ASSEMBLY MAPS WITH COEFFICIENTS IN TOPOLOGICAL ALGEBRAS AND THE INTEGRAL K-THEORETIC NOVIKOV CONJECTURE

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ABSTRACT. We prove that any countable discrete and torsion free subgroup of a general linear group over an arbitrary field or a similar subgroup of an almost connected Lie group satisfies the integral algebraic K-theoretic (split) Novikov conjecture over $\mathbb{K}$ and $\mathcal{S}$, where $\mathbb{K}$ denotes the C^*-algebra of compact operators and $\mathcal{S}$ denotes the algebra of Schatten class operators. We prove that such a group also satisfies the algebraic K-theoretic (split) Novikov conjecture over $\overline{\mathbb{Q}}$ and $\mathbb{C}$ with finite coefficients. For all Gromov hyperbolic groups G, we show that the canonical algebra homomorphism $\mathbb{K}[G] \to C^*_r(G)\hat{\otimes}\mathbb{K}$ induces an isomorphism between their algebraic K-theory groups. We end with a discussion of a recent conjecture of Yu about the algebraic K-theory of the group algebra $\mathcal{S}[G]$.

Introduction

For any group discrete G and a unital ring R the algebraic K-theoretic Novikov conjecture for G over R asserts that a canonically defined Loday assembly map

$$(\mu^{\mathrm{L}}_R)_* : \mathrm{H}_*(BG; \mathbf{K}_R) \to \mathrm{K}_*(R[G])$$

is rationally injective. Here $\mathbf{K}_R$ denotes the nonconnective algebraic K-theory spectrum of R. The stronger integral K-theoretic Novikov conjecture asserts that the same map is injective. Using standard excision arguments and the fact that H-unital $\mathbb{Q}$-algebras in the sense of [60] satisfy excision in algebraic K-theory [58], the Loday assembly map can be extended to H-unital coefficient $\mathbb{Q}$-algebras R. Conjectures of this nature in K-theory and L-theory can be traced back to Hsiang [27]. In this article the term K-theory without any adjective will always refer to algebraic K-theory and will be denoted by K_*.

For any countable discrete and torsion free group G the Baum–Connnes conjecture asserts that a canonically defined Baum–Connes assembly map

$$\mu^{\mathrm{BC}}_* : \mathrm{K}^g_*(BG) \to \mathrm{K}^{\mathrm{top}}_*(C^*_r(G))$$

is an isomorphism [5]. Here K^g denotes the Baum–Douglas picture of geometric K-homology [8, 7] and $\mathrm{K}^{\mathrm{top}}_*$ denotes the topological K-theory of C^*-algebras. The Baum–Connes assembly map can be constructed for any second countable locally compact and Hausdorff group. The conjecture can even be generalized to incorporate coefficients in a separable G-C^*-algebra by using universal proper G-spaces and Kasparov's bivariant K-theory [6]. The Baum–Connes conjecture has implications in many areas of mathematics and the status of this conjecture is also well documented in the literature (see, e.g., [45]). Recently Lafforgue announced the proof of the Baum–Connes conjecture with coefficients for all Gromov hyperbolic groups [37].

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The original Novikov conjecture about the homotopy invariance of higher signatures is known to follow from the rational injectivity of the Baum–Connes assembly map. Presumably, motivated by this observation Kasparov formulated the Strong Novikov Conjecture with trivial coefficients, which asserts that μ_*^{BC} is a rationally injective (see, e.g., page 192 of [36]).

Since we are dealing with two different Novikov conjectures, in order to avoid confusion, we refer to the Novikov conjecture in algebraic K*-theory as the Hsiang–Novikov conjecture. Similarly the Strong Novikov Conjecture in* KK*-theory is referred to as the Kasparov–Novikov conjecture. Predictably the integral* K*-theoretic Hsiang–Novikov (resp. integral Kasparov–Novikov) conjecture asserts that* $(\mu_R^{\mathrm{L}})_*$ *(resp.* μ_*^{BC}*) is injective. Finally, the integral* K*-theoretic split Hsiang–Novikov (resp. the integral split Kasparov–Novikov) conjecture asserts that the assembly map* $(\mu_R^{\mathrm{L}})_*$ *(resp.* μ_*^{BC}*) is split injective.*

In this article we prove the following two results (*reduction principles*):

Theorem 0.1. *(see Theorem 4.3 and Theorem 4.2) If a countable discrete and torsion free group satisfies the integral Kasparov–Novikov (resp. the split Kasparov–Novikov) conjecture with trivial coefficients, then it satisfies the* K*-theoretic Hsiang–Novikov (resp. split Hsiang–Novikov) conjecture over* $\overline{\mathbb{Q}}$ *and* $\mathbb{C}$ *with finite coefficients.*

Theorem 0.2. *(see Theorem 6.4 and Theorem 8.4) If a countable discrete and torsion free group satisfies the integral Kasparov–Novikov (resp. the split Kasparov–Novikov) conjecture with trivial coefficients, then it satisfies the integral* K*-theoretic Hsiang–Novikov (resp. split Hsiang–Novikov) conjecture over* $\mathbb{K}$ *and* $\mathcal{S}$*, where* $\mathbb{K}$ *denotes the* C^**-algebra of compact operators and* $\mathcal{S}$ *denotes the algebra of Schatten class operators.*

Furthermore, if μ_*^{BC} *is only rationally injective then so is* $(\mu_{\mathbb{K}}^{\mathrm{L}})_*$*. (The rational injectivity of* $(\mu_{\mathcal{S}}^{\mathrm{L}})_*$ *is known for all groups without any further hypothesis* [61]*).*

Theorem 0.1 goes in a direction that is different from the rational injectivity question. Admittedly, this result is not optimal and work is in progress to strengthen it. Building upon a result of [16], Bökstedt–Hsiang–Madsen proved the rational injectivity of the Loday assembly map for $R = \mathbb{Z}$ under the assumption that $\mathrm{H}_*(G, \mathbb{Z})$ is finitely generated in [11]. Carlsson–Goldfarb proved the integral K-theoretic split Hsiang–Novikov conjecture for all geometrically finite groups with finite asymptotic dimension over arbitrary unital coefficient rings in [13]. The integral K-theoretic Hsiang–Novikov conjecture can be refined to a statement that predicts a certain assembly map to be an isomorphism. This is the Farrell–Jones isomorphism conjecture in algebraic K-theory. For simplicity, let us suppose that G is a torsion free group and R is a regular Noetherian $\mathbb{Q}$-algebra. Then the Farrell–Jones conjecture simply asserts that the above-mentioned Loday assembly map is actually an isomorphism. Bartels–Lück–Reich proved the Farrell–Jones conjecture for all Gromov hyperbolic groups over arbitrary unital coefficient rings in [3]. For an update on the status of these conjectures we refer the readers to [29, 41]. Motivated by the Connes–Moscovici work on the Novikov conjecture in topology [18], recently Yu proved that for any discrete group G the assembly map in the Farrell–Jones isomorphism conjecture in algebraic K-theory is rationally injective over $\mathcal{S}$, where $\mathcal{S}$ denotes the algebra of Schatten class operators [61]. However, our techniques differ from that of ibid. in a significant manner (explained in the following paragraph). In Theorem 7.2 we prove that, for all Gromov hyperbolic groups [23], the canonical algebra homomorphism $\mathbb{K}[G] \to C_r^*(G)\hat{\otimes}\mathbb{K}$ induces an isomorphism between their algebraic K-theory groups. Moreover, the Whitehead groups of G over $\mathbb{K}$, which are the homotopy groups of the homotopy cofibre of the Loday assembly map $\mu^{\mathrm{L}}_{\mathbb{K}}$, vanish for any Gromov hyperbolic group G. The preprint of Yu [61] appeared while the article was under preparation and it is clear that this result follows easily from the circle of ideas in ibid. We end the article with a brief discussion of a conjecture of Yu (Conjecture 5.1 of ibid.) and propose a variant of the conjecture (see Conjecture 8.6).

The works of Connes–Moscovici [18] and Yu [61] make use of the periodic cyclic (co)homology valued Chern–Connes character. Roughly, one relates the K-theoretic assembly map to a periodic cyclic homological assembly map, which is designed to retain information only up to torsion. Therefore, such techniques are very useful for proving rational injectivity statements. However, in order to prove integral statements it is important to work directly with the K-theoretic assembly maps. The main strategy in our proof of the *reduction principle* is to interconnect the various K-theoretic assembly maps involved. In order to do so we make use of the unified perspective of Davis–Lück on the various isomorphism conjectures [20]. Davis–Lück construct an assembly map in any G-homology theory, which in turn is constructed from a spectrum that is a module over the orbit category of G in a suitable sense. By choosing different spectra, e.g, nonconnective algebraic K-theory or topological K-theory, one obtains different G-homology theories and assembly maps therein. Under favourable circumstances these assembly maps can be identified with the Loday assembly map and with the Baum–Connes assembly map with trivial coefficients [25] respectively. In order to prove Theorem 0.2 we extend the identification of the Baum–Connes assembly map with the Davis–Lück assembly map to the case where the coefficient algebra is $\mathbb{K}$, equipped with the trivial G-action (see Proposition 6.2). The domain of the Davis–Lück isomorphism conjectures is always a G-homology theory. However, if G is torsion free and the coefficient algebra satisfies some reasonable hypotheses, then one can avoid equivariant homology theories and work with ordinary homology theories. Therefore, we restrict our attention to torsion free groups. In this case, the domain of the Davis–Lück assembly map looks like $\mathrm{H}_*(BG;\mathrm{E})$, where E is the spectrum defining the homology theory. These groups are computable using the Atiyah–Hirzebruch spectral sequence in generalized homology theories. One aim of the isomorphism conjectures is to predict the values of the codomains of the assembly maps, which are typically hard to compute, in terms these generalized homology groups of BG. In Section 5 we construct Künneth type spectral sequences using the machinery of [21] to compute these homology groups, although an extremely trivial case of that is used in this article. We hope that these spectral sequences will be of independent interest.

