SYMBOLIC POWERS VERSUS REGULAR POWERS OF IDEALS OF GENERAL POINTS IN $\mathbb{P}^1 \times \mathbb{P}^1$

ELENA GUARDO, BRIAN HARBOURNE, AND ADAM VAN TUYL

ABSTRACT. Let $I \subseteq R = k[x_0, \ldots, x_n]$ be a homogeneous ideal. A current research theme is to compare the symbolic powers of I with the regular powers of I. In this paper, we investigate which ordinary powers I^r contain given symbolic powers $I^{(m)}$ for radical ideals $I \subset k[x_0, \ldots, x_3]$ defining a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. We focus on the case that the points are general.

1. INTRODUCTION

In this paper, we take the first step towards understanding the results of [1, 2, 6, 19, 20] in a multi-graded context by comparing the symbolic powers and regular powers of an ideal of a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Other aspects of ideals of such points have been studied by the first and third authors, among others; see, for example [11, 13, 15, 23, 27, 28].

Throughout this paper, we work over an algebraically closed field k, of arbitrary characteristic. Let I be a homogeneous ideal in the homogeneous coordinate ring $k[\mathbb{P}^N] = k[x_0, \ldots, x_N]$ of $\mathbb{P}^N$. Then

$$\rho(I) = \sup\left\{m/r \ : \ I^{(m)} \not\subseteq I^r\right\}$$

is called the *resurgence* of I [1] (see section 2 below for the definiton of the symbolic power $I^{(m)}$). By definition, if m and r are positive integers such that $m/r > \rho(I)$, then $I^{(m)} \subseteq I^r$. However, $\rho(I)$ is often very hard to compute. Even showing that $\rho(I)$ is finite is not easy, but it follows from [6, 20] that $\rho(I) \leq N$, and more generally that $\rho(I) \leq h_I$ where h_I is the maximum height of an associated prime of I.

Lower bounds for $\rho(I)$ involving numerical invariants of I other than h_I were given in [1]. For each h, these bounds show that ideals with $h_I = h$ can be found where $\rho(I)$ is arbitrarily close to h (in particular, the supremum of the values of $\rho(I)$ over all homogeneous ideals I with $h_I = h$ is h, but no examples are known for which $\rho(I)$ is equal to h when $h > 1$). In some situations, [1] gives an exact value for $\rho(I)$ by giving an upper bound on $\rho(I)$ which coincides with the lower bound. However, these upper bounds require that I define a zero-dimensional subscheme of $\mathbb{P}^N$ (or a cone over a finite set of points; see [1, Proposition 2.5.1(b)]). The only other case where upper bounds other than

2000 *Mathematics Subject Classification.* 13F20, 13A15,14C20.
Key words and phrases. symbolic powers, multigraded, points.
Version: July 23, 2011 (b).

$\rho(I) \leq h_I$ are known is that of complete intersections (in which case the resurgence is 1; see section 2.4). Thus much less is known about values of $\rho(I)$ when the subscheme that I defines is positive dimensional but not a cone over a finite set of points nor a complete intersection.

Given the difficulty in finding $\rho(I)$, one may instead wish to look at variants of the resurgence. For a homogeneous ideal $I \subseteq k[\mathbb{P}^N]$, we introduce an *asymptotic resurgence*, which we define as

$$\rho_a(I) = \sup\{m/r : I^{(mt)} \not\subseteq I^{rt}, t \gg 0\}.$$

We will also consider an alternate asymptotic version of the resurgence, which we define as

$$\rho'_a(I) = \limsup_{t\to\infty} \rho(I,t),$$

where $\rho(I,t) = \{m/r : I^{(m)} \not\subseteq I^r, m \geq t, r \geq t\}$. In Theorem 2.5.1 we give both upper and lower bounds on these asymptotic versions of the resurgence for ideals of certain finite sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

What makes this especially interesting is that the upper bounds in Theorem 2.5.1, albeit on the asymptotic resurgence, break new ground by applying to ideals of certain positive dimensional subschemes which are not complete intersections or cones over points. This is a consequence of the fact that the ideal of a finite set of points in a product of projective spaces is actually also the homogeneous ideal of a positive dimensional subscheme in a single projective space, as we now explain in more detail.

The multi-homogeneous coordinate ring $k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]$ of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ is

$$k[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{t,0}, \ldots, x_{t,n_t}].$$

It has a multi-grading given by

$$\deg(x_{i,j}) = e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^t,$$

where the 1 is in the ith position. The ring $k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]$ is a direct sum of its multi-homogeneous components $k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]_{(a_1,\ldots,a_t)}$, where $k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]_{(a_1,\ldots,a_t)}$ is the k-vector space span of the monomials of multi-degree $(a_1, \ldots, a_t)$. An ideal $I \subseteq k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]$ is multi-homogeneous if it is the direct sum of its multi-homogeneous components (i.e., of $k[\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}]_{(a_1,\ldots,a_t)} \cap I$). Note that a multi-homogeneous ideal I can be regarded as a homogeneous ideal in $k[\mathbb{P}^N]$, $N = n_1 + \cdots + n_t + t - 1$, where a monomial of multi-degree $(a_1, \ldots, a_t)$ has degree $d = a_1 + \cdots + a_t$ and the homogeneous component of I of degree d is $I_d = \bigoplus_{\sum_i a_i = d} I_{(a_1,\ldots,a_t)}$. However, when $t > 1$, a multi-homogeneous ideal I when regarded as being homogeneous never defines a 0-dimensional subscheme of $\mathbb{P}^N$, even if I defines a zero-dimensional subscheme of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$. For example, the multi-homogeneous ideal I of a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ defines a finite set of lines in $\mathbb{P}^3$, which are skew (and thus not a cone) if no two of the points lie on the same horizontal or vertical rule of $\mathbb{P}^1 \times \mathbb{P}^1$ (see Remark 2.1.2), and not a complete intersection unless the points comprise a rectangular array in $\mathbb{P}^1 \times \mathbb{P}^1$ (see section 2.4).

Thus except for special configurations of the points, the results of [1] giving upper bounds on the resurgence of ideals of finite sets of points of projective space do not apply to multi-homogeneous ideals defining finite sets of points in products of projective space.

Our goal, therefore, is to take a first step in extending the results of [1] to a multi-graded environment (and to thereby obtain results for certain homogeneous ideals defining positive dimensional subschemes of projective space) by studying $\rho(I)$, $\rho'_a(I)$ and $\rho_a(I)$ when I is a radical ideal defining a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Even in this restricted setting, we will require extensive machinery coming from both commutative algebra (involving multi-homogeneous ideals) and algebraic geometry (involving rational surfaces). As an application of our results, we obtain the following theorem (see the end of section 3 for the proof):

Theorem 1.1. *Let I be the ideal of a set Z of s general points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then $I^m = I^{(m)}$ for all $m > 0$ if and only if s is $1, 2, 3$ or 5. In particular, $\rho(I) = \rho'_a(I) = \rho_a(I) = 1$ if s is $1, 2, 3$ or 5. Moreover, except for $s = 1, 2, 3, 5$, we have $1 < \rho_a(I) \leq \rho'_a(I) \leq \rho(I)$.*

It may also be of interest to note that since the ideal of a set of s general points in $\mathbb{P}^1 \times \mathbb{P}^1$ is a complete intersection if and only if $s = 1$, Theorem 1.1 gives examples of ideals which are not complete intersections which nonetheless have $\rho(I) = 1$.

Our paper is structured as follows. In Section 2 we develop the required background and we apply it to obtain results on the various variants of the resurgence. In Section 3, we prove the results which lead to a proof of Theorem 1.1.

Whereas most of our focus in this paper is on general sets of points, points not in general position can also be of interest. In a forthcoming paper we will study the same sort of problems for finite sets of points which are arithmetically Cohen-Macaulay subschemes of $\mathbb{P}^1 \times \mathbb{P}^1$.

Acknowledgments. We would like to thank Irena Swanson for answering some of our questions. This work was facilitated by the Shared Hierarchical Academic Research Computing Network (SHARCNET:www.sharcnet.ca) and Compute/Calcul Canada. Computer experiments carried out on `CoCoA` [3] and *Macaulay2* [12] were very helpful in guiding our research. The second author's work on this project was sponsored by the National Security Agency under Grant/Cooperative agreement "Advances on Fat Points and Symbolic Powers," Number H98230-11-1-0139. The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notice. The third author acknowledges the support provided by NSERC.

2. Background

2.1. **Points in $\mathbb{P}^1 \times \mathbb{P}^1$ and their ideals.** For the convenience of the reader, we review some of the properties of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $R = k[\mathbb{P}^1 \times \mathbb{P}^1]$, where we will use the standard multi-grading for R. That is, $R = k[x_0, x_1, y_0, y_1]$, with $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$.

Let $I \subseteq R$ be a multi-homogeneous ideal (because R is bigraded, we sometimes say I is bihomogeneous). Then I has a multi-homogeneous primary decomposition, i.e., a primary decomposition $I = \bigcap_i Q_i$ where each $\sqrt{Q_i}$ is a multi-homogeneous prime ideal, and Q_i is multi-homogeneous and $\sqrt{Q_i}$-primary [4]. We define the m-th *symbolic power* of I to be the ideal $I^{(m)} = \bigcap_j P_{i_j}$, where $I^m = \bigcap_i P_i$ is a multi-homogeneous primary decomposition, and the intersection $\bigcap_j P_{i_j}$ is over all components P_i such that $\sqrt{P_i}$ is contained in an associated prime of I. In particular, we see that $I^{(1)} = I$ and that $I^m \subseteq I^{(m)}$.

Remark 2.1.1. It is clear that $\rho_a(I) \leq \rho'_a(I) \leq \rho(I)$. For any ideal $(0) \neq J \subseteq k[\mathbb{P}^N]$, let $\alpha(J)$ denote the degree of a non-zero element of J of least degree. Given a homogeneous ideal $(0) \neq I \subsetneq k[\mathbb{P}^N]$, then $\alpha(I^m) = m\alpha(I)$ (since I is homogeneous) and $\alpha(I) > 0$ (since $I \subsetneq k[\mathbb{P}^N]$). Therefore, $I^{(m)} \subseteq I^r$ implies $I^m \subseteq I^{(m)} \subseteq I^r$ hence $m\alpha(I) = \alpha(I^m) \geq \alpha(I^r) = r\alpha(I)$ so $m/r \geq 1$ and thus $1 \leq \rho_a(I) \leq \rho'_a(I) \leq \rho(I)$, so if $\rho(I) \leq 1$, then $1 = \rho_a(I) = \rho'_a(I) = \rho(I)$.

Of particular interest to this paper is the case that I is the ideal of a set Z of s distinct reduced points of $\mathbb{P}^1 \times \mathbb{P}^1$, i.e., $Z = \{P_1, \ldots, P_s\}$. A point has the form $P = [a_0 : a_1] \times [b_0 : b_1] \in \mathbb{P}^1 \times \mathbb{P}^1$ and its defining ideal $I(P)$ in R is a prime ideal of the form $I(P) = (F, G)$ where $\deg F = (1, 0)$ and $\deg G = (0, 1)$. The ideal $I(Z)$ is then given by $I(Z) = \bigcap_{i=1}^s I(P_i)$. Furthermore, the m-th symbolic power of $I(Z)$ has the form $I(Z)^{(m)} = \bigcap_{i=1}^s I(P_i)^m$. The scheme defined by $I(Z)^{(m)}$ is sometimes referred to as a *fat point scheme*, and denoted $mP_1 + \cdots + mP_s$.

Remark 2.1.2. Note that $k[\mathbb{P}^1 \times \mathbb{P}^1] = k[x_0, x_1, y_0, y_1] = k[\mathbb{P}^3]$. The irrelevant ideals (x_0, x_1) and (y_0, y_1) corresponding to the two factors of $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$ define a pair of skew lines $L_1 \cong \mathbb{P}^1$ and $L_2 \cong \mathbb{P}^1$ in $\mathbb{P}^3$, where $I(L_1) = (y_0, y_1)$ and hence $k[L_1] = k[x_0, x_1]$, and similarly $I(L_2) = (x_0, x_1)$ and $k[L_2] = k[y_0, y_1]$. Thus the point $P = [a_0 : a_1] \times [b_0 : b_1] \in \mathbb{P}^1 \times \mathbb{P}^1$ defines a pair of points $P_1 = [a_0 : a_1] \in L_1$ and $P_2 = [b_0 : b_1] \in L_2$ and the ideal $I(P)$ defines the line L_P in $\mathbb{P}^3$ through the points P_1 and P_2. Given distinct points $P, Q \in \mathbb{P}^1 \times \mathbb{P}^1$, the lines L_P and L_Q meet if and only if either $P_1 = Q_1$ or $P_2 = Q_2$; i.e., if and only if P and Q are both on the same horizontal rule or both on the same vertical rule of $\mathbb{P}^1 \times \mathbb{P}^1$.

2.2. **Hilbert functions and points in 1-generic position.** Let $Z \subseteq \mathbb{P}^N$ be the subscheme defined by a homogeneous ideal I in $k[\mathbb{P}^N]$. We recall that the *Hilbert function* H_Z of Z is defined to be $H_Z(t) = \dim k[\mathbb{P}^N]_t - \dim I_t$, where for a graded module M, M_t denotes the homogeneous piece of degree t. Similarly, recall that the Hilbert function H_Z of a subscheme $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is defined to be $H_Z(i, j) = \dim k[\mathbb{P}^1 \times \mathbb{P}^1]_{(i,j)} - \dim I(Z)_{(i,j)}$.

