

Distance sets of universal and Urysohn metric spaces

N. W. Sauer

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Abstract

A metric space $M = (M; d)$ is *homogeneous* if for every isometry f of a finite subspace of M to a subspace of M there exists an isometry of M onto M extending f . A metric space \mathbf{U} is an *Urysohn* metric space if it is homogeneous and separable and complete and if it isometrically embeds every separable metric space M with $\text{dist}(M) \subseteq \text{dist}(\mathbf{U})$. (With $\text{dist}(M)$ being the set of distances between points in M .)

The main results are: (1) A characterization of the sets $\text{dist}(\mathbf{U})$ for Urysohn metric spaces \mathbf{U} . (2) If R is the distance set of an Urysohn metric space and M and N are two metric spaces, of any cardinality with distances in R , then they amalgamate disjointly to a metric space with distances in R . (3) The completion of a homogeneous separable metric space M which embeds isometrically every finite metric space F with $\text{dist}(F) \subseteq \text{dist}(M)$ is homogeneous.

1 Introduction

There exists a homogeneous separable complete metric space, the classical *Urysohn metric space* $\mathbf{U}_{\mathfrak{R}_{\geq 0}}$, which embeds every separable metric space, see [1]. In this paper we will discuss the question: For which subsets R of the reals \mathfrak{R} and for which subsets of the set of properties of being homogeneous, separable, complete, embedding every finite metric space with distances in R or even embedding every separable metric space with distances in R , does there exist a metric space M with $\text{dist}(M) = R$? It is for example well known that there exists a unique homogeneous, separable, complete metric

space which isometrically embeds every separable metric space with set of distances a subset of the interval $[0, 1]$. This space is the *Urysohn sphere* $\mathbf{U}_{[0,1]}$.

In [8], Kechris, Pestov and Todorćević established a connection between structural Ramsey theory and automorphism groups of homogeneous relational structures and there and in particular also in [14] the notion of oscillation stability of such groups is defined. It is shown that this notion is equivalent to a partition problem in the case of homogeneous metric spaces. A metric space $M = (M; d)$ being *oscillation stable* if for $\epsilon > 0$ and $f : M \rightarrow \mathfrak{R}$ bounded and uniformly continuous there exists a copy $M^* = (M^*; d)$ of M in M so that:

$$\sup\{|f(x) - f(y)| \mid x, y \in M^*\} < \epsilon.$$

Prompted by results of V. Milman [9] it is shown in [11] that the Hilbert sphere in ℓ_2 is not oscillation stable and in [12] that the Urysohn sphere $\mathbf{U}_{[0,1]}$ is oscillation stable. In particular it was shown in [12] that the Urysohn metric spaces \mathbf{U}_n are indivisible, which due to the main result in [16] implies the oscillation stability of \mathbf{U}_n . (\mathbf{U}_n is the Urysohn metric space with $\text{dist}(\mathbf{U}_n) = \{0, 1, 2, \dots, n - 1\}$.) It is shown in [15] that all Urysohn metric spaces with a finite distance set are indivisible. In order to determine if Urysohn metric spaces, or which Urysohn metric spaces, are oscillation stable we characterize in this paper the distance sets of Urysohn metric spaces; providing a baseline for the set of spaces to be investigated for oscillation stability. (According to Theorem 1 below, any two Urysohn metric spaces with the same set of distances being isometric.) See [13] for details and additional references on oscillation stability.

J. Clemens in [10] proved that: Given a set of non-negative reals, $R \subseteq \mathfrak{R}_{\geq 0}$, the set R is the set of distances for some complete and separable metric space if and only if R is an analytic set containing 0 and either R is countable or 0 is a limit point of A . Clemens then asks to determine distance sets of metric spaces which are homogeneous. The following three definitions describe the metric spaces under consideration and define the basic tool for their characterization:

Definition 1.1. *A metric space M is universal if it is homogeneous and isometrically embeds every finite metric space F with $\text{dist}(F) \subseteq \text{dist}(M)$.*

Definition 1.2. *A metric space \mathbf{U} is an Urysohn metric space if it is homogeneous and separable and complete and if it isometrically embeds every separable metric space M with $\text{dist}(M) \subseteq \text{dist}(\mathbf{U})$*

Note that every countable universal metric space with a finite set of distances is an Urysohn metric space.

Definition 1.3. *A set $R \subseteq \mathfrak{R}_{\geq 0}$ satisfies the 4-values condition if: For all pairs of metric spaces $A = (\{x, y, z\}; d_A)$ and $B = (\{u, y, z\}; d_B)$ with $\text{dist}(A) \cup \text{dist}(B) \subseteq R$ and with $d_A(y, z) = d_B(y, z)$ exists a metric space $C = (\{x, y, z, u\}; d_C)$ with $\text{dist}(C) \subseteq R$ and so that the subspace of C induced by $\{x, y, z\}$ is equal to A and the subspace of C induced by $\{u, y, z\}$ is equal to B .*

The notion 4-values condition was first formulated in [17] and will be discussed and used at length in this paper.

We will prove:

Theorem 1. *(See Theorem 4.4 and Theorem 2.2.) Let $0 \in R \subseteq \mathfrak{R}_{\geq 0}$ with 0 as a limit. Then there exists an Urysohn metric space \mathbf{U}_R if and only if R is a closed subset of \mathfrak{R} which satisfies the 4-values condition.*

Let $0 \in R \subseteq \mathfrak{R}_{\geq 0}$ which does not have 0 as a limit. Then there exists an Urysohn metric space \mathbf{U}_R if and only if R is a countable subset of \mathfrak{R} which satisfies the 4-values condition.

Any two Urysohn metric spaces having the same set of distances are isometric.

It follows from Theorem 1.4 in [17] and is stated in this paper as Theorem 3.3 that if $0 \in R \subseteq \mathfrak{R}_{\geq 0}$ is a countable set of numbers which satisfies the 4-values condition then there exists a unique countable universal metric space \mathbf{U}_R with $\text{dist}(\mathbf{U}_R) = R$. Note that this space \mathbf{U}_R is not an Urysohn metric space if 0 is a limit of R and R is not closed in \mathfrak{R} .

Theorem 2. *(See Theorem 3.2.) The set of distances of a universal metric space satisfies the 4-values condition.*

Proposition 10 of [13] provides an example of a countable homogeneous metric space whose completion is not homogeneous. But, in the case of universal metric spaces we have the following:

Theorem 3. *(See Theorem 4.2.) The completion of a universal separable metric space M is homogeneous.*

On the other hand, according to example 5.3, the completion of a universal separable metric space \mathbf{U} need not be universal. That is, the completion

M might not embed every finite metric space with distances in $\text{dist}(M)$. The next theorem characterizes the finite metric spaces which have an embedding into M . In particular it follows from the next Theorem that $\text{dist}(M)$ is the closure of $\text{dist}(\mathbf{U})$ in \mathfrak{R} , Corollary 4.1. See Example 5.2 for a metric space for which the distance set of its completion is not closed.

Theorem 4. (See theorem 4.3.) *Let $0 \in R \subseteq \mathfrak{R}_{\geq 0}$ be countable, satisfy the 4-values condition and have 0 as a limit. Let $M = (M; d)$ be the completion of \mathbf{U}_R .*

A finite metric space $A = (A; d_A)$ with $A = \{a_i \mid i \in m\}$ has an isometric embedding into M if and only if for every $\epsilon > 0$ there exists a metric space B with $B = \{b_i \mid i \in m\}$ and distances in R so that $|\text{d}(a_i, a_j) - \text{d}(b_i, b_j)| < \epsilon$ for all $i, j \in m$.

Extending Urysohn's original result M. Katětov in [2], using "Katětov functions", generalized Urysohn's construction to metric spaces which are " κ -homogeneous" and have weight κ for κ an inaccessible cardinal number. The distance sets of the such constructed Urysohn type spaces are either $\mathfrak{R}_{\geq 0}$ or the unit interval. More recently those Urysohn spaces attracted attention because of interesting properties of their isometry group, $\text{Iso}(\mathbf{U})$. For example Uspenskij's result [3] that the isometry group of the Urysohn space is a universal Polish group and the connection of $\text{Iso}(\mathbf{U}_{[0,1]})$ to minimal topological groups, [4]. See also Mbombo and Pestove [5] and Melleray [6] for further discussion.

We do not follow Katětov's method but lean instead on the general Fraïssé theory, see [7], for our results. Fraïssé theory being particularly well suited for investigating partition problems of separable metric spaces, our main interest. Nevertheless it turned out to be easy to extend the arguments to obtain a general amalgamation result, Theorem 3.1, for metric spaces whose distance sets are subsets of a closed set of reals satisfying the 4-values condition. This then implies, by extending the Fraïssé constructions in an obvious way, (see for example [18] or more recently [19] or many other recent generalizations,) the existence of Urysohn type metric spaces \mathbf{U} which are " κ -homogeneous" and have weight κ for κ an inaccessible cardinal. The distance sets of those spaces \mathbf{U} are closed subsets of \mathfrak{R} satisfying the 4-values condition. Providing another construction for the "Katětov type metric spaces" with sets of distances all of \mathfrak{R} or the unit interval.