Upshot: Let us mention that the integral (split) Kasparov–Novikov conjecture is known to be true in numerous examples. For instance, let G be a countable discrete group with a proper left-invariant metric. Thanks to Skandalis–Tu–Yu we know that if G admits a uniform embedding into a Hilbert space, then G satisfies the split Kasparov–Novikov conjecture (even with coefficients in any separable G-C^*-algebra) [62, 55]. Using the results of ibid., Guentner–Higson–Weinberger showed that if G is a countable discrete subgroup of $\mathrm{GL}(n, F)$ for any field F or of any almost connected Lie group, then G satisfies the split Kasparov–Novikov conjecture (with coefficients in any separable G-C^*-algebra) [24]. Using the above-mentioned *reduction principle*, we arrive at the main result of this article.

Theorem. *Any countable discrete and torsion free subgroup of a general linear group over an arbitrary field or a similar subgroup of an almost connected Lie group satisfies the integral* K*-theoretic split Hsiang–Novikov conjecture over* $\mathbb{K}$ *and* $\mathcal{S}$*, i.e., the Loday assembly maps* $(\mu^{\mathrm{L}}_{\mathbb{K}})_* : \mathrm{H}_*(BG;\mathbf{K}_{\mathbb{K}}) \to \mathrm{K}_*(\mathbb{K}[G])$ *and* $(\mu^{\mathrm{L}}_{\mathcal{S}})_* : \mathrm{H}_*(BG;\mathbf{K}_{\mathcal{S}}) \to \mathrm{K}_*(\mathcal{S}[G])$ *are split injective. They also satisfy the* K*-theoretic split Hsiang–Novikov conjecture with finite coefficients over* $\overline{\mathbb{Q}}$ *and* $\mathbb{C}$*, i.e., the Loday assembly maps* $(\mu^{\mathrm{L}}_{\mathbb{K}})_*$ *and* $(\mu^{\mathrm{L}}_{\mathcal{S}})_*$ *are split injective with finite coefficients.*
For a countable discrete and torsion free group with a proper left-invariant metric, the above assertions continue to hold if the group admits a uniform embedding into a Hilbert Space, which is a rather mild condition.

To the best of the author's knowledge these results are new at this level of generality (although with restricted coefficients).

Notations and conventions: A discrete group is tacitly assumed to be countable and all C^*-algebras are assumed to be separable. A space will always mean a compactly generated and Hausdorff space and $\hat{\otimes}$ will denote the minimal tensor product between C^*-algebras. Throughout this article $\mathbf{K}$ will denote the nonconnective Gersten–Wagoner algebraic K-theory spectrum and $\mathbf{K}^{\mathrm{top}}$ will denote the complex topological K-theory spectrum. Occasionally we are going to work with specific models for these spectra and we shall comment on them when the need arises. Unless otherwise stated, while dealing with a space like BG (the classifying space of a discrete group G) we implicitly work with a CW model, e.g., the geometric realization of the simplicial model of BG.

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1. Baum–Connes assembly map for discrete and torsion free groups

Let G be any discrete and torsion free group. Let $\mathbf{K}^{\mathrm{top}}_{\mathbb{C}}$ denote the (complex) topological K-theory spectrum. It defines a (reduced) generalized homology theory on the category of pointed compactly generated and Hausdorff spaces as follows [59]: For any unpointed space X, set

$$\mathrm{K}^{\mathrm{top}}_*(X) = \pi_*(X_+ \wedge \mathbf{K}^{\mathrm{top}}_{\mathbb{C}}),$$

where X_+ denotes the disjoint union of X and a basepoint. This is called the topological K-homology of X. Let BG denote the classifying space of G and let $C \subset BG$ be any compact subset. There is a canonical map

$$\mu(C) : \mathrm{K}^{\mathrm{top}}_*(C) \to \mathrm{K}^{\mathrm{top}}_*(C^*_r(G)),$$

where $\mathrm{K}^{\mathrm{top}}$ on the right hand side denotes the topological K-theory of (complex) Banach algebras and $C^*_r(G)$ denotes the reduced group C^*-algebra of G. It is defined in terms of a canonical 'Mischenko line bundle' and Kasparov product in KK-theory. For the details we refer the readers to pages 97 and 98 of [17], where the connection to the L-theory assembly map is also explained. Let us remind the readers that the L-theoretic assembly map is the relevant one for the original Novikov conjecture about the homotopy invariance of higher signatures; in fact, it is equivalent to the rational injectivity of the L-theoretic assembly map.

Being a compactly generated space, BG is a filtered colimit of its compact subsets. Here the set of compact subsets of BG is canonically filtered by inclusion. Since generalized homology theory commutes with filtered colimits, one concludes

$$\mathrm{K}_*^{\mathrm{top}}(BG) \cong \varinjlim_C \mathrm{K}_*^{\mathrm{top}}(C),$$

where $C \subset BG$ runs through compact subsets. The assembly map construction is compatible with the inductive system of $\mathrm{K}^{\mathrm{top}}$-homology groups in the following sense: Whenever $C \subset C'$ there is a commutative diagram

$$\begin{array}{ccc} \mathrm{K}_*^{\mathrm{top}}(C') & \xrightarrow{\mu(C')} & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G)) \\ \uparrow & \nearrow_{\mu(C)} & \\ \mathrm{K}_*^{\mathrm{top}}(C) & & \end{array}$$

As a result one obtains the assembly map

$$\mu_* : \mathrm{K}_*^{\mathrm{top}}(BG) \cong \varinjlim_{C \subset BG} \mathrm{K}_*^{\mathrm{top}}(C) \longrightarrow \mathrm{K}_*^{\mathrm{top}}(C_r^*(G)), \tag{1}$$

and the Baum–Connes conjecture (with trivial or complex coefficients) [5, 6] asserts that this map is an isomorphism.

There is a geometric picture of K-homology, denoted by K^g, due to Baum–Douglas on the category of a (pairs of) finite CW complexes [8, 7]. Each geometric K-cycle over X is a triple (M, E, f), where M is a compact spin^c n-manifold, E is a complex vector bundle over M and $f : M \to X_+$ is a continuous map sending ∂M to the basepoint in X_+. There are some canonical equivalence relations imposed on such K-cycles and the equivalences classes, denoted by $[M, E, f]$, comprise $\mathrm{K}^g(X)$. There is a natural map $\alpha_X : \mathrm{K}_*^{\mathrm{top}}(X) = \pi_*(X_+ \wedge \mathbf{K}_{\mathbb{C}}^{\mathrm{top}})$, which turns out to be an isomorphism between generalized homology theories (see, e.g., [28, 4]). The map α_X simply sends a K-class $[M, E, f]$ to $f_*([E] \cap [M])$, where $[M]$ denotes the canonical K-homological fundamental class determined by the spin^c structure on M. In [9] it is shown that, for any finite CW complex X, the geometric K-homology $\mathrm{K}_*^g(X)$ is naturally isomorphic to Kasparov's analytic K-homology $\mathrm{KK}_*(C(X), \mathbb{C})$, whence K^g is a generalized homology theory on the category of finite CW complexes. In the sequel we shall use geometric and analytic K-homology interchangeably without further explanation. The definition of geometric K-homology is extended to all compactly generated and Hausdorff spaces by setting $\mathrm{K}_*^g(X) = \varinjlim_C \mathrm{K}_*^g(C)$, where $C \subset X$ runs through all compact subsets. Since $\mathrm{K}_*^g(X) \cong \mathrm{K}_*^{\mathrm{top}}(X)$ for every finite CW complex, it follows that

$$\mathrm{K}_*^g(BG) \cong \mathrm{K}_*^{\mathrm{top}}(BG),$$

where one may choose a CW model for BG and use the fact that every compact subset of a CW complex is contained in a finite CW subcomplex.

The original definition of the Baum–Connes assembly map μ_*^{BC} used $\mathrm{K}_*^g(BG)$ as the domain for discrete and torsion free groups. Given any K-cycle $[M, E, f]$ one gets an induced map $f_* : \pi_1(M) \to G$; now form the Dirac operator on M with coefficients in E and map it to the G-index in $\mathrm{K}_*^{\mathrm{top}}(C_r^*(G))$ along f_*. For a more general parametrized version of this construction see Theorem 4 of [5]. The two assembly maps μ_*^{BC} and μ_* are equivalent in the sense that there is a commutative diagram:

$$
\begin{array}{ccc}
\mathrm{K}_*^{g}(BG) & \xrightarrow{\mu_*^{\mathrm{BC}}} & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G)) \\
\downarrow{\cong} & \nearrow{\mu_*} & \\
\mathrm{K}_*^{\mathrm{top}}(BG) & &
\end{array}
$$

Thanks to this equivalence, in the sequel we are going to use μ_* and μ_*^{BC} interchangeably.

