BRUHAT GRAPHS AND PATTERN AVOIDANCE

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ABSTRACT. We characterize permutations whose Bruhat graphs can be drawn in the plane. In particular, we show this property, as well as that of having having length at most a fixed ℓ, is characterized by avoiding finitely many permutations.

1. INTRODUCTION

The set of all permutations (of an arbitrary finite number of elements) admits a partial order known as pattern containment order. This partial order is known to admit infinite antichains [17, 14]. On the other hand, in almost all cases where the set of permutations satisfying some property has been characterized by pattern containment, the number of permutations involved is finite. For some properties, such as in [19, 6, 4], the number of permutations is moderate to quite large, so this phenomenon seems to involve more than merely the natural inclination of mathematicians to study simpler examples. Our goal in this paper is to begin the exploration of one possible explanation for this finiteness.

Associated to each permutation is a directed graph known as the Bruhat graph. (For definitions see Section 2.) Whenever a permutation π is contained in a permutation σ (so $\pi \leq \sigma$ in pattern containment order), the Bruhat graph of π is a subgraph of the Bruhat graph of σ. While not all properties characterized by pattern containment can be reduced to properties of Bruhat graphs, many of the properties that have been so characterized, especially those coming from algebraic geometry or representation theory, have a graph theoretic description. For example, the permutations w avoiding 3412 and 4231 are exactly the ones whose Bruhat graphs are regular, meaning that they have the same number of edges at each vertex; these are also the ones associated to smooth Schubert varieties [13, 10].

In this paper we show that the property of having Coxeter length at most ℓ is characterized by avoidance of finitely many patterns for any ℓ. Having length at most ℓ is a property of the Bruhat graph, since the Coxeter length is the length of the longest directed path in the graph. We were first drawn to this question by the our characterization of permutations whose Bruhat graphs are planar. We show these are the permutations that avoid 321 and have length at most 3, and their Bruhat graphs are either a single

point, a single edge, (the edge graph of) a square, or a cube. Using specifics of our theorem on length, we show by computer calculation that having a planar Bruhat graph is characterized by avoiding 29 permutations.

The motivation for considering Bruhat graphs is the Graph Minor Theorem, a deep and surprising result of Robertson and Seymour [16] stating that graph minor order, the partial order on graphs generated by deleting and contracting edges, has no infinite antichains. In particular, the graphs satisfying any property which is preserved under deletion and contraction are characterized by avoiding finitely many graphs. An important special case of the Graph Minor Theorem is the one involving the class of graphs which can be drawn without crossings on a surface of genus g [15], and the original inspiration for graph minor theory was Kuratowski's Theorem, which characterizes graphs that cannot be drawn in the plane. Our results in this paper can therefore be seen as potential first steps towards using graph minor theory to establish either directly or by analogy a possible finiteness result for pattern avoidance under yet unknown hypotheses involving Bruhat graphs.

We were particularly inspired by the question of Billey and Weed [6] concerning whether, for a fixed integer k, having Kazhdan–Lusztig polynomial $P_{id,\sigma}(1) \leq k$ is characterized by the permutation σ avoiding some finite list of patterns. (Billey and Braden previously showed in [3] that this property is characterized by avoiding a possibly infinite set of patterns.) This Kazhdan–Lusztig polynomial is known to be a property depending only on the Bruhat graph of σ [12, 9, 11], although the aforementioned property is not preserved by deletion and contraction on Bruhat graphs (nor is its negation).

Our work is in some sense orthogonal to earlier work of Atkinson, Murphy, and Ruskuc [1, 2] on finitely generated order ideals in pattern containment order. Their work takes the viewpoint that a permutation is a string consisting of distinct integers. Our viewpoint considers permutations as elements of a Coxeter group. Therefore our results are complementary to theirs. Furthermore, pattern avoidance has alternative definitions within the framework of Coxeter groups [5]. We believe our work can be easily extended to the other Coxeter groups.

Section 2 is devoted to definitions, while Sections 3 and 4 respectively give proofs for our theorems about permutations of at most length ℓ and permutations whose Bruhat graphs are planar.

2. Definitions

We begin with definitions from the combinatorics of Coxeter groups applied to the specific case of the symmetric group S_n; a standard reference for this material is [7].

By a **transposition** t we mean some 2-cycle $(i\ j)$ in the symmetric group. An **adjacent transposition** is one of the form $(i\ i+1)$.

Let $\pi \in S_n$ be a permutation. The **length** of π, denoted $\ell(\pi)$, is the minimum number of adjacent transpositions $s_{i_1}, \ldots, s_{i_\ell}$ such that π can be written as their product $s_{i_1} \cdots s_{i_\ell}$. An **inversion** of π is a pair of indices $i < j$ with $\pi(i) > \pi(j)$; note that $\ell(\pi)$ is also equal to the number of inversions of π.