Let $U_s \subset (\mathbb{P}^1 \times \mathbb{P}^1)^s$ be the open set of all sets of distinct points $P_1, \ldots, P_s \in \mathbb{P}^1 \times \mathbb{P}^1$. When one says that some fact is true for s *general* points, or that it is true if the points $P_1, \ldots, P_s$ are general, it means that the locus of points for which the fact holds contains a non-empty open subset of U_s. Thus it is not meaningful to refer to a specific choice $P_1, \ldots, P_s$ of s points as being general. To be specific about a case of relevance here, consider a finite set of points $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ (regarded as a reduced subscheme). We will say Z has *generic* Hilbert function if

$$H_Z(i,j) = \min\{\dim R_{(i,j)}, |Z|\} = \min\{(i+1)(j+1), |Z|\}.$$

It is well known that general points have generic Hilbert function; i.e, for each $s \geq 1$, there is a non-empty open subset of U_s consisting of distinct ordered sets of s points of $\mathbb{P}^1 \times \mathbb{P}^1$ with generic Hilbert function (see, for example, [28]). In particular, those sets of s distinct points Z for which every subset of Z has generic Hilbert function contains a non-empty open subset of U_s. It is traditional to express this by saying: every subset of a set of s general points $P_1, \dots, P_s$ in $\mathbb{P}^1 \times \mathbb{P}^1$ has generic Hilbert function, even though it does not make sense to refer to any specific set $P_1, \dots, P_s$ of s points as being general.

We will say that a set of s distinct points $P_1, \dots, P_s$ are *1-generic* or are in *1-generic position* if for every subscheme $Z = m_1P_1 + \cdots + m_sP_s$ with $0 \leq m_i \leq 1$, Z has generic Hilbert function. Thus s general points are 1-generic. Note that points $P_1, \dots, P_s \in \mathbb{P}^1 \times \mathbb{P}^1$ being generic is not the same as being general or 1-generic. To explain, let $\mathbb{K} \subseteq k$ be a subfield. Then there is a natural inclusion $\mathbb{P}^1_{\mathbb{K}} \subseteq \mathbb{P}^1_k$, and we say that $P_1, \dots, P_s \in \mathbb{P}^1_k \times \mathbb{P}^1_k = (\mathbb{P}^1 \times \mathbb{P}^1)_k$ are *generic* if $P_i \in (\mathbb{P}^1 \times \mathbb{P}^1)_{k_i} \setminus (\mathbb{P}^1 \times \mathbb{P}^1)_{k_{i-1}}$ for each i, where $k_0 \subsetneq k_1 \subsetneq \cdots \subsetneq k_s = k$ is a tower of algebraically closed fields such that k_0 is the algebraic closure $\overline{k'}$ of the prime field k' of k. Thus for example, if $C \subset \mathbb{P}^2$ is an irreducible reduced cubic with a double point, and if we pick points $p_1, \dots, p_8 \in C$ such that no three are collinear and no six lie on a conic but such that p_1 is the double point, then the points are 1-generic but not general nor generic. On the other hand, s generic points are 1-generic.

Example 2.2.1. Any single point of $\mathbb{P}^1 \times \mathbb{P}^1$ is in 1-generic position. Two points of $\mathbb{P}^1 \times \mathbb{P}^1$ are in 1-generic position if and only if they are not both on the same horizontal or vertical rule of $\mathbb{P}^1 \times \mathbb{P}^1$. As a consequence, if $s \geq 3$ points are in 1-generic position, then no two of them lie on the same horizontal or vertical rule. For $s = 3$, the converse is also true (since any such three points are equivalent under an isomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$), but for $s \geq 4$ points the condition that no two lie on the same horizontal or vertical rule is not sufficient to ensure that the points are in 1-generic position. (This is because given three points in 1-generic position, there is, up to multiplication by scalars, a unique form of degree $(1,1)$ which vanishes on the three points. In order for four points to be in 1-generic position, the fourth point cannot be in the zero-locus of the $(1,1)$-form associated to the other three points.)

2.3. **Divisors on blow ups and a connection to $\mathbb{P}^2$.** Given a finite set of distinct points $P_1, \dots, P_s \in \mathbb{P}^1 \times \mathbb{P}^1$, let $\pi : X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the birational morphism obtained by blowing up the points P_i. Let $\mathrm{Cl}(X)$ be the divisor class group of X. Let H and V be the pullback to X of general members of the rulings on $\mathbb{P}^1 \times \mathbb{P}^1$ (horizontal and vertical, respectively), and for each point P_i let E_i be the exceptional divisor of the blow up of P_i. Every divisor is linearly equivalent to a unique divisor of the form $aH + bV - m_1E_1 - \cdots - m_sE_s$. Because of this, we can regard $\mathrm{Cl}(X)$ as the free abelian group on the set $\{H, V, E_1, \dots, E_s\}$. This basis is called an *exceptional configuration*. In particular, when we have a divisor of the form $aH+bV-m_1E_1-\cdots-m_sE_s$, we will leave it to context whether we really mean a divisor or its linear equivalence class in $\mathrm{Cl}(X)$. We also recall that the intersection form on $\mathrm{Cl}(X)$ is determined by $H\cdot E_i = V\cdot E_i = H^2 = V^2 = E_i\cdot E_j = 0$ for all $i\neq j$, and $-H\cdot V = E_i^2 = -1$ for $i>0$.

Given a divisor F on X, it will be convenient to write $h^i(X,F)$ in place of $h^i(X,\mathcal{O}_X(F))$, and we will refer to a divisor class as being *effective* if it is the class of an effective divisor. We also sometimes say by ellipsis that a divisor is effective when we mean only that it is linearly equivalent to an effective divisor. (If we were ever to mean that a divisor is actually effective and not just linearly equivalent to an effective divisor, we would say the divisor is strictly effective.) We denote the subsemigroup of classes of effective divisors by $\mathrm{EFF}(X)\subseteq \mathrm{Cl}(X)$. We recall that a divisor or divisor class D is *nef* if $D\cdot C\geq 0$ for every effective divisor C, and we denote the subsemigroup of classes of nef divisors by $\mathrm{NEF}(X)\subseteq\mathrm{Cl}(X)$.

Problems involving fat points $Z=\sum_i m_iP_i$ with support at distinct points $P_i\in\mathbb{P}^1\times\mathbb{P}^1$ can be translated into problems involving divisors on X. Given $I=I(Z)$ and (i,j), then as a vector space $I(Z)_{(i,j)}$ can be identified with $H^0(X, iH+jV-\sum_i m_iE_i)$, which itself can be regarded as a vector subspace of the space of sections $H^0(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(i,j))$. Thus given (i,j), it is convenient to define the divisor $F(Z,(i,j)) = iH+jV-\sum_i m_iE_i$, in which case we have, under the identifications above,

$$I(Z)=\bigoplus_{i,j} I(Z)_{(i,j)} = \bigoplus_{i,j} H^0(X,F(Z,(i,j))).$$

Remark 2.3.1. It can be useful to reinterpret problems involving points of $\mathbb{P}^1\times\mathbb{P}^1$ as problems involving points of $\mathbb{P}^2$. Let Y be a finite set of points $p_1,\ldots,p_s$ of $\mathbb{P}^2$. Let Z be the image of Y under the birational transformation from $\mathbb{P}^2$ to $\mathbb{P}^1\times\mathbb{P}^1$ given by blowing up two points $p_{s+1},p_{s+2}\in\mathbb{P}^2$ such that none of the points p_i, $i<s+1$ is on the line A through p_{s+1} and p_{s+2} and blowing down the proper transform E of A. The divisors $L,E_1,\ldots,E_{s+2}$, where L is a line and E_i is the exceptional curve obtained by blowing up the point p_i, give a basis of the divisor class group $\mathrm{Cl}(X)$ for the surface X obtained by blowing up the points p_i, also called an exceptional configuration. The birational transformation from $\mathbb{P}^2$ to $\mathbb{P}^1\times\mathbb{P}^1$ described above induces a birational morphism $X\to\mathbb{P}^1\times\mathbb{P}^1$ given by contracting $E_1,\ldots,E_s, L-E_{s+1}-E_{s+2}$. We also have an exceptional configuration on X coming from blowing up points $P_0,P_1,\ldots,P_s\in\mathbb{P}^1\times\mathbb{P}^1$ to obtain X; this basis is given by $H=L-E_{s+1}, V=L-E_{s+2}, E_1,\ldots,E_s, E=L-E_{s+1}-E_{s+2}$ where H and V give the rulings on $\mathbb{P}^1\times\mathbb{P}^1$. We can identify P_i with p_i for $i=1,\ldots,s$; P_0 is the point obtained by contracting the proper transform of the line through p_{s+1} and p_{s+2}. Thus $H^0(X,aH+bV-m(E_1+\cdots+E_s)) = H^0(X,(a+b)L-m(E_1+\cdots+E_n)-aE_{s+1}-bE_{s+2})$. If I is the ideal of the fat points $mP_1+\cdots+mP_s$, we note that $\alpha(I^{(m)})$ is then the least t such that $t=a+b$ and $h^0(X,(a+b)L-m(E_1+\cdots+E_s)-aE_{s+1}-bE_{s+2})>0$.

Alternatively, suppose $P_1, \dots, P_s \in \mathbb{P}^1 \times \mathbb{P}^1$ are such that no two of the points P_i lie on the same horizontal or vertical rule. Let $X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the birational morphism obtained by blowing up the points P_i. Then there is also a birational morphism $X \to \mathbb{P}^2$. If $H, V, E_1, \dots, E_s$ is the exceptional configuration for $X \to \mathbb{P}^1 \times \mathbb{P}^1$, the exceptional configuration for $X \to \mathbb{P}^2$ can be taken to be $L = H + V - E_s$, $E'_1 = E_1, \dots, E'_{s-1} = E_{s-1}$, $E'_s = H - E_s$ and $E'_{s+1} = V - E_s$.

2.4. **Complete Intersections.** Whenever $I^{(m)} = I^m$ for all $m \geq 1$ for a homogeneous ideal I, we have $I^{(m)} = I^m \subseteq I^r$ whenever $m \geq r$, and hence $\rho(I) \leq 1$. If also $(0) \neq I \neq (1)$, then $\rho(I) = \rho'_a(I) = \rho_a(I) = 1$ by Remark 2.1.1. One situation for which $I^{(m)} = I^m$ for all m occurs is the case that I is a complete intersection, meaning that I has a set of t generators, where t is the codimension. For example, suppose I is the ideal of a finite set Z of points of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1 = (\mathbb{P}^1)^t = Y$. Then $\mathrm{codim}_Y(Z) = t$, so I is a complete intersection if it is generated by t elements of I. As noted in [11, Remark 1.3] for $t = 2$ (but which extends naturally to all $t \geq 2$), an ideal I of a finite set of points $Z \subset Y$ is a complete intersection if and only if Z is a rectangular array of points (i.e., $Z = X_1 \times \cdots \times X_t$ for finite sets $X_i \subset \mathbb{P}^1$).

Proposition 2.4.1. *Let $X_1, \dots, X_t \subseteq \mathbb{P}^1$ be finite sets of points, and let I be the ideal of $Z = X_1 \times X_2 \times \cdots \times X_t \subseteq \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. Then $I^m = I^{(m)}$ for all $m \geq 1$ and $\rho(I) = \rho'_a(I) = \rho_a(I) = 1$.*

Proof. Under these hypotheses, $I = I(X_1)R + \cdots + I(X_t)R$ with $R = k[\mathbb{P}^1 \times \cdots \times \mathbb{P}^1]$ and $I(X_i)$ is the defining ideal of X_i in $k[\mathbb{P}^1]$. The ideal I is then a complete intersection. For any complete intersection I, we have $I^m = I^{(m)}$ for all $m \geq 1$ (see [30, Lemma 5, Appendix 6]). Thus $\rho(I) = 1$, and hence also $\rho'_a(I) = \rho_a(I) = 1$ by Remark 2.1.1. $\square$

2.5. **Bounds on the resurgence.** In Theorem 2.5.1 we give bounds on the resurgence. Bounds (2) and (3), which are along the lines of bounds given in [1], require some notation. Given a homogeneous ideal $I \neq (0)$, we denote the least integer t such that I is generated in degrees t or less by $\omega(I)$, and we define, as in [1],

$$\gamma(I) = \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m}.$$

Bound (4) generalizes the fact that $\rho'_a(I) = 1$ if $I^{(m)} = I^m$ for all $m \geq 1$ for a homogeneous ideal $(0) \neq I \neq (1)$. This generalization applies when there is a c such that $I^{(cm)} = (I^{(c)})^m$ for all $m \geq 1$. This occurs whenever the symbolic power algebra $\bigoplus_j I^{(j)}$ is Noetherian; see [26, Proposition 2.1] or [25]. (See [19, Proposition 3.5, Remark 3.12] and [2, Example 5.1] for examples for various values of c.)