2 Notation and Fraïssé theory

For another and more detailed introduction to Fraïssé theory in the context of metric spaces see [13]. The exposition here is complete and self contained but might require some, indeed very limited, familiarity with simple model theoretic constructions.

A pair $H = (H, d)$ is a *premetric space* if $d : H^2 \rightarrow \mathfrak{R}_{\geq 0}$, the *distance function* of H , is a function with $d(x, y) = 0$ if and only if $x = y$ and $d(x, y) = d(y, x)$ for all $x, y \in H$. For $A \subseteq H$ we denote by $H \upharpoonright A$ the *substructure of H generated by A* , that is the premetric space on A with distance function the restriction of d to A^2 . The *skeleton* of H is the set of finite induced subspaces of H and the *age* of H is the class of finite premetric spaces isometric to some element of the skeleton of H . Let $\text{dist}(H) = \{d(x, y) \mid x, y \in H\}$.

A function $\mathbf{t} : F \rightarrow \mathfrak{R}_{>0}$ with F a finite subset of H is a *type function* of H . For \mathbf{t} a type function let $\text{Sp}(\mathbf{t})$ be the premetric space on $F \cup \{\mathbf{t}\}$ for which:

1. $\text{Sp}(\mathbf{t}) \upharpoonright \text{dom}(\mathbf{t}) = H \upharpoonright \text{dom}(\mathbf{t})$.
2. $\forall x \in F \ (d(\mathbf{t}, x) = \mathbf{t}(x))$.

Note that $\text{Sp}(\mathbf{t})$ is a metric space if and only if $H \upharpoonright \text{dom}(\mathbf{t})$ is a metric space and if for all $x, y \in \text{dom}(\mathbf{t})$:

$$|\mathbf{t}(x) - \mathbf{t}(y)| \leq d(x, y) \leq \mathbf{t}(x) + \mathbf{t}(y). \quad (1)$$

For \mathbf{t} a type function let

$$\text{tset}(\mathbf{t}) = \{y \in H \setminus \text{dom}(\mathbf{t}) : \forall x \in \text{dom}(\mathbf{t}) \ (d(y, x) = \mathbf{t}(x))\},$$

the *typeset* of \mathbf{t} . Every element $y \in \text{tset}(\mathbf{t})$ is a *realization* of \mathbf{t} in H . Let $\text{dist}(\mathbf{t}) = \{\mathbf{t}(x) \mid x \in \text{dom}(\mathbf{t})\}$.

Definition 2.1. *Let $M = (M; d)$ be a metric space. A type function \mathfrak{k} of M is metric if $\text{Sp}(\mathfrak{k})$ is a metric space and it is a Katětov function of M if $\text{Sp}(\mathfrak{k})$ is an element of the age of M .*

A type function of M is restricted if it is metric and if $\text{dist}(\mathfrak{k}) \subseteq \text{dist}(M)$.

Note that a type function \mathfrak{k} of a universal metric space M is a Katětov function if and only if it is restricted.

Lemma 2.1. *If every restricted type function of a metric space $M = (M; d)$ has a realization in M then every countable metric space $N = (N; d)$ with $\text{dist}(N) \subseteq \text{dist}(M)$ has an isometric embedding into M .*

If every Katětov function of a metric space $M = (M; d)$ has a realization in M then every countable metric space $N = (N; d)$ whose age is a subset of the age of M has an isometric embedding into M .

Proof. Enumerate N into an ω sequence $(v_i; i \in \omega)$ and for $n \in \omega$ let $N_n = \{v_i \mid i \in n\}$. If f_n is an isometry of $N \upharpoonright N_n$ into M let f_{n+1} be the extension of f_n to an isometry of $N \upharpoonright N_{n+1}$ into M constructed as follows: Let \mathfrak{k} be the type function of M with $\text{dom}(\mathfrak{k}) = f_n[N_n]$ and $\mathfrak{k}(f_n(x)) = d(x, v_n)$. Then \mathfrak{k} is a restricted type function of M and hence has a realization, say a , in M . Let $f_{n+1}(v_n) = a$.

Then $f = \bigcup_{n \in \omega} f_n$ with f_0 the empty function is an isometry of N into M .

The proof of the second part of the Lemma is analogous. □

Lemma 2.2. *Let M and N be two countable metric spaces with $\text{dist}(M) = \text{dist}(N)$ and so that every restricted type function of M has a realization in M and every restricted type function of N has a realization in N .*

Or, let M and N be two countable metric spaces with equal ages and so that every Katětov function of M has a realization in M and every Katětov function of N has a realization in N .

Then every isometry of a finite subspace of M into N has an extension to an isometry of M onto N .

Proof. Extend the proof of Lemma 2.1 to a back and forth argument by alternating the extension of finite isometries between M and N . (As in the standard proof that every countable dense and unbounded linear order is order isomorphic to the rationals.) □

Corollary 2.1. *Let M be a countable metric space so that every Katětov function of M has a realization in M . Then M is homogeneous. If every restricted type function of M has a realization in M then M is universal.*

Lemma 2.3. *Every Katětov function of a homogeneous metric space M has a realization in M . Every restricted type function of a universal metric space M has a realization in M .*

Proof. If M is homogeneous and \mathfrak{k} is a Katětov function of M , there exists an isometry f of $\text{Sp}(\mathfrak{k})$ into M . Let g be the restriction of f to $\text{dom}(\mathfrak{k})$. Then g^{-1} is an isometry of a finite subspace of M to a finite subspace of M , which has, because M is homogeneous, an extension, say h , to an isometry of M onto M . The point $h(f(\mathfrak{k}))$ is a realization of \mathfrak{k} . \square

Lemma 2.4. *Let $M = (M; d)$ be a homogeneous metric space and $A = (A; d_A)$ a countable metric space with $A \cap M$ finite whose age is a subset of the age of M and for which $d(x, y) = d_A(x, y)$ for all $x, y \in A \cap M$. Then there exists a realization of A in M , that is a subset $B \subseteq M \setminus (A \cap M)$ for which there an isometry of A onto $M \upharpoonright (B \cup (M \cap A))$ which fixes $A \cap M$ pointwise.*

Proof. By induction on $A \setminus M$ or a recursive construction realizing Katětov functions step by step. \square

Lemma 2.5. *Let $M = (M; d)$ be a separable metric space and T a countable subset of M . If M realizes all of its restricted type functions then it contains a countable dense subspace S with $T \subseteq S$, which realizes all of its restricted type functions. If M realizes all of its Katětov functions then it contains a countable dense subspace S with $T \subseteq S$, which realizes all of its Katětov functions.*

Proof. Let M be separable and realize all of its restricted type functions. For $A \subseteq M$ let $\text{spec}(A)$ be the set of distances between points of A and $\mathcal{K}(A)$ the set of restricted type functions \mathfrak{k} of M with $\text{dom}(\mathfrak{k}) \subseteq A$.

Let S_0 be a countable dense subset of M . If for $n \in \omega$ a countable set S_n has been determined, choose a realization $\bar{\mathfrak{k}}$ for every restricted type function $\mathfrak{k} \in \mathcal{K}(S_n)$. Let $S_{n+1} = S_n \cup \{\bar{\mathfrak{k}} \mid \mathfrak{k} \in \mathcal{K}(S_n)\}$. The set S_{n+1} is countable because $\mathcal{K}(S_n)$ is countable. Then $S = \bigcup_{n \in \omega} S_n$ is countable and every restricted type function $\mathfrak{k} \in \mathcal{K}(S)$ has a realization in S .

The proof in the case M homogeneous is analogues. \square

Hence we obtain from Corollary 2.1:

Corollary 2.2. *Every separable universal metric space $M = (M; d)$ contains a countable dense universal subspace. Every separable homogeneous metric space $M = (M; d)$ contains a countable dense homogeneous subspace.*

Theorem 2.1. *Every Katětov function of a homogeneous metric space M has a realization in M . Every restricted type function of a universal metric space M has a realization in M .*

If a metric space M is countable and every Katětov function has a realization in M then M is homogeneous. If a metric space M is countable and every restricted type function has a realization in M then M is universal.

If a metric space M is complete and separable and every Katětov function has a realization in M then M is homogeneous. If a metric space M is complete and separable and every restricted type function has a realization in M then M is an Urysohn metric space.

Proof. On account of Corollary 2.1 and Lemma 2.3 it remains to consider the case that M is complete and separable.

