EHRHART POLYNOMIALS OF INTEGRAL SIMPLICES WITH PRIME VOLUMES

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ABSTRACT. For an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$ of dimension d, we call $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ its δ-vector and $\mathrm{vol}(\mathcal{P}) = \sum_{i=0}^{d} \delta_i$ its normalized volume. In this paper, we will establish the new equalities and inequalities on δ-vectors for integral simplices whose normalized volumes are prime. Moreover, by using those, we will classify all the possible δ-vectors of integral simplices with normalized volume 5 and 7.

INTRODUCTION

One of the most fascinating problems on enumerative combinatorics is to characterize the δ-vectors of integral convex polytopes.

Let $\mathcal{P} \subset \mathbb{R}^N$ be an *integral* convex polytope of dimension d, which is a convex polytope any of whose vertices has integer coordinates. Let $\partial\mathcal{P}$ denote the boundary of $\mathcal{P}$. Given a positive integer n, we define

$$i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^N|, \qquad i^*(\mathcal{P}, n) = |n(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|,$$

where $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$ and $|X|$ is the cardinality of a finite set X. The enumerative function $i(\mathcal{P}, n)$ is called the *Ehrhart polynomial* of $\mathcal{P}$, which was studied originally in the work of Ehrhart [1]. The Ehrhart polynomial has the following fundamental properties:

- $i(\mathcal{P}, n)$ is a polynomial in n of degree d. (Thus, in particular, $i(\mathcal{P}, n)$ can be defined for *every* integer n.)
- $i(\mathcal{P}, 0) = 1$.
- (loi de réciprocité) $i^*(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n)$ for every integer $n > 0$.

We refer the reader to [2, Part II] and [7, pp. 235–241] for the introduction to the theory of Ehrhart polynomials.

We define the sequence $\delta_0, \delta_1, \delta_2, \ldots$ of integers by the formula

$$(1) \qquad (1-\lambda)^{d+1} \left(\sum_{n=0}^{\infty} i(\mathcal{P}, n)\lambda^n \right) = \sum_{i=0}^{\infty} \delta_i \lambda^i.$$

Then, from a fundamental result on generating function ([7, Corollary 4.3.1]), we know that $\delta_i = 0$ for every $i > d$. We call the integer sequence

$$\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d),$$

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which appears in (1), the δ-*vector* of $\mathcal{P}$. In addition, by the reciprocity law, one has

$$\sum_{n=1}^{\infty} i^*(\mathcal{P}, n)\lambda^n = \frac{\sum_{i=0}^{d} \delta_{d-i}\lambda^{i+1}}{(1-\lambda)^{d+1}}.$$

The δ-vector has the following fundamental properties:

- $\delta_0 = 1$ and $\delta_1 = |\mathcal{P} \cap \mathbb{Z}^N| - (d+1)$.
- $\delta_d = |(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|$. Hence, we have $\delta_1 \geq \delta_d$.
- Each δ_i is nonnegative ([8]).
- If $(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N$ is nonempty, then one has $\delta_1 \leq \delta_i$ for every $1 \leq i \leq d-1$ ([3]).
- When $d = N$, the leading coefficient $(\sum_{i=0}^{d} \delta_i)/d!$ of $i(\mathcal{P}, n)$ is equal to the usual volume of $\mathcal{P}$ ([7, Proposition 4.6.30]). In general, the positive integer $\mathrm{vol}(\mathcal{P}) = \sum_{i=0}^{d} \delta_i$ is said to be the *normalized volume* of $\mathcal{P}$.

Recently, the δ-vectors of integral convex polytopes have been studied intensively. For example, see [6], [10] and [11].

There are two well-known inequalities on δ-vectors. Let $s = \max\{i : \delta_i \neq 0\}$. One is

$$\delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_s + \delta_{s-1} + \cdots + \delta_{s-i}, \qquad 0 \leq i \leq \lfloor s/2 \rfloor, \tag{2}$$

which is proved in Stanley [9], and another one is

$$\delta_d + \delta_{d-1} + \cdots + \delta_{d-i} \leq \delta_1 + \delta_2 + \cdots + \delta_i + \delta_{i+1}, \qquad 0 \leq i \leq \lfloor (d-1)/2 \rfloor, \tag{3}$$

which appears in Hibi [3, Remark (1.4)].

When $\sum_{i=0}^{d} \delta_i \leq 3$, the above inequalities (2) and (3) characterize the possible δ-vectors completely ([5]). Moreover, when $\sum_{i=0}^{d} \delta_i = 4$, the possible δ-vectors are determined completely ([4, Theorem 5.1]) by (2) and (3) together with an additional condition. Furthermore, by the proofs of [5, Theorem 0.1] and [4, Theorem 5.1], we know that all the possible δ-vectors can be realized as the δ-vectors of integral simplices when $\sum_{i=0}^{d} \delta_i \leq 4$. However, unfortunately, it is not true when $\sum_{i=0}^{d} \delta_i = 5$. (See [4, Remark 5.2].) Therefore, for the further classifications of the δ-vectors with $\sum_{i=0}^{d} \delta_i \geq 5$, it is natural to investigate the δ-vectors of integral simplices. In this paper, in particular, we establish some new constraints on δ-vectors for integral simplices whose normalized volumes are prime numbers. The following theorem is our main result of this paper.