It follows from the natural equivalence of equivariant geometric K-homology and equivariant KK-theory [10] that this formulation of the assembly Baum–Connes assembly map is equivalent to the more familiar KK-theoretic one. More generally, for any G-C^*-algebra there is an assembly map $\mathrm{KK}_*(BG; A) \to \mathrm{K}_*^{\mathrm{top}}(A \rtimes_r G)$ and the Baum–Connes conjecture with coefficients in A asserts that this map is an isomorphism [6]. For our purposes, we do not need to know the details of this construction or that of Kasparov's KK-theory. Interested readers may refer to Kasparov's original papers [34, 33, 35]. We simply remark that both μ_*^{BC} and μ_* are equivalent to the KK-theoretic assembly map, when $A = \mathbb{C}$, i.e.,

$$\mathrm{KK}_*(BG; \mathbb{C}) \to \mathrm{K}_*^{\mathrm{top}}(\mathbb{C} \rtimes_r G) \cong \mathrm{K}_*^{\mathrm{top}}(C_r^*(G)).$$

Recall that the integral (split) Kasparov–Novikov conjecture asserts that this map is (split) injective.

2. Algebraic assembly maps

Let G be any discrete group (not necessarily torsion free) and let R be any unital complex algebra. There is a canonical inclusion of groups $G \hookrightarrow \mathrm{GL}_1(\mathbb{Z}[G])$ as units. Composing with the map $\mathrm{GL}_1(\mathbb{Z}[G]) \to \mathrm{GL}(\mathbb{Z}[G]) = \varinjlim_n \mathrm{GL}_n(\mathbb{Z}[G])$ and applying the classifying space functor $B(-)$, we obtain $j : BG \to B\mathrm{GL}(\mathbb{Z}[G])$. Now the functorial plus construction gives rise to the following commutative diagram of spaces:

$$
\begin{array}{ccc}
BG & \xrightarrow{j} & B\mathrm{GL}(\mathbb{Z}[G]) \\
\downarrow & & \downarrow \\
BG^+ & \xrightarrow{j^+} & B\mathrm{GL}(\mathbb{Z}[G])^+,
\end{array}
$$

whence we get a continuous map of spaces

$$BG_+ \wedge B\mathrm{GL}(R)^+ \stackrel{j^+\wedge \mathrm{id}}{\longrightarrow} B\mathrm{GL}(\mathbb{Z}[G])^+ \wedge B\mathrm{GL}(R)^+. \tag{2}$$

Composing it with the Loday product map

$$B\mathrm{GL}(\mathbb{Z}[G])^+ \wedge B\mathrm{GL}(R])^+ \to B\mathrm{GL}(\mathbb{Z}[G] \otimes R)^+ \cong B\mathrm{GL}(R[G])^+,$$

we obtain $BG_+ \wedge B\mathrm{GL}(R)^+ \to B\mathrm{GL}(R[G])^+$. This map is actually compatible with the infinite loop space structure on $B\mathrm{GL}(-)^+$ and hence can be upgraded to a map of spectra, giving rise to Loday's *assembly map* (with coefficients in R)

$$\mu_R^{\mathrm{L}} : BG_+ \wedge \mathbf{K}_R \to \mathbf{K}_{R[G]}. \tag{3}$$

Here $\mathbf{K}$ denotes the Gersten–Wagoner nonconnective algebraic K-theory spectrum (see, e.g., [48]). For the details of the construction of the assembly map we refer the readers to Chapter 4 of [39]. Following Loday one calls the homotopy cofibre of μ^{L}_R as the *Whitehead spectrum of G over R* [38].

Remark 2.1. *Carlsson–Pedersen used a different assembly map to prove the integral* K*-theoretic Hsiang–Novikov conjecture under some assumptions using techniques from controlled topology* [14]*. It is known that their assembly map is naturally equivalent to the Loday assembly map* [25, 56]*. The Carlsson–Pedersen assembly map has built-in naturality.*

Let A be a separable and unital C^*-algebra, on which G acts trivially. In this case the reduced crossed product $A\rtimes_r G$ simply becomes $A\hat{\otimes}C^*_r(G)$. In the sequel we denote $A\hat{\otimes}C^*_r(G)$ by $C^*_r(G,A)$. Recall that $C^*_r(G,A)$ is defined as a suitable completion of $C_c(G,A)=A[G]$, so that there is a canonical complex algebra homomorphism $\iota_A : A[G]\to C^*_r(G,A)$. Let $\iota_A:\mathbf{K}_{A[G]}\to\mathbf{K}_{C^*_r(G,A)}$ denote the induced map of K-theory spectra.

Remark 2.2. *The above discussion can be extended to nonunital C^*-algebras by defining* $\mathrm{GL}(A)=\varinjlim_n \ker[\mathrm{GL}_n(\tilde{A})\to\mathrm{GL}_n(\mathbb{C})]$*, where $\tilde{A}$ denotes the (complex) unitization of A (note that any C^*-algebra satisfies excision in algebraic* K*-theory). Indeed, by the naturality of the assembly map there is a commutative diagram in the homotopy category of spectra:*

$$\begin{array}{ccc} BG_+\wedge \mathbf{K}_{\tilde A} & \xrightarrow{\mu^{\mathrm{L}}_{\tilde A}} & \mathbf{K}_{\tilde A[G]} \\ \downarrow & & \downarrow \\ BG_+\wedge\mathbf{K}_{\mathbb{C}} & \xrightarrow{\mu^{\mathrm{L}}_{\mathbb{C}}} & \mathbf{K}_{\mathbb{C}[G]}. \end{array}$$

Using excision, we conclude that the induced map between the homotopy fibres (well-defined up to homotopy) $\mu^{\mathrm{L}}_A : BG_+\wedge\mathbf{K}_A\to\mathbf{K}_{A[G]}$ is the desired assembly map. It follows that if both $\mu^{\mathrm{L}}_{\tilde A}$ and $\mu^{\mathrm{L}}_{\mathbb{C}}$ are weak homotopy equivalences, then so is μ^{L}_A. The same argument allows us to construct the Loday assembly map with coefficients in any H*-unital $\mathbb{Q}$-algebra, since such algebras also satisfy excision in algebraic* K*-theory (see* [58]*).*

Definition 2.3. *We define the* algebraic reduced assembly map $\mu^{\mathrm{alg}}_A : BG_+\wedge\mathbf{K}_A\to\mathbf{K}_{C^*_r(G,A)}$ *by the following commutative diagram of spectra:*

$$\begin{array}{ccc} & & \mathbf{K}_{A[G]} \\ & \overset{\mu^{\mathrm{L}}_A}{\nearrow} & \downarrow{\scriptstyle \iota_A} \\ BG_+\wedge \mathbf{K}_A & \xrightarrow[\mu^{\mathrm{alg}}_A]{} & \mathbf{K}_{C^*_r(G,A)}. \end{array}$$

Remark 2.4. *Due to the octahedral property, there is an exact triangle in the triangulated homotopy category of spectra:*

$$\mathrm{Cone}(\mu^{\mathrm{L}}_A)\to\mathrm{Cone}(\mu^{\mathrm{alg}}_A)\to\mathrm{Cone}(\iota_A),$$

where Cone *denotes the mapping cone. We mentioned earlier that* $\mathrm{Cone}(\mu^{\mathrm{L}}_A)$ *is taken to be the definition of the Whitehead spectrum of G over A.*

Remark 2.5. *One can easily insert another level of factorization by realizing that there is an interesting intermediate (Banach) algebra $A[G] \to L^1(G,A) \to C^*_r(G,A)$. This will produce maps of spectra $\mathbf{K}_{A[G]} \to \mathbf{K}_{L^1(G,A)} \to \mathbf{K}_{C^*_r(G,A)}$ and one needs to construct in a similar vein the intermediate $\mathbf{K}_{L^1(G,A)}$-valued assembly map.*

3. The relation to the Baum–Connes assembly map

There is no reason to expect that the algebraic reduced assembly map μ^{alg}_A possesses good properties. Now we this map to the more interesting Baum–Connes assembly map. In order to do so, we need the general framework of Davis–Lück on isomorphism conjectures in K-theory, L-theory and $\mathrm{K}^{\mathrm{top}}$-theory.

The Davis–Lück isomorphism conjecture [20] provides a unified perspective on the various conjectures that are related to assembly maps, e.g., those of Baum–Connes, Bost, Farrell–Jones. More precisely, they construct an assembly map in a G-homology theory and show that the aforementioned assembly maps can be seen as specific cases by choosing the G-homology theory appropriately. We focus our attention to the Baum–Connes conjecture for discrete groups from this viewpoint. One obtains the Farrell–Jones conjecture in algebraic K-theory, roughly, by replacing the topological K-theory spectrum by the algebraic K-theory spectrum.

Let G be a discrete group and $\mathtt{Fin}$ denote the family of finite subgroups of G. Let $E_{\mathtt{Fin}}(G)$ denote the classifying space of G associated to $\mathtt{Fin}$, which is uniquely characterized up to G-homotopy by the properties:

(1) $E_{\mathtt{Fin}}(G)$ is a G-CW complex,
(2) $E_{\mathtt{Fin}}(G)^H$ is contractible if $H \in \mathtt{Fin}$ and empty otherwise.