Let F be a finite subset of M and f an isometry of $M \upharpoonright F$ into M . Lemma 2.5 yields a dense countable subspace $S = (S; d)$ of M , with $F \subseteq S$, which in the case of Katětov functions realizes all of its Katětov functions and hence is homogeneous on account of Corollary 2.1. It follows that there is an extension g of f to an isometry of S onto S . Because M is complete the isometry g has an extension to an isometry of M to M . It follows that M is homogeneous.

In the case of reduced type functions the metric space M is homogeneous as well because every reduced type function is a Katětov function. Let $N = (N; d)$ be a separable metric space with $\text{dist}(N) \subseteq \text{dist}(M)$. Let T be a countable dense subset of N . According to Lemma 2.1 there exists an isometry f of $N \upharpoonright T$ into M , which because M is complete, has an extension to an isometry of N into M . Hence M is Urysohn. □

Corollary 2.3. *Every separable, complete and universal metric space M is an Urysohn metric space.*

Proof. It follows from Theorem 2.1 that M realizes all of its reduced type functions and hence it follows again from Theorem 2.1 that M is Urysohn. □

Theorem 2.2. *Any two homogeneous and separable and complete metric spaces M and N with the same age are isometric. Any two Urysohn metric spaces M and N with $\text{dist}(M) = \text{dist}(N)$ are isometric.*

Proof. Let S_0 be a countable dense subset of M and T_0 a countable dense subset of N . There exists an isometry f of $M \upharpoonright S_0$ into N and then a dense

countable homogeneous subspace $T = (T; d)$ of N with $f[S_0] \cup T_0 \subseteq N$. There exists an isometry g of T into M with $g(f(x)) = x$ for all $x \in S_0$. Then $g[T]$ is dense in M and because M and N are complete there exists an extension of g to an isometry of N onto M . \square

Definition 2.2. A pair of metric spaces (A, B) of metric spaces is an amalgamation instance if $d_A(x, y) = d_B(x, y)$ for all $x, y \in A \cap B$. Then $\Pi(A, B)$ is the set of metric spaces with:

$$\Pi(A, B) = \{C = (A \cup B; d_C) \mid C \upharpoonright A = A \text{ and } C \upharpoonright B = B\}.$$

For $R \subseteq \mathfrak{R}_{\geq 0}$ let

$$\Pi_R(A, B) = \{C \in \Pi(A, B) \mid \text{dist}(C) \subseteq R\}.$$

Definition 2.3. An age of metric spaces is a class of finite metric spaces closed under subspaces and isometric copies and which is updirected, that is for all metric spaces A and B in the class exists a metric space C in the class which isometrically embeds both spaces A and B . An age is countable if it has countably many isometry classes.

Definition 2.4. A Fraïssé class \mathcal{A} of metric spaces is a countable age of metric spaces which is closed under amalgamation, that is for all amalgamation instances (A, B) with $A, B \in \mathcal{A}$ exists a metric space $C \in \mathcal{A} \cap \Pi(A, B)$.

Theorem 2.3. [Fraïssé] For every Fraïssé class \mathcal{A} of metric spaces exists a unique countable homogeneous metric space $\mathbf{U}_{\mathcal{A}}$, the Fraïssé limit of \mathcal{A} , whose age is equal to \mathcal{A} .

Note that Theorem 2.3 implies that two countable universal metric spaces with set of distances are isometric and together with Theorem 2.2 that any two homogeneous and countable or separable and complete metric spaces with the same age are isometric. Hence we can define:

Definition 2.5. An Urysohn metric space or countable universal metric space with set of distances equals to R will be denoted by \mathbf{U}_R . A homogeneous metric space with age \mathcal{A} which is countable or separable and complete will be denoted by $\mathbf{U}_{\mathcal{A}}$.

3 The 4-values condition

A triple (a, b, c) of non negative numbers is *metric* if $a \leq b+c$ and $b \leq a+c$ and $c \leq a+b$. Note that a triple (a, b, c) is metric if and only if $|a-b| \leq c \leq a+b$. For (a, b, c, d) a quadruple of numbers and x a number write $x \rightsquigarrow (a, b, c, d)$ for: The triples (x, a, b) and (x, c, d) are metric and $a \geq \max\{b, c, d\}$.

Lemma 3.1. *If $x \rightsquigarrow (a, b, c, d)$ then $|a-d| \leq b+c$ and $|b-c| \leq a+d$.*

Proof. If $a \geq b$ then $|b-c| \leq a+d$ and $a-b \leq x \leq c+d$ implying $|a-d| \leq b+c$. If $b \geq a$ then $|a-d| \leq b+c$ and $b-a \leq x \leq c+d$ implying $|b-c| \leq a+d$. \square

Definition 3.1. *For $R \subseteq \mathfrak{R}_{\geq 0}$ let $\mathbf{Q}(R)$ be the set of quadruples (a, b, c, d) of numbers in R for which there exists a number $x \in R$ with $x \rightsquigarrow (a, b, c, d)$.*

Definition 3.2. *A set $R \subseteq \mathfrak{R}_{\geq 0}$ satisfies the 4-values condition if $(a, d, c, b) \in \mathbf{Q}(R)$ for every quadruple $(a, b, c, d) \in \mathbf{Q}(R)$*

Note: The set R satisfies the 4-values condition if and only if for every quadruple $(a, b, c, d) \in \mathbf{Q}(R)$ there exists a number $y \in R$ so that $y \rightsquigarrow (a, d, c, b)$ that is the triples (d, a, y) and (b, c, y) are metric.

Lemma 3.2. *A set $R \subseteq \mathfrak{R}_{\geq 0}$ satisfies the 4-values condition if and only if for any two metric spaces of the form $A = (\{p, v, w\}; d_A)$ and $B = (\{q, v, w\}; d_B)$ with $\text{dist}(A) \subseteq R$ and $\text{dist}(B) \subseteq R$ and $R \ni x = d_A(v, w) = d_B(v, w)$ the set $\Pi_R(A, B) \neq \emptyset$.*

Proof. Let R satisfy the 4-values condition and assume that the spaces A and B with $x = d_A(v, w) = d_B(v, w)$ are given. Let:

$$a = d_A(p, v) \quad b = d_A(p, w) \quad c = d_B(q, w) \quad d = d_B(q, v). \quad (2)$$

Then $x \rightsquigarrow (a, b, c, d)$. Hence there is a number $y \in R$ with $y \rightsquigarrow (a, d, c, b)$ implying that the space $C = (\{p, q, v, w\}; d_C) \in \Pi_R(A, B)$ with $d_C(p, q) = y$.

For the other direction of the proof let $x, a, b, c, d \in R$ and $x \rightsquigarrow (a, b, c, d)$. Let then $A = (\{p, v, w\}; d_A)$ and $B = (\{q, v, w\}; d_B)$ be metric spaces with distances as in (2) and with $x = d_A(v, w) = d_B(v, w)$. Let $C = (\{p, q, v, w\}; d_C) \in \Pi_R(A, B)$. Then $R \ni y \rightsquigarrow (a, d, c, b)$ for $y = d_C(p, q)$. \square

Definition 3.3. For two metric spaces $A = (\{p, v, w\}; d_A)$ and $B = (\{q, v, w\}; d_B)$ with distances as in (2) let:

$$u(A, B) = \max\{|a - d|, |b - c|\} \quad l(A, B) = \min\{a + d, b + c\}$$

Lemma 3.3. Let $A = (\{p, v, w\}; d_A)$ and $B = (\{q, v, w\}; d_B)$ be two metric spaces with $d_A(v, w) = d_B(v, w)$, then $u(A, B) \leq l(A, B)$ and

$$[u(A, B), l(A, B)] = \{d_C(p, q) \mid C \in \Pi(A, B)\}.$$

Let $R \subseteq \mathfrak{R}_{\geq 0}$ satisfy the 4-values condition and $\text{dist}(A) \cup \text{dist}(B) \subseteq R$. Then there exists a number $y \in R \cap [u(A, B), l(A, B)]$.

Proof. Let the numbers (a, b, c, d) be given by Condition (2).

The inequality $u(A, B) \leq l(A, B)$ follows from Lemma 3.1. If $y \in [u(A, B), l(A, B)]$ then $|a - d| \leq y \leq a + d$ and $|b - c| \leq y \leq b + c$ and hence the triples (y, a, d) and (y, b, c) are metric. If $y = d_C(p, q)$ for $C \in \Pi(A, B)$, then the triples (y, a, d) and (y, b, c) are metric and hence $|a - d| \leq y \leq a + d$ and $|b - c| \leq y \leq b + c$.