Theorem 2.5.1. *Consider a homogeneous ideal $(0) \neq I \subsetneq k[\mathbb{P}^N]$.*

1. *If $\alpha(I^{(m)}) < r\alpha(I)$, then $I^{(m)} \not\subseteq I^r$.*
2. *We have $\alpha(I)/\gamma(I) \leq \rho_a(I) \leq \rho'_a(I) \leq \rho(I)$.*

(3) *If I is the ideal of a non-empty finite set Z of points in $\mathbb{P}^1 \times \mathbb{P}^1$, no two of which are on the same horizontal or vertical rule, then*

$$\rho_a(I) \leq \frac{\omega(I)}{\gamma(I)} \leq \frac{\mathrm{reg}(I)}{\gamma(I)}.$$

In particular, if Z is in 1-generic position, then $\rho_a(I) \leq \frac{|Z|}{\gamma(I)}$.

(4) *If for some positive integer c we have $I^{(cm)} = (I^{(c)})^m$ for all $m \geq 1$, and if $I^{(c)} \subseteq I^b$ for some positive integer b, then*

$$\rho'_a(I) \leq \frac{c}{b}.$$

Proof. (1) This is [1, Lemma 2.3.2(a)]: $\alpha(I^{(m)}) < r\alpha(I) = \alpha(I^r)$ so $I^{(m)} \not\subseteq I^r$.

(2) The bound $\alpha(I)/\gamma(I) \leq \rho(I)$ is [1, Lemma 2.3.2(b)]. The proof, which we now recall, actually shows $\alpha(I)/\gamma(I) \leq \rho_a(I)$ (as mentioned in Remark 2.1.1, $\rho_a(I) \leq \rho'_a(I) \leq \rho(I)$ is clear from the definitions). It is enough to show that $m/r < \alpha(I)/\gamma(I)$ implies $I^{(mt)} \not\subseteq I^{rt}$ for $t \gg 0$ and hence that $m/r \leq \rho_a(I)$. But $m/r < \alpha(I)/\gamma(I)$ implies $m\gamma(I) < r\alpha(I)$, so for all $t \gg 0$ we have $m\alpha(I^{(mt)})/(mt) < r\alpha(I)$, hence $\alpha(I^{(mt)}) < rt\alpha(I)$ so $I^{(mt)} \not\subseteq I^{rt}$ by (1).

(3) It is well known that $\alpha(I^{(m)})/m \geq \gamma(I)$ for all $m \geq 1$. (This is because $\gamma(I) = \lim_{t\to\infty} \alpha(I^{(tm)})/(tm)$, but clearly $\alpha(I^{(tm)}) \leq t\alpha(I^{(m)})$, so $\alpha(I^{(tm)})/(tm) \leq \alpha(I^{(m)})/m$.) We also observe that $\omega(I)/\gamma(I) \geq 1$. (Note that $\omega(I) \geq \alpha(I)$. But $I^m \subseteq I^{(m)}$ implies $\alpha(I) \geq \alpha(I^{(m)})/m$ for all m, and hence $\alpha(I) \geq \gamma(I)$. Moreover, since Z is non-empty, for any $P \in Z$ we have $\alpha(I^{(m)}) \geq \alpha(I(P)^m) = m$. Thus $\gamma(I) \geq 1$ and we have $\omega(I)/\gamma(I) \geq \alpha(I)/\gamma(I) \geq 1$.)

Since no two points of Z are on the same horizontal or vertical rule, I defines a disjoint set of lines in $\mathbb{P}^3$ and hence the only possible associated prime of I^r in addition to the minimal primes is the irrelevant ideal. Thus $I^{(r)}$ is the saturation $\mathrm{sat}(I^r)$ of I^r. By [21, Corollary 3], the regularity of I^r is bounded above by a linear function $\lambda_I r + c_I$ of r and moreover $\lambda_I \leq \omega(I) \leq \mathrm{reg}(I)$. (Unfortunately the constant term c_I may be positive so we know only that the regularity is bounded above by $\lambda_I r + c_I$ for some c_I; we do not know that $\lambda_I r$ is an upper bound.) Also, as noted in [8], the saturation degree $\mathrm{satdeg}(I^r)$ of I^r is bounded above by the regularity of I^r (where $\mathrm{satdeg}(I^r)$ is the least i such that $(\mathrm{sat}(I^r))_j = (I^r)_j$ for all $j \geq i$).

Thus if $m/r > \omega(I)/\gamma(I)$, then $m\gamma(I) > r\omega(I)$. This means for any constant c, we have $mt\gamma(I) > rt\omega(I) + c$ for $t \gg 0$. (To see this, since $m\gamma(I) = r\omega(I) + \epsilon$, we pick t so that $t\epsilon > c$.) So, in particular, for $t \gg 0$ we have

$$\alpha(I^{(mt)}) \geq mt\gamma(I) > rt\omega(I) + c_I \geq \mathrm{reg}(I^{rt}) \geq \mathrm{satdeg}(I^{rt}).$$

Thus $(I^{(mt)})_l = 0$ when $l < \mathrm{satdeg}(I^{rt})$ (since $\mathrm{satdeg}(I^{rt}) < \alpha(I^{(mt)})$), but $I^{(mt)} \subseteq I^{(rt)}$ (since $m \geq r$) so $(I^{(mt)})_l \subseteq (I^{(rt)})_l = (I^{rt})_l$ when $l \geq \mathrm{satdeg}(I^{rt})$; i.e., we have $(I^{(mt)})_l \subseteq (I^{rt})_l$ for all l when $t \gg 0$. Thus $\rho_a(I) \leq \omega(I)/\gamma(I)$.

For the last statement, when Z is in 1-generic position, no two points are on the same horizontal or vertical rule. Moreover, by [16, Theorem 2.4], $\mathrm{reg}(I) = |Z|$.

(4) Note that $I^{(c)} \subseteq I^b$ implies $b \leq c$. Suppose for some $m \geq 0$ we have $s \geq c(m+1)$ and $r \leq b(m+1)$. Then $I^{(s)} \subseteq I^{(c(m+1))} = (I^{(c)})^{m+1} \subseteq I^{b(m+1)} \subseteq I^r$.

Let s, r be positive integers such that $r \leq bs/c - b$, and hence $s \geq c(r+b)/b$. Let m be the biggest integer such that $r \geq bm$, so $r < b(m+1)$. Then $bs/(cm) - b/m \geq r/m \geq b$, hence $s \geq (m+1)c$. Thus $I^{(s)} \subseteq I^r$.

In particular, if $s, r \geq 1$ give $I^{(s)} \not\subseteq I^r$, then we must have $r > bs/c - b$. For all ordered pairs (i, j) satisfying $j \geq bi/c - b$ and $i, j \geq t$ for any given $t \geq 0$, it is easy to see (keeping in mind that $c \geq b$) that i/j is greatest when $j = t$ and $j = bi/c - b$ (and hence $i = c(t+b)/b$). Thus $s/r \leq c(t+b)/(bt) = (c/b)(1 + (b/t))$, so $\rho(I, t) \leq (c/b)(1 + (b/t))$, and hence $\rho'_a(I) \leq c/b$. □

Lemma 2.5.2. *Let $P_1, \dots, P_s \in \mathbb{P}^1 \times \mathbb{P}^1$ be distinct points and let $X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the birational morphism obtained by blowing these points up. Furthermore, if $s \leq 7$ assume the points are either general or generic, while if $s = 8$ assume the points are generic. Then a divisor $C \subset X$ is a prime divisor with $C^2 < 0$ if and only if $C^2 = C \cdot K_X = -1$. If $s \leq 7$, then in terms of the exceptional configuration for $X \to \mathbb{P}^1 \times \mathbb{P}^1$ the classes of these curves C are (up to permutations of the E_i and swapping H and V) precisely*

$E_1,$
$H - E_1,$
$H + V - E_1 - E_2 - E_3,$
$2H + V - E_1 - \cdots - E_5,$
$2H + 2V - 2E_1 - E_2 - \cdots - E_6,$
$3H + V - E_1 - \cdots - E_7,$
$3H + 2V - 2E_1 - 2E_2 - E_3 - \cdots - E_7,$
$3H + 3V - 2E_1 - \cdots - 2E_4 - E_5 \cdots - E_7,$
$4H + 3V - 2E_1 - \cdots - 2E_6 - E_7,$ *and*
$4H + 4V - 3E_1 - 2E_2 - \cdots - 2E_7.$

Proof. Since $s \leq 8$ and the points are either general or generic, we can regard $X \to \mathbb{P}^2$ as being the blow up of $s + 1 \leq 9$ points $p_1, \dots, p_{s+1}$ in $\mathbb{P}^2$, and that there is a smooth cubic curve $D \subset \mathbb{P}^2$ passing through these points. Thus up to linear equivalence we have $D = -K_X = 3L - E'_1 - \cdots - E'_s$ with respect to the exceptional configuration $L, E'_1, \dots, E'_{s+1}$ of the morphism $X \to \mathbb{P}^2$. Since D is irreducible with $D^2 \geq 0$, D is nef, so for any prime divisor C we have $D \cdot C \geq 0$. By the adjunction formula $C^2 - C \cdot D = 2p_C - 2$ we see $C^2 \geq -2$, with $C \cdot D = 1$ if $C^2 = -1$ and $C \cdot D = 0$ if $C^2 = -2$.

There are only finitely many possible classes of reduced, irreducible curves C with $C \cdot D = 0$ when $s \leq 7$ (see [9, Proposition 4.1]). For each of these classes, C is not effective if the points p_i are general, so in fact no such C is effective if $s \leq 7$ and the points p_i are general. (For example, $(L - E'_1 - E'_2 - E'_3) \cdot D = 0$; if $L - E'_1 - E'_2 - E'_3$ is the class of a strictly effective divisor C, then the points p_1, p_2, p_3 are collinear and hence not general.) For $s = 8$ there are infinitely many possible such classes so it is not enough to assume the points are general, but if the points are generic then there are no prime divisors $C \neq D$ with $C \cdot D = 0$ (since $C \cdot D = 0$ implies the coordinates of the points satisfy an algebraic relation coming from the group law on D). Thus the only prime divisors C with $C^2 < 0$ are those that satisfy $C^2 = C \cdot K_X = -1$. Conversely, if C is a divisor with $C^2 = C \cdot K_X = -1$, then by Serre duality $h^2(X, C) = h^0(X, -D - C)$ but $h^0(X, -D - C) = 0$ since $D \cdot (-D - C) < 0$. Now by Riemann-Roch for surfaces we have $h^0(X, C) - h^1(X, C) = 1 + (C^2 + D \cdot C)/2 = 1$ so C is effective. Up to linear equivalence, if F is a prime divisor with $F \cdot D = 0$, then $F = D$ (otherwise, as above, we would get an algebraic condition on the points p_i) and so $D^2 = 0$ (hence $s = 8$). Now if C is not a prime divisor, then from $D \cdot C = 1$ it follows that $C = G + rD$ with $r > 0$ and $D^2 = 0$, where G is the unique component of C with $D \cdot G = 1$. But then $G^2 = (C - rD)^2 = -1 - 2r < -1$, contrary to what is proved above.

Finally, suppose $s \leq 7$. Let C be a prime divisor on X with $C^2 = C \cdot K_X = -1$. Let Y be the surface obtained by blowing up an arbitrary point $P_{s+1} \in \mathbb{P}^1 \times \mathbb{P}^1$. Then denoting the pullback of C to Y also by C we have $(C - E_{s+1}) \cdot K_Y = 0$ and $(C - E_{s+1})^2 = -2$. It is not hard to check that the subgroup $K_Y^{\perp}$ of classes orthogonal to K_Y is, for $s < 7$, negative definite, and, if $s = 7$, negative semi-definite with the only classes F having $F \cdot K_Y = F^2 = 0$ being the multiples of K_Y. Thus for $s < 7$ it follows by negative definiteness that there are only finitely many classes C with $(C - E_{s+1}) \cdot K_Y = 0$ and $(C - E_{s+1})^2 = -2$ and it is not hard to find them all. For $s = 7$, the quotient $K_Y^{\perp}/\langle K_Y \rangle$ is negative definite so, modulo K_Y, there are only finitely many classes C with $(C - E_{s+1}) \cdot K_Y = 0$ and $(C - E_{s+1})^2 = -2$. But C must satisfy $C \cdot K_Y = -1$ and $C^2 = -1$, so there is at most one such representative in each coset of $K_Y^{\perp}/\langle K_Y \rangle$. Again it is not hard to find all C. $\square$

Note that a prime divisor C with $C^2 = C \cdot K_X = -1$ is called an *exceptional curve*. Exceptional curves are smooth rational curves.