For the existence of $E_{\mathtt{Fin}}(G)$ (for more general families) interested readers may refer to [40]. Let A be a G-C^*-algebra. One needs to construct a spectrum $\mathbf{K}^{\mathrm{top}}_{G,A}$ *as a module in a suitable sense over the orbit category of* G, which has the property that, for all subgroups $H \subset G$, one has $\mathrm{H}^G_*(G/H; \mathbf{K}^{\mathrm{top}}_{G,A}) \cong \mathrm{K}^{\mathrm{top}}_*(A \rtimes_r H)$; in particular, $\mathrm{H}^G_*(\mathrm{pt}; \mathbf{K}^{\mathrm{top}}_{G,A}) \cong \mathrm{K}^{\mathrm{top}}_*(A \rtimes_r G)$. Then one can define the Davis–Lück *assembly map* μ^{DL}_A *(at the level of spectra) in the* G*-homology theory determined by* $\mathbf{K}^{\mathrm{top}}_{G,A}$, which is induced by the canonical G-projection $E_{\mathtt{Fin}}(G)_+ \to \mathrm{pt}_+ = S^0$, i.e.,

$$(\mu^{\mathrm{DL}}_A)_* : \mathrm{H}^G_*(E_{\mathtt{Fin}}(G); \mathbf{K}^{\mathrm{top}}_{G,A}) \to \mathrm{H}^G_*(\mathrm{pt}; \mathbf{K}^{\mathrm{top}}_{G,A}) \cong \mathrm{K}^{\mathrm{top}}_*(A \rtimes_r G). \tag{4}$$

The *isomorphism conjecture in topological* K*-theory* asserts that the above assembly map $(\mu^{\mathrm{DL}}_A)_*$ is an isomorphism. For the details of the construction of the spectrum $\mathbf{K}^{\mathrm{top}}_{G,A}$ we refer the readers to Section 2 of [20] and [30]. Presumably the article [46] is also relevant in this context. A result of Hambleton–Pedersen says that for a discrete group (not necessarily torsion free) the Davis–Lück assembly map $\mu^{\mathrm{DL}}_{\mathbb{C}}$ agrees with the Baum–Connes assembly map $\mu^{\mathrm{BC}}_{\mathbb{C}}$ (see Corollary 8.4 of [25]). In other words, the Davis–Lück isomorphism conjecture is equivalent to the Baum–Connes conjecture with trivial coefficients.
In the same spirit, there is an algebraic K-theory spectrum $\mathbf{K}_{G,R}$ (R unital) giving rise to an assembly map

$$(\mu^{\mathrm{DL}}_R)_* : \mathrm{H}^G_*(E_{\mathtt{VC}}(G); \mathbf{K}_{G,R}) \to \mathrm{H}^G_*(\mathrm{pt}; \mathbf{K}_{G,R}) \cong \mathrm{K}^{\mathrm{top}}_*(R[G]), \tag{5}$$

where $\mathtt{VC}$ stands for the family of virtually cyclic subgroups of G. Once again the classifying space $E_{\mathtt{VC}}(G)$ is characterized uniquely up to G-homotopy by the two properties. The *Farrell–Jones isomorphism conjecture in algebraic* K*-theory* asserts that the above assembly map is an isomorphism.

For any ring R set $\mathrm{NK}_n(R)$ to be the cokernel of the split inclusion $\mathrm{K}_n(R) \to \mathrm{K}_n(R[t])$. It is well-known that if R is a regular Noethering ring then $\mathrm{NK}_n(R) \cong \{0\}$ for all $n \in \mathbb{Z}$. If R is regular Noetherian, then so is $R[t]$ and hence for all $m \in \mathbb{N}$ the split inclusion $\mathrm{K}_n(R) \to \mathrm{K}_n(R[t_1, \cdots, t_m])$ is actually an isomorphism. More generally, a ring R is called K*-regular* if the split inclusion $\mathrm{K}_n(R) \to \mathrm{K}_n(R[t_1, \cdots, t_m])$ is an isomorphism for all $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. In the sequel we shall only consider K-regular rings.

Example 3.1. *Since $\mathbb{C}$ is a regular Noetherian ring, it is* K*-regular. More interestingly, it is known that any stable C^*-algebra is* K*-regular (see Theorem 3.4 of* [50]*). In particular, the C^*-algebra of all compact operators on a separable Hilbert space $\mathbb{K}$ is* K*-regular.*

If G is torsion free then $\mathtt{Fin}$ consists of only the trivial subgroup and hence one observes that EG is a model for $E_{\mathtt{Fin}}(G)$. So the domain of μ^{DL}_A reduces as $\mathrm{H}^G_*(EG; \mathbf{K}^{\mathrm{top}}_{G,A}) \cong \mathrm{H}_*(BG; \mathbf{K}^{\mathrm{top}}_{G,A})$, which begins to look like the domain of the Baum–Connes assembly map μ^{BC}_A, i.e., the topological K-homology of BG with coefficients in A. It is known that with trivial coefficients, i.e., $A = \mathbb{C}$ the Davis–Lück assembly map is equivalent to the Baum Connes assembly map (see Corollary 8.4 of [25]). In fact, some simplifications also occur in the Farrell–Jones isomorphism conjecture in algebraic K-theory.

Remark 3.2. *If G is a discrete and torsion free group and R is a unital,* K*-regular $\mathbb{Q}$-algebra, then EG can be taken as a model for $E_{\mathtt{VC}}(G)$. In this case the Davis–Lück assembly map*

$$(\mu^{\mathrm{DL}}_R)_* : \mathrm{H}^G_*(E_{\mathtt{VC}}(G); \mathbf{K}_{G,R}) \to \mathrm{H}^G_*(\mathrm{pt}; \mathbf{K}_{G,R})$$

is naturally equivalent to the Loday assembly map μ^{L}_* that we saw in the previous section (see Corollary 67 (ii) of [42]*). In ibid. the above assertion is stated only for regular Noetherian $\mathbb{Q}$-algebras R; however, Remark 15 in ibid. clarifies that* K*-regularity in the above sense suffices.*

The Farrell–Jones isomorphism conjecture in algebraic K-theory was stated in [22] for $R = \mathbb{Z}$ and in it appeared in the general form in [2]. Thanks to the above Remark the Farrell–Jones isomorphism conjecture in algebraic K-theory can be extended to incorporate H-unital K-regular $\mathbb{Q}$-algebras as coefficients for discrete and torsion free groups. Indeed, one can identify the Davis–Lück assembly map with the Loday assembly map and then argue as in Remark 2.2.

Now let $\iota_{\mathbb{C}} : \mathbb{C}[G] \to C^*_r(G)$ denote the canonical complex algebra homomorphism inducing a map $\mathbf{K}_{\mathbb{C}[G]} \overset{\iota_{\mathbb{C}}}{\to} \mathbf{K}_{C^*_r(G)}$ between their nonconnective K-theory spectra. There is a natural comparison map between the algebraic K-theory and the topological K-theory spectra of a (complex) Banach algebra [31]. For any unital Banach algebra A, it is induced by the canonical continuous map $c(A) : \mathrm{GL}(A) \to \mathrm{GL}^{\mathrm{top}}(A)$. Here $\mathrm{GL}(A)$ is the algebraic inductive limit of the discrete groups $\mathrm{GL}_n(A)$ as before, whereas $\mathrm{GL}^{\mathrm{top}}(A)$ is that of $\mathrm{GL}^{\mathrm{top}}_n(A)$, each of which inherits its topology from the Banach space $\mathrm{M}_n(A)$. In fact, this map can be promoted to a natural map of spectra $c(A) : \mathbf{K}_A \to \mathbf{K}^{\mathrm{top}}_A$, where $\mathbf{K}^{\mathrm{top}}_A$ denotes the (complex) topological K-theory spectrum of A (see Theorem 2.1. of [50]).

Lemma 3.3. *There is a commutative diagram:*

$$
\begin{array}{ccc}
\mathrm{H}_*(BG;\mathbf{K}_{\mathbb{C}}) & \xrightarrow{\mu_*^{\mathrm{alg}}} & \mathrm{K}_*(C_r^*(G)) \\
\Big\downarrow{\scriptstyle c(\mathbb{C})_*} & & \Big\downarrow{\scriptstyle c(C_r^*(G))_*} \\
\mathrm{H}_*(BG;\mathbf{K}_{\mathbb{C}}^{\mathrm{top}}) & \xrightarrow{\mu_*^{\mathrm{BC}}} & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G)),
\end{array}
$$

where the left vertical arrow $c(\mathbb{C})_*$ is induced by the map of spectra $c(\mathbb{C}) : \mathbf{K}_{\mathbb{C}} \to \mathbf{K}_{\mathbb{C}}^{\mathrm{top}}$.