Theorem 0.1. *Let $\mathcal{P}$ be an integral simplex of dimension d and $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ its δ-vector. Suppose that $\sum_{i=0}^{d} \delta_i = p$ is an odd prime number. Let $i_1, \ldots, i_{p-1}$ be the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_{p-1}}$ with $1 \leq i_1 \leq \cdots \leq i_{p-1} \leq d$. Then,*

(a) *one has*

$$i_1 + i_{p-1} = i_2 + i_{p-2} = \cdots = i_{(p-1)/2} + i_{(p+1)/2} \leq d+1;$$

(b) *one has*

$$i_k + i_\ell \geq i_{k+\ell} \ \text{ for } \ 1 \leq k \leq \ell \leq p-1 \ \text{ with } \ k+\ell \leq p-1.$$

We prove Theorem 0.1 in Section 1 via the languages of elementary group theory.

As an application of Theorem 0.1, we give a complete characterization of the possible δ-vectors of integral simplices when $\sum_{i=0}^{d} \delta_i = 5$ and 7.

Theorem 0.2. *Given a finite sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ of nonnegative integers, where $\delta_0 = 1$ and $\sum_{i=0}^{d} \delta_i = 5$, there exists an integral simplex $\mathcal{P} \subset \mathbb{R}^d$ of dimension d whose δ-vector coincides with $(\delta_0, \delta_1, \ldots, \delta_d)$ if and only if $i_1, \ldots, i_4$ satisfy $i_1 + i_4 = i_2 + i_3 \leq d + 1$ and $i_k + i_\ell \geq i_{k+\ell}$ for $1 \leq k \leq \ell \leq 4$ with $k + \ell \leq 4$, where $i_1, \ldots, i_4$ are the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_4}$ with $1 \leq i_1 \leq \cdots \leq i_4 \leq d$.*

Theorem 0.3. *Given a finite sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ of nonnegative integers, where $\delta_0 = 1$ and $\sum_{i=0}^{d} \delta_i = 7$, there exists an integral simplex $\mathcal{P} \subset \mathbb{R}^d$ of dimension d whose δ-vector coincides with $(\delta_0, \delta_1, \ldots, \delta_d)$ if and only if $i_1, \ldots, i_6$ satisfy $i_1 + i_6 = i_2 + i_5 = i_3 + i_4 \leq d+1$ and $i_k + i_\ell \geq i_{k+\ell}$ for $1 \leq k \leq \ell \leq 6$ with $k + \ell \leq 6$, where $i_1, \ldots, i_6$ are the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_6}$ with $1 \leq i_1 \leq \cdots \leq i_6 \leq d$.*

By virtue of Theorem 0.1, the "Only if" parts of Theorem 0.2 and 0.3 are obvious. A proof of the "If" part of Theomre 0.2 is given in Section 2 and that of Theorem 0.3 is given in Section 3.

Finally, we note that we cannot characterize the possible δ-vectors of integral simplices with higher prime normalized volumes only by Theorem 0.1. In fact, since the volume of an integral convex polytope containing a unique integer point in its interior has an upper bound, if p is a sufficiently large prime number, then the integer sequence $(1, 1, p-3, 1)$ cannot be a δ-vector of some integral simplex of dimension 3, although $(1, 1, p-3, 1)$ satisfies all the conditions of Theorem 0.1.

1. A proof of Theorem 0.1

The goal of this section is to give a proof of Theorem 0.1.

First of all, we recall from [2, Part II] the well-known combinatorial technique how to compute the δ-vector of an integral simplex.

Given an integral simplex $\mathcal{F}$ in $\mathbb{R}^N$ of dimension d with the vertices $v_0, v_1, \ldots, v_d$, we set

$$\widetilde{\mathcal{F}} = \left\{ (\alpha, 1) \in \mathbb{R}^{N+1} \, : \, \alpha \in \mathcal{F} \right\},$$

which is an integral simplex in $\mathbb{R}^{N+1}$ of dimension d with the vertices $(v_0, 1), (v_1, 1), \ldots, (v_d, 1)$. Clearly, we have $i(\mathcal{F}, n) = i(\widetilde{\mathcal{F}}, n)$ for all n. Let

$$\mathcal{C}(\widetilde{\mathcal{F}}) = \{ r\beta : \beta \in \widetilde{\mathcal{F}}, 0 \leq r \in \mathbb{Q} \}.$$

Then one has

$$i(\mathcal{F}, n) = \left| \left\{ (\alpha, n) \in \mathcal{C}(\widetilde{\mathcal{F}}) : \alpha \in \mathbb{Z}^N \right\} \right|.$$

Each rational point $\alpha \in \mathcal{C}(\widetilde{\mathcal{F}})$ has a unique expression of the form $\alpha = \sum_{i=0}^{d} r_i (v_i, 1)$ with $0 \leq r_i \in \mathbb{Q}$. Let S be the set of all points $\alpha \in \mathcal{C}(\widetilde{\mathcal{F}}) \cap \mathbb{Z}^{N+1}$ of the form $\alpha = \sum_{i=0}^{d} r_i (v_i, 1)$, where $r_i \in \mathbb{Q}$ with $0 \leq r_i < 1$. We define the degree of $\alpha = \sum_{i=0}^{d} r_i (v_i, 1) \in \mathcal{C}(\widetilde{\mathcal{F}}) \cap \mathbb{Z}^{N+1}$ with $\deg(\alpha) = \sum_{i=0}^{d} r_i$, i.e., the last coordinate of α.