If R satisfies the 4-values condition and $\text{dist}(A) \cup \text{dist}(B) \subseteq R$ it follows from Lemma 3.2 that $\Pi_R(A, B) \neq \emptyset$ and hence $\emptyset \neq \{d_C(p, q) \mid C \in \Pi_R(A, B)\} = R \cap \{d_C(p, q) \mid C \in \Pi(A, B)\} = R \cap [u(A, B), l(A, B)]$. \square

Lemma 3.4. Let the set $R \subseteq \mathfrak{R}_{\geq 0}$ satisfy the 4-values condition and let $(A = (A; d_A), B = (B; d_B))$ with $\text{dist}(A) \cup \text{dist}(B) \subseteq R$ be an amalgamation instance. Let $A \setminus B = \{p\}$ and $B \setminus A = \{q\}$.

Then $\Pi_R(A, B) \neq \emptyset$ if $A \cup B$ is finite or if R is closed.

Proof. For $v, w \in A \cap B$ let $A_{v,w} = A \upharpoonright \{p, v, w\}$ and $B_{v,w} = B \upharpoonright \{q, v, w\}$. Note that $(A_{v,w}, B_{v,w})$ is an amalgamation instance. We have to prove that there is a number $y \in R$ so that the premetric space $C = (A \cup B; d)$ with $d(p, q) = y$ and $d(p, v) = d_A(p, v)$ and $d(q, v) = d_B(q, v)$ and $d(v, w) = d_A(v, w) = d_B(v, w)$ for all $v, w \in A \cap B$ is a metric space. That is, for

$$\mathcal{S} := \bigcap_{v, w \in A \cap B} [u(A_{v,w}, B_{v,w}), l(A_{v,w}, B_{v,w})],$$

we have to prove, according to Lemma 3.3, that $\mathcal{S} \cap R \neq \emptyset$. Let

$$\hat{u} := \sup\{|d_A(p, v) - d_B(q, v)| \mid v \in A \cap B\}, \quad \hat{l} := \inf\{d_A(p, v) + d_B(q, v)\}.$$

Then $\hat{u} \leq \hat{l}$ according to Lemma 3.1 and $[\hat{u}, \hat{l}] \subseteq \mathcal{S}$ according to Definition 3.3.

Let R be closed. There exists for every $\epsilon > 0$ a point $v \in A \cap B$ with $\hat{u} - \epsilon < |d_A(p, v) - d_B(q, v)| \leq \hat{u}$ and a point $w \in A \cap B$ with $\hat{l} \leq d_A(p, w) + d_B(q, w) \leq \hat{l} + \epsilon$. Then:

$$\begin{aligned} \hat{u} - \epsilon &< |d_A(p, v) - d_B(q, v)| \leq \mathbf{u}(A_{v,w}, B_{v,w}) \leq \hat{u} \leq \\ &\leq \hat{l} \leq d_A(p, w) + d_B(q, w) \leq \mathbf{l}(A_{v,w}, B_{v,w}) < \hat{l} + \epsilon. \end{aligned}$$

Because R satisfies the 4-values condition it follows that for every $\epsilon > 0$ there exists a number $y_\epsilon \in R$ with

$$\hat{u} - \epsilon < \mathbf{u}(A_{v,w}) \leq y_\epsilon \leq \mathbf{l}(A_{v,w}, B_{v,w}) < \hat{l} + \epsilon.$$

Hence, because R is closed, there exists a number $y \in R \cap [\hat{u}, \hat{l}] \subseteq R \cap S$.

If $A \cup B$ is finite let $v \in A \cap B$ such that $\hat{u} = |d_A(p, v) - d_B(q, v)|$ and $w \in A \cap B$ such that $\hat{l} = d_A(p, w) + d_B(q, w)$. Then:

$$\hat{u} = \mathbf{u}(A_{v,w}, B_{v,w}) \leq \mathbf{l}(A_{v,w}, B_{v,w}) = \hat{l}.$$

Because R satisfies the 4-values condition there exists a number $y \in R \cap [\hat{u}, \hat{l}] \subseteq R \cap S$. \square

Lemma 3.5. *Let the set $R \subseteq \mathfrak{R}_{\geq 0}$ satisfy the 4-values condition and let $(A = (A; d_A), B = (B; d_B))$ with $\text{dist}(A) \cup \text{dist}(B) \subseteq R$ be an amalgamation instance. Let $A \setminus B = \{p\}$.*

Then $\Pi_R(A, B) \neq \emptyset$ if $A \cup B$ is finite or if R is closed.

Proof. Note that for $A \cap B \subseteq C \subseteq B$ and $C = B \upharpoonright C$ the pair (A, C) is an amalgamation instance. Let:

$$\mathcal{M} = \bigcup_{A \cap B \subseteq C \subseteq B} \Pi_R(A, B \upharpoonright C).$$

Then $(\mathcal{M}; \preceq)$ is a partial order for $L = (L; d) \preceq N = (N; d)$ if $L \subseteq N$ and $N \upharpoonright L = L$. Every chain in the partial order $(\mathcal{M}; \preceq)$ has an upper bound and hence using Zorn's Lemma the partial order $(\mathcal{M}; \preceq)$ has a maximal element $M = (M; d_M)$. If $M = A \cup B$ then $M \in \Pi_R(A, B)$. Otherwise let $b \in (A \cup B) \setminus M$ and let $D = (M \setminus \{p\}) \cup \{b\}$ and $D = B \upharpoonright D$. Lemma 3.4 applied to the amalgamation instance (M, D) results in a metric space contradicting the maximality of M . \square

Theorem 3.1. *Let the set $R \subseteq \mathfrak{R}_{\geq 0}$ satisfy the 4-values condition and let $(A = (A; d_A), B = (B; d_B))$ with $\text{dist}(A) \cup \text{dist}(B) \subseteq R$ be an amalgamation instance.*

Then $\Pi_R(A, B) \neq \emptyset$ if $A \cup B$ is finite or if R is closed.

Proof. Let:

$$\mathcal{M} = \bigcup_{A \cap B \subseteq C \subseteq B} \Pi_R(A, B \upharpoonright C).$$

Then $(\mathcal{M}; \preceq)$ is a partial order for $L = (L; d) \preceq N = (N; d)$ if $L \subseteq N$ and $N \upharpoonright L = L$. Every chain in the partial order $(\mathcal{M}; \preceq)$ has an upper bound and hence using Zorn's Lemma the partial order $(\mathcal{M}; \preceq)$ has a maximal element $M = (M; d_M)$. If $M = A \cup B$ then $M \in \Pi_R(A, B)$. Otherwise let $b \in (A \cup B) \setminus M$ and let $D = (M \cap B) \cup \{b\}$ and $D = B \upharpoonright D$. Lemma 3.5 applied to the amalgamation instance (M, D) results in a metric space contradicting the maximality of M . \square

Theorem 3.2. *The set of distances of a universal metric space satisfies the 4-values condition.*

Proof. Let $M = (M; d)$ be a universal metric space with $R = \text{dist}(M)$. Let $A = (\{p, v, w\}; d_A)$ and $B = (\{q, v, w\}; d_B)$ with $\text{dist}(A) \subseteq R$ and $\text{dist}(B) \subseteq R$ and $R \ni x = d_A(v, w) = d_B(v, w)$. There exists an isometric copy with points $\{p', v', w'\} \subseteq M$ in M . Let \mathfrak{t} be the restricted type function with $\text{dom}(\mathfrak{t}) = \{v', w'\}$ and with $\mathfrak{t}(v') = d_B(q, v)$ and $\mathfrak{t}(w') = d_B(q, w)$. Let q' be a realization of \mathfrak{t} . Then the metric space $C = (\{p, v, w, q\}; d_C)$ with $C \upharpoonright \{p, v, w\} = A$ and $C \upharpoonright \{q, v, w\} = B$ and $d_C(p, q) = d(p', q')$ is a metric space in $\Pi_R(A, B)$. Hence the Theorem follows from Lemma 3.2. \square

Definition 3.4. *Let $R \subseteq \mathfrak{R}_{\geq 0}$, then \mathcal{F}_R is the class of finite metric spaces M with $\text{dist}(M) \subseteq R$.*

Theorem 3.3. *Let $0 \in R \subseteq \mathfrak{R}_{\geq 0}$ be a countable set of numbers which satisfies the 4-values condition. Then there exists a countable universal metric space \mathbf{U}_R .*

If there exists a countable universal metric space \mathbf{U}_R then R satisfies the 4-values condition.