Proof. It follows after setting $A = \mathbb{C}$ (with trivial G-action) in the large commutative diagram in section 1.6 (page 47) of [1] that there is a commutative diagram

$$
\begin{array}{ccc}
\mathrm{H}_*^G(E_{\mathtt{VC}}(G);\mathbf{K}_{G,\mathbb{C}}) & \longrightarrow & \mathrm{K}_*(\mathbb{C}[G]) \\
\Big\downarrow & & \Big\downarrow \\
\mathrm{H}_*^G(E_{\mathtt{Fin}}(G);\mathbf{K}_{G,\mathbb{C}}^{\mathrm{top}}) & \longrightarrow & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G)),
\end{array}
$$

where the top (resp. bottom) horizontal arrow is the Davis–Lück assembly map in algebraic (resp. topological) K-theory and the left vertical map is induced by the change of theory map or the comparison map from algebraic to topological K-theory. Under the assumptions the top horizontal map can be identified with the Loday assembly map $\mu_*^{\mathrm{L}} : \mathrm{H}_*(BG;\mathbf{K}_{\mathbb{C}}) \to \mathrm{K}_*(\mathbb{C}[G])$ (see Remark 3.2), whereas the bottom horizontal map is known to be equivalent to the Baum–Connes assembly map (see Corollary 8.4 of [25]). Consequently, we get the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{H}_*(BG;\mathbf{K}_{\mathbb{C}}) & \xrightarrow{\mu_*^{\mathrm{L}}} & \mathrm{K}_*(\mathbb{C}[G]) \\
\Big\downarrow{\scriptstyle c(\mathbb{C})_*} & & \Big\downarrow \\
\mathrm{H}_*(BG;\mathbf{K}_{\mathbb{C}}^{\mathrm{top}}) \cong \mathrm{K}_*^{\mathrm{top}}(BG) & \xrightarrow{\mu_*^{\mathrm{BC}}} & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G)),
\end{array}
$$

where the left vertical map is induced by the comparison map. The right vertical map can be factorized as

$$\mathrm{K}_*(\mathbb{C}[G]) \xrightarrow{(\iota_{\mathbb{C}})_*} \mathrm{K}_*(C_r^*(G)) \xrightarrow{c(C_r^*(G))_*} \mathrm{K}_*^{\mathrm{top}}(C_r^*(G)),$$

where the first map in algebraic K-theory is induced by the canonical algebra homomorphism $\mathbb{C}[G] \overset{\iota_{\mathbb{C}}}{\to} C_r^*(G)$. Incorporating this factorization into the above diagram, we get

$$
\begin{array}{ccccc}
\mathrm{H}_*(BG;\mathbf{K}_{\mathbb{C}}) & \xrightarrow{\mu_*^{\mathrm{L}}} & \mathrm{K}_*(\mathbb{C}[G]) & \xrightarrow{(\iota_{\mathbb{C}})_*} & \mathrm{K}_*(C_r^*(G)) \\
\Big\downarrow{\scriptstyle c(\mathbb{C})_*} & & & & \Big\downarrow{\scriptstyle c(C_r^*(G))_*} \\
\mathrm{H}_*(BG;\mathbf{K}_{\mathbb{C}}^{\mathrm{top}}) & & \xrightarrow{\mu_*^{\mathrm{BC}}} & & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G)).
\end{array}
$$

Now observe that the composition of the top two horizontal arrows is μ_*^{alg}. $\square$

4. K-theoretic Hsiang–Novikov conjecture with finite coefficients

Let us first observe that

Lemma 4.1. *Any discrete and torsion free group satisfies the integral* K*-theoretic split Hsiang–Novikov conjecture in negative degrees, i.e.,* $\mathrm{H}_*(BG; \mathbf{K}_{\mathbb{C}}) \to \mathrm{K}_*(\mathbb{C}[G])$ *is split injective for all* $* < 0$*.*

Proof. Indeed, since the negative algebraic K-theory groups of $\mathbb{C}$ vanish, an inspection of the Atiyah–Hirzebruch spectral sequence $E^2_{p,q} = \mathrm{H}_p(BG; \mathrm{K}_q(\mathbb{C})) \Rightarrow \mathrm{H}_{p+q}(BG; \mathbf{K}_{\mathbb{C}})$ reveals that $\mathrm{H}_*(BG; \mathbf{K}_{\mathbb{C}}) = \{0\}$ for $* < 0$. □

For a pointed space X one obtains the notion of its homotopy group with coefficients in $\mathbb{Z}/n$ by replacing the spheres S^i in the definition of homotopy groups $\pi_i(X) = [S^1, X]$ by the Moore space $M^i_n := S^{i-1} \cup_n e^i$, i.e., an i-cell attached to S^{i-1} by a map $\partial B^i = S^{i-1} \to S^{i-1}$ of degree n. This enables us to construct K-*theory with finite coefficients* [12], which enjoys many good functorial properties like ordinary K-theory. For a recent survey of the theory one may refer to [47]. Karoubi studied that algebraic and topological K-theory of Banach algebras with finite coefficients and obtained some striking results in [32]. It is known that if $A = \mathbb{C}$ or $\mathbb{K}$, the comparison map with finite coefficients $\mathrm{K}(A, \mathbb{Z}/n) \to \mathrm{K}^{\mathrm{top}}_*(A, \mathbb{Z}/n)$ is an isomorphism for all $n \geqslant 2$ and $* \geqslant 0$ (see Theorem 4.2 of [49]). Using this result we are able to prove

Theorem 4.2. *If a discrete and torsion free group satisfies the integral (split) Kasparov–Novikov conjecture with trivial coefficients, then* G *and* $\mathbb{C}$ *satisfy the* K*-theoretic (split) Hsiang–Novikov conjecture with finite coefficients.*

Proof. By the previous Lemma we may concentrate only on nonnegative degrees. Attaching the Loday assembly map in algebraic K-theory to the top of the commutative diagram in Lemma 3.3, we get

$$\begin{array}{ccc}
 & & \mathrm{K}_*(\mathbb{C}[G]) \\
 & \nearrow^{\mu^{\mathrm{L}}_*} & \downarrow{\scriptstyle (\iota_{\mathbb{C}})_*} \\
\mathrm{H}_*(BG; \mathbf{K}_{\mathbb{C}}) & \xrightarrow{\mu^{\mathrm{alg}}_*} & \mathrm{K}_*(C^*_r(G)) \\
{\scriptstyle c(\mathbb{C})_*}\downarrow & & \downarrow{\scriptstyle c(C^*_r(G))_*} \\
\mathrm{H}_*(BG; \mathbf{K}^{\mathrm{top}}_{\mathbb{C}}) & \xrightarrow{\mu^{\mathrm{BC}}_*} & \mathrm{K}^{\mathrm{top}}_*(C^*_r(G)),
\end{array}$$

The bottom horizontal arrow is (split) injective, since G is assumed to satisfy the (split) Kasparov–Novikov conjecture with trivial coefficients. Since $c(\mathbb{C}) : \mathbf{K}_{\mathbb{C}} \to \mathbf{K}^{\mathrm{top}}_{\mathbb{C}}$ induces an isomorphism $\mathrm{K}_*(\mathbb{C}, \mathbb{Z}/n) \to \mathrm{K}^{\mathrm{top}}_*(\mathbb{C}, \mathbb{Z}/n)$ in nonnegative degrees, the assertions follows. □

A Theorem of Suslin says that if $F \hookrightarrow L$ is an extension of algebraically closed fields, then the induced map $\mathbf{K}_F \to \mathbf{K}_L$ produces an isomorphism between the algebraic K-theory groups with finite coefficients [57]. Using this result for the extension $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ we improve the coefficients in the above result to $\overline{\mathbb{Q}}$.

Theorem 4.3. *If a discrete and torsion free group* G *satisfies the integral (split) Kasparov–Novikov conjecture with trivial coefficients, then* G *and* $\overline{\mathbb{Q}}$ *satisfy the* K*-theoretic (split)*

Kasparov–Novikov conjecture with finite coefficients, i.e., the Loday assembly map with finite coefficients at the level of homotopy groups $\mathrm{H}_*((BG;\mathbf{K}_{\overline{\mathbb{Q}}}),\mathbb{Z}/n)\to\mathrm{K}_*(\overline{\mathbb{Q}}[G],\mathbb{Z}/n)$ *is split injective.*

Proof. Once again we may concentrate only on nonnegative degrees. By the naturality of the Loday assembly map, there is a commutative diagram induced by $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$:

$$\begin{array}{ccc}\mathrm{H}_*(BG;\mathbf{K}_{\overline{\mathbb{Q}}}) & \longrightarrow & \mathrm{K}_*(\overline{\mathbb{Q}}[G])\\ \downarrow & & \downarrow\\ \mathrm{H}_*(BG;\mathbf{K}_{\mathbb{C}}) & \longrightarrow & \mathrm{K}_*(\mathbb{C}[G]).\end{array}$$

With finite coefficients the bottom horizontal arrow is split injective (by the previous Proposition) and so is the left vertical arrow by Suslin's Theorem. □

Remark 4.4. *Both Theorem 4.2 and Theorem 4.3 would continue to hold if* μ_*^{BC} *is only (split) injective with finite coefficients.*