Lemma 2.5.3. *Let $P_1, \ldots, P_s \in \mathbb{P}^1 \times \mathbb{P}^1$ be distinct points, $I \subset k[\mathbb{P}^1 \times \mathbb{P}^1]$ the ideal generated by all bi-homogeneous forms that vanish at all of the points P_i. Let X be the blow up of these s points of $\mathbb{P}^1 \times \mathbb{P}^1$, with exceptional configuration $H, V, E_1, \ldots, E_s$. If for some λ and m we have an effective divisor $C = \lambda(H + V) - m(E_1 + \cdots + E_s)$, then*

$$\gamma(I) \leq \frac{2\lambda}{m}.$$

If moreover for some t and r we have a nef divisor $D = t(H + V) - r(E_1 + \cdots + E_s)$ with $C \cdot D = 0$, then

$$\gamma(I) = \frac{2\lambda}{m} = \frac{sr}{t}.$$

Proof. If C is effective, so is lC and thus $\alpha(I^{(lm)}) \le 2\lambda l$ for all $l \ge 1$ and therefore

$$\frac{\alpha(I^{(lm)})}{lm} \le \frac{2\lambda l}{lm} = \frac{2\lambda}{m}.$$

Now assume D is nef. From $C \cdot D = 0$ we get

$$\frac{2\lambda}{m} = \frac{sr}{t}.$$

Now, given $\alpha(I^{(j)})$, we can find $a \ge 0$ and $b \ge 0$ with $\alpha(I^{(j)}) = a+b$ such that $(I^{(j)})_{(a,b)} \ne 0$. Moreover, $C' = aH + bV - r(E_1 + \cdots + E_s)$ is effective so $C' \cdot D = t(a+b) - jrs \ge 0$, hence

$$\frac{\alpha(I^{(j)})}{j} \ge \frac{rs}{t}$$

and therefore

$$\frac{rs}{t} \le \frac{\alpha(I^{(lm)})}{lm} \le \frac{2\lambda l}{lm} = \frac{rs}{t}.$$

Taking the limit as $l \to \infty$ gives the conclusion. □

We now give bounds on $\rho_a(I)$ for the ideal I of s general points of $\mathbb{P}^1 \times \mathbb{P}^1$ for each $s \ge 1$.

Corollary 2.5.4. *Let I be the ideal of $s \ge 1$ general points of $\mathbb{P}^1 \times \mathbb{P}^1$.*

- *If $s = 1$ or 2, then $\rho_a(I) = 1$.*
- *If $s = 3$, then $1 \le \rho_a(I) \le 3/2$ (but we will see in Theorem 3.1.2 that $\rho_a(I) = 1$).*
- *If $s = 4$, then $9/8 \le \rho_a(I) \le 3/2$ (but see Remark 3.2.2).*
- *If $s = 5$, then $1 \le \rho_a(I) \le 5/3$ (but we will see in Theorem 3.1.4 that $\rho_a(I) = 1$).*
- *If $s = 6$, then $7/6 \le \rho_a(I) \le 7/4$.*
- *If $s = 7$, then $15/14 \le \rho_a(I) \le 15/8$.*
- *If $s = 8$, then $5/4 \le \rho_a(I) \le 2$.*
- *If $9 \le s \le 11$, then $1 < 5/\sqrt{2s} \le \rho_a(I) \le 2$.*
- *If $s \ge 12$, then $1 < 2(\sqrt{s} - 1)/\sqrt{2s} < \rho_a(I) \le 2$.*

Proof. We note that $\mathrm{reg}(I) = s$ by [16, Theorem 2.4]. Let X be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at the s points with exceptional configuration $H, V, E_1, \ldots, E_s$.

The case $s = 1$ follows from Proposition 2.4.1, so consider $s = 2$. Then $C = D = H + V - E_1 - E_2$ is effective (since $C = (H - E_1) + (V - E_2)$ is a sum of effective divisors) and nef (since $D = (H - E_1) + (V - E_2)$ is a sum of prime divisors, each of which D meets non-negatively). Since $C \cdot D = 0$, we have $\gamma(I) = 2$ by Lemma 2.5.3. In this case $\mathrm{reg}(I) = \alpha(I) = 2$, so we have $1 = \alpha(I)/\gamma(I) \le \rho_a(I) \le \mathrm{reg}(I)/\gamma(I) = 1$ by Theorem 2.5.1(2, 3).

Consider $s = 3$. Then $C = H + V - E_1 - E_2 - E_3$ is effective (being exceptional, by Lemma 2.5.2) and $D = 3H + 3V - 2(E_1 + E_2 + E_3) = H + V + 2C$ is nef with $C \cdot D = 0$

so $\gamma(I) = 2$. In this case $\mathrm{reg}(I) = 3$ and $\alpha(I) = 2$, so we have $1 = \alpha(I)/\gamma(I) \leq \rho_a(I) \leq \mathrm{reg}(I)/\gamma(I) = 3/2$.

Consider $s = 4$. Then $C = 4(H + V) - 3(E_1 + E_2 + E_3 + E_4) = C_1 + C_2 + C_3 + C_4$ is effective (being the sum of the four exceptional curves C_i, where $C_i = (H+V-E_1-E_2-E_3-E_4)+E_i$) and $D = 3H + 3V - 2(E_1 + E_2 + E_3 + E_4) = 2C_4 + (H - E_4) + (V - E_4)$ is nef with $C \cdot D = 0$ so $\gamma(I) = 8/3$. In this case $\mathrm{reg}(I) = 4$ and $\alpha(I) = 3$, so we have $9/8 = \alpha(I)/\gamma(I) \leq \rho_a(I) \leq \mathrm{reg}(I)/\gamma(I) = 3/2$.

Consider $s = 5$. Then $C = 3(H + V) - 2(E_1 + \cdots + E_5) = (2H + V - E_1 - \cdots - E_5) + (H + 2V - E_1 - \cdots - E_5)$ is effective (being the sum of two exceptional curves) and $D = 10(H + V) - 6(E_1 + \cdots + E_5) = D_1 + \cdots + D_5$ is nef (since each $D_i = 2H + 2V - (E_1 + \cdots + E_5) - E_i$ is a sum of two exceptionals, each of which D meets non-negatively; for example, $D_1 = (H + V - E_1 - E_2 - E_3) + (H + V - E_1 - E_4 - E_5)$). Since $C \cdot D = 0$ we have $\gamma(I) = 3$. In this case $\mathrm{reg}(I) = 5$ and $\alpha(I) = 3$, so we have $1 = \alpha(I)/\gamma(I) \leq \rho_a(I) \leq \mathrm{reg}(I)/\gamma(I) = 5/3$.

Consider $s = 6$. Then $C = 12(H + V) - 7(E_1 + \cdots + E_6) = C_1 + \cdots + C_6$ is effective (since each $C_i = 2(H + V) - (E_1 + \cdots + E_6) - E_i$ is exceptional) and $D = 7(H + V) - 4(E_1+\cdots+E_6) = (4H+3V-2(E_1+\cdots+E_6))+(3H+4V-2(E_1+\cdots+E_6))$ is nef (since $4H+3V-2(E_1+\cdots+E_6) = (2H+V-(E_1+\cdots+E_5))+(2(H+V)-(E_1+\cdots+E_5)-2E_6)$ is a sum of two exceptional curves, and likewise for $(3H + 4V - 2(E_1 + \cdots + E_6))$, each of which D meets non-negatively). Since $C \cdot D = 0$ we have $\gamma(I) = 24/7$. In this case $\mathrm{reg}(I) = 6$ and $\alpha(I) = 4$, so we have $7/6 = \alpha(I)/\gamma(I) \leq \rho_a(I) \leq \mathrm{reg}(I)/\gamma(I) = 7/4$.

Consider $s = 7$. Then $C = 28(H + V) - 15(E_1 + \cdots + E_7) = C_1 + \cdots + C_7$ is effective (since each $C_i = 4(H + V) - 2(E_1 + \cdots + E_7) - E_i$ is exceptional) and $D = 15(H + V) - 8(E_1 + \cdots + E_7) = H + V + D_1 + \cdots + D_7$ is nef (since $4D = 2C + (3H + V - (E_1 + \cdots + E_7)) + (H + 3V - (E_1 + \cdots + E_7))$ is a sum of exceptionals, each of which D meets non-negatively). Since $C \cdot D = 0$ we have $\gamma(I) = 56/15$. In this case $\mathrm{reg}(I) = 7$ and $\alpha(I) = 4$, so we have $15/14 = \alpha(I)/\gamma(I) \leq \rho_a(I) \leq \mathrm{reg}(I)/\gamma(I) = 15/8$.

Consider $s = 8$. Applying Theorem 2.5.1 gives the bounds $1 = \alpha(I)/\gamma(I) \leq \rho_a(I) \leq \mathrm{reg}(I)/\gamma(I) = 2$. (In this case $C = D = 2(H + V) - (E_1 + \cdots + E_8) = -K_X$ is effective, since 8 points impose at most 8 conditions on the 9 dimensional space of forms of degree $(2, 2)$. Since the blow up X of $\mathbb{P}^1 \times \mathbb{P}^1$ at 8 general points is a blow up of $\mathbb{P}^2$ at 9 general points, and since there is an irreducible cubic through 9 general points of $\mathbb{P}^2$, we see that $-K_X$ is nef. Since $C \cdot D = 0$, we have $\gamma(I) = 4$. But $\mathrm{reg}(I) = 8$ and $\alpha(I) = 4$, so we have $1 = \alpha(I)/\gamma(I) \leq \rho_a(I) \leq \mathrm{reg}(I)/\gamma(I) = 2$.) This is not very informative, since we have $1 \leq \rho_a(I) \leq \rho(I) \leq 2$ for any ideal I of points in $\mathbb{P}^1 \times \mathbb{P}^1$, where the lower bound $1 \leq \rho_a(I)$ is elementary (see Remark 2.1.1), and the upper bound (as discussed in the introduction) is due to [6, 20]. However, we can obtain a better lower bound with a little more work. It turns out (as we will show in a moment) that $(I^{(m)})_{(2m+1,2m+1)}$ has gcd 1 while for $(I^r)_{(a,b)}$ to have gcd 1 we must have $a + b \geq 5r$. Clearly, I^r cannot contain $I^{(m)}$ if $(I^r)_{(2m+1,2m+1)}$ does not contain $(I^{(m)})_{(2m+1,2m+1)}$, and $(I^r)_{(2m+1,2m+1)}$ cannot contain $(I^{(m)})_{(2m+1,2m+1)}$ if

$(I^{(m)})_{(2m+1,2m+1)}$ has gcd 1 but $(I^r)_{(2m+1,2m+1)}$ does not. Thus $I^{(m)} \not\subseteq I^r$ if $4m+2 < 5r$; i.e., if $m/r < 5/4 - 1/(2r)$. But if $m'/r' < 5/4$, then $m/r < 5/4 - 1/(2r)$ for $m = m't$, $r = r't$ for all $t \gg 0$. Thus $\rho_a(I) \geq 5/4$.

We now show that $(I^{(m)})_{(2m+1,2m+1)}$ has gcd 1. Let $E = E_1 + \cdots + E_8$. Then $I_{(2,2)} = H^0(X, \mathcal{O}_X(2H+2V-E))$. We can regard X as a blow up of 9 general points of $\mathbb{P}^2$ with respect to the exceptional configuration $L = H + V - E_8$, $E'_i = E_i$ for $i = 1, \dots, 7$, and $E'_8 = H - E_8$, $E'_9 = V - E_9$. Then $2H+2V-E = -K_X = 3L - E'_1 - \cdots - E'_9$, and since the points are general, we have $h^0(X, \mathcal{O}_X(2H+2V-E)) = h^0(X, \mathcal{O}_X(3L - E'_1 - \cdots - E'_9)) = 1$ and the unique curve in the linear system $|-K_X|$ is a smooth elliptic curve C. The image C' of C in $\mathbb{P}^1 \times \mathbb{P}^1$ is defined by the basis element F of the 1-dimensional vector space $I_{(2,2)}$. On X we have the short exact sequence of line bundles

$$0 \to \mathcal{O}_X((m-1)C + H + V) \to \mathcal{O}_X(mC + H + V) \to \mathcal{O}_C(mC + H + V) \to 0.$$

Since $|H|$ and $|V|$ are base point free, if for any $m \geq 0$ the linear system $|mC+H+V|$ has fixed components, the fixed components must consist of a multiple of C. But in that case we would have $h^0(X, \mathcal{O}_X(mC+H+V)) = h^0(X, \mathcal{O}_X((m-1)C+H+V))$ which we now show is not true. Note that $C \cdot C = 0$, and $C \cdot H = C \cdot V = 2$. By Riemann-Roch for curves we have $h^0(C, \mathcal{O}_C(mC+H+V)) = 4$, since duality tells us that $h^1(C, \mathcal{O}_C(mC+H+V)) = 0$. We also know that $h^1(X, \mathcal{O}_X(H+V)) = h^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(H+V)) = 0$. By induction on m, using the fact that $h^0(X, \mathcal{O}_X(H+V)) = h^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(H+V)) = 4$, we get $h^0(X, \mathcal{O}_X(mC+H+V)) = 4 + 4m$. Thus $h^0(X, \mathcal{O}_X(mC+H+V)) = 4(m+1) > 4m = h^0(X, \mathcal{O}_X((m-1)C+H+V))$ for $m > 0$. Thus $|mC+H+V|$ is fixed component free so $(I^{(m)})_{(2m+1,2m+1)}$ has gcd $= 1$.

Finally we show that F divides every element of $(I^r)_{(a,b)}$ whenever $a+b < 5r$, and hence $(I^r)_{(a,b)}$ does not have gcd 1 unless $a + b \geq 5r$. It is easy to check that the only bi-homogeneous elements of I of total degree 4 or less are scalar multiples of F. But I^r is spanned by products of r bi-homogeneous elements of I. If such a product P has total degree less than $5r$, then at least one of the r bi-homogeneous factors A of P has total degree less than 5, and hence is divisible by F. Thus any bi-homogeneous element of total degree less than $5r$ is divisible by F. In particular, if $a + b < 5r$, then F divides every element of $(I^r)_{(a,b)}$.