Proof. The class \mathcal{F}_R of finite metric spaces is closed under isometric copies and substructures and it follows from Theorem 3.1 and Definition 2.2 that $\emptyset \neq \Pi_R(A, B) \subseteq \mathcal{F}_R$ for all $A, B \in \mathcal{F}_R$. Hence \mathcal{F}_R is updirected and closed

under amalgamation and hence a Fraïssé class. According to Theorem 2.3 there exists a countable homogeneous metric space $\mathbf{U}_{\mathcal{F}_R}$ whose age is equal to \mathcal{F}_R . It follows that $\mathbf{U}_{\mathcal{F}_R}$ is the countable universal metric space \mathbf{U}_R . \square

Lemma 3.6. *Let $R \subseteq \mathfrak{R}_{\geq 0}$ be a set of numbers which satisfies the 4-values condition. For every countable subset T of R exists a dense countable subset $C \supseteq T$ of R which satisfies the 4-values condition.*

Proof. Let $S \supseteq T$ be a countable dense subset of R . There are countably many instances of the form $x \rightsquigarrow (a, b, c, d)$ with numbers in S . Because R does satisfy the 4-values condition there is a countably set $S' \subseteq R$ so that for all those quadruples there is a $y \in S'$ with $y \rightsquigarrow (a, d, c, b)$. Repeating this process countably often leads to a countable subset $C \subseteq R$ which satisfies the 4-values condition. \square

In order to verify the 4-values condition the following Lemma is often useful.

Lemma 3.7. *If $a \leq b + c$ or $a \leq b + d$ or $a \leq c + d$ and $(a, b, c, d) \in \mathbf{Q}(R)$ then there exists $y \in \{a, b, c, d\}$ with $y \rightsquigarrow (a, d, c, b)$.*

Proof. If $a \leq b + c$ then $a \rightsquigarrow (a, d, c, b)$. If $a \leq b + d$ then $b \rightsquigarrow (a, d, c, b)$ unless $2b < c$ in which case $c \rightsquigarrow (a, d, c, b)$. If $a \leq c + d$ then $c \rightsquigarrow (a, d, c, b)$ unless $2c < b$ in which case $b \rightsquigarrow (a, d, c, b)$. \square

Hence, in order to verify that R satisfies the 4-values condition, it suffices to consider quadruples for which a is larger than the sum of any two of the other three numbers.

4 Completion of universal metric spaces

Definition 4.1. *Let $A = (A; d_A)$ and $B = (B; d_B)$ be two metric spaces with $A = \{a_i \mid i \in m \in \omega\}$ and $B = \{b_i \mid i \in m \in \omega\}$ and $A \cap B = \emptyset$. A metric space $P = (A \cup B; d_P)$ with $P \upharpoonright A = A$ and $P \upharpoonright B = B$ is an h -join of A and B if $d_P(a_i, b_i) < h$ for all $i \in m$. (The h -join P depends explicitly on the enumeration of A and B .)*

Lemma 4.1. *Let $R \subseteq \mathfrak{R}_{\geq 0}$ satisfy the 4-values condition and have 0 as a limit and let \mathcal{F}_R be the class of finite metric spaces F with $\text{dist}(F) \subseteq R$.*

Let $A = (A; d_A)$ and $B = (B; d_B)$ be two metric spaces in \mathcal{F}_R for which $A = \{a_0, a_1, a_2, \dots, a_m\}$ and $B = \{b_0, b_1, b_2, \dots, b_m\}$ and $A \cap B = \emptyset$. Let $A' = \{a_i \mid i \in m\}$ and $B' = \{b_i \mid i \in m\}$ and A', B' metric spaces with $A' = A \upharpoonright A'$ and $B' = B \upharpoonright B'$. Let $k \geq \max\{|d_A(a_m, a_i) - d_B(b_m, b_i)| \mid i \in m\}$ and $h \in R$ and $l \in R$ with :

$$l + k \leq h \leq \min(\{d_A(a_m, a_i) \mid i \in m\} \cup \{d_B(b_m, b_i) \mid i \in m\}).$$

Then if there exists an l -join $Q = (A', B'); d_Q \in \mathcal{F}_R$ of A' and B' , there exists an h -join $P \in \mathcal{F}_R$ of A and B .

Proof. There exist, according to Theorem 3.1, a metric space $A^* = (A^*; d_{A^*}) \in \Pi_R(A, Q)$ and a metric space $B^* = (A^*; d_{B^*}) \in \Pi_R(B, Q)$. Note $|d_A(a_m, a_i) - d_{A^*}(a_m, a_i)| < l$ and $|d_B(b_m, b_i) - d_{B^*}(b_m, a_i)| < l$ for all $i \in m$. Let $P = (A \cup B; d)$ be the premetric space with $P \upharpoonright (A \cup B') = A^*$ and $P \upharpoonright (B \cup A') = B^*$ and with $d(a_m, b_m) = h$.

In order to see that P is a metric space we have to check the triples of the form (a_m, b_m, a_i) and (a_m, b_m, b_i) for all $i \in m$. Indeed:

$$\begin{aligned} & |d(a_m, a_i) - d(b_m, a_i)| \leq \\ & |d(a_m, a_i) - d(b_m, b_i)| + |d(b_m, b_i) - d(b_m, a_i)| \leq k + l \leq h = d(a_m, b_m). \end{aligned}$$

This verifies that the triple (a_m, b_m, a_i) is metric because $h \leq d(a_m, a_i)$ and hence the distance $d(a_m, b_m)$ is not larger than the other two distances in the triple (a_m, b_m, a_i) . Similar, the triangles of the form (a_m, b_m, b_i) are metric. \square

Lemma 4.2. *Let $R \subseteq \mathfrak{R}_{\geq 0}$ satisfy the 4-values condition and have 0 as a limit. Then, for every $m \in \omega$ and $\mathbf{r} > 0$ and $h \in \mathfrak{R}_{> 0}$ exists a number $\gamma(h) < h$ so that:*

For all metric spaces $A = (\{a_i \mid i \in m\}; d_A)$ and $B = (\{b_i \mid i \in m\}; d_B)$ in \mathcal{F}_R with $|d_A(a_i, a_j) - d_B(b_i, b_j)| < \gamma(h)$ for all $i, j \in m$ and with $\min(\text{dist}(A) \cup \text{dist}(B)) \geq \mathbf{r}$ and $A \cap B = \emptyset$ exists an h -join of A and B in \mathcal{F}_R .

Proof. For every $0 < x \in R$ let $0 < x^* \in R$ with $2 \cdot x^* < x$. Given h and \mathbf{r} let $0 < h_{m-1} < \min\{h, \mathbf{r}\}$ and for all integers $m-1 > i \geq 0$ let $h_i = h_{i+1}^*$ and $\gamma(h) = h_0$. The Lemma follows from Lemma 4.1 via induction on i . \square

Note: Let a, b, a', b' be four points in a metric space, then

$$\begin{aligned} |d(a, b) - d(a', b')| &\leq \\ |d(a, b) - d(a, b')| + |d(a, b') - d(a', b')| &\leq d(b, b') + d(a, a'). \end{aligned} \quad (3)$$

Theorem 4.1. *Let $0 \in R \subseteq \mathfrak{R}_{\geq 0}$ be countable, satisfy the 4-values condition and have 0 as a limit. Let \mathbf{U}_R be the countable universal metric space with R as set of distances given by Theorem 3.3. Then $M = (M; d)$, the completion of \mathbf{U}_R , is homogeneous and separable and complete.*

Proof. The universal space $\mathbf{U}_R = (U; d)$ is dense in M . It follows from Theorem 2.1 that M is homogeneous if every Katětov function of M has a realization in M .

Let \mathfrak{k} be a Katětov function of M with $\text{dom}(\mathfrak{k}) = \{a_i \mid i \in m \in \omega\} := A$. There exists a subset $B = \{b_i \mid i \in m\} \cup \{b_m\} \subseteq M$ with $d(a_i, a_j) = d(b_i, b_j)$ and $\mathfrak{k}(a_i) = d(b_m, b_i)$ for all $i, j \in m$. Let $k = \min(\text{dist}(\text{Sp}(\mathfrak{k})))$.