Lemma 1.1. *Let δ_i be the number of integer points $\alpha \in S$ with $\deg(\alpha) = i$. Then,*

$$\delta(\mathcal{F}) = (\delta_0, \delta_1, \ldots, \delta_d).$$

Notice that the elements of S form an abelian group with a unit $(0,\dots,0)\in S$. For α and β in S with $\alpha=\sum_{i=0}^d r_i(v_i,1)$ and $\beta=\sum_{i=0}^d s_i(v_i,1)$, where $r_i,s_i\in\mathbb{Q}$ with $0\le r_i,s_i<1$, we define the operation in S by setting $\alpha\oplus\beta:=\sum_{i=0}^d\{r_i+s_i\}(v_i,1)$, where $\{r\}=r-\lfloor r\rfloor$ denotes the fractional part of a rational number r. (Throughout this paper, in order to distinguish the operation in S from the usual addition, we use the notation $\oplus$, which is not a direct sum.)

We prove Theorem 0.1 by using the above notations.

A proof of Theorem 0.1. Let $v_0,v_1,\dots,v_d$ be the vertices of the integral simplex $\mathcal{P}$ and S the group appearing above. Then, since $\mathrm{vol}(\mathcal{P})=p$ is prime, it follows from Lemma 1.1 that the order of S is also prime. In particular, S is a cyclic group.

(a) Write $g_{i_1},\dots,g_{i_{p-1}}\in S\setminus\{(0,\dots,0)\}$ for $(p-1)$ distinct elements with $\deg(g_{i_j})=i_j$ for $1\le j\le p-1$, that is, $S=\{(0,\dots,0),g_{i_1},\dots,g_{i_{p-1}}\}$. Then, for each g_{i_j}, there exists its inverse $-g_{i_j}$ in $S\setminus\{(0,\dots,0)\}$. Let $-g_{i_j}=g_{i'_j}$. If g_{i_j} has the expression $g_{i_j}=\sum_{q=0}^d r_q(v_q,1)$, where $r_q\in\mathbb{Q}$ with $0\le r_q<1$, then its inverse has the expression $g_{i'_j}=\sum_{q=0}^d\{1-r_q\}(v_q,1)$. Thus, one has

$$\deg(g_{i_j})+\deg(g_{i'_j})=\sum_{q=0}^d(r_q+\{1-r_q\})\le\sum_{q=0}^d(r_q+1-r_q)=d+1$$

for all $1\le j\le p-1$.

For $j_1,j_2\in\{1,\dots,p-1\}$ with $j_1\ne j_2$, let $g_{i_{j_1}}=\sum_{q=0}^d r_q^{(1)}(v_q,1)$ and $g_{i_{j_2}}=\sum_{q=0}^d r_q^{(2)}(v_q,1)$. Since S is a cyclic group with a prime order, $g_{i_{j_1}}$ generates S, which implies that we can write $g_{i_{j_2}}$ and $g_{i'_{j_2}}$ as follows:

$$g_{i_{j_2}}=\underbrace{g_{i_{j_1}}\oplus\cdots\oplus g_{i_{j_1}}}_{t},\quad g_{i'_{j_2}}=\underbrace{g_{i'_{j_1}}\oplus\cdots\oplus g_{i'_{j_1}}}_{t}$$

for some integer $t\in\{2,\dots,p-1\}$. Thus, we have

$$\sum_{q=0}^d(r_q^{(2)}+\{1-r_q^{(2)}\})=\deg(g_{i_{j_2}})+\deg(g_{i'_{j_2}})$$
$$=\deg(\underbrace{g_{i_{j_1}}\oplus\cdots\oplus g_{i_{j_1}}}_{t})+\deg(\underbrace{g_{i'_{j_1}}\oplus\cdots\oplus g_{i'_{j_1}}}_{t})=\sum_{q=0}^d(\{tr_q^{(1)}\}+\{t(1-r_q^{(1)})\}).$$

Moreover, $\underbrace{g_{i_{j_1}}\oplus\cdots\oplus g_{i_{j_1}}}_{p}=(0,\dots,0)$ holds. Thus, we have $\{pr_q^{(1)}\}=0$ for all $0\le q\le d$. Again, since p is prime, it follows that the denominator of each rational number $r_q^{(1)}$ must be p. Hence, if $0<r_q^{(1)}<1$ (resp. $0<\{1-r_q^{(1)}\}<1$), then $0<\{tr_q^{(1)}\}<1$ (resp. $0<\{t(1-r_q^{(1)})\}<1$), so $r_q^{(1)}+\{1-r_q^{(1)}\}=\{tr_q^{(1)}\}+\{t(1-r_q^{(1)})\}=1$. In addition, obviously, if $r_q^{(1)}=\{1-r_q^{(1)}\}=0$, then $\{tr_q^{(1)}\}=\{t(1-r_q^{(1)})\}=0$, so $r_q^{(1)}+\{1-r_q^{(1)}\}=\{tr_q^{(1)}\}+\{t(1-r_q^{(1)})\}=0$. Thus, $\deg(g_{i_{j_1}})+\deg(g_{i'_{j_1}})=\deg(g_{i_{j_2}})+\deg(g_{i'_{j_2}})$, i.e.,

$i_{j_1} + i'_{j_1} = i_{j_2} + i'_{j_2}$. Hence, we obtain

$$i_1 + i'_1 = \cdots = i_{(p-1)/2} + i'_{(p-1)/2} (= i_{(p+1)/2} + i'_{(p+1)/2} = \cdots = i_{p-1} + i'_{p-1}) \leq d+1.$$

Our work is to show that $i'_j = i_{p-j}$ for all $1 \leq j \leq (p-1)/2$.