Now assume $s \geq 9$. Let $C = d(H+V) - m(E_1 + \cdots + E_s)$. If $C^2 > 0$, then tC is effective for $t \gg 0$, so by Lemma 2.5.3 we have $\gamma(I) \leq 2d/m$. It follows that $\gamma(I) \leq \sqrt{2s}$ and thus $\alpha(I)/\sqrt{2s} \leq \rho_a(I)$. It is easy to compute $\alpha(I)$ for any given s. For example, we have: for $s = 9, 10, 11$, $\alpha(I) = 5$; for $s = 12$, $\alpha(I) = 6$; etc. In each case, $s = 9, 10, 11$, one checks directly that $1 < \alpha(I)/\sqrt{2s}$. In fact, since the points are general, they impose independent conditions on forms of every bi-degree (i,j); i.e., there are forms of bi-degree (i,j) vanishing at the s points if and only if $(i+1)(j+1) > s$. But for a given degree $t = i+j$, the maximum value of $(i+1)(j+1)$ occurs when $i = j$, and so there are no forms in I of total degree t if $(t/2+1)^2 \leq s$. But $(t/2+1)^2 \leq s$ is equivalent to $t \leq 2(\sqrt{s} - 1)$. Thus $\alpha(I) > 2(\sqrt{s} - 1)$, hence we get $2(\sqrt{s}-1)/\sqrt{2s} < \rho_a(I)$, and it is easy to check that $1 < 2(\sqrt{s}-1)/\sqrt{2s}$ for $s \geq 12$. Finally, as noted in the introduction, we always have $\rho_a(I) \leq \rho(I) \leq 2$ because the height of the largest associated prime of I is two. $\square$

3. Additional results for general points of $\mathbb{P}^1 \times \mathbb{P}^1$

In this section, we consider the problem of whether $I^m = I^{(m)}$ for all m when I is the ideal of s general points of $\mathbb{P}^1 \times \mathbb{P}^1$. For $s = 1, 2, 3, 5$, we verify $I^m = I^{(m)}$ for all m. For $s \geq 6$, we prove that $I^2 \neq I^{(2)}$. For $s = 4$, computer calculations suggest that $I^2 = I^{(2)}$, but we show that $I^3 \neq I^{(3)}$, and by a similar method we obtain lower bounds on the resurgence when s is a square or the product of consecutive integers. In some cases, these bounds improve upon the bounds of Corollary 2.5.4.

3.1. **Two or three points in 1-generic position.** We first compute $\rho(I)$ when $I = I(Z)$ for a set of two points $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ in 1-generic position. For this case, the problem reduces to a question of monomial ideals.

Theorem 3.1.1. *Let $I = I(Z)$ where $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ consists of two points in 1-generic position. Then $I^{(m)} = I^m$ for all $m \geq 1$, and in particular, $\rho(I) = \rho'_a(I) = \rho_a(I) = 1$.*

Proof. Let $Z = P_1 + P_2$. We can assume, after a change of coordinates, that $I(P_1) = (x_0, y_0)$ and $I(P_2) = (x_1, y_1)$. Clearly $I(P_1)^{m_1} \cdot I(P_2)^{m_2} \subseteq I(P_1)^{m_1} \cap I(P_2)^{m_2}$. Moreover, all of these ideals are monomial ideals. If $x_0^a y_0^b x_1^c y_1^d \in I(P_1)^{m_1} \cap I(P_2)^{m_2}$, then $a + b \geq m_1$ and $c+d \geq m_2$, and hence $x_0^a y_0^b x_1^c y_1^d \in I(P_1)^{m_1} \cdot I(P_2)^{m_2}$, so $I(P_1)^{m_1} \cdot I(P_2)^{m_2} = I(P_1)^{m_1} \cap I(P_2)^{m_2}$. In particular, $I^m = (I(P_1) \cap I(P_2))^m = (I(P_1) \cdot I(P_2))^m = I(P_1)^m \cdot I(P_2)^m = I(P_1)^m \cap I(P_2)^m = I^{(m)}$. Since $\rho(I) \geq 1$ by Remark 2.1.1, but for $m \geq r$ we have $I^{(m)} \subseteq I^{(r)} = I^r$, we see $\rho(I) \leq 1$. Hence $\rho(I) = 1$ and so also $\rho'_a(I_Z) = \rho_a(I_Z) = 1$ by Remark 2.1.1. $\square$

We now consider three points in 1-generic position.

Theorem 3.1.2. *Let $I = I(Z)$ where $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ consists of three points in 1-generic position. Then $I^{(m)} = I^m$ for all $m \geq 1$, and in particular, $\rho(I) = \rho'_a(I) = \rho_a(I_Z) = 1$.*

Proof. For specificity say that the three points are $P_i = P_{i1} \times P_{i2}$, $i = 1, 2, 3$, for points $P_{ij} \in \mathbb{P}^1$ and that $k[\mathbb{P}^1 \times \mathbb{P}^1] = k[a, b, c, d] = k[a, b] \otimes_k k[c, d] = k[\mathbb{P}^1] \otimes k[\mathbb{P}^1]$. Up to change of coordinates, we may as well assume $P_{11} = P_{12} = [0 : 1]$, $P_{21} = P_{22} = [1 : 1]$, and $P_{31} = P_{32} = [1 : 0]$.

Since the points are 1-generic, we know $\dim I_{(1,1)} = 1$, so there is (up to scalar multiples) a unique form F of degree $(1, 1)$ in I. We will show that $I^{(m)} \subseteq I^{(m-1)}I + FI^{(m-1)}$ for each $m \geq 2$. Formally, we can write the right hand side as $I^{(m-1)}(I + F)$. Iterating $m - 1$ times gives $I^{(m)} \subseteq I(I + F)^{m-1} = I^m + FI^{m-1} + \cdots + F^{m-1}I$. Since $F \in I$, we see that $F^i I^{m-i} \subseteq I^m$, hence $I^{(m)} \subseteq I^m$. But $I^m \subseteq I^{(m)}$, so we have $I^{(m)} = I^m$. Now, as in the proof of Theorem 3.1.1, the remaining conclusions follow.

We now show $I^{(m)} \subseteq I^{(m-1)}I+FI^{(m-1)}$. This is clear if $m = 1$, so assume $m \geq 2$. We will consider $(I^{(m)})_{(i,j)}$ for various cases. If $(I^{(m)})_{(i,j)} = 0$, then clearly $I^{(m)} \subseteq I^{(m-1)}I+FI^{(m-1)}$ so we may assume $(I^{(m)})_{(i,j)} \neq 0$.

If $i + j < 3m$, then apply Bézout's theorem: for any element $G \in (I^{(m)})_{(i,j)}$ the sum of the intersection multiplicities of F with G over all points $P \in \mathbb{P}^1\times\mathbb{P}^1$ is at least $3m$ since G vanishes at each point P_i with order at least m while F vanishes with order 1, so summing over the three points gives at least $3m$. But G has degree (i, j) and F has degree $(1, 1)$, so at most $i + j$ common zeros are possible unless F divides G. Since $i + j < 3m$, we see F divides G, say $G = FH$. Then H has degree $(i-1, j-1)$ and vanishes at least $m-1$ times at each of the three points (since G vanishes at least m times and F vanishes once at each point). Thus $H \in (I^{(m-1)})_{(i-1,j-1)}$, so $(I^{(m)})_{(i,j)} \subseteq F(I^{(m-1)})_{(i-1,j-1)} \subset I^{(m-1)}I + FI^{(m-1)}$.

Hereafter assume $i + j \geq 3m$. If $j = 0$, then $(I^{(m)})_{(i,j)}$ is the space of polynomials in a and b of degree $(i, 0)$ divisible by $a^m b^m (a - b)^m$. Thus $(I^{(m)})_{(i,j)} = (I_{(3,0)})^m I_{(i-3m,0)}$, hence $(I^{(m)})_{(i,j)} \subseteq I^m \subseteq I^{(m-1)}I$. Similarly, if $i = 0$, swapping c and d for a and b we again have $(I^{(m)})_{(i,j)} \subseteq I^m \subseteq I^{(m-1)}I$.

Now assume $i > 0$ and $j > 0$, in addition to $i + j \geq 3m$. The cases $i \geq j$ and $j \geq i$ are symmetric, so assume $i \geq j$. We work on the surface X obtained by blowing up the points P_i. We have the birational morphism $\pi : X \to \mathbb{P}^1 \times \mathbb{P}^1$ with exceptional configuration H, V, E_1, E_2, E_3, with respect to which we can identify $(I^{(m)})_{(i,j)}$ with $H^0(X, iH + jV - (m-1)E)$, where $E = E_1 + E_2 + E_3$.

If $1 \leq j < m$, then we can write $iH + jV - mE = (i - 3m + j)H + j(2H + V - E) + (m-j)(3H - E)$. Note that $3H - E = (H - E_1) + (H - E_2) + (H - E_3)$ is a sum of three disjoint exceptional curves, disjoint also from $(i - 3m + j)H$ and $j(2H + V - E)$. Thus $(i - 3m + j)H + j(2H + V - E)$ is the nef part (with $|(i - 3m + j)H + j(2H + V - E)|$ non-empty and fixed component free) and $(m - j)(3H - E)$ is the negative (and fixed) part of a Zariski decomposition of $iH + jV - mE$. The unique element of $|3H - E|$ corresponds to an element $Q \in I_{(3,0)}$, and since $m - j > 0$ and $|3H - E|$ is the fixed part of $|iH + jV - mE|$, Q is a factor of every element of $(I^{(m)})_{(i,j)}$. Since Q vanishes with order 1 at each point P_1, P_2, P_3, we have $(I^{(m)})_{(i,j)} = Q(I^{(m-1)})_{(i-3,j)} \subset I^{(m-1)}I$, as we wanted to show.

So now we may assume that $i \geq j \geq m > 1$ and $i + j \geq 3m$. We will show that under multiplication we have a surjection $\mu : (I^{(m-1)})_{(i-2,j-1)} \otimes_k (I)_{(2,1)} \to (I^{(m)})_{(i,j)}$ and hence $(I^{(m)})_{(i,j)} \subset I^{(m-1)}I$. But surjectivity of μ is equivalent to surjectivity of the corresponding map $\lambda : H^0(X, (i-2)H+(j-1)V-(m-1)E)\otimes H^0(X, 2H+V-E) \to H^0(X, iH+jV-mE)$.

Under our assumptions, we have $(i - m) + (j - m) \geq m$ and $i - m \geq j - m \geq 0$, so we can pick integers $0 \leq s \leq r \leq i - m$ and $s \leq j - m$ such that $r + s = m$. Thus $iH + jV - mE = r(2H + V - E) + s(H + 2V - E) + (i - m - r)H + (j - m - s)V$, and moreover $r \geq 1$ (since $r \geq m/2 > 0$). Note also that $|2H + V - E|$ is non-empty and fixed component free (since we can write $2H + V - E$ as a sum of three exceptional curves

$(H-E_u)+(H-E_v)+(V-E_w)$ in three different ways using various permutations of $\{u,v,w\}=\{1,2,3\}$, showing that none of the curves occurring as summands is a fixed component), and likewise for $H+2V-E$. Since $|2H+V-E|$, $|H+2V-E|$, $|H|$ and $|V|$ are non-empty and fixed component free, $2H+V-E$, $H+2V-E$, H and V are nef. Since $r\geq 1$ and $m\geq 2$, $|(i-2)H+(j-1)V-(m-1)E| = |(r-1)(2H+V-E)+s(H+2V-E)+(i-m-r)H+(j-m-s)V|$ is also non-empty and fixed component free, so $(i-2)H+(j-1)V-(m-1)E$ is nef.

As discussed in Remark 2.3.1, we have a birational morphism $p: X\to\mathbb{P}^2$ with exceptional configuration $L'=H+V-E_3$, $E_1'=E_1$, $E_2'=E_2$, $E_3'=H-E_3$ and $E_4'=V-E_3$, so $H=L'-E_4'$, $V=L'-E_3'$, $E_1=E_1'$, $E_2=E_2'$ and $E_3=L'-E_3'-E_4'$. Let $p_1,\dots,p_4\in\mathbb{P}^2$ be the points such that $E_l'=p^{-1}(p_l)$. Because the points P_1,P_2,P_3 are 1-generic, no three of the points p_l are collinear. Thus the proper transform E_{uv}' of the line through the points p_u and p_v for $u\neq v$ is an exceptional curve and by contracting E_{14}', E_{24}', E_{12}' and E_3' we get another birational morphism $X\to\mathbb{P}^2$ obtained by blowing up four distinct general points p_u'', this one having exceptional configuration $L''=2L'-E_1'-E_2'-E_4'$, $E_1''=E_{14}'$, $E_2''=E_{24}'$, $E_3''=E_{34}'$, and $E_4''=E_3'$. Note that $2H+V-E=2L'-E_1'-E_2'-E_4'=L''$.