Let $0 < e \in \mathfrak{R}$ and $B' = \{b'_i \mid 0 \leq i \leq m\} \subseteq U$ with $d(b_i, b'_i) < e$. Then from Inequality (3): $|d(b_i, b_j) - d(b'_i, b'_j)| < 2e$ for all $0 \leq i, j \leq m$. Note that if $e < \frac{k}{4}$ then $\min\{d(b_i, b_j) \mid 0 \leq i, j \leq m\} > \frac{k}{2}$. If there is a set of points $A' = \{a'_i \mid i \in m\} \subseteq U$ with $d(a_i, a'_i) < e$ then: $|d(a_i, a_j) - d(a'_i, a'_j)| < 2e$ for all $i, j \in m$. Hence, because $d(a_i, a_j) = d(b_i, b_j)$:

$$|d(a'_i, a'_j) - d(b'_i, b'_j)| < 4e.$$

It follows from Lemma 4.2 that if $4e < \min(\gamma(h), k)$ then there exists an h -join $C' = (\{B' \cup A'\}; d_{C'}) \in \mathcal{F}_R$ of $\mathbf{U}_R \upharpoonright B'$ and $\mathbf{U}_R \upharpoonright A'$. It follows from Lemma 2.3 that there exists a realization $C = (\{c_i \mid i \in m\}; d)$ of C' in \mathbf{U}_R with $d(a'_i, c_i) < h$ and $d(c_i, c_j) = d(b'_i, b'_j)$ for all $i, j \in m$. Also:

$$d(c_i, a_i) \leq d(c_i, a'_i) + d(a'_i, a_i) < h + e < 2h \text{ for all } i \in m. \quad (4)$$

Let $(h_n; n \in \omega)$ and $(e_n; n \in \omega)$ be a sequence of numbers in R with $h_n > 2 \cdot h_{n+1}$ and $4e_n < \min(\gamma(\frac{1}{2}h_n), k)$. Then there exist sets of points:

$$\begin{aligned} B'_n &= \{b'_{n,i} \mid 0 \leq i \leq m\} \text{ with } d(b'_{n,i}, b_i) < e_n < \frac{1}{2}h_n \text{ and} \\ C_n &= \{c_{n,i} \mid i \in m\} \text{ with } d(c_{n,i}, a_i) < h_n \text{ and with} \\ d(b'_{n,i}, b'_{n,j}) &= d(c_{n,i}, c_{n,j}) \text{ and } d(c_i, a_i) < h_n \text{ for all } i, j \in m \text{ and} \\ d(b'_{n,m}, b'_{n+1,m}) &\leq d(b'_{n,m}, b_{n,m}) + d(b_{n,m}, b'_{n+1,m}) < h_n. \end{aligned} \quad (5)$$

Using the fact that Katětov functions of \mathbf{U}_R have realizations in U construct recursively points $c_{n,m} \in U$ so that for all $i \in m$ and $n \in \omega$:

$$d(c_{n,m}, c_{n,i}) = d(b'_{n,m}, b'_{n,i}) \quad d(c_{n,m}, c_{n+1,m}) = d(b'_{n,m}, b'_{n+1,m}) < h_n.$$

That is, the function f with $f(b'_{n,i}) = c_{n,i}$ for $0 \leq i \leq m$ and $n \in \omega$ is an isometry of a subset of U to a subset of U .

It follows from Inequality (4) that for every $i \in m$ the sequence $(c_{n,i})$ converges to a_i and from Inequality (5) that the sequence $(c_{n,m})$ is Cauchy converging to, say c_m . For every $0 \leq i \leq m$ the sequence $(b'_{n,i})$ converges to b_i and hence for every $i \in m$:

$$\lim_{n \rightarrow \infty} d(b'_{n,m}, b_{n,i}) = d(b_m, b_i) = \mathfrak{k}(a_i).$$

It follows that $d(c_m, a_i) = \lim_{n \rightarrow \infty} d(c_{n,m}, c_{n,i}) = \mathfrak{k}(a_i)$ implying that c_m is a realization of the Katětov function \mathfrak{k} . □

Theorem 4.2. *The completion of a homogeneous separable metric space M which embeds isometrically every finite metric space F with $\text{dist}(F) \subseteq \text{dist}(M)$ is homogeneous.*

Proof. The space M contains a countable dense universal subspace N according to Corollary 2.2. Let $R = \text{dist}(N)$. It follows from Theorem 2.2 that we can take N to be the universal metric space \mathbf{U}_R . The Theorem follows from Theorem 4.1 because the completion of \mathbf{U}_R is equal to the completion of M . □

Theorem 4.3. *Let $0 \in R \subseteq \mathfrak{R}_{\geq 0}$ be countable, satisfy the 4-values condition and have 0 as a limit and let $M = (M; d)$ be the completion of $\mathbf{U}_R = (U; d)$.*

A finite metric space $A = (A; d_A)$ with $A = \{a_i \mid i \in m\}$ has an isometric embedding into M if and only if for ever $\epsilon > 0$ there exists a metric space $B = (B; d_B) \in \mathcal{F}_R$ with $B = \{b_i \mid i \in m\}$ so that $|d(a_i, a_j) - d(b_i, b_j)| < \epsilon$ for all $i, j \in m$.

Proof. The condition is clearly necessary.

Let $k = \min(\text{dist}(A))$. Let (h_n) be a sequence of positive numbers in R so that $h_0 < \frac{k}{4}$ and $2h_{n+1} < h_n$ for all $n \in \omega$. Let (e_n) be a sequence of positive numbers in R so that $\gamma(h_n) < 2e_n$ and $e_{n+1} < e_n$, with γ given by Lemma 4.2.

For $n \in \omega$ let $B_n = (B_n; d_{B_n})$ be a metric space with $B_n = \{b_{n,i} \mid i \in m\} \in \mathcal{F}_R$ and with

$$|d_A(a_i, a_j) - d_{B_n}(b_{n,i}, b_{n,j})| < e_n < h_n. \quad (6)$$

Then:

$$|d_{B_{n+1}}(b_{n+1,i}, b_{n+1,j}) - d_{B_n}(b_{n,i}, b_{n,j})| < e_{n+1} + e_n < 2e_n.$$

It follows from Lemma 4.2 that there exists, for every $n \in \omega$, an h_n -join $P_n = (B_n \cup B_{n+1}; d_P) \in \mathcal{F}_R$ of B_n with B_{n+1} .

The space \mathbf{U}_R is universal and hence each of the finite metric spaces P_n has an isometric embedding into \mathbf{U}_R . It follows from Lemma 2.4 via a recursive construction that there exist isometric copies $B'_n = \{b'_{n,i} \mid i \in m\}$ of the sets B_n in U so that, for all $n \in \omega$:

$$d(b'_{n,i}, b'_{n+1,i}) = d_{P_n}(b_{n,i}, b_{n+1,i}) < h_n.$$

It follows that for every $i \in m$ the sequence $(b_{n,i})$ is Cauchy and hence has a limit, say $b_i \in M$. Also for all $i, j \in m$:

$$\lim_{n \rightarrow \infty} d(b'_{n,i}, b'_{n,j}) = \lim_{n \rightarrow \infty} d(b_{n,i}, b_{n,j}) = d_A(a_i, a_j),$$

with the last equality implied by Inequality (6). □

Corollary 4.1. *Let $0 \in R \subseteq \mathfrak{R}_{\geq 0}$ be countable, satisfy the 4-values condition and have 0 as a limit and let $M = (M; d)$ be the completion of $\mathbf{U}_R = (U; d)$. Then $\text{dist}(M)$ is the closure of R . The set of distances of the completion N of a universal separable metric space is a closed subset of \mathfrak{R} .*

Proof. Let $\epsilon > 0$ be given and a in the closure of $\text{dist}(M)$. There exists a number $b \in \text{dist}(M)$ with $|a - b| < \frac{\epsilon}{2}$. There exists a number $c \in R$ with $|b - c| < \frac{\epsilon}{2}$. That $\text{dist}(N)$ is closed follows as in the proof of Theorem 4.2. □

Note that in general the distance set of the completion of a metric space need not be closed. (See Example 5.2.)

Definition 4.2. *For $R \subseteq \mathfrak{R}$ let*

$$R^0 = \{x \in R \mid \exists \epsilon > 0 \ ((x, x + \epsilon) \cap R = \emptyset)\}.$$

Lemma 4.3. *Let $R \subseteq \mathfrak{R}_{\geq 0}$, satisfy the 4-values condition and have 0 as a limit. If $\{x, y, z\} \subseteq R$ with $z = y + x$ and $x \in R^0$ then $z \in R^0$.*

Proof. Let $\{x, y, z\} \subseteq R$ with $z = y + x$ and $x \in R^0$ and $\epsilon > 0$ so that $(x, x + \epsilon) \cap R = \emptyset$ and let $0 < \delta < \min\{\epsilon, x\}$. If $z \notin R^0$ there exists $z < z' \in R$ with $z' - z < \delta$. Then $z \rightsquigarrow (z', \delta, x, y)$. If $R \ni u \rightsquigarrow (z', y, x, \delta)$ then the triple (δ, x, u) is metric and hence $u \leq x + \delta$, which implies, because $(x, \delta] \cap R = \emptyset$, that $u \leq x$. It follows that $u + y \leq x + y = z < z'$ and hence that the triple (u, y, z') is not metric. In contradiction to R satisfying the 4-values condition. \square

Lemma 4.4. *Let $R \subseteq \mathfrak{R}_{\geq 0}$, satisfy the 4-values condition and have 0 as a limit. If $\{x, y, z\} \subseteq R$ with $z = y + x$ and $\{x, y\} \subseteq R^0$ then both x and y are isolated points of R and $z \in R^0$.*

Proof. It follows from Lemma 4.3 that $z \in R^0$. Let $z = y + x$ and $\{x, y\} \subseteq R^0$. If, say x , is not isolated in R , let $\epsilon > 0$ be such that $(y, y + \epsilon) \cap R = \emptyset$. Let $R \ni \delta < x$ with $0 < \delta < \epsilon$ and let $0 < u < x$ with $u \in R$ such that $x - u \leq \delta$. Note that $x \rightsquigarrow (z, y, \delta, u)$.