The **absolute length** of π, denoted $a(\pi)$, is the minimum number of transpositions $t_{i_1}, \ldots, t_{i_a}$, not necessarily adjacent transpositions, such that $\pi = t_{i_1} \cdots t_{i_a}$. If $\pi \in S_n$ has c disjoint cycles (counting fixed points as 1-cycles), then $a(\pi) = n - c$. By definition, $a(\pi) \leq \ell(\pi)$ for any permutation π.

Example 2.1. *Permutations are written in one line notation unless stated otherwise. The permutation $\pi = 3412$ has length $\ell(\pi) = 4$, and absolute length $a(\pi) = 2$.*

The symmetric group S_n has a partial ordering known as **Bruhat order**. Define $\pi \prec \sigma$ if $\ell(\pi) < \ell(\sigma)$ and there is a transposition t such that $t\pi = \sigma$ (or equivalently a transposition t' such that $\pi t' = \sigma$). Bruhat order is the transitive closure of $\prec$, so $\pi \leq \tau$ if there exist permutations $\sigma_1, \ldots, \sigma_k \in S_n$ such that $\pi \prec \sigma_1 \prec \cdots \prec \sigma_k \prec \tau$.

The **Bruhat graph** for S_n is the directed graph whose vertices are the elements of S_n, with edges defined as follows. Given permutations σ and τ, there is an edge $\sigma \to \tau$ if $\sigma \prec \tau$, meaning that $\ell(\sigma) < \ell(\tau)$ and there exists a transposition t with $t\sigma = \tau$. (Note $\ell(\tau) - \ell(\sigma)$ need not equal 1.) Given a permutation π, the **Bruhat graph for π**, denoted $\mathcal{B}(\pi)$, is the induced subgraph whose vertices are those labeled by permutations σ with $\sigma \leq \pi$; this is the largest subgraph with unique sink π.

The length of π is the length of the longest directed path (necessarily from the identity to π) in $\mathcal{B}(\pi)$, and the absolute length of π is the length of the shortest directed path in $\mathcal{B}(\pi)$ from the identity (which is graph theoretically the unique source) to π (the unique sink).

Now we give various definitions related to the notion of **pattern containment**; a standard reference for this subject is [8]. Let $\pi \in S_k$ and $\tau \in S_n$ with $k \leq n$. We say that τ **(pattern) contains** π if there exist indices $1 \leq i_1 < \ldots < i_k \leq n$ such that $\tau(i_a) < \tau(i_b)$ if and only if $\pi(a) < \pi(b)$. We say τ **(pattern) avoids** π if τ does not contain π.

Example 2.2. *The permutation 5736241 contains the permutation 3412 (both written in one-line notation) three different ways using the bolded entries:* **573**6**2**41, **57**3**6**2**4**1, *and* **5736**2**4**1.

In contrast, 135246 avoids 3412.

The following proposition is immediate.

Proposition 2.3. *If τ contains π, then $B(\pi)$ is isomorphic to a subgraph of $B(\tau)$ which includes the sink vertex τ.*

Proof. Let $i_1 < \cdots < i_k$ be the indices of by which π is contained in τ. Consider the induced subgraph given by the vertices of $B(\tau)$ labelled by permutations σ for which $\sigma(j) = \tau(j)$ for all j not among the containment indices, meaning that $j \neq i_a$ for all a. $\square$

Pattern containment is a partial order relation; the poset of permutations under pattern containment is sometimes called **pattern order**.

Finally, we need one definition from graph theory. A graph is **planar** if it can be drawn in the plane without edges crossing. The following theorem is classical and known as Kuratowski's Theorem.

Theorem 2.4. *A graph is planar if and only if it contains no subgraph isomorphic to a subdivision of $K_{3,3}$ (the complete bipartite graph with 3 vertices on each side) or K_5 (the complete graph on 5 vertices).*

3. Permutations of less than a fixed length

In this section we prove the following theorem.

Theorem 3.1. *Given a permutation $\sigma \in S_m$, $\ell(\sigma) \geq n$ if and only if σ contains some $\tau \in S_k$ where $k \leq 2n$ and $\ell(\tau) \geq n$.*

One can restate this theorem to say that $\ell(\sigma) < n$ if and only if σ avoids all permutations $\tau \in S_k$ with $k \leq 2n$ for which $\ell(\tau) \geq n$. Of course, the minimal avoidance set will be much smaller. Nevertheless, this shows the avoidance set for the property $\ell(\pi) < n$ is finite.

Example 3.2. *Our bound of $2n$ is sharp in that there is a permutation in S_{2n} of length n which contains no other permutation of length n, namely the permutation $214365\cdots$.*

We begin by proving the special case where σ has a fixed point, meaning some i for which $\sigma(i) = i$.