Thus λ can be written as $\lambda: H^0(X,G)\otimes H^0(X,L'')\to H^0(X,L''+G)$ where $G=(i-2)H+(j-1)V-(m-1)E$ is nef. Since X is the blow up of four points p_u'' and therefore $|2L''-E_1''-E_2''-E_3''-E_4''|\neq\varnothing$, it follows by [2, Proposition 2.4] that λ is surjective, as claimed. $\square$

Remark 3.1.3. Li and Swanson have given a criterion under which a radical ideal I in a reduced Noetherian domain has the property that $I^{(m)}=I^m$ for all $m\geq 1$; see [22, Theorem 3.6]. It is possible that the criterion applies for ideals of any sets of two, three or five 1-generic points of $\mathbb{P}^1\times\mathbb{P}^1$ in any characteristic, but it seems difficult to verify. However, for a specific choice of ground field and a specific choice of points one can use *Macaulay2* to check the criterion. Irena Swanson, for example, shared with us such a *Macaulay2* script, which shows over $\mathbb{Q}$ that the ideal I of a reduced set of three points in 1-generic position in $\mathbb{P}^1\times\mathbb{P}^1$ does satisfy the conditions of [22, Theorem 3.6], whence $I^{(m)}=I^m$ for all $m\geq 1$.

Let I be the ideal of five 1-generic points $P_1,\dots,P_5\in\mathbb{P}^1\times\mathbb{P}^1$. We will show that $I^{(m)}=I^m$ for all $m\geq 1$. The basic argument is the same as we used for three points in general position, but it is now more complicated.

Theorem 3.1.4. *Let $I=I(Z)$ with $Z\subseteq\mathbb{P}^1\times\mathbb{P}^1$ be five 1-generic points. Then $I^{(m)}=I^m$ for all $m\geq 1$, and in particular, $\rho(I)=\rho_a'(I)=\rho_a(I_Z)=1$.*

Proof. We will show that $(I^{(m)})_{(i,j)}\subset I^{(m-1)}I$ for all i and j, and hence that $I^{(m)}\subseteq I^m$. Since we know $I^m\subseteq I^{(m)}$, this shows equality. By symmetry, we may assume $i\geq j$. We also know $I_{(5,0)}$ is 1-dimensional, whose single basis element is the form $G=H_1\cdots H_5$, where H_s is a form of bi-degree $(1,0)$ defining the horizontal rule through the point P_s. Any form $F\in(I^{(m)})_{(i,j)}$ restricts for each s to a form of degree j on H_s, but with order of vanishing at least m. If $j < m$, then F must vanish on the entire horizontal rule through each P_s, and hence each H_s divides F, so G divides F. I.e., if $j < m$, then $(I^{(m)})_{(i,j)} = G(I^{(m-1)})_{(i-5,j)} \subset I^{(m-1)}I$.

We also know that $I_{(2,1)}$ is 1-dimensional, with basis a form D defining a smooth rational curve C vanishing with order 1 at each point P_s. Likewise, if $i + 2j < 5m$, then any form $F \in (I^{(m)})_{(i,j)}$ vanishes on C, and hence D divides F, so $(I^{(m)})_{(i,j)} = D(I^{(m-1)})_{(i-2,j-1)} \subset I^{(m-1)}I$.

We now may assume that $i \geq j \geq m \geq 2$ and $i + 2j \geq 5m$. This implies $2i + j \geq i + 2j \geq 5m$, and it also implies $i + j > 3m$. (To see the latter, given $m \geq 2$, consider the system of inequalities $i \geq j$, $j \geq m$, $i + j \leq 3m$. The solution set is a triangular region in the (i, j)-plane with vertices $(3m/2, 3m/2)$, (m, m) and $(2m, m)$. Since each vertex has $i + 2j < 5m$, we see $i \geq j \geq m \geq 2$ and $i + 2j \geq 5m$ imply $i + j > 3m$.)

There is a natural map $\mu_{(i,j)} : (I^{(m-1)})_{(i-3,j-1)} \otimes I_{(3,1)} \to (I^{(m)})_{(i,j)}$. Since $\mathrm{Im}(\mu_{(i,j)}) = (I^{(m-1)})_{(i-3,j-1)}I_{(3,1)}$, to finish it is enough to show $(I^{(m)})_{(i,j)} \subseteq I^{(m-1)}I$ whenever $\mu_{(i,j)}$ is not surjective. We can identify $(I^{(m)})_{(i,j)}$ with $H^0(X, A)$, and $I_{(3,1)}$ with $H^0(X, L)$, where $A = iH + jV - mE$, $L = 3H + V - E$ and $E = E_1 + \cdots + E_5$ are divisors on the blow up X of $\mathbb{P}^1 \times \mathbb{P}^1$ at the points $P_1, \ldots, P_5$ with respect to the usual exceptional configuration $H, V, E_1, \ldots, E_5$. Surjectivity of $\mu_{(i,j)}$ is equivalent to surjectivity of the map $H^0(X, A - L) \otimes H^0(X, L) \to H^0(X, A)$, which we will also denote by $\mu_{(i,j)}$.

Using Lemma 2.5.2, the inequalities $i \geq j \geq m \geq 2$, $i + 2j \geq 5m$, $2i + j \geq 5m$, and $i + j > 3m$ show that $A \cdot B \geq 0$ for every exceptional curve B on X, and hence A is effective and nef (since for a blow up X of $\mathbb{P}^1 \times \mathbb{P}^1$ at five 1-generic points, and thus 6 general points of $\mathbb{P}^2$, using the results of [9] one checks that the only prime divisors of negative self-intersection are the exceptional curves, but any divisor meeting every exceptional curve non-negatively is effective and nef [9, Proposition 4.1]).

Note that the exceptional configuration $L, E'_1 = H - E_1, E'_2 = H - E_2, E'_3 = H - E_3, E'_4 = H - E_4, E'_5 = H - E_5, E'_6 = 2H + V - E$ corresponds to a birational morphism $X \to \mathbb{P}^2$ obtained by blowing up 6 general points of $\mathbb{P}^2$, and that L is the pullback of a line in $\mathbb{P}^2$. By [14], $\mu_{(i,j)}$ always has maximal rank. Determining whether $\mu_{(i,j)}$ is surjective or injective is now purely numerical, and by [7, Theorem 3.4], $\mu_{(i,j)}$ is surjective if $A - L$ is nef, unless either $A - L = 5L - 2E'_1 - \cdots - 2E'_6 = H + 3V - E$ or $A - L = t(-K_X - E'_s)$ for $t > 0$. Note that $-K_X - E'_s = H + 2V - E + E_s$ for $1 \leq s \leq 5$ while $-K_X - E'_6 = V$. Since each term E_s of $A - L$ has the same coefficient, $A - L = t(-K_X - E'_s)$ is impossible for $s \neq 6$. Thus $\mu_{(i,j)}$ is surjective if $A - L$ is nef, unless either $A - L = H + 3V - E$ or $A - L = tV$ for $t > 0$; i.e., unless either $A = 4H + 4V - 2E$ or $A = 3H + tV - E$ for $t > 1$. But $A = 3H + tV - E$ is not relevant since we are interested in cases with $m > 1$. For the case $A = 4H + 4V - 2E = -2K_X$, we have surjectivity of $H^0(X, -K_X)^{\otimes 2} \to H^0(X, A)$ by [17, Proposition 3.1(a)]. Thus $(I^{(2)})_{(4,4)} = (I_{(2,2)})^2 \subset I^2$.

So now it suffices to show that $(I^{(m)})_{(i,j)} \subseteq I^{(m-1)}I$ whenever $A - L$ is not nef but A is nef and $m \geq 2$. First we must find all such A.

Either by hand or using software such as Normaliz [5], we can find generators for the semigroup of all (i, j, m) such that $i \geq j \geq m \geq 0$ and $i + 2j \geq 5m$. The result is that every such (i, j, m) is a non-negative integer linear combination of $(1, 0, 0)$, $(1, 1, 0)$, $(2, 2, 1)$, $(3, 1, 1)$, $(4, 3, 2)$, and $(5, 5, 3)$. So consider $A = a(1, 0, 0) + b(1, 1, 0) + c(2, 2, 1) + d(3, 1, 1) + e(4, 3, 2) + f(5, 5, 3)$, where here we use (i, j, m) as shorthand for $iH + jV - mE$.

Note $A - L$ is nef for any $A = a(1, 0, 0) + b(1, 1, 0) + c(2, 2, 1) + d(3, 1, 1) + e(4, 3, 2) + f(5, 5, 3)$ with $d > 0$, since $(3, 1, 1) = L$. So we may assume $d = 0$. However, $f(5H + 5V - 3E) - L = (t - 1)(5H + 5V - 3E) + 2(H + 2V - E)$, where $H + 2V - E$ is an exceptional curve by Lemma 2.5.2 with $(5H + 5V - 3E) \cdot (H + 2V - E) = 0$, so $A - L$ is effective but never nef for $A = f(5H + 5V - 3E)$.

In contrast, $(e(4H + 3V - 2E) - L) \cdot (H + 2V - E) < 0$ for $e = 1$, but for $e > 1$ we have $e(4H + 3V - 2E) - L = (5H + 5V - 3E) + (e - 2)(4H + 3V - 2E)$ so, for $e > 0$, $A - L$ is not nef for $A = e(4H + 3V - 2E)$ if and only if $e = 1$. In particular, if $e > 1$, then $A - L$ is nef for $A = a(1, 0, 0) + b(1, 1, 0) + c(2, 2, 1) + e(4, 3, 2) + f(5, 5, 3)$ regardless of the values of a, b, c, and f. However, $((4H + 3V - 2E) + f(5H + 5V - 3E) - L) \cdot (H + 2V - E) < 0$ for all $f \geq 0$, $A - L$ is never nef for $A = (4H + 3V - 2E) + f(5H + 5V - 3E)$.

Similarly, for $c \geq 0$, $c(2H + 2V - E) - L$ is nef if and only if $c > 1$, and $(2H + 2V - E) + f(5H + 5V - 3E) - L$ is never nef, but $(2H + 2V - E) + (4H + 3V - 2E) - L$ is nef. Thus the only cases with $A = c(2, 2, 1) + d(3, 1, 1) + e(4, 3, 2) + f(5, 5, 3)$ for which $A - L$ is not nef but $m \geq 2$ are: $A = f(5, 5, 3)$, $f \geq 1$; $A = (4, 3, 2) + f(5, 5, 3)$, $f \geq 0$; and $A = (2, 2, 1) + f(5, 5, 3)$, $f \geq 1$.

The only other possible cases are obtained from these by adding on to one of these multiples of either $(1, 0, 0)$ or $(1, 1, 0)$. But $(A - L) + (1, 0, 0)$ for any of these A is nef, so we do not get any additional cases by allowing $a > 0$ or $b > 0$. I.e., we must check that $(I^{(m)})_{(i,j)} \subseteq I^{(m-1)}I$ only when (i, j, m) is either $(2, 2, 1) + f(5, 5, 3)$, $(4, 3, 2) + f(5, 5, 3)$ or $f(5, 5, 3)$.

First, we show $(I^{(m)})_{(i,j)} \subseteq I^{(m-1)}I$ holds for the cases $f(5, 5, 3)$. Let $F = 5H + 5V - 3E$. The divisor $E_6' = 2H + V - E$ is linearly equivalent to the exceptional curve which is the proper transform C' of the curve above denoted as C. Likewise, $H + 2V - E$ is linearly equivalent to an exceptional curve; denote this exceptional curve by C''. Note that $F = 2C' + (H + 3V - E) = 2C'' + (3H + V - E)$. Thus $(I_{(2,1)})^2 I_{(1,3)} \subseteq (I^{(3)})_{(5,5)}$ and $(I_{(1,2)})^2 I_{(3,1)} \subseteq (I^{(3)})_{(5,5)}$, but $\dim I_{(1,2)} = \dim I_{(2,1)} = 1$ and $\dim I_{(3,1)} = \dim I_{(1,3)} = 3$, while $\dim(((I_{(2,1)})^2 I_{(1,3)}) \cap ((I_{(1,2)})^2 I_{(3,1)})) = 0$ since $F - 2C' - 2C''$ is not linearly equivalent to an effective divisor. Thus $\dim(((I_{(2,1)})^2 I_{(1,3)}) + ((I_{(1,2)})^2 I_{(3,1)})) = 6 = \dim(I^{(3)})_{(5,5)}$, hence $(I^{(3)})_{(5,5)} \subset I^3$. Moreover, $F = 5H + 5V - 3E$ is normally generated by [17, Proposition 3.1(a)], which means that $H^0(X, F)^{\otimes n} \to H^0(X, nF)$ is surjective. Thus $(I^{(3f)})_{(5f,5f)} = ((I^{(3)})_{(5,5)})^f$ and hence $(I^{(3f)})_{(5f,5f)} \subset (I^3)^f = I^{3f}$, as we needed to show.

Now consider $(I^{(2)})_{(4,3)}$. We have $I_{(1,2)} I_{(3,1)} \subseteq (I^{(2)})_{(4,3)}$ and $I_{(2,1)} I_{(2,2)} \subseteq (I^{(2)})_{(4,3)}$, but $\dim I_{(1,2)} I_{(3,1)} = 3$, $\dim I_{(2,1)} I_{(2,2)} = \dim I_{(2,2)} = 4$, and $\dim((I_{(1,2)} I_{(3,1)}) \cap (I_{(2,1)} I_{(2,2)})) =$

$\dim H^0(X,H) = 2$, so $\dim((I_{(1,2)}I_{(3,1)}) + (I_{(2,1)}I_{(2,2)})) = 4+3-2 = 5 = \dim(I^{(2)})_{(4,3)}$, hence $(I^{(2)})_{(4,3)} = ((I_{(1,2)}I_{(3,1)}) + (I_{(2,1)}I_{(2,2)})) \subset I^2$, as we needed to show.