If $R \ni r \rightsquigarrow (z, u, \delta, y)$ then $r \leq y + \delta$ because the triple (r, y, δ) is metric and hence $r \leq y$ because of the choice of δ . Then $r + u \leq y + u < y + x = z$ and hence the triple (r, u, z) is not metric in contradiction to $r \rightsquigarrow (z, u, \delta, y)$. \square

Lemma 4.5. *Let $0 \in R \subseteq \mathfrak{R}_{\geq 0}$ satisfy the 4-values condition and have 0 as a limit. Let S be a dense subset of R .*

Then there exists, for every metric space $A = (A; d_A) \in \mathcal{F}_R$ with $A = \{a_i \mid i \in m\}$ and every $\epsilon > 0$, a metric space $B = (B; d_B) \in \mathcal{F}_S$ with $B = \{b_i \mid i \in m\}$ so that $|d_A(a_i, a_j) - d_B(b_i, b_j)| < \epsilon$ for all $i, j \in m$.

Proof. Let:

$$\Delta = \frac{1}{3} \min\{y + x - z \mid z < y + x \text{ and } \{x, y, z\} \subseteq \text{dist}(A)\}.$$

Let $I \subseteq \text{dist}(A)$ be the set of isolated points of R which are elements of $\text{dist}(A)$. Note that $I \subseteq S$.

Let E be the set of positive numbers in $\text{dist}(A) \setminus R^0$ with $e_0 < e_1 < e_2 < \dots < e_{n-1} < e_n$ an enumeration of E . Let $e_0 < \hat{e}_0 \in S$ so that $\hat{e}_0 - e_0 < \min\{\Delta, \epsilon\}$. The numbers $\hat{e}_i \in S$ are determined recursively so that always $\hat{e}_i > e_i$ and $\frac{1}{3}(\hat{e}_i - e_i) > \hat{e}_{i+1} - e_{i+1}$ for all indices $i \in n$.

Let K be the set of positive numbers in $\text{dist}(A) \cap R^0$ which are not isolated and let $k_0 > k_1 > k_2 > \dots > k_{r-1} > k_r$ be an enumeration of K . Let $k_0 > \hat{k}_0 \in S$ so that $k_0 - \hat{k}_0 < \frac{1}{3}(\hat{e}_n - e_n)$. The numbers $\hat{k}_i \in S$ are determined recursively so that always $k_i > \hat{k}_i$ and $\frac{1}{3}(k_i - \hat{k}_i) > k_{i+1} - \hat{k}_{i+1}$ for all indices $i \in r$.

For every $x \in I$ let $\hat{x} = x$ and let $\hat{0} = 0$. Note that the inequalities above imply for $\{x, y, z\} \subseteq \text{dist}(A)$ with $x, y \neq 0$ and $z = x + y$ and $x \in E$ or $y \in E$ that $\hat{z} \leq \hat{x} + \hat{y}$.

CLAIM: If (x, y, z) is a metric triple of numbers with entries in $\text{dist}(A)$ then $(\hat{x}, \hat{y}, \hat{z})$ is a metric triples of numbers with entries in S .

Proof: Let $z \geq \max\{y, x\}$. It follows from the choice of Δ and the definition of the function $\hat{\cdot}$ that $\hat{z} \geq \max\{\hat{y}, \hat{x}\}$ and if $z < y + x$ then $\hat{z} < \hat{y} + \hat{x}$ and hence that the triple $(\hat{z}, \hat{y}, \hat{x})$ is metric.

Let $z = y + x$ with $x, y \neq 0$. If at least one of x and y are in E then $\hat{z} \leq \hat{x} + \hat{y}$. If both are not in E then they are both in R^0 and it follows from Lemma 4.4 that both x and y are isolated in R and $z \in R^0$. If z is isolated then $\hat{z} = z = x + y = \hat{x} + \hat{y}$. If z is not isolated then $\hat{z} < z = x + y = \hat{x} + \hat{y}$. End proof of CLAIM.

It follows that the premetric space $B = (B; d_B)$ with $B = \{b_i \mid i \in m\}$ and $d_B(b_i, b_j) = \hat{x}_{i,j}$ for $x_{i,j} = d_A(a_i, a_j)$ is a metric space with $|d_A(a_i, a_j) - d_B(b_i, b_j)| < \epsilon$ for all $i, j \in m$. \square

Corollary 4.2. *Let $0 \in R \subseteq \mathfrak{R}_{\geq 0}$ satisfy the 4-values condition and have 0 as a limit. Let S be a dense subset of R .*

Then there exists, for every metric space $A = (A; d_A) \in \mathcal{F}_R$ with $A = \{a_i \mid i \in m\}$ and every $\epsilon > 0$, a metric space $B = (B; d_B) \in \mathcal{F}_S$ with $B = \{b_i \mid i \in m\}$ so that $d_A(a_i, b_i) < \epsilon$ for all $i \in m$.

Proof. Follows from Lemma 4.5 and Lemma 4.2. \square

Lemma 4.6. *Let $U_R = (U; d)$ be an Urysohn metric space for which 0 is a limit of R . Then R is a closed subset of \mathfrak{R} .*

Proof. Let c in the closure of R . Let (a_n) be a sequence of numbers in R converging monotonic to 0 with $a_0 < \frac{1}{2}c$. Let (c_n) be a sequence of numbers in R for which $|c - c_n|$ converges monotonic to 0 with $|c - c_0| < \frac{1}{2}c$. For every $n \in \omega$ let $\bar{n} \in \omega$ with $n \leq \bar{n}$ and $2|c - c_{\bar{n}}| < a_n$ and $\bar{n} < \bar{m}$ for $n < m$.

Let $V = (V; d)$ be a premetric space with $V = \{v_n \mid n \in \omega\} \cup \{w\}$ so that $d(v_n, v_m) = a_n$ for $n < m$ and $d(v_n, w) = c_{\bar{n}}$. Then $V \upharpoonright (\{v_n \mid n \in \omega\})$

is a metric space and every triple (a_n, \bar{n}, \bar{m}) is metric because $|c_{\bar{n}} - c_{\bar{m}}| \leq |c - c_{\bar{n}}| + |c - c_{\bar{m}}| \leq 2|c_{\bar{n}} - c| < a_n$ and $a_n < \bar{n}$. It follows that V is a metric space.

There exists an isometric embedding f of V into \mathbf{U}_R . The sequence $(f(v_n))$ is Cauchy with limit, say $v \in U$. Then:

$$d(f(w), v) = \lim_{n \rightarrow \infty} d(f(w), f(v_n)) = \lim_{n \rightarrow \infty} c_{f(n)} = \lim_{n \rightarrow \infty} c_n = c.$$

□

Theorem 4.4. *Let $0 \in R \subseteq \mathfrak{R}_{\geq 0}$ with 0 as a limit. Then there exists an Urysohn metric space \mathbf{U}_R if and only if R is a closed subset of \mathfrak{R} which satisfies the 4-values condition.*

Let $0 \in R \subseteq \mathfrak{R}_{\geq 0}$ which does not have 0 as a limit. Then there exists an Urysohn metric space \mathbf{U}_R if and only if R is a countable subset of \mathfrak{R} which satisfies the 4-values condition.

Proof. Let $0 \in R \subseteq \mathfrak{R}_{\geq 0}$ with 0 as a limit.

If there exists an Urysohn metric space \mathbf{U}_R it follows from Theorem 3.2 that \mathbf{U}_R satisfies the 4-values condition because Urysohn metric spaces are universal. It follows from Lemma 4.6 that R is a closed subset of \mathfrak{R} .

Let R be closed and satisfy the 4-values condition. It follows from Lemma 3.6 that R has a countable dense subset T which satisfies the 4-values condition. Let the countable universal metric space \mathbf{U}_T be given by Theorem 3.3. The completion M of \mathbf{U}_T is homogeneous and separable and complete according to Theorem 4.1. It follows from Corollary 4.1 that $\text{dist}(M)$ is the closure of T which implies because T is dense in R and R is closed that $\text{dist}(M) = R$. According to Lemma 2.3 it remains to prove that M is universal, that is that every finite metric space $A \in \mathcal{F}_R$ has an isometric embedding into M . This then indeed follows from Lemma 4.5 and Theorem 4.3

Let $0 \in R \subseteq \mathfrak{R}_{\geq 0}$ which does not have 0 as a limit.