First, we consider i'_1. Suppose that $i'_1 \neq i_{p-1}$. Then, there is $m \in \{1, \ldots, p-2\}$ with $i'_1 = i_m < i_{p-1}$. Thus, it follows that

$$i_{p-1} + i'_{p-1} = i_1 + i'_1 = i_1 + i_m < i_1 + i_{p-1} \leq i'_{p-1} + i_{p-1},$$

a contradiction. Thus, i'_1 must be i_{p-1}. Next, we consider i'_2. Since $g_{i'_2} \neq g_{i_1}$ and $g_{i'_2} \neq g_{i_{p-1}}$, we may consider i'_2 among $\{i_2, \ldots, i_{p-2}\}$. Then, the same discussion can be done. Hence, $i'_2 = i_{p-2}$. Similarly, we have $i'_3 = i_{p-3}, \ldots, i'_{(p-1)/2} = i_{(p+1)/2}$.

Therefore, we obtain the desired conditions

$$i_1 + i_{p-1} = i_2 + i_{p-2} = \cdots = i_{(p-1)/2} + i_{(p+1)/2} \leq d+1.$$

(b) Write $g_{i_1}, \ldots, g_{i_\ell} \in S \setminus \{(0, \ldots, 0)\}$ for ℓ distinct elements with $\deg(g_{i_j}) = i_j$ for $1 \leq j \leq \ell$. Let $A = \{g_{i_1}, \ldots, g_{i_\ell}\}$. Then there are k distinct elements $h_{i_1}, \ldots, h_{i_k}$ in A with $\deg(h_{i_j}) = i_j$ for $1 \leq j \leq k$ satisfying $|A| + |B| = k + \ell \leq p-1$, where $B = \{h_{i_1}, \ldots, h_{i_k}\} \subset A$. Moreover, for each $g \in A \oplus B = \{a \oplus b : a \in A, b \in B\}$, g satisfies $\deg(g) \leq i_k + i_\ell$. In fact, for $g_{i_j} \in A$ and $h_{i_{j'}} \in B$, if they have the expressions

$$g_{i_j} = \sum_{q=0}^{d} r_q(v_q, 1) \quad \text{and} \quad h_{i_{j'}} = \sum_{q=0}^{d} r'_q(v_q, 1),$$

where $r_q, r'_q \in \mathbb{Q}$ with $0 \leq r_q, r'_q < 1$, then one has

$$\deg(g_{i_j} \oplus h_{i_{j'}}) = \sum_{q=0}^{d} \{r_q + r'_q\} \leq \sum_{q=0}^{d} (r_q + r'_q) = i_j + i_{j'} \leq i_k + i_\ell.$$

Now, Lemma 1.2 below guarantees that there exist at least k elements in $A \oplus B \setminus A \cup \{(0, \ldots, 0)\}$. In addition, each g_{i_j} in A satisfies $\deg(g_{i_j}) \leq i_\ell \leq i_k + i_\ell$. Thus, we can say that there exist at least $(k+\ell)$ distinct elements in $S \setminus \{(0, \ldots, 0)\}$ whose degrees are at most $i_k + i_\ell$. From the definition of $i_1, \ldots, i_{p-1}$, this means that $i_k + i_\ell \geq i_{k+\ell}$, as desired. □

Lemma 1.2. *Let G be a group with prime order p, where its operation is denoted by $+$, and let $G^* = G \setminus \{0\}$, where 0 is the unit of G. We choose two subsets (not subgroups) A and B of G satisfying $B \subset A \subset G^*$ and $|A| + |B| \leq p-1$ and we set $C = G^* \setminus A$. Then one has*

$$|(A+B) \cap C| \geq |B|, \tag{4}$$

where $A + B = \{a + b : a \in A, b \in B\}$.

Proof. Let $A = \{a_1, \ldots, a_\ell\}$ and $B = \{b_1, \ldots, b_k\}$. We show the assertion by induction on k.

First, we consider $k = 1$, i.e., $B = \{b_1\}$. Then, $\ell + 1 \leq p - 1$. For $1 \leq i \leq \ell$, let $a_i + b_1 = a'_i \in G$. Then we have

$$(0 + b_1) + \sum_{i=1}^{\ell}(a_i + b_1) = \left(0 + \sum_{i=1}^{\ell} a_i\right) + \underbrace{b_1 + \cdots + b_1}_{\ell+1} = b_1 + \sum_{i=1}^{\ell} a'_i.$$

If we suppose that $A + \{b_1\} \subset A \cup \{0\}$, then we have $\{b_1, a'_1, a'_2, \ldots, a'_\ell\} \subset A \cup \{0\}$. Since $b_1, a'_1, a'_2, \ldots, a'_\ell$ are distinct, one has $\{b_1, a'_1, a'_2, \ldots, a'_\ell\} = A \cup \{0\}$. Thus, $\underbrace{b_1 + \cdots + b_1}_{\ell+1} = 0$ from the above equality. However, since $|G|$ is prime and $l+1 < p$, $\underbrace{b_1 + \cdots + b_1}_{\ell+1} \in G$ cannot be 0, a contradiction. Hence, $A + \{b_1\} \not\subset A \cup \{0\}$, which implies that $|(A + \{b_1\}) \cap C| \geq 1$.