Note that $4H+3V-2E = 3L-E_1'-\cdots-E_5'$. Since the points are general, $|3L-E_1'-\cdots-E_5'|$ and hence $|4H+3V-2E|$ contains the class of a smooth elliptic curve, Q. Let $F = 5H+5V-3E$. Tensoring $0 \to \mathcal{O}_X(-Q) \to \mathcal{O}_X \to \mathcal{O}_Q \to 0$ by $\mathcal{O}_X(Q+fF)$ and taking global sections gives $0 \to H^0(X, fF) \to H^0(X, Q+fF) \to H^0(Q, \mathcal{O}_Q(Q+fF)) \to 0$. Tensoring by $H^0(X,F) = \Gamma_X(F)$ and applying the natural multiplication maps gives the following commutative diagram (see [24], or [10, Lemma 2.3.1]):

$$\begin{array}{ccccccccc}
0 & \to & H^0(X,fF)\otimes\Gamma_X(F) & \to & H^0(X,Q+fF)\otimes\Gamma_X(F) & \to & H^0(Q,\mathcal{O}_Q(Q+fF))\otimes\Gamma_X(F) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^0(X,(f+1)F) & \to & H^0(X,Q+(f+1)F) & \to & H^0(Q,\mathcal{O}_Q(Q+(f+1)F)) & \to & 0
\end{array}$$

Since $F-Q$ is linearly equivalent to an exceptional curve and hence $h^1(X,F-Q)=0$, the sequence $0 \to \mathcal{O}_X(F-Q) \to \mathcal{O}_X(F) \to \mathcal{O}_Q(F) \to 0$ is exact on global sections. Thus the map $H^0(Q,\mathcal{O}_Q(Q+fF))\otimes\Gamma_X(F) \to H^0(Q,\mathcal{O}_Q(Q+(f+1)F))$ has the same image as $H^0(Q,\mathcal{O}_Q(Q+fF))\otimes\Gamma_Q(F) \to H^0(Q,\mathcal{O}_Q(Q+(f+1)F))$, and the latter is surjective by [24, Theorem 6] (or see [18, Proposition II.5(c)]). We saw above that F is normally generated, and hence that the map $H^0(X,fF)\otimes\Gamma_X(F) \to H^0(X,(f+1)F)$ is surjective. Now apply the snake lemma to the above diagram to conclude that $H^0(X,Q+fF)\otimes\Gamma_X(F) \to H^0(X,Q+(f+1)F)$ is surjective. By induction, we have surjectivity for all $f \geq 0$ and hence $(I^{(2+3f)})_{(4+5f,3+5f)} = (I^{(2)})_{(4,3)}((I^{(3)})_{(5,5)})^f \subset I^2 I^{3f} = I^{2+3f}$.

Finally we consider the case of $(2,2,1)+f(5,5,3)$. The proof here is the same as for $(4,3,2)+f(5,5,3)$, except now Q is a smooth elliptic curve linearly equivalent to $-K_X = 3L-E_1'-\cdots-E_6'$ and $F-Q$ is linearly equivalent to the sum $C'+C''$ of two disjoint exceptional curves, so as before we have $h^1(X,F-Q)=0$. Thus $(I^{(1+3f)})_{(2+5f,2+5f)} = (I)_{(2,2)}((I^{(3)})_{(5,5)})^f \subset I^{1+3f}$. $\square$

3.2. s **points in 1-generic position with** s **a perfect square.** While computer calculations suggest that $I^{(2)} = I^2$ for the ideal I of four 1-generic points in $\mathbb{P}^1\times\mathbb{P}^1$, it is not hard to see that $I^{(3)} \neq I^3$. This is because $\alpha(I)=3$, so $\alpha(I^3)=9$, but there is a unique curve of bi-degree $(1,1)$ through any three of the four points (corresponding to the divisors $H+V-E_1-E_2-E_3-E_4+E_i$ in Lemma 2.5.2), hence the sum of these four curves corresponds to a non-trivial form in $(I^{(3)})_{(4,4)}$. Thus $\alpha(I^{(3)}) \leq 8$, so $I^{(3)} \not\subseteq I^3$.

In fact, the case of four 1-generic points is part of a much larger family, namely a set Z of s points in 1-generic position when $s=t^2$ for some integer $t\geq 2$. For this family, we can in a similar way verify failures of containments of certain symbolic powers of the ideal $I(Z)$ of the points in various ordinary powers of the ideal, and thereby obtain lower bounds on $\rho_a(I(Z))$ bigger than 1.

Theorem 3.2.1. *Let $I = I(Z)$ where $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a set of $s = t^2$ points in 1-generic position with $t \geq 2$. Then for all integers $n \geq 1$,*

$$I^{((s-1)(2t-1)n)} \not\subseteq I^{2s(t-1)n+1}.$$

In particular, $\rho(I) \geq \rho'_a(I) \geq \rho_a(I) \geq \frac{(s-1)(2t-1)}{2s(t-1)} = \frac{(t+1)(2t-1)}{2t^2} = 1 + \frac{1}{2t} - \frac{1}{2s}$.

Proof. We begin by showing that the symbolic power $I^{((s-1)(2t-1)n)}$ has a nonzero element of bidgree $((t-1)s(2t-1)n, (t-1)s(2t-1)n)$. For each point $P_i \in Z$, let $Y_i = Z \setminus \{P_i\}$. Then Y_i is a set of $s-1$ points in 1-generic position for each $i = 1, \ldots, s$ and hence

$$\dim(I(Y_i)_{(t-1,t-1)}) = \max\{t^2 - |Y_i|, 0\} = \max\{s - (s-1), 0\} = 1.$$

Thus, for each $i = 1, \ldots, s$, there is a form F_i (unique up to scalar multiplication) that vanishes at all of the points of Y_i. Moreover, F_i does not vanish at P_i. Indeed, if $F_i(P_i) = 0$, then $F_i \in I(Z)_{(t-1,t-1)}$, but $I(Z)_{(t-1,t-1)} = 0$ since $\dim(I_{(t-1,t-1)}) = \max\{t^2 - |Z|, 0\} = 0$.

Set $F = \prod_{i=1}^s F_i$. The form F has degree $((t-1)s, (t-1)s)$ and passes through all the points of Z with multiplicity at least $s-1$, so $F \in I^{(s-1)}$. Thus $F^{(2t-1)n} \in (I^{(s-1)})^{(2t-1)n} \subseteq I^{((s-1)(2t-1)n)}$ and $\deg F^{(2t-1)n} = ((t-1)s(2t-1)n, (t-1)s(2t-1)n)$ for each $n \geq 1$.

To show $I^{((s-1)(2t-1)n)} \not\subseteq I^{2s(t-1)n+1}$, it is now enough to check that

$$(I^{2s(t-1)n+1})_{((t-1)s(2t-1)n,(t-1)s(2t-1)n)} = 0.$$

Because the points of Z are in 1-generic position, then for $i + j = 2(t-1)$, $i, j \geq 0$, we have $(i+1)(j+1) \leq t^2 = |Z|$, so $\dim(I_{(i,j)}) = 0$. Thus, viewing I as a singly graded ideal, we have $\alpha(I) \geq 2t - 1$, hence

$$\alpha(I^{2s(t-1)n+1}) \geq (2s(t-1)n + 1)(2t-1) > 2s(t-1)n(2t-1)$$

and so $(I^{2s(t-1)n+1})_{(s(t-1)n(2t-1),s(t-1)n(2t-1))} = 0$. Taking the supremum of $\frac{(s-1)(2t-1)n}{2s(t-1)n+1}$ over all n gives the bound. $\square$

Remark 3.2.2. In the case of 4 general points, the preceding result gives the same lower bound, $9/8 \leq \rho_a(I) \leq \rho(I)$ that we obtained in Corollary 2.5.4. Computational evidence using *Macaulay2* in this case suggests that $I^{(3t)} = (I^{(3)})^t$ for all $t > 0$ and also that $I^{(9)} \subseteq I^8$. This would imply by Theorem 2.5.1(4), that $\rho'_a(I) \leq 9/8$, and hence $\rho_a(I) = \rho'_a(I) = 9/8$.

[In fact, evidence from *Macaulay2* also suggests that $I^{(3t+i)} = (I^{(3)})^t I^i$ for $i = 1, 2$ and all $t > 0$ and that $I^{(3)} \subseteq I^2$ and $I^{(6)} \subseteq I^5$. Given that these hold, one can prove that $I^{(m)} \subseteq I^r$ whenever $m/r \geq 9/8$, which by definition implies that $\rho(I) \leq 9/8$ and thus that $\rho_a(I) = \rho'_a(I) = \rho(I) = 9/8$. The proof is to check cases modulo 3. For example, given any $m \geq 1$, we can write $m = 3t+i$, for some t and $0 \leq i \leq 2$, and we can write $t = 3j+q$ for some $0 \leq q \leq 2$. Then $I^{(m)} = I^{(9j+3q+i)} = (I^{(3)})^{3j+q} I^i = (I^{(9)})^j (I^{(3)})^q I^i$. Also note that $m/r \geq 9/8$ is equivalent to $8j + 8q/3 + 8i/9 = 8m/9 \geq r$. If $q = 0$, then $I^{(9)} \subseteq I^8$ implies $I^{(m)} = (I^{(9)})^j I^i \subseteq I^{8j} I^i$, but $I^{8j} I^i \subseteq I^r$, since $8j + i \geq 8j + 8q/3 + 8i/9 \geq r$. If $q = 1$, then $I^{(9)} \subseteq I^8$ and $I^{(3)} \subseteq I^2$ imply $I^{(m)} = (I^{(9)})^j (I^{(3)})^q I^i \subseteq I^{8j} I^2 I^i$, but $I^{8j} I^2 I^i \subseteq I^r$ since

$8j+8/3+8i/9 = 8j+8q/3+8i/9 \geq r$ implies $8j+2+i = \lfloor 8j+8/3+8i/9 \rfloor \geq r$. If $q = 2$, then $I^{(9)} \subseteq I^8$ and $I^{(6)} \subseteq I^5$ imply $I^{(m)} = (I^{(9)})^j (I^{(3)})^q I^i \subseteq I^{8j} I^5 I^i$, but $I^{8j} I^5 I^i \subseteq I^r$ since $8j+5+i = \lfloor 8j+8q/3+8i/9 \rfloor \geq r$.]

By adapting the proof of Theorem 3.2.1 slightly, we can prove a similar result for s points in 1-generic position when $s = t^2 - t = t(t-1)$ for some integer t.

Theorem 3.2.3. *Let $I = I(Z)$ where $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a set of $s = t^2 - t$ points in 1-generic position with $t \geq 2$. Then for all integers $n \geq 1$,*

$$I^{(2(s-1)(t-1)n)} \not\subseteq I^{s(2t-3)n+1}.$$

In particular, $\rho(I) \geq \rho'_a(I) \geq \rho_a(I) \geq \frac{2(s-1)(t-1)}{s(2t-3)} = \frac{2(t^2-t-1)}{t(2t^2-3)} = 1 + \frac{1}{2t} - \frac{1}{2t(2t-3)}$.

Proof. For each point P of Z, let $Y_P = Z \setminus \{P\}$. As in the proof of Theorem 3.2.1, there exists a unique form (up to scalar multiplication) of degree $(t-1, t-2)$ that vanishes at all points of Z except P, and likewise for degree $(t-2, t-1)$.

Since s is even, we can partition Z into two disjoint sets of the same size, i.e., $Z = Z_1 \cup Z_2$, where $Z_1 = \{P_1, \ldots, P_{\frac{s}{2}}\}$ and $Z_2 = \{P_{\frac{s}{2}+1}, \ldots, P_s\}$. For each $P_i \in Z_1$, let G_i be the form of degree $(t-2, t-1)$ vanishing at all points of Z except P_i, and for each $P_i \in Z_2$, let H_i be the form of degree $(t-1, t-2)$ vanishing at all points of Z except P_i. We then set

$$F = \left(\prod_{i=1}^{\frac{s}{2}} G_i \right) \left(\prod_{i=\frac{s}{2}+1}^{s} H_i \right).$$

The form F has bidegree $\left(\frac{s(2t-3)}{2}, \frac{s(2t-3)}{2} \right)$, and furthermore, $F \in I^{(s-1)}$.

For any integer $n \geq 1$, we have $F^{2(t-1)n} \in (I^{(s-1)})^{2(t-1)n} \subseteq I^{(2(s-1)(t-1)n)}$. If we view $I^{(2(s-1)(t-1)n)}$ as a homogeneous ideal, then this ideal has an element of degree $2s(2t-3)(t-1)n$.