Lemma 3.3. *Suppose $m > 2n$, and $\sigma \in S_m$ is a permutation with at least one fixed point. Then $\ell(\sigma) \geq n$ if and only if σ strictly contains some τ with $\ell(\tau) \geq n$.*

Proof. Clearly if σ strictly contains some τ with $\ell(\tau) \geq n$ then $\ell(\sigma) \geq n$. Assume $\ell(\sigma) \geq n$. Consider two cases:

Case 1: Assume there exists a fixed point i (so $\sigma(i) = i$) and i is not involved in any 321 pattern in σ. In such a case i is not involved in any inversions, so the permutation obtained by removing the i-th entry (in 1-line notation) has the same length as σ. Thus σ strictly contains a pattern of length at least n, as desired.

Case 2: Assume all of the j fixed points in σ are involved in a 321 pattern contained in σ. Then every fixed point in σ must represent the middle element (the 2) in a 321 pattern for the following reason: if i is an fixed point, and there are 2 elements after i that are less than i, then there must be 2 elements before i greater than i. This means each of the j fixed points is involved in at least 2 inversions, so the total number of inversions involving the j fixed points is at least $2j$. Note no inversion can involve two fixed points.

Furthermore, every entry of σ that is not a fixed point must be involved in an inversion. Since every fixed point of σ is the middle element of a 321 pattern, every entry that is not a fixed point is actually involved in an inversion with another entry that is not a fixed point. There are $m - j > 2n - j$ non-fixed points in σ, so there are more than $n - \frac{j}{2}$ inversions not involving any fixed points.

Let τ be the permutation created by deleting one fixed point from σ. It has more than $n - \frac{j}{2}$ inversions from the non-fixed points of σ, and $2j - 2$ inversions from the fixed points of σ which were not deleted. Therefore,

$$\begin{aligned}
\ell(\tau) &> n - \frac{j}{2} + (2j - 2) \\
&= n - 2 + \frac{3}{2}j \\
&> n - 2 + \frac{3}{2} \\
&= n - \frac{1}{2},
\end{aligned}$$

as $j \geq 1$, so $\ell(\tau) \geq n$, as desired. $\qquad\square$

Now we deal with the case of cycles.

Lemma 3.4. *Every m-cycle in S_m contains some $\tau \in S_{m-1}$ with $\ell(\tau) \geq m - 2$.*

Proof. Let σ be an m-cycle, so every element in σ is involved in at least one inversion. Let j be the smallest number of inversions that involve any one element. Let $\tau \in S_{m-1}$ be obtained by removing an element from σ with j inversions. Note that τ is contained in σ. If $j = 1$ then $\ell(\tau) \geq m - 2$, since the fact that σ is an m-cycle implies $\ell(\sigma) \geq m - 1$.

Assume $j \geq 2$. Since every element is involved in at least j inversions, $\ell(\sigma) \geq \frac{jm}{2}$, and $\ell(\tau) \geq \frac{jm}{2} - j = \frac{j}{2}(m - 2)$. Since $j \geq 2$, $\ell(\tau) \geq m - 2$, as desired. $\qquad\square$

Finally we treat the last case, that where our permutation is a product of disjoint (non-trivial) cycles.

Lemma 3.5. *Suppose $m > 2n$, and let $\sigma \in S_m$ be a product of disjoint (non-trivial) cycles. Then σ contains some $\tau \in S_{m'}$, where $m' \leq m - 2$, with $\ell(\tau) \geq n$.*

Proof. Let j be the size of the smallest cycle in σ. Note that removing all entries of any given cycle does not change the relative order of the other elements, so removing all the entries of a cycle produces a pattern contained in σ. Furthermore, the entries of any single cycle also forms a pattern contained in σ.

Now, if $j > n$, then let τ be the permutation formed by removing all the entries other than the ones in a minimal size cycle. Note that the absolute length $a(\tau) = j - 1 \geq n$, so $\ell(\tau) \geq n$. Also, $\tau \in S_j$, and $j \leq m - 2$ since the cycles which were removed have at least 2 elements.

If $j \leq n$, let τ be the permutation formed by removing all entries in a minimal size cycle. The absolute length of τ is $m - j - c$, where c is the number of disjoint cycles in τ. Since j is the size of the smallest cycle of σ, $c \leq (m - j)/j$. Therefore,

$$a(\tau) \geq m - j - \frac{m-j}{j}$$
$$= \frac{(j-1)(m-j)}{j}$$
$$> \frac{(j-1)(2n-j)}{j}$$
$$= \frac{(n-j)(j-1) + n(j-1) - (n-1)j}{j} + n - 1$$
$$= \frac{(n-j)(j-1) + j - n}{j} + n - 1$$
$$= \frac{(n-j)(j-2)}{j} + n - 1.$$

Since $j \leq n$ and $j \geq 2$, $a(\tau) > n - 1$, so $a(\tau) \geq n$ and $\ell(\tau) \geq n$. Note $\tau \in S_{m-j}$, and $j \geq 2$.