Next, we consider $k \geq 2$. Let $B' = \{b_1, \ldots, b_{k-1}\}$. Then, by the hypothesis of induction, one has $|(A + B') \cap C| \geq k - 1$. When $|(A + B') \cap C| > k - 1$, the assertion holds. Thus, we assume that $|(A + B') \cap C| = k - 1$. Let $(A + B') \cap C = \{c_1, \ldots, c_{k-1}\}$, where $c_1, \ldots, c_{k-1}$ are $(k - 1)$ distinct elements, $A' = A \cup \{c_1, \ldots, c_{k-1}\}$ and $C' = G^* \setminus A'$. Then, again by the hypothesis of induction, one has $|(A' + \{b_k\}) \cap C'| \geq 1$. This implies that there exists at least one element c_k in C' such that $a + b_k = c_k$ for some $a \in A'$. When $a \in A$, then $c_k \in (A + B) \cap C'$, which says that the assertion holds. Hence, we assume that $a \in \{c_1, \ldots, c_{k-1}\}$, say, $a = c_1$.

Now, again by the hypothesis of induction, it is easy to see that we have the following equalities by renumbering $c_1, \ldots, c_{k-1} \in (A + B') \cap C$ if necessary:

$$\begin{cases} c_1 = a_{i_1} + b_1, \\ c_2 = a_{i_2} + b_2, \\ \vdots \\ c_{k-1} = a_{i_{k-1}} + b_{k-1}, \end{cases} \tag{5}$$

where $a_{i_1}, \ldots, a_{i_{k-1}} \in A$. Suppose that the inequality

$$|(A + B) \cap C| \geq k \tag{6}$$

is not satisfied. From (5), one has

$$c_k = a + b_k = c_1 + b_k = a_{i_1} + b_1 + b_k.$$

Set $c'_1 = a_{i_1} + b_k$. When $c'_1 \in A$, since $c'_1 + b_1 \in A + B$ and $c'_1 + b_1 = c_k \in C'$, one has $c_k \in (A + B) \cap C'$, which means that (6) holds. When $c'_1 \in C'$, since $c'_1 = a_{i_1} + b_k \in A + B$, one has $c'_1 \in (A + B) \cap C'$, which also means that (6) holds. Moreover, c'_1 cannot be 0 since $c_k \neq b_1$. In addition, c'_1 cannot be c_1 since $b_1 \neq b_k$. Hence, it must be $c'_1 \in \{c_2, \ldots, c_{k-1}\}$, say, $c'_1 = c_2$. Then, again from (5),

$$c_k = c_1 + b_k = c_2 + b_1 = a_{i_2} + b_2 + b_1.$$

Set $c'_2 = a_{i_2} + b_1$. Similarly, when $c'_2 \in A$ or $c'_2 \in C'$, (6) holds. Moreover, c'_2 cannot be 0, c_1 and c_2. Hence, it must be $c'_2 \in \{c_3, \ldots, c_{k-1}\}$, say, $c'_2 = c_3$. By repeating these discussions, we obtain

$$c_k = c_1 + b_k = c_2 + b_1 = \cdots = c_{k-1} + b_{k-2} = a_{i_{k-1}} + b_{k-1} + b_{k-2}.$$

Set $c'_{k-1} = a_{i_{k-1}} + b_{k-2}$. However, we have

$$c'_{k-1} \notin A \cup C' \cup \{0, c_1, c_2, \dots, c_{k-1}\} = G,$$

a contradiction. Thus, the inequality (6) must be satisfied.

Therefore, we obtain the required inequality (4). □

Remark 1.3. (a) When $i_1 + i_{p-1} = \cdots = i_{(p-1)/2} + i_{(p+1)/2} = d+1$, the δ-vector is shifted symmetric. Shifted symmetric δ-vectors are studied in [6]. Moreover, the theorem [6, Theorem 2.3] says that if $i_1 + i_{p-1} = d+1$, then we have $i_1 + i_{p-1} = \cdots = i_{(p-1)/2} + i_{(p+1)/2} = d+1$.

(b) The inequalities $i_1 + i_\ell \geq i_{\ell+1}$ are not new. In fact, for example, when $i_1 < \cdots < i_{p-1}$, by (2), one has

$$\delta_0 + \cdots + \delta_{i_1} \leq \delta_{i_{p-1}} + \cdots + \delta_{i_{p-1}-i_1}.$$

Thus, we obtain $i_{p-1} - i_1 \leq i_{p-2}$, i.e., $i_1 + i_{p-2} \geq i_{p-1}$. Similarly, one has

$$\delta_0 + \cdots + \delta_{i_2} \leq \delta_{i_{p-1}} + \cdots + \delta_{i_{p-1}-i_2}.$$

Thus, we obtain $i_{p-1} - i_2 \leq i_{p-3}$. Since $i_1 + i_{p-1} = i_2 + i_{p-2}$, this is equivalent to $i_1 + i_{p-3} \geq i_{p-2}$. In the same way, we can obtain all inequalities $i_1 + i_\ell \geq i_{\ell+1}$. On the other hand, when $k \geq 2$, there are many new inequalities.

2. The possible δ-vectors of integral simplices with $\sum_{i=0}^{d} \delta_i = 5$

In this section, we give a proof of the "If" part of Theorem 0.2, i.e., we classify all the possible δ-vectors of integral simplices whose normalized volume is 5.

