx} - x^2(1+2u)v_x + 2(x + 2xu - 1)v = 0,$$

does not have a solution of the form (6) but it has the solution of the form (12), namely

$$v = x^2.$$

Hence, equation (17) is not quasi self-adjoint, but it is self-adjoint in the sense of Definition 2. Equation (16) provides the multiplier $\mu = x^2/u$. Hence the Kompaneets equation becomes self-adjoint in the sense of Definition 1 if we write it in the form

$$\frac{x^2}{u}\, u_t = \frac{1}{u}\, D_x\left[x^4(u_x + u + u^2)\right]. \tag{18}$$

3 Time-dependent conservation laws for the KP equation

We will use the KP equation (Kadomtsev-Petviashvili [8])

$$u_{tx} - uu_{xx} - u_x^2 - u_{xxxx} = u_{yy} \tag{19}$$

written as the system (see, e.g. [9], p. 241, and the references therein)

$$u_t - uu_x - u_{xxx} - \omega_y = 0, \quad \omega_x - u_y = 0. \tag{20}$$

3.1 Self-adjointness

The formal Lagrangian (4) for equations (20) is written

$$\mathcal{L} = v(u_t - uu_x - u_{xxx} - \omega_y) + z(\omega_x - u_y) \tag{21}$$

and equation (3) yields the following adjoint system to the system (20):

$$v_t - uv_x - v_{xxx} - z_y = 0, \quad z_x - v_y = 0. \tag{22}$$

Equations (22) become identical with the KP equations (20) upon the substitution

$$v = u, \quad z = \omega. \tag{23}$$

It means that the system (20) is self-adjoint.

3.2 Symmetries

The system (20) admits the infinite-dimensional Lie algebra spanned by the operators

$$\begin{aligned} X_f = {} & 3f\frac{\partial}{\partial t} + (f'x + \frac{1}{2}\, f''y^2)\frac{\partial}{\partial x} + 2f'y\frac{\partial}{\partial y} \\ & - \big[2f'u + f''x + \frac{1}{2}\, f'''y^2\big]\frac{\partial}{\partial u} - \big[3f'\omega + f''yu + f'''xy + \frac{1}{6}f^{(4)}y^3\big]\frac{\partial}{\partial \omega}\,, \end{aligned} \tag{24}$$

$$X_g = 2g\frac{\partial}{\partial y} + g'y\frac{\partial}{\partial x} - g''y\frac{\partial}{\partial u} - \big[g'u + g''x + \frac{1}{2}\, g'''y^2\big]\frac{\partial}{\partial \omega}\,, \tag{25}$$

$$X_h = h\frac{\partial}{\partial x} - h'\frac{\partial}{\partial u} - h''y\frac{\partial}{\partial \omega}\,, \tag{26}$$

where f, g, h are three arbitrary functions of t. We will ignore the obvious symmetry

$$X_\alpha = \alpha(t)\frac{\partial}{\partial \omega}$$

describing the addition to ω an arbitrary function of t.

Note, that the operators (24)-(26) considered without the term $\frac{\partial}{\partial\omega}$ span the infinite-dimensional Lie algebra of symmetries of the KP equation (19). They coincide (up to normalizing coefficients) with the symmetries of the KP equation that were first obtained by F. Schwarz in 1982 (see also [10], [11] and the references therein).

3.3 Conservation laws

Noether's theorem is not applicable to the system (20). But Theorem 1 from Section 1.4 is applicable. Applying formula (9) to the formal Lagrangian (21) and to the symmetry (24), then eliminating v, z by the substitution (23) we obtain the conservation law

$$\big[D_t(C^1) + D_x(C^2) + D_y(C^3)\big]_{(20)} = 0 \tag{27}$$

with the following components of the conserved vector $C = (C^1, C^2, C^3)$:

$$\begin{aligned}
C^1 &= -\frac{1}{2}\,f'u^2 - \left(xf'' + \frac{1}{2}\,y^2 f'''\right) u,\\
C^2 &= \left(uu_{xx} + \frac{1}{3}\,u^3 - \frac{1}{2}\,u_x^2 - \frac{1}{2}\,\omega^2\right) f' + \left(xu_{xx} + \frac{1}{2}\,xu^2 - u_x\right) f''\\
&\quad + \frac{1}{4}\,\left(y^2u^2 + 2y^2u_{xx} - 4xy\omega\right) f''' - \frac{1}{6}\,y^3\omega f^{(4)},\\
C^3 &= u\omega f' + x\omega f'' + \left(xyu + \frac{1}{2}\,y^2\omega\right) f''' + \frac{1}{6}\,y^3 u f^{(4)}.
\end{aligned} \tag{28}$$

The conservation equation (27) for the vector (28) has the form

$$\begin{aligned}
&D_t(C^1) + D_x(C^2) + D_y(C^3)\\
&= \left(uf' + xf'' + \frac{1}{2}\,y^2 f'''\right)(u_{xxx} + uu_x + \omega_y - u_t)\\
&+ \left(\omega f' + xyf''' + \frac{1}{6}\,y^3 f^{(4)}\right)(u_y - \omega_x).
\end{aligned} \tag{29}$$