5. Some spectral sequences

For any C^*-algebra A, there is symmetric spectrum (in the sense of [26]) model of $\mathbf{K}_A^{\mathrm{top}}$, which is, in addition, a (left) module spectrum over a (commutative) symmetric ring spectrum model of $\mathbf{K}_{\mathbb{C}}^{\mathrm{top}}$ (see Theorem B of [30]). Furthermore, there is a unit map from the sphere spectrum $\mathbb{S}$ to $\mathbf{K}_{\mathbb{C}}^{\mathrm{top}}$, which is a homomorphism of commutative symmetric ring spectra. After passing to functorial cofibrant replacements (in the $\mathbb{S}$-model structures [54] *or the flat stable model structures as in Schwede's book project on symmetric spectra*) on the categories of (left) module spectra over the symmetric ring spectra, we may assume that all spectra are cofibrant. Now apply the functorial left Quillen construction, which produces a (cofibrant) $\mathbb{S}$-algebra (resp. $\mathbb{S}$-module) from a (cofibrant) symmetric ring spectrum (resp. symmetric spectrum) as explained in [53]. Thus we obtain a model of $\mathbf{K}_A^{\mathrm{top}}$ as an $\mathbb{S}$-module over an $\mathbb{S}$-algebra model of $\mathbf{K}_{\mathbb{C}}^{\mathrm{top}}$, where all $\mathbb{S}$-algebras (resp. $\mathbb{S}$-modules) are cofibrant. For the details about $\mathbb{S}$-algebras and $\mathbb{S}$-modules the readers may refer to [21]. Now one may write

$$BG_+\wedge\mathbf{K}_A^{\mathrm{top}}\simeq(BG_+\wedge\mathbf{K}_{\mathbb{C}}^{\mathrm{top}})\wedge_{\mathbf{K}_{\mathbb{C}}^{\mathrm{top}}}\mathbf{K}_A^{\mathrm{top}},$$

using a cofibrant CW model of BG, e.g., the geometric realization of the simplicial model of BG. If R is a (cofibrant) commutative $\mathbb{S}$-algebra and M,N are (cofibrant) R-modules, then there is a strongly convergent natural (both in M and N) spectral sequence (see Theorem 4.1 of ibid.)

$$E^2_{p,q}=\mathrm{Tor}^{\pi_*(R)}_{p,q}(\pi_*(M),\pi_*(N))\Rightarrow\pi_{p+q}(M\wedge_R N). \tag{6}$$

Here p is the resolution degree of M and q is the internal degree of the graded modules whence it is a right half plane homological spectral sequence.

Remark 5.1. *The symmetric spectra constructed in* [30] *take values in pointed simplicial sets, whereas* $\mathbb{S}$*-modules are spectra valued in based spaces. However, it is known that there is a Quillen equivalence between the category of symmetric spectra valued in pointed simplicial sets and that of symmetric spectra valued in based spaces (see Section 18 of* [43]*).*

Setting $R = \mathbf{K}^{\mathrm{top}}_{\mathbb{C}}$, $M = BG_+ \wedge \mathbf{K}^{\mathrm{top}}_{\mathbb{C}}$ and $N = \mathbf{K}^{\mathrm{top}}_{A}$ and observing that $\pi_*(BG_+ \wedge \mathbf{K}^{\mathrm{top}}_{\mathbb{C}}) = \mathrm{K}^{\mathrm{top}}_*(BG)$ we get:

Lemma 5.2. *There is a right half plane homological strongly convergent natural spectal sequence*

$$E^2_{p,q} = \mathrm{Tor}^{\mathbb{Z}[u,u^{-1}]}_{p,q}(\mathrm{K}^{\mathrm{top}}_*(BG), \mathrm{K}^{\mathrm{top}}_*(A)) \Rightarrow \pi_{p+q}(BG_+ \wedge \mathbf{K}^{\mathrm{top}}_{A}), \tag{7}$$

where the degree of u is 2.

Proposition 5.3. *If A is* KK*-equivalent to $\mathbb{K}$, then there is an identification of $\mathbb{Z}/2$-graded theories*

$$\mathrm{K}^{\mathrm{top}}_*(BG) \otimes \mathrm{K}^{\mathrm{top}}_*(A) \cong \pi_*(BG_+ \wedge \mathbf{K}^{\mathrm{top}}_{A}) = \mathrm{H}_*(BG; \mathbf{K}^{\mathrm{top}}_{A}),$$

which is natural in both G and A.

Proof. Under the assumption on A, one knows that $\mathrm{K}^{\mathrm{top}}_*(A) \simeq \mathrm{K}^{\mathrm{top}}_*(\mathbb{K}) \simeq \mathbb{Z}[u, u^{-1}]$. The assertion is now evident from the above spectral sequence (7). □

Remark 5.4. *Setting $R = \mathbf{K}_{\mathbb{C}}$, $M = BG_+ \wedge \mathbf{K}_{\mathbb{C}}$ and $N = \mathbf{K}_A$ in the spectral sequence* (6)*, we get*

$$E^2_{p,q} = \mathrm{Tor}^{\mathrm{K}_*(\mathbb{C})}_{p,q}(\mathrm{H}_*(BG; \mathbf{K}_{\mathbb{C}}), \mathrm{K}_*(A)) \Rightarrow \pi_{p+q}(BG_+ \wedge \mathbf{K}_A) = \mathrm{H}_{p+q}(BG; \mathbf{K}_A).$$

Apart from the standard Atiyah–Hirzebruch spectral sequences, these Künneth type spectral sequences can potentially be useful for computational purposes for the domain of the Davis–Lück assembly map in certain situations (compare [51]).

6. Assembly maps with coefficients in topological algebras

Now we are going to study assembly maps with coefficients in certain topological algebras. We use the extra knowledge about the K-theory of these algebras to gain more information about such assembly maps. In rather technical terms, we are going to identify $(\mu^{\mathrm{alg}}_{\mathbb{K}})_*$ with $(\mu^{\mathrm{BC}}_{\mathbb{K}})_*$.

Lemma 6.1. *There are canonical isomorphisms*

1. $\mathrm{H}_*(BG; \mathbf{K}_{G,\mathbb{K}}) \xrightarrow{\cong} \mathrm{H}_*(BG; \mathbf{K}^{\mathrm{top}}_{G,\mathbb{K}})$ *induced by the change of theory morphism from the algebraic to topological* K*-theory, where $\mathbb{K}$ is equipped with trivial G-action.*
2. $\mathrm{H}_*(BG; \mathbf{K}^{\mathrm{top}}_{G,\mathbb{C}}) \xrightarrow{\cong} \mathrm{H}_*(BG; \mathbf{K}^{\mathrm{top}}_{B,\mathbb{K}})$ *induced by the algebra homomorphism $\mathbb{C} \to \mathbb{K}$.*

Proof. For (1) observe that, for every subgroup $H \subset G$, by construction $\mathrm{H}^G_*(G/H; \mathbf{K}_{G,\mathbb{K}}) \cong \mathrm{K}_*(C^*_r(H, \mathbb{K}))$ and $\mathrm{H}^G_*(G/H; \mathbf{K}^{\mathrm{top}}_{G,\mathbb{K}}) \cong \mathrm{K}^{\mathrm{top}}_*(C^*_r(H, \mathbb{K}))$. Since $C^*_r(H, \mathbb{K})$ is stable, the change of theory morphism induces an isomorphism $\mathrm{K}_*(C^*_r(H, \mathbb{K})) \cong \mathrm{K}^{\mathrm{top}}_*(C^*_r(H, \mathbb{K}))$. The assertion now follows from Theorem 6.3 of [20].
The argument for (2) is similar; one simply needs to use the C^*-stability of topological K-theory, i.e., $\mathrm{K}^{\mathrm{top}}_*(C^*_r(H)) \cong \mathrm{K}^{\mathrm{top}}_*(C^*_r(H, \mathbb{K}))$ for any subgroup $H \subset G$. □

Let us set $\mathrm{K}^{\mathrm{top}}_*(G; A) = \mathrm{KK}^G_*(\underline{E}G; A)$, where $\underline{E}G$ is the universal proper G-space, which is uniquely characterized up to G-homotopy. If G is torsion free, one may take $\underline{E}G = EG$. The classifying space $\underline{E}G$ may not be locally compact. So one defines $\mathrm{KK}^G_*(\underline{E}G; A) = \varinjlim_C \mathrm{KK}^G_*(C_0(X), A)$, where $C \subset \underline{E}G$ runs through the set of all G-compact subspaces canonically ordered by inclusion. Let X be a locally compact proper G-space and let A be a G-C^*-algebra (with not necessarily trivial G-action). For any C^*-algebra B with trivial G-action, there is a canonical homomorphism

$$\alpha_X : \mathrm{KK}^G_*(C_0(X), A) \otimes \mathrm{K}^{\mathrm{top}}_*(B) \to \mathrm{KK}^G_*(C_0(X), A\hat{\otimes}B),$$

which is constructed by first identifying $\mathrm{K}^{\mathrm{top}}_*(B) \cong \mathrm{KK}^G_*(\mathbb{C}, B)$ and then applying Kasparov product $\otimes_{\mathbb{C}}$. This map is compatible with inclusions of G-compact subspaces of $\underline{E}G$ and hence defines a morphism

$$\alpha_G : \mathrm{K}^{\mathrm{top}}_*(G; A) \otimes \mathrm{K}^{\mathrm{top}}_*(B) \to \mathrm{K}^{\mathrm{top}}_*(G; A\hat{\otimes}B).$$