Let $(\delta_0, \delta_1, \dots, \delta_d)$ be a nonnegative integer sequence with $\delta_0 = 1$ and $\sum_{i=0}^{d} \delta_i = 5$ which satisfies $i_1 + i_4 = i_2 + i_3 \leq d+1$, $2i_1 \geq i_2$ and $i_1 + i_2 \geq i_3$, where $i_1, \dots, i_4$ are the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_4}$ with $1 \leq i_1 \leq \cdots \leq i_4 \leq d$. Since $i_1 + i_4 = i_2 + i_3$, we notice that $i_1 + i_3 \geq i_4$ (resp. $2i_2 \geq i_4$) is equivalent to $2i_1 \geq i_2$ (resp. $i_1 + i_2 \geq i_3$). From the conditions $\delta_0 = 1$, $\sum_{i=0}^{d} \delta_i = 5$ and $i_1 + i_4 = i_2 + i_3$, the possible sequences are only the following forms:

(i) $(1, 0, \dots, 0, 4, 0, \dots, 0)$;
(ii) $(1, 0, \dots, 0, 2, 0, \dots, 0, 2, 0, \dots, 0)$;
(iii) $(1, 0, \dots, 0, 1, 0, \dots, 0, 2, 0, \dots, 0, 1, 0, \dots, 0)$;
(iv) $(1, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$.

Our work is to find integral simplices whose δ-vectors are of the above forms.

To construct integral simplices, we define the following integer matrix, which is called the *Hermite normal form*:

$$A_5(d_1, \dots, d_4) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ * & \cdots & * & 5 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \tag{7}$$

where there are d_j j's among the $*$'s for $j = 1, \ldots, 4$ and the rest of the entries are all 0. Then, clearly, it must be $d_j \geq 0$ and $d_1 + \cdots + d_4 \leq d - 1$. By determining $d_1, \ldots, d_4$, we obtain an integer matrix $A_5(d_1, \ldots, d_4)$ and we define the integral simplex $\mathcal{P}_5(d_1, \ldots, d_4)$ from the matrix as follows:

$$\mathcal{P}_5(d_1, \ldots, d_4) = \mathrm{conv}(\{(0, \ldots, 0), v_1, \ldots, v_d\}) \subset \mathbb{R}^d,$$

where v_i is the ith row vector of $A_5(d_1, \ldots, d_4)$. The following lemma enables us to compute $\delta(\mathcal{P}_5(d_1, \ldots, d_4))$ easily.

Lemma 2.1 ([4, Corollary 3.1]). *If $\delta(\mathcal{P}_5(d_1, \ldots, d_4)) = (\delta_0, \delta_1, \ldots, \delta_d)$, then we have*

$$\sum_{i=0}^{d} \delta_i t^i = 1 + \sum_{i=1}^{4} t^{1-s_i},$$

where

$$s_i = \left\lfloor \frac{i}{5} - \sum_{j=1}^{4} \left\{ \frac{ij}{5} \right\} d_j \right\rfloor, \qquad \textit{for } i = 1, \ldots, 4.$$

2.1. **The case (i).** Let $i_1 = i_2 = i_3 = i_4 = i$. Thus, one has $i - 1 \geq 0$ and $2i - 2 \leq d - 1$ from our conditions. Hence, we can define $\mathcal{P}_5(0, i-1, i-1, 0)$. Then, by Lemma 2.1, $\delta(\mathcal{P}_5(0, i-1, i-1, 0))$ coincides with (i) since $s_1 = s_2 = s_3 = s_4 = -i + 1$.

2.2. **The case (ii).** Let $i_1 = i_2 = i$ and $i_3 = i_4 = j$. Thus, one has $2i \geq j$, $2j - 2i - 2 \geq 0$ and $i + j - 2 \leq d - 1$. Hence, we can define $\mathcal{P}_5(0, i, 2i - j, 2j - 2i - 2)$ and its δ-vector coincides with (ii) since $s_1 = s_2 = -j + 1$ and $s_3 = s_4 = -i + 1$.

2.3. **The case (iii).** Let $i_1 = i, i_2 = i_3 = j$ and $i_4 = k$. Thus, one has $2i \geq j$, $3j - 3i - 2 \geq 0$ and $2j - 2 \leq d - 1$. Hence, we can define $\mathcal{P}_5(0, 2i-j, i, 3j-3i-2)$ and its δ-vector coincides with (iii) since $s_1 = -2j + i + 1 = -k + 1$, $s_2 = s_3 = -j + 1$ and $s_4 = -i + 1$.

2.4. **The case (iv).** In this case, one has $2i_1 \geq i_2, i_1 + i_2 \geq i_3, i_2 + 2i_3 - 3i_1 - 2 \geq 0$ and $i_2 + i_3 - 2 \leq d - 1$. Hence, we can define $\mathcal{P}_5(0, 2i_1 - i_2, i_1 + i_2 - i_3, i_2 + 2i_3 - 3i_1 - 2)$ and its δ-vector coincides with (iv) since $s_1 = i_1 - i_2 - i_3 + 1 = -i_4 + 1, s_2 = -i_3 + 1, s_3 = -i_2 + 1$ and $s_4 = -i_1 + 1$.

Remark 2.2. (a) The classification of the case (iv) is essentially given in [6, Lemma 4.3]. (b) The inequalities $2i_1 \geq i_2$ and $i_1 + i_2 \geq i_3$ can be obtained from (2) as we mentioned in Remark 1.3 (b). Thus, the possible δ-vectors of integral simplices with normalized volume 5 can be essentially characterized only by Theorem 0.1 (a) and the inequalities (2).

3. The possible δ-vectors of integral simplices with $\sum_{i=0}^{d} \delta_i = 7$

In this section, similarly to the previous one, we give a proof of the "If" part of Theorem 0.3, i.e., we classify all the possible δ-vectors of integral simplices whose normalized volume is 7.