We now show that $I^{s(2t-3)n+1}$ has no elements of bidegree $(s(2t-3)(t-1)n, s(2t-3)(t-1)n)$. We have $\alpha(I) \geq 2t-2$, so $\alpha(I^{s(2t-3)n+1}) \geq (2t-2)(s(2t-3)n+1) > (2t-2)s(2t-3)n$. As in the proof of Theorem 3.2.1, the conclusion now follows. $\square$

3.3. **Six or more points in 1-generic position.** In this section we compare the symbolic squares and ordinary squares of ideals of six or more points in 1-generic position.

Proposition 3.3.1. *Let $I = I(Z)$ with $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of 6 points in 1-generic position. Then $I^2 \neq I^{(2)}$.*

Proof. We look at the bigraded pieces of I^2 and $I^{(2)}$. Since Z imposes at most $6\binom{2+1}{2} = 18$ conditions on forms of bidegree $(3,4)$, we see $\dim((I^{(2)})_{(3,4)}) \geq 2$. Thus $\alpha(I^{(2)}) \leq 7$, but using the fact that I is 1-generic we compute that $\alpha(I) = 4$ so $\alpha(I^2) = 8$, and hence $I^2 \subsetneq I^{(2)}$. $\square$

To extend this result to 7 or more points, we require [29, Theorem 1], which describes the Hilbert function of a set of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ whose support is in 1-generic position. We state only the part we need:

Lemma 3.3.2. *Let $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of s points in 1-generic position, with defining ideal $I = I(Z)$. If $(i,j) \notin \{(2, s-1), (s-1, 2)\}$, then*

$$\dim(I^{(2)})_{(i,j)} = \max\{0, (i+1)(j+1) - 3s\}.$$

We now proceed to the case of 7 or more points:

Theorem 3.3.3. *Let $I = I(Z)$ with $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of $s = |Z| \geq 7$ points in 1-generic position. Then $I^2 \neq I^{(2)}$.*

Proof. Let $I = I(Z)$. To show that $I^2 \neq I^{(2)}$, we find a bidegree (i,j) where $(I^2)_{(i,j)} \neq (I^{(2)})_{(i,j)}$, which we verify by showing that the two graded pieces have different dimensions.

We divide s by 2 and by 3 using the division algorithm to write s as $s = 2q_1 + r_1$ and $s = 3q_2 + r_2$ where $0 \leq r_1 \leq 1$ and $0 \leq r_2 \leq 2$. Because Z is in 1-generic position,

$$\begin{aligned} H_Z(1, q_1) &= \min\{2(q_1+1), 2q_1 + r_1\} = 2q_1 + r_1 \text{ and} \\ H_Z(2, q_2) &= \min\{3(q_2+1), 3q_2 + r_2\} = 3q_2 + r_2. \end{aligned}$$

It then follows from the Hilbert function that $\dim(I_{(1,q_1)}) = 2(q_1+1) - H_Z(1,q_1) = 2 - r_1$ and $\dim(I_{(2,q_2)}) = 3(q_2+1) - H_Z(2,q_2) = 3 - r_2$.

We will use this information, and Lemma 3.3.2, to compare the ideals I^2 and $I^{(2)}$ in bidegree $(3, q_1+q_2)$. We require two claims.

Claim 1. $\dim((I^2)_{(3,q_1+q_2)}) \leq (2-r_1)(3-r_2)$.

Proof of Claim 1. We first note that

$$(I^2)_{(3,q_1+q_2)} = \sum_{\substack{0 \leq a,b,c,d \\ a+c=3,\ b+d=q_1+q_2}} I_{(a,b)} I_{(c,d)}.$$

The claim will follow if we show that whenever $(a,b) \notin \{(1,q_1), (2,q_2)\}$, then $I_{(a,b)} I_{(c,d)} = 0$. This would then show that $(I^2)_{(3,q_1+q_2)} = I_{(1,q_1)} I_{(2,q_2)}$, and thus

$$\dim((I^2)_{(3,q_1+q_2)}) \leq \dim(I_{(1,q_1)}) \dim(I_{(2,q_2)}) = (2-r_1)(3-r_2).$$

If $a = 0$, then $I_{(a,b)} = 0$ since Z is in 1-generic position and $0 \leq b \leq q_1 + q_2 \leq s - 1$. Likewise, $I_{(c,d)} = 0$ if $c = 0$.

If $a = 1$ and $b \neq q_1$, then there are two cases. If $b < q_1$, then $I_{(a,b)} = 0$, since $H_Z(a,b) = \min\{(a+1)(b+1), 2q_1 + r_1\} = (a+1)(b+1)$. On the other hand, if $b > q_1$, then $I_{(c,d)} = 0$ since $c = 2$ and $d = q_1 + q_2 - b < q_2$, so $(c+1)(d+1) \leq 3q_2 \leq 3q_2 + r_2$, whence $H_Z(c,d) = (c+1)(d+1)$. Likewise, $I_{(a,b)} I_{(c,d)} = 0$ if $c = 1$ and $d \neq q_1$.

Finally, if $a \geq 2$, then $c \leq 1$, so the same arguments apply. $\square$

Claim 2. $\dim(I^{(2)})_{(3,q_1+q_2)} = q_2 + 4 - 2r_1 - r_2$.

Proof of Claim 2. By Lemma 3.3.2, we have

$$\dim(I^{(2)})_{(3,q_1+q_2)} = \max\{0, 4(q_1 + q_2 + 1) - 3s\}.$$

By the definition of q_1 and q_2, we have $s \leq 2q_1 + 1$ and $s \leq 3q_2 + 2$. So

$$\begin{aligned} 4(q_1 + q_2 + 1) - 3s &= 4q_1 + 4q_2 + 4 - 3s \\ &= (2q_1 + 1) + (2q_1 + 1) + (3q_2 + 2) + q_2 - 3s \geq 0. \end{aligned}$$

Thus $\dim(I^{(2)})_{(3,q_1+q_2)} = 4(q_1 + q_2 + 1) - 3s$. Now, using the fact that $s = 2q_1 + r_1$ and $s = 3q_2 + r_2$, we get

$$4(q_1 + q_2 + 1) - 3s = 4q_1 + 4q_2 + 4 - 2(2q_1 + r_1) - (3q_2 + r_2) = q_2 + 4 - 2r_1 - r_2.$$

□

To complete the proof, it suffices to show that

$$\dim(I^{(2)})_{(3,q_1+q_2)} = q_2 + 4 - 2r_1 - r_2 > (2 - r_1)(3 - r_2) \geq \dim(I^2)_{(3,q_1+q_2)}.$$

But $q_2 + 4 - 2r_1 - r_2 > (2 - r_1)(3 - r_2)$ is equivalent to $q_2 - 1 > (r_1 - 1)(r_2 - 1)$. The maximum value of $(r_1-1)(r_2-1)$ is 1, and it occurs only for $r_1 = r_2 = 0$, whereas $q_2-1 > 1$ unless $s = 7$ or 8, and in both of these cases we have $q_1 - 1 = 1 \geq 0 \geq (r_1 - 1)(r_2 - 1)$. □

Remark 3.3.4. We cannot use the above proof for the case $s = 6$ because $q_2+4-2r_1-r_2 = (2 - r_1)(3 - r_2)$ when $s = 6$ but the proof needs $q_2 + 4 - 2r_1 - r_2 > (2 - r_1)(3 - r_2)$.

Now, we are able to prove the main result of this paper:

Proof of Theorem 1.1. That $I^{(m)} = I^m$ for all $m \geq 1$ for s general points for $s = 1, 2, 3, 5$, follows from Theorems 2.4.1, 3.1.1, 3.1.2 and 3.1.4, respectively. That $I^{(m)} \neq I^m$ for some m for all other s follows for $s = 4$ by Theorem 3.2.1 (apply the theorem with $t = 2$, and the fact that we would have $\rho(I) = 1$ if $I^{(m)} = I^m$ for all $m \geq 1$), for $s = 6$ by Proposition 3.3.1 and for $s > 6$ by Theorem 3.3.3. The rest of the statement of the theorem follows from Corollary 2.5.4. □

References

[1] C. Bocci and B. Harbourne, Comparing powers and symbolic power of ideals. J. Algebraic Geometry **19** (2010), 399–417.

[2] C. Bocci and B. Harbourne, The resurgence of ideals of points and the containment problem. Proc. Amer. Math. Soc. **138** (2010), 1175–1190.

[3] CoCoATeam, CoCoA: a system for doing Computations in Commutative Algebra. Available at `http://cocoa.dima.unige.it`

[4] J. H. de Boer, Localization in a graded ring. Proc. Amer. Math. Soc. **12** (1961), 764-772.

[5] W. Bruns, B. Ichim and C. Sger, Normaliz. Algorithms for rational cones and affine monoids. Available from `http://www.math.uos.de/normaliz`

[6] L. Ein, R. Lazarsfeld, and K. E. Smith, Uniform behavior of symbolic powers of ideals. Invent. Math., **144** (2001), 241–252.

[7] S. Fitchett, Maps of linear systems on blow-ups of the projective plane. J. Pure Appl. Algebra **156** (2001), 1–14.

[8] A. V. Geramita, A. Gimigliano and Y. Pitteloud, Graded Betti numbers of some embedded rational n-folds. Math. Annalen **301** (1995), 363–380.

[9] A. Geramita, B. Harbourne and J. Migliore, Classifying Hilbert functions of fat point subschemes in $\mathbb{P}^2$. Collect. Math. **60** (2009), 159–192.

[10] A. Gimigliano, B. Harbourne and M. Idà, Betti numbers for fat point ideals in the plane: a geometric approach. Trans. Amer. Math. Soc. **361** (2009), 1103–1127.

[11] S. Giuffrida, R. Maggioni and A. Ragusa, On the postulation of 0-dimensional subschemes on a smooth quadric. Pacific J. Math. **155** (1992), 251–282.

[12] D. R. Grayson and M. E. Stillman, Macaulay 2, a software system for research in algebraic geometry. `http://www.math.uiuc.edu/Macaulay2/`.

[13] E. Guardo, Fat point schemes on a smooth quadric. J. Pure Appl. Algebra **162** (2001), 183–208.

[14] E. Guardo and B. Harbourne, Resolutions of ideals of six fat points in $\mathbb{P}^2$. J. Algebra **318** (2007), 619–640.

[15] E. Guardo and A. Van Tuyl, Fat points in $\mathbb{P}^1 \times \mathbb{P}^1$ and their Hilbert functions. Canad. J. Math. **56** (2004), 716–741.

[16] H. T. Hà and A. Van Tuyl, The regularity of points in multi-projective spaces. J. Pure Appl. Algebra **187** (2004), 153–167.

[17] B. Harbourne, Birational models of rational surfaces. J. Algebra **190** (1997), 145–162.

[18] B. Harbourne, Free Resolutions of Fat Point Ideals on $\mathbf{P}^2$. J. Pure Appl. Algebra **125** (1998), 213–234.

[19] B. Harbourne and C. Huneke, Are symbolic powers highly evolved? Preprint (2011). `arXiv:1103.5809v1`

[20] M. Hochster and C. Huneke, Comparison of symbolic and ordinary powers of ideals. Invent. Math. **147** (2002), 349–369.

[21] V. Kodiyalam, Asymptotic behaviour of Castelnuovo-Mumford regularity. Proc. Amer. Math. Soc. **128** (2000), 407–411.

[22] A. Li and I. Swanson, Symbolic powers of radical ideals. Rocky Mountain J. Math. **36** (2006), 997–1009.

[23] L. Marino, Conductor and separating degrees for sets of points in $\mathbb{P}^r$ and in $\mathbb{P}^1 \times \mathbb{P}^1$. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **9** (2006), 397–421.

[24] D. Mumford, Varieties defined by quadratic equations. In: *Questions on algebraic varieties*, Corso C.I.M.E. 1969 Rome: Cremonese, 1970, 30–100.

[25] D. Rees, On a problem of Zariski. Illinois J. Math. **2** (1958), 145–149.

[26] P. Schenzel, Filtrations and Noetherian symbolic blow-up rings. Proc. Amer. Math. Soc. **102** (1988), no. 4, 817–822.

[27] J. Sidman and A. Van Tuyl, Multigraded regularity: syzygies and fat points. Beiträge Algebra Geom. **47** (2006), 67–87.

[28] A. Van Tuyl, The defining ideal of a set of points in multi-projective space. J. London Math. Soc. (2) **72** (2005), 73–90.

[29] A. Van Tuyl, An appendix to a paper of Catalisano, Geramita, Gimigliano: The Hilbert function of generic sets of 2-fat points in $\mathbb{P}^1 \times \mathbb{P}^1$. In *Projective Varieties with Unexpected Properties*. de Gruyter (2005), 109–112.

[30] O. Zariski and P. Samuel, *Commutative algebra. Vol. II.* The University Series in Higher Mathematics, D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto-London-New York, 1960.

Dipartimento di Matematica e Informatica, Viale A. Doria, 6 - 95100 - Catania, Italy

E-mail address: guardo@dmi.unict.it

URL: http://www.dmi.unict.it/~guardo/

Department of Mathematics, University of Nebraska–Lincoln, Lincoln, NE 68588-0130, USA

E-mail address: bharbour@math.unl.edu

URL: http://www.math.unl.edu/~bharbourne1/

Department of Mathematical Sciences, Lakehead University, Thunder Bay, ON P7B 5E1, Canada

E-mail address: avantuyl@lakeheadu.ca

URL: http://flash.lakeheadu.ca/~avantuyl/