If R is uncountable then there does not exist an Urysohn metric space \mathbf{U}_R because Urysohn metric spaces are separable. If R is countable then there exists universal metric space \mathbf{U}_R according to Theorem 3.3. The space \mathbf{U}_R is an Urysohn metric space because the completion of \mathbf{U}_R is equal to \mathbf{U}_R . □

5 Examples

Example 5.1. It is not difficult to check that the set of reals in the intervals $[0, \infty)$ and $[0, 1]$ satisfy the 4-values condition. Hence there exist according

to Theorem 4.4 an Urysohn space $\mathbf{U}_{[0,\infty)}$, the classical *Urysohn space* and $\mathbf{U}_{[0,1]}$, the *Urysohn sphere*.

Example 5.2. The set of distances of the completion of a metric space need not be closed:

Let R be the set of rationals in the interval $[0, 1]$ and $V = \{a_i \mid i \in R\} \cup \{b_i \mid i \in R\}$. Let $\mathbf{V} = (V; d)$ be the metric space with $d(a_i, b_i) = i$ and $d(a_i, a_j) = d(a_i, b_j) = 1$ for all $i, j \in R$ with $i \neq j$. The completion of \mathbf{V} is \mathbf{V} .

Example 5.3. The completion of a universal metric space \mathbf{U}_R need not be universal but is homogeneous according to Theorem 4.1. The age of the completion consists of all finite metric spaces which can be “approximated” by metric spaces with distances in R , Theorem 4.3 and Lemma 4.5.

Let R be the set of rationals in the interval $[0, 1)$ together with the number 2. Then R satisfies the 4-values condition because: Let $x \rightsquigarrow (a, b, c, d)$ with $x, a, b, c, d \in R$. According to Lemma 3.7 it suffices to assume $a > \max\{b + c, b + d, c + d\}$. If $a \in [0, 1)$ then $b + c \rightsquigarrow (a, d, c, b)$. If $a = 2$ then $b = 2$ or $x = 2$. If $b = 2$ then $2 \rightsquigarrow (a, d, c, b)$. If $x = 2$ then $c = 2$ or $d = 2$. If $c = 2$ then $2 \rightsquigarrow (a, d, c, b)$ and if $d = 2$ then $b + c \rightsquigarrow (a, d, c, b)$.

Hence, according to Theorem 3.3, there exists a universal countable metric space \mathbf{U}_R . Let \mathbf{M} be the completion of \mathbf{U}_R . According to Corollary 4.1 $\text{dist}(\mathbf{M}) = [0, 1] \cup \{2\} := T$, which does not satisfy the 4-values condition because $1 \rightsquigarrow (2, 1, \frac{1}{2}, \frac{1}{2})$ but there is no number $y \in T$ with $y \rightsquigarrow (2, \frac{1}{2}, \frac{1}{2}, 1)$. The class \mathcal{F}_T contains a triangle with distance set $\{2, 1, 1\}$. The class \mathcal{R} does not contain any triangle with distance set of the form $\{a, b, c\}$ with $2 - \frac{1}{4} < 2 < 2 + \frac{1}{4}$ and $1 - \frac{1}{4} < b, c < 1 + \frac{1}{4}$. Hence \mathbf{M} does not contain a triangle with distance set $\{2, 1, 1\}$ and is therefore not universal. Theorem 4.3 characterizes the finite metric spaces in the age of \mathbf{M} .

Indeed, it is not difficult to check that \mathbf{M} consists of countably many copies of the Urysohn space $\mathbf{U}_{[0,1]}$ for which any two points in different copies have distance 2.

Example 5.4. Some additional examples of subsets R of the reals satisfying the 4-values condition. For examples of finite sets R see [20].

Let R be the union of the closed intervals

$$R = \{0\} \cup \bigcup_{n \in \omega} \left[\frac{1}{2^{2n+1}}, \frac{2^8 - 1}{2^{2n+4}} \right].$$

Then R is closed has 0 as a limit and it is not difficult to check that it satisfies the 4-values condition. Hence there exists an Urysohn metric space \mathbf{U}_R .

Every sum closed set $R \subseteq \mathfrak{R}$ containing 0 satisfies the 4-values condition. The intersection of a sum closed set $R \subseteq \mathfrak{R}$ containing 0 with an interval of the form $[0, a]$ or $[0, a)$ satisfies the 4-values condition.

The set ω and every initial interval of ω satisfies the 4-values condition.

The sets $R = [0, 1] \cup [3, 4] \cup [9, \infty)$ and $R = [0, 1] \cup [3, 4] \cup (8, \infty)$ satisfy the 4-values condition. The set $R = [0, 1] \cup [3, 4] \cup [8, \infty)$ does not satisfy the 4-values condition.

It is a bit more challenging to prove that the set of Cantor numbers C with finitely many digits 2 in the ternary expansion satisfy the 4-values condition, manuscript. Hence there exists a unique countable universal metric space \mathbf{U}_C according to Theorem 3.3. The set of all Cantor numbers does not satisfy the 4-values condition but it follows from Theorem 4.2 that there exists separable complete homogeneous metric space $\mathbf{U}_{\bar{C}}$ whose set of distances is the set of Cantor numbers. It follows from Theorem 2.2 that $\mathbf{U}_{\bar{C}}$ is the unique separable complete metric space whose age is the set of finite metric spaces which can be approximated by finite metric spaces with distances in C as described in Theorem 4.3.

References

- [1] P.S. Urysohn, Sur un espace mtrique universel, *C. R. Acad. Sci. Paris* **180** (1925), 803806.
- [2] M. Katětov, On universal metric spaces, in: *Gen. Topology and its Relations to Modern Analysis and Algebra VI*, Proc. Sixth Prague Topol. Symp. 1986, Z. Frolik, ed., Heldermann Verlag (1988), 323330.
- [3] V.V. Uspenskij, On the group of isometries of the Urysohn universal metric space, *Comment. Math. Univ. Carolinae* **31**(1) (1990), 181182.
- [4] V.V. Uspenskij, On subgroups of minimal topological groups, *Topology Appl.* **155** (2008), 15801606.

- [5] B. Mbombo, V. Pestov, Subgroups of isometries of Urysohn-Katětov metric spaces of uncountable density, *arXiv:1012.1056v6 [math.GN]*, 20 May 2011
- [6] J. Melleray, Topology of the isometry group of the Urysohn space, *Fund. Math.* **207** (2010), 273287.
- [7] R. Fraïssé, Theory of relations, Studies in Logic and the Foundations of Mathematics, 145, North-Holland Publishing Co., Amsterdam, 2000.
- [8] A. S. Kechris, V. Pestov and S. Todorcevic, Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups, *Geom. Funct. Anal.* **15** (2005) 106-189.
- [9] V. D. Milman, A new proof of A. Dvoretzky's theorem on cross-sections of convex bodies, *Funkcional. Anal. i Priložen.*, 5 (4), 28-37, 1971 (in Russian).
- [10] J. Clemens, The set of distances in a Polish metric space, accepted to *Fundamenta Mathematicae*.
- [11] E. Odell, T. Schlumprecht, The distortion problem, *Acta Mathematica* **173** (1994) 259-281.
- [12] L. Nguyen Van Thé, N. Sauer, The Urysohn Sphere is oscillation stable, *Geometric and Functional Analysis* **19** Issue 2 (2009) 536-557.
- [13] L. Nguyen Van Thé, Structural Ramsey theory of metric spaces and topological dynamics of isometry groups, *Memoirs of the Amer. Math. Soc.*, 968 (206), 155 pages, 2010.
- [14] V. Pestov, Dynamics of infinite-dimensional groups. The Ramsey-Dvoretzky-Milman phenomenon. Revised edition of Dynamics of infinite-dimensional groups and Ramsey-type phenomena [Inst. Mat. Pura. Apl. (IMPA), Rio de Janeiro, 2005]. University Lecture Series, 40. American Mathematical Society, Providence, RI, 2006.

- [15] N. Sauer, Vertex partitions of metric spaces with finite distance sets, *Discrete Mathematics*, accepted for publication in print.
- [16] J. Lopez-Abad and L. Nguyen Van Thé, The oscillation stability problem for the Urysohn sphere: A combinatorial approach, *Topology Appl.* **155** Issue 14 (2008) 1516-1530.
- [17] C. Delhommé, C. Laflamme, M. Pouzet, N. Sauer, Divisibility of countable metric spaces, *European Journal of Combinatorics*, Volume **28** Issue 6 (2007) 1746-1769.
- [18] B. Jónsson, Homogeneous Universal Relational Systems, *Math. Scand.*, **8** (1960), 137-142.
- [19] S. Barbina, D. Zambella, A viewpoint on amalgamation classes, *Commentationes Mathematicae Universitatis Carolinae*, **51** (2010), issue 4, pp. 681-691 (Also: arXiv:1009.1789v1 [math.LO] 9 Sep 2010)
- [20] L. Nguyen Van Thé, Théorie de Ramsey structurale des espaces métriques et dynamique topologique des groupes d'isométries, Ph.D. Thesis, Université Paris 7, 2006 (available in English).