Let $(\delta_0, \delta_1, \ldots, \delta_d)$ be a nonnegative integer sequence with $\delta_0 = 1$ and $\sum_{i=0}^{d} \delta_i = 7$ which satisfies $i_1 + i_6 = i_2 + i_5 = i_3 + i_4 \leq d + 1$, $i_1 + i_l \geq i_{l+1}$ for $1 \leq l \leq 3$ and $2i_2 \geq i_4$, where $i_1, \ldots, i_6$ are the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_6}$

with $1 \le i_1 \le \cdots \le i_6 \le d$. Since $i_1 + i_6 = i_2 + i_5 = i_3 + i_4$, we need not consider the inequalities $i_1 + i_4 \ge i_5$, $i_1 + i_5 \ge i_6$, $i_2 + i_3 \ge i_5$, $i_2 + i_4 \ge i_6$ and $2i_3 \ge i_6$. From the conditions $\delta_0 = 1$, $\sum_{i=0}^{d} \delta_i = 7$ and $i_1 + i_6 = i_2 + i_5 = i_3 + i_4$, the possible sequences are only the following forms:

(i) $(1, 0, \dots, 0, 6, 0, \dots, 0)$;
(ii) $(1, 0, \dots, 0, 3, 0, \dots, 0, 3, 0, \dots, 0)$;
(iii) $(1, 0, \dots, 0, 1, 0, \dots, 0, 4, 0, \dots, 0, 1, 0, \dots, 0)$;
(iv) $(1, 0, \dots, 0, 2, 0, \dots, 0, 2, 0, \dots, 0, 2, 0, \dots, 0)$;
(v) $(1, 0, \dots, 0, 1, 0, \dots, 0, 2, 0, \dots, 0, 2, 0, \dots, 0, 1, 0, \dots, 0)$;
(vi) $(1, 0, \dots, 0, 2, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0, 2, 0, \dots, 0)$;
(vii) $(1, 0, \dots, 0, 1, 0, \dots, 0, 1, \dots, 0, 2, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$;
(viii) $(1, 0, \dots, 0, 1, 0, \dots, 0, 1, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$.

In the same way as the previous section, we define the following integer matrix:

$$A_7(d_1, \dots, d_6) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ * & \cdots & * & 7 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \tag{8}$$

where there are d_j j's among the $*$'s for $j = 1, \dots, 6$ and the rest of the entries are all 0. Then it must be $d_j \ge 0$ and $d_1 + \cdots + d_6 \le d - 1$. By determining $d_1, \dots, d_6$, we obtain the integral simplex

$$\mathcal{P}_7(d_1, \dots, d_6) = \mathrm{conv}(\{(0, \dots, 0), v_1, \dots, v_d\}) \subset \mathbb{R}^d,$$

where v_i is the ith row vector of $A_7(d_1, \dots, d_6)$. Similarly, the following lemma enables us to compute $\delta(\mathcal{P}_7(d_1, \dots, d_6))$ easily.

Lemma 3.1 ([4, Corollary 3.1]). *If $\delta(\mathcal{P}_7(d_1, \dots, d_6)) = (\delta_0, \delta_1, \dots, \delta_d)$, then we have*

$$\sum_{i=0}^{d} \delta_i t^i = 1 + \sum_{i=1}^{6} t^{1-s_i},$$

where

$$s_i = \left\lfloor \frac{i}{7} - \sum_{j=1}^{6} \left\{ \frac{ij}{7} \right\} d_j \right\rfloor, \quad \textit{for } i = 1, \dots, 6.$$

3.1. **The case (i).** Let $i_1 = \cdots = i_6 = i$. Thus, one has $i - 1 \ge 0$ and $2i - 2 \le d - 1$ from our conditions. Hence, we can define $\mathcal{P}_7(0, 0, i-1, i-1, 0, 0)$. Then, by Lemma 3.1, $\delta(\mathcal{P}_7(0, 0, i-1, i-1, 0, 0))$ coincides with (i) since $s_1 = \cdots = s_6 = -i + 1$.

3.2. **The case (ii).** Let $i_1 = \cdots = i_3 = i$ and $i_4 = \cdots = i_6 = j$. Thus, one has $j - i \ge 0, 2i \ge j, 2j - 2i - 2 \ge 0$ and $i + j - 2 \le d - 1$. Hence, we can define $\mathcal{P}_7(0, j - i, 2i - j, 2i - j, 0, 2j - 2i - 2)$ and its δ-vector coincides with (ii) since $s_1 = s_2 = s_3 = -j + 1$ and $s_4 = s_5 = s_6 = -i + 1$.

3.3. **The case (iii).** Let $i_1 = i, i_2 = \cdots = i_5 = j$ and $i_6 = k$. Thus, one has $i + j \geq k, k - j \geq 0, k - i - 1 \geq 0, i - 1 \geq 0$ and $i + k - 2 \leq d - 1$. Hence, we can define $\mathcal{P}_7(i + j - k, k - j, k - i - 1, 0, 0, i - 1)$ and its δ-vector coincides with (iii) since $s_1 = \frac{-4i+j-4k}{7} + 1 = -j + 1, s_2 = \frac{-i+2j-8k}{7} + 1 = -k + 1, s_3 = \frac{-5i+3j-5k}{7} + 1 = -j + 1, s_4 = \frac{-2i-3j-2k}{7} + 1 = -j + 1, s_5 = \frac{-6i-2j+k}{7} + 1 = -i + 1$ and $s_6 = \frac{-3i-j-3k}{7} + 1 = -j + 1$.