□

Proof of Theorem 3.1: If σ contains τ with $\ell(\tau) \geq n$, then $\ell(\sigma) \geq n$ since σ must have the inversions of τ at the embedding indices.

Suppose $\ell(\sigma) \geq n$. We prove that $\sigma \in S_m$ contains $\tau \in S_k$ for some $k \leq 2n$ with $\ell(\tau) \geq n$ by induction on m. If $m \leq 2n$, then the theorem holds tautologically with $\tau = \sigma$. Otherwise, we consider separately the cases where σ has a fixed point, where σ consists of a single m-cycle, and where σ is a product of disjoint non-trivial cycles. Lemmas 3.3, 3.4, and 3.5 show that σ contains some permutation σ' where $\ell(\sigma') \geq n$ and $\sigma' \in S_{m'}$ for some $m' < m$. By the induction hypothesis, σ' contains a permutation $\tau \in S_k$ for some $k \leq 2n$ with $\ell(\tau) \geq n$. Since pattern containment is transitive, σ contains τ and the theorem is proven.   □

4. Planar Bruhat graphs

In this section we give a number of characterizations of permutations with planar Bruhat graphs, beginning with the following.

Theorem 4.1. *Let σ be a permutation. Then the Bruhat graph $\mathcal{B}(\sigma)$ is planar if and only if σ avoids 321 and $\ell(\sigma) \leq 3$.*

Proof. The Bruhat graph $\mathcal{B}(321)$ is the complete bipartite graph $K_{3,3}$, which is not planar. Furthermore, $\mathcal{B}(3412)$ contains (as a subgraph) $\mathcal{B}(1432)$ since $1432 < 3412$ in Bruhat order, and 1432 contains 321, so $\mathcal{B}(3412)$ is also not planar. Therefore, if σ contains 321 or 3412, then $\mathcal{B}(\sigma)$ is not planar.

On the other hand, Tenner showed [18, Thm. 5.3] that if σ avoids both 321 and 3412, then the interval in Bruhat order between the identity and σ is isomorphic to the Boolean lattice. In this case, all the edges in the Bruhat graph come from covering relations in

the Bruhat order, so the Bruhat graph is the edge graph for a cube of dimension $\ell(\sigma)$. (One way to see this is by noting that σ avoids 3412 and 4231, so its Bruhat graph has exactly $\ell(\sigma)$ edges at each vertex, and the covering relations in the Boolean lattice already provide $\ell(\sigma)$ edges.) The edge graph of an n-dimensional cube is planar if and only if $n \leq 3$. Therefore, $\mathcal{B}(\sigma)$ is planar if and only if σ avoids 321 and 3412 and $\ell(\sigma) \leq 3$. Since $\ell(3412) = 4$, the condition that σ avoid 3412 is encompassed in the condition that $\ell(\sigma) \leq 3$. $\qquad\square$

We also have a longer proof of the above theorem which is independent of [18].

The following characterization in terms of the Bruhat graph itself follows immediately from the above proof.

Corollary 4.2. *The graph $\mathcal{B}(\sigma)$ is planar if and only if it is a point, a single edge, or the edge graph of a square or a cube.*

Combining Theorems 3.1 and 4.1 produces the following corollary.

Corollary 4.3. *The graph $\mathcal{B}(\sigma)$ is planar if and only if σ avoids 321 and all permutations $\pi \in S_m$ where $m \leq 8$ and $\ell(\pi) \geq 4$.*

A computer calculation to reduce the set of permutations given by Corollary 4.3 to a minimal avoidance set produces the following.

Corollary 4.4. *The graph $\mathcal{B}(\sigma)$ is planar if and only if σ avoids all of the permutations in the following list: $\{321, 3412, 23451, 23514, 24153, 25134, 31425, 31524, 41253, 51234, 234165, 231564, 231645, 241365, 214563, 214635, 215364, 216345, 314265, 312564, 312645, 412365, 2315476, 2143675, 2143756, 2145376, 2153476, 3125476, 21436587\}$*

Finally, we record the number of permutations with planar Bruhat graphs by length, which follows from [18, Cor. 5.5]:

Corollary 4.5. *For any $m \geq 1$, there are*

(1) *1 permutation in S_m of length 0,*
(2) *$(m-1)$ permutations in S_m of length 1,*
(3) *$\frac{(m+1)(m-2)}{2}$ permutations in S_m of length 2, and*
(4) *$\frac{(m+4)(m-1)(m-3)}{6}$ permutations in S_m of length 3*

which have planar Bruhat graphs.

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