Since $f = f(t)$ is an arbitrary function, (28) provides an infinite set of conserved vectors. Note that the subscript (20) in Eq. (27) refers to restriction on the solution manifold of Eqs. (20). I did not find the conserved vector (28) with arbitrary $f(t)$ in previous publications, e.g. in [12]. The symmetries (25) and (26) lead to the conserved vectors

$$
\begin{aligned}
C^1 &= yug'',\\
C^2 &= \left(x\omega - yu_{xx} - \frac{1}{2}\,yu^2\right)g'' + \frac{1}{2}\,y^2\omega g''',\\
C^3 &= -(xu + y\omega)g'' - \frac{1}{2}\,y^2ug'''
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
C^1 &= uh',\\
C^2 &= y\omega h'' - \left(u_{xx} + \frac{1}{2}\,u^2\right)h',\\
C^3 &= -\omega h' - yuh'',
\end{aligned}
\tag{31}
$$

respectively.

4 Conservation laws for linear equations

One can obtain conserved vector by formula (9) for any linear equation because linear equations are self adjoint according to Proposition 2. Consider, e.g. the heat equation

$$
u_t - \Delta u = 0 \tag{32}
$$

with any number of spatial variables $x = (x^1, \ldots, x^n)$. Applying formula (9) to $X = u\frac{\partial}{\partial u}$ we obtain the conservation law $[D_t(\tau) + \nabla \cdot \chi]_{(32)} = 0$ with

$$
\tau = \varphi(t,x)u, \quad \chi = u\nabla\varphi(t,x) - \varphi(t,x)\nabla u, \tag{33}
$$

where $v = \varphi(t,x)$ is an arbitrary solution of the adjoint equation $v_t + \Delta v = 0$ to equation (32). The conserved vector (33) embraces the conserved vectors associated with all other symmetries of equation (32). In particular, the projective symmetry

$$
X = t^2\frac{\partial}{\partial t} + tx^i\frac{\partial}{\partial x^i} - \frac{|x|^2 + 2nt}{4}\,u\frac{\partial}{\partial u}
$$

of equation (32) gives the conserved vector

$$
\tau = \frac{|x|^2 - 2nt}{4}\,u, \quad \chi^i = \frac{x^i}{2}\,u - \frac{|x|^2 - 2nt}{4}\,u_i\,.
$$

It corresponds to (33) with the particular solution $v = (|x|^2 - 2nt)/4$ of the adjoint equation. In one-dimensional case ($n = 1$), it is shown in [13] by direct calculation that all conserved vectors for the heat equation $u_t = u_{xx}$ have the form (33). See also [14].

Bibliography

[1] Ibragimov N H 2006 Integrating factors, adjoint equations and Lagrangians *J. Math. Anal. Appl.* **318** 742–57

[2] Ibragimov N H 2007 A new conservation theorem *J. Math. Anal. Appl.* **333** 311–28

[3] Ibragimov N H 2007 Quasi self-adjoint differential equations *Archives of ALGA* **4** 55–60

[4] Ibragimov N H 1983 *Transformation groups applied to mathematical physics* (Moscow: Nauka). English transl. Dordrecht: Reidel 1985

[5] Rogers C 1985 Application of a reciprocal transformation to a two phase Stefan problem *J. Phys. A: Math. Gen.* **18** L105-L109

[6] Rogers C 1986 On a class of moving boundary value problems in nonlinear heat conduction: application of a Bäcklund transformation *Int. J. Nonlinear Mechanics* **21** 249-256

[7] Kompaneets A S 1956 The establishment of thermal equilibrium between quanta and electrons *Zh. Eksp. Teor. Fiz* **31** 876–85
Kompaneets A S 1957 *Sov. Phys. - JETP* **4** 730 (Engl. Transl.)

[8] Kadomtsev B B and Petviashvili V I 1970 On stability of solitary waves in weakly dispersive media *Dokl. Akad. Nauk SSSR* **192** 753–56
Engl. Transl.: Kadomtsev B B and Petviashvili V I 1970 *Sov. Phys. Dokl.* **15** 539-42

[9] Novikov S, Manakov S V, Pitaevskii L P and Zakharov V E 1984 *Theory of solitons: The inverse scattering method* (New York: Consultants Bureau)

[10] David D, Kamran N, Levi D and Winternitz P 1985 Subalgebras of loop algebras and symmetries of the Kadomtsev-Petviashvili equation *Phys. Rev. Letters* **55** 2111 -2113

[11] Rogers C and Ames W F 1989 *Nonlinear boundary value problems in science and engineering* (Boston: Academic Press)

[12] Oevel W and Fuchsssteiner B 1982 Explicit formulas for symmetries and conservation laws of the Kadomtsev-Petviashvili equation *Phys. Letters* **88A** 323–27

[13] Dorodnitsyn V A and Svirshchevskii S R 1983 On Lie-Bäcklund groups admitted by the heat equation with a source *Preprint 101* (Moscow: Inst. Appl. Math. USSR Acad. Sci.)

[14] Steinberg S and Wolf K B 1981 Symmetry, conserved quantities and moments in diffusive equations *J. Math. Anal. Appl.* **80** 36–45