3.4. **The case (iv).** Let $i_1 = i_2 = i, i_3 = i_4 = j$ and $i_5 = i_6 = k$. Thus, one has $i - 1 \geq 0, i + j \geq k, 3k - 3j - 1 \geq 0$ and $2i - 2j + 2k - 2 = i + k - 2 \leq d - 1$. Hence, we can define $\mathcal{P}_7(0, 0, i - 1, i + j - k, 0, 3k - 3j - 1)$ and its δ-vector coincides with (iv) since $s_1 = s_2 = -i + 2j - 2k + 1 = -k + 1, s_3 = s_4 = -i + j - k + 1 = -j + 1$ and $s_5 = s_6 = -i + 1$.

3.5. **The case (v).** Let $i_1 = k_1, i_2 = i_3 = k_2, i_4 = i_5 = k_3$ and $i_6 = k_4$. Thus, one has $2k_1 \geq k_2, k_2 - k_1 \geq 0, k_1 + k_2 \geq k_3, 2k_3 - 2k_1 - 2 \geq 0$ and $k_2 + k_3 - 2 \leq d - 1$. Hence, we can define $\mathcal{P}_7(0, 2k_1 - k_2, 0, k_2 - k_1, k_1 + k_2 - k_3, 2k_3 - 2k_1 - 2)$ and its δ-vector coincides with (v) since $s_1 = k_1 - k_2 - k_3 + 1 = -k_4 + 1, s_2 = s_3 = -k_3 + 1, s_4 = s_5 = -k_2 + 1$ and $s_6 = -k_1 + 1$.

3.6. **The case (vi).** Let $i_1 = i_2 = k_1, i_3 = k_2, i_4 = k_3$ and $i_5 = i_6 = k_4$. Thus, one has $k_3 - k_2 - 1 \geq 0, k_1 + k_2 \geq k_3, 2k_1 \geq k_3, k_2 + 2k_3 - 3k_1 - 1 \geq 0$ and $k_2 + k_3 - 2 \leq d - 1$. Hence, we can define $\mathcal{P}_7(0, k_3 - k_2 - 1, k_1 + k_2 - k_3, 2k_1 - k_3, 0, k_2 + 2k_3 - 3k_1 - 1)$ and its δ-vector coincides with (vi) since $s_1 = s_2 = k_1 - k_2 - k_3 + 1 = -k_4 + 1, s_3 = -k_3 + 1, s_4 = -k_2 + 1$ and $s_5 = s_6 = -k_1 + 1$.

3.7. **The case (vii).** Let $i_1 = k_1, i_2 = k_2, i_3 = i_4 = k_3, i_5 = k_4$ and $i_6 = k_5$. Thus, one has $2k_1 \geq k_2, k_1 + k_2 \geq k_3, k_2 - k_1 \geq 0, 3k_3 - 2k_1 - k_2 - 2 \geq 0$ and $2k_3 - 2 \leq d - 1$. Hence, we can define $\mathcal{P}_7(0, 0, 2k_1 - k_2, k_1 + k_2 - k_3, k_2 - k_1, 3k_3 - 2k_1 - k_2 - 2)$ and its δ-vector coincides with (vii) since $s_1 = k_1 - 2k_3 + 1 = -k_5 + 1, s_2 = k_2 - 2k_3 + 1 = -k_4 + 1, s_3 = s_4 = -k_3 + 1, s_5 = -k_2 + 1$ and $s_1 = -k_1 + 1$.

3.8. **The case (viii).** In this case, one has $i_1 + i_2 \geq i_3, 2i_2 \geq i_4, i_3 + 2i_4 - 2i_1 - i_2 - 2 \geq 0, 2i_1 \geq i_2, i_1 + i_3 \geq i_4$ and $i_3 + i_4 - 2 \leq d - 1$. Hence, we can define $\mathcal{P}_7(0, i_1 + i_2 - i_3, i_1 + i_3 - 2i_2, 0, 2i_2 - i_4, i_3 + 2i_4 - 2i_1 - i_2 - 2)$ if $i_1 + i_3 \geq 2i_2$ and $\mathcal{P}_7(0, 2i_1 - i_2, 0, 2i_2 - i_1 - i_3, i_1 + i_3 - i_4, i_3 + 2i_4 - 2i_1 - i_2 - 2)$ $i_1 + i_3 \leq 2i_2$. Moreover, each of δ-vectors of them coincides with (viii) since $s_1 = i_1 - i_3 - i_4 + 1 = -i_6 + 1$, $s_2 = i_2 - i_3 - i_4 + 1 = -i_5 + 1$, $s_3 = -i_4 + 1$, $s_4 = -i_3 + 1$, $s_5 = -i_2 + 1$ and $s_6 = -i_1 + 1$.

Remark 3.2. When we discuss the cases of (vi) and (viii), we need the new inequality $2i_2 \geq i_4$. In fact, for example, the sequence $(1, 0, 2, 0, 1, 1, 0, 2, 0)$ cannot be the δ-vector of an integral simplex, although this satisfies $i_1 + i_l \geq i_{l+1}, l = 1, \ldots, 3$. Similarly, the sequence $(1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0)$ also cannot be the δ-vector of an integral simplex, although this satisfies $i_1 + i_l \geq i_{l+1}, l = 1, \ldots, 3$.

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