In [15] the authors define the following class $\mathcal{N}_G$ of G-C^*-algebras: $A \in \mathcal{N}_G$ if and only if the above morphism α_G is an isomorphism for every C^*-algebra B with trivial G-action, such that $\mathrm{K}^{\mathrm{top}}_*(B)$ is torsion free. The class $\mathcal{N}_G$ is fairly large; for instance, it contains all type I C^*-algebras and if $A \in \mathcal{N}_G$ and B is KK^G-equivalent to A, then $B \in \mathcal{N}_G$ (see Lemma 4.7 and Theorem 0.1 of ibid.). Clearly, $\mathbb{K} \in \mathcal{N}_G$.
Let A be a G-C^*-algebra. We quote the following commutative diagram from Section 1.6 (page 47) of [1]:

$$\begin{array}{ccc} \mathrm{H}^G_*(E_{\mathtt{VC}}(G); \mathbf{K}_{G,A}) & \xrightarrow{(\mu^{\mathrm{DL}}_A)_*} & \mathrm{K}_*(A \rtimes G) \\ \downarrow & & \downarrow \\ \mathrm{H}^G_*(E_{\mathtt{Fin}}(G); \mathbf{K}^{\mathrm{top}}_{G,A}) & \xrightarrow{(\mu^{\mathrm{DL}}_A)_*} & \mathrm{K}^{\mathrm{top}}_*(A \rtimes_r G), \end{array}$$

where the top (resp. bottom) horizontal arrow $(\mu^{\mathrm{DL}}_A)_*$ denotes the Davis–Lück assembly map in algebraic (resp. topological) K-theory. Putting $A = \mathbb{K}$ with trivial G-action, where G is a discrete and torsion free, we get

$$\begin{array}{ccc} \mathrm{H}_*(BG; \mathbf{K}_{G,\mathbb{K}}) & \xrightarrow{(\mu^{\mathrm{DL}}_{\mathbb{K}})_*} & \mathrm{K}_*(\mathbb{K}[G]) \\ \cong\downarrow & & \downarrow c(C^*_r(G,\mathbb{K}))_*\circ(\iota_{\mathbb{K}})_* \\ \mathrm{H}_*(BG; \mathbf{K}^{\mathrm{top}}_{G,\mathbb{K}}) & \xrightarrow{(\mu^{\mathrm{DL}}_{\mathbb{K}})_*} & \mathrm{K}^{\mathrm{top}}_*(C^*_r(G, \mathbb{K})), \end{array}$$

where the left vertical arrow is an isomorphism due to Lemma 6.1 part 1. Now we use the fact that $\mathbb{K}$ is K-regular, to replace $(\mu^{\mathrm{DL}}_{\mathbb{K}})_*$ by $(\mu^{L}_{\mathbb{K}})_*$ (cf. Remark 3.2) and obtain

(8)
$$\begin{array}{ccc} \mathrm{H}_*(BG; \mathbf{K}_{\mathbb{K}}) & \xrightarrow{(\mu^{\mathrm{L}}_{\mathbb{K}})_*} & \mathrm{K}_*(\mathbb{K}[G]) \\ \cong\downarrow & & \downarrow c(C^*_r(G,\mathbb{K}))_*\circ(\iota_{\mathbb{K}})_* \\ \mathrm{H}_*(BG; \mathbf{K}^{\mathrm{top}}_{G,\mathbb{K}}) & \xrightarrow{(\mu^{\mathrm{DL}}_{\mathbb{K}})_*} & \mathrm{K}^{\mathrm{top}}_*(C^*_r(G, \mathbb{K})) \end{array}$$

The Hambleton–Pedersen result on the equivalence of μ^{DL}_* with μ^{BC}_* is encapsulated in the following commutative diagram:

(9)
$$\begin{array}{ccc} \mathrm{K}_*^{\mathrm{top}}(BG) \cong \mathrm{H}_*(BG;\mathbf{K}_{\mathbb{C}}^{\mathrm{top}}) & \xrightarrow{\mu_*^{\mathrm{BC}}} & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G)) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{H}_*(BG;\mathbf{K}_{G,\mathbb{C}}^{\mathrm{top}}) & \xrightarrow{\mu_*^{\mathrm{DL}}} & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G)), \end{array}$$

where the vertical maps are isomorphisms. Note that the actual result identifies both μ_*^{DL} and μ_*^{BC} with a continuously controlled assembly map; however, those details are irrelevant for our purposes. The naturality of Davis–Lück assembly map produces the following commutative diagram:

(10)
$$\begin{array}{ccc} \mathrm{H}_*(BG;\mathbf{K}_{G,\mathbb{C}}^{\mathrm{top}}) & \xrightarrow{\mu_*^{\mathrm{DL}}} & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G)) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{H}_*(BG;\mathbf{K}_{G,\mathbb{K}}^{\mathrm{top}}) & \xrightarrow{(\mu_{\mathbb{K}}^{\mathrm{DL}})_*} & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G,\mathbb{K})), \end{array}$$

where the left vertical arrow is an isomorphism due to Lemma 6.1 part 2. From diagrams (9) and (10) we get the following commutative diagram:

(11)
$$\begin{array}{ccc} \mathrm{H}_*(BG;\mathbf{K}_{\mathbb{C}}^{\mathrm{top}}) & \xrightarrow{\mu_*^{\mathrm{BC}}} & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G)) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{H}_*(BG;\mathbf{K}_{G,\mathbb{K}}^{\mathrm{top}}) & \xrightarrow{(\mu_{\mathbb{K}}^{\mathrm{DL}})_*} & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G,\mathbb{K})) \end{array}$$

We tensor the diagram (11) with $\mathrm{K}_*^{\mathrm{top}}(\mathbb{K})$ and make the following simplifications:

(1) We identify $\mu_*^{\mathrm{BC}}\otimes\mathrm{id} : \mathrm{K}_*^{\mathrm{top}}(BG)\otimes\mathrm{K}_*^{\mathrm{top}}(\mathbb{K}) \to \mathrm{K}_*^{\mathrm{top}}(C_r^*(G))\otimes\mathrm{K}_*^{\mathrm{top}}(\mathbb{K})$ with the Baum–Connes assembly map with coefficient in the G-C^*-algebra $\mathbb{K}$ with trivial G-action $(\mu_{\mathbb{K}}^{\mathrm{BC}})_* : \mathrm{K}_*^{\mathrm{top}}(BG;\mathbb{K}) \to \mathrm{K}_*^{\mathrm{top}}(C_r^*(G,\mathbb{K}))$ using Proposition 4.9 of [15].
(2) Since $\mathrm{K}_*^{\mathrm{top}}(\mathbb{K})$ is torsion free we use Künneth formula in topological K-theory [52] to identify $\mathrm{K}_*^{\mathrm{top}}(C_r^*(G,\mathbb{K}))\otimes\mathrm{K}_*^{\mathrm{top}}(\mathbb{K}) \cong \mathrm{K}_*^{\mathrm{top}}(C_r^*(G,\mathbb{K}))$. Now from Proposition 5.3 we conclude that
$$\mathrm{H}_*(BG;\mathbf{K}_{\mathbb{K}}^{\mathrm{top}})\otimes\mathrm{K}_*^{\mathrm{top}}(\mathbb{K}) \cong \mathrm{H}_*(BG;\mathbf{K}_{\mathbb{K}}^{\mathrm{top}}).$$
Observe that $\mathrm{H}_*(BG;\mathbf{K}_{\mathbb{K}}^{\mathrm{top}}) \cong \mathrm{K}_*^{\mathrm{top}}(BG)$ via the homotopy equivalence $\mathbf{K}_{\mathbb{C}}^{\mathrm{top}} \simeq \mathbf{K}_{\mathbb{K}}^{\mathrm{top}}$ induce by the C^*-algebra homomorphism $\mathbb{C}\to\mathbb{K}$. Therefore the bottom horizontal arrow in the diagram (11) does not change.

Thus we have proved the following result:

Proposition 6.2. *There is a commutative diagram:*

$$
\begin{array}{ccc}
\mathrm{K}_*^{\mathrm{top}}(BG;\mathbb{K}) & \xrightarrow{(\mu_{\mathbb{K}}^{\mathrm{BC}})_*} & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G,\mathbb{K})) \\
\cong\big\downarrow & & \big\downarrow\cong \\
\mathrm{H}_*(BG;\mathbf{K}_{G,\mathbb{K}}^{\mathrm{top}}) & \xrightarrow{(\mu_{\mathbb{K}}^{\mathrm{DL}})_*} & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G,\mathbb{K}))
\end{array}
\tag{12}
$$

expressing the equivalence of the Davis–Lück assembly map and the Baum–Connes assembly map with coefficients in $\mathbb{K}$*, when* G *acts on it trivially.*

Combining the diagram (8) with the above Proposition, we get

Lemma 6.3. *There is a commutative diagram:*

$$
\begin{array}{ccc}
 & & \mathrm{K}_*(\mathbb{K}[G]) \\
 & \nearrow^{(\mu_{\mathbb{K}}^{\mathrm{L}})_*} & \big\downarrow (\iota_{\mathbb{K}})_* \\
\mathrm{H}_*(BG;\mathbf{K}_{\mathbb{K}}) & \xrightarrow{(\mu_{\mathbb{K}}^{\mathrm{alg}})_*} & \mathrm{K}_*(C_r^*(G,\mathbb{K})) \\
\cong\big\downarrow & & \cong\big\downarrow c(C_r^*(G,\mathbb{K}))_* \\
\mathrm{K}_*^{\mathrm{top}}(BG;\mathbb{K}) & \xrightarrow{(\mu_{\mathbb{K}}^{\mathrm{BC}})_*} & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G,\mathbb{K}))
\end{array}
\tag{13}
$$

Observe that, by definition, $(\mu_{\mathbb{K}}^{\mathrm{alg}})_* = (\iota_{\mathbb{K}})_* \circ (\mu_{\mathbb{K}}^{\mathrm{L}})_*$.

Theorem 6.4. *Let* G *be a discrete and torsion free group. If* G *satisfies the (split) Kasparov–Novikov conjecture with trivial coefficients, then* G *and* $\mathbb{K}$ *satisfy the integral* K*-theoretic (split) Hsiang–Novikov conjecture, i.e., the Loday assembly map* $(\mu_{\mathbb{K}}^{\mathrm{L}})_* : \mathrm{H}_*(BG;\mathbf{K}_{\mathbb{K}}) \to \mathrm{K}_*(\mathbb{K}[G])$ *is (split) injective.*
Furthermore, if μ_*^{BC} *is only rationally injective then so is* $(\mu_{\mathbb{K}}^{\mathrm{L}})_*$.

Proof. For any G-C^*-algebra A and any other C^*-algebra B with a trivial G-action the authors of [15] construct the following commutative diagram in Proposition 4.9:

$$
\begin{array}{ccc}
\mathrm{K}_*^{\mathrm{top}}(BG;A)\otimes \mathrm{K}_*^{\mathrm{top}}(B) & \xrightarrow{(\mu_A^{\mathrm{BC}}\otimes\mathrm{id})_*} & \mathrm{K}_*^{\mathrm{top}}(A\rtimes_r G)\otimes \mathrm{K}_*^{\mathrm{top}}(B) \\
\alpha_G\big\downarrow & & \big\downarrow \\
\mathrm{K}_*^{\mathrm{top}}(BG;A\hat{\otimes}B) & \xrightarrow{(\mu_{A\hat{\otimes}B}^{\mathrm{BC}})_*} & \mathrm{K}_*^{\mathrm{top}}((A\hat{\otimes}B)\rtimes_r G)
\end{array}
$$

where the right vertical arrow is the Künneth map is topological K-theory. Putting $A=\mathbb{C}$ and $B=\mathbb{K}$ we get

$$
\begin{array}{ccc}
\mathrm{K}_*^{\mathrm{top}}(BG)\otimes \mathrm{K}_*^{\mathrm{top}}(\mathbb{K}) & \xrightarrow{(\mu^{\mathrm{BC}}\otimes\mathrm{id})_*} & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G))\otimes \mathrm{K}_*^{\mathrm{top}}(\mathbb{K}) \\
\alpha_G\big\downarrow\cong & & \big\downarrow\cong \\
\mathrm{K}_*^{\mathrm{top}}(BG;\mathbb{K}) & \xrightarrow{(\mu_{\mathbb{K}}^{\mathrm{BC}})_*} & \mathrm{K}_*^{\mathrm{top}}(C_r^*(G,\mathbb{K}))
\end{array}
$$

Since $\mathrm{K}_*^{\mathrm{top}}(\mathbb{K})$ is torsion free and $\mathbb{K}$ is in the bootstrap class both vertical arrows are isomorphisms. Using flatness of $\mathrm{K}_*^{\mathrm{top}}(\mathbb{K})$ we conclude that if μ_*^{BC} is (split) injective, then so is $(\mu^{\mathrm{BC}} \otimes \mathrm{id})_* \cong (\mu_{\mathbb{K}}^{\mathrm{BC}})_*$. The assertion now follows from the previous Lemma.
The statement about the rational injectivity is obvious. □

7. Algebraic K-theory of certain group algebras

Recall that the homotopy cofibre of Loday assembly map $\mu_R^{\mathrm{L}} : BG_+ \wedge \mathbf{K}_R \to \mathbf{K}_{R[G]}$ is defined to be the *Whitehead spectrum of G over R* and its homotopy groups are called the *Whitehead groups of G over R*. By excision in algebraic K-theory this notion carries over to H-unital coefficient $\mathbb{Q}$-algebras (see Remark 2.2).

Lemma 7.1. *Let G satisfy the Baum–Connes conjecture with trivial coefficients. Then the obstruction to the algebra homomorphism $\mathbb{K}[G] \to C_r^*(G, \mathbb{K})$ inducing a weak homotopy equivalence at the level of nonconnective algebraic* K*-theory spectra lies in the Whitehead spectrum of G over $\mathbb{K}$ (up to a shift).*

Proof. Since the bijectivity of μ_*^{BC} implies that of $(\mu_{\mathbb{K}}^{\mathrm{BC}})_*$ (see Corollary 5.2 of [15]) and the algebraic reduced assembly map $(\mu_{\mathbb{K}}^{\mathrm{alg}})_*$ at the level of homotopy groups agrees with $(\mu_{\mathbb{K}}^{\mathrm{BC}})_*$ (see Lemma 6.3), the assertion follows from Remark 2.4. □

As a consequence we deduce the following result:

Theorem 7.2. *Let G be a discrete and torsion free Gromov hyperbolic group and let G act on $\mathbb{K}$ trivially. Then canonical algebra homomorphism $\iota_{\mathbb{K}} : \mathbb{K}[G] \to C_r^*(G, \mathbb{K})$ induces a weak homotopy equivalence between their nonconnective algebraic* K*-theory spectra and the Whitehead groups of G over $\mathbb{K}$ vanish.*

Proof. Under the assumptions on the group G it is known that it satisfies the Baum–Connes conjecture with trivial coefficients [44]. By the previous Lemma it suffices to show that the Whitehead groups of G over $\mathbb{K}$ vanish. A result of Bartels–Lück–Reich says that all Gromov hyperbolic groups satisfy the Farrell–Jones isomorphism conjecture in algebraic K-theory for every associative and unital ring R (see Corollary 1.2 of [3]); more precisely, the authors prove that for torsion free groups $\mathrm{K}_n(R[G]) \cong \mathrm{H}_n(BG; \mathbf{K}_R) \oplus (\oplus_I(\mathrm{NK}_n(R) \oplus \mathrm{NK}_n(R)))$, where I denotes the set of conjugacy classes of maximal infinite cyclic subgroups of G. Using the naturality of the decomposition (in R), Remark 2.2 and excision in NK-theory, one concludes

$$\mathrm{K}_n(\mathbb{K}[G]) \cong \mathrm{H}_n(BG; \mathbf{K}_{\mathbb{K}}) \oplus (\oplus_I(\mathrm{NK}_n(\mathbb{K}) \oplus \mathrm{NK}_n(\mathbb{K}))) \,. \tag{14}$$

It is known that $\mathrm{NK}_n(\mathbb{K})$ vanishes for all n, since $\mathbb{K}$ is a stable C^*-algebra (see Theorem 3.4 of [50]). It follows that the Whitehead groups of G over $\mathbb{K}$ vanish. □

Remark 7.3. *Since the C^*-algebra $C_r^*(G, \mathbb{K})$ is stable, its nonconnective algebraic* K*-theory is the same as its topological* K*-theory, which in turn is that same as its topological* K*-homology, i.e., $\mathrm{K}_*(\mathbb{K}[G]) \simeq \mathrm{K}_*^{\mathrm{top}}(BG)$.*

8. On a conjecture of Yu

For any $p \geqslant 1$ let $\mathcal{S}_p$ denote the ring operators of Schatten p-class, i.e., $T \in B(H)$ is an element of $\mathcal{S}_p$ if and only if $\mathrm{tr}(T^*T)^{\frac{1}{2}} < \infty$, where the trace is defined as $\mathrm{tr}(T) =$

$\sum_n \langle Te_n, e_n\rangle$ with respect to an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of H (the definition of tr turns out to be independent of the choice of the orthonormal basis). The algebra of Schatten class operators is defined to be $\mathcal{S} = \cup_{p\geqslant 1}\mathcal{S}_p$. There is a canonical sequence of $\mathbb{C}$-algebra inclusions $\mathcal{S}\subset\mathbb{K}\subset B(H)$. It follows from the results of [19] that $\mathcal{S}[G]$ is H-unital for any discrete group G (see, e.g., Theorem 2.2 of [61]).

Conjecture 8.1 (Yu [61])**.** *For any discrete group G the canonical algebra homomorphism $i : \mathcal{S}[G]\cong\mathbb{C}[G]\otimes_{\mathbb{C}}\mathcal{S}\to C^*_r(G)\hat{\otimes}\mathbb{K} = C^*_r(G,\mathbb{K})$ induces an isomorphism between their algebraic* K*-theory groups*

$$i_* : \mathrm{K}_n(\mathcal{S}[G])\to\mathrm{K}_n(C^*_r(G,\mathbb{K})).$$

The algebra homomorphism $i:\mathcal{S}[G]\to C^*_r(G,\mathbb{K})$ can be factorized as $\mathcal{S}[G]\to\mathbb{K}[G]\to C^*_r(G,\mathbb{K})$. Now one can separately investigate these homomorphisms and this leads us to two separate conjectures:

Conjecture 8.2. *For any discrete group G, the canonical algebra homomorphism $\mathcal{S}[G]\to\mathbb{K}[G]$ induces a weak equivalence between their nonconnective algebraic* K*-theory spectra.*

Conjecture 8.3. *For any discrete group G, the canonical algebra homomorphism $\mathbb{K}[G]\to C^*_r(G,\mathbb{K})$ induces a weak equivalence between their nonconnective algebraic* K*-theory spectra.*

The Farrell–Jones isomorphism conjecture in algebraic K-theory should imply conjecture 8.2. The Theorem 7.2 above gives an affirmative answer to conjecture 8.3 for all Gromov hyperbolic groups (note that such groups are known to satisfy both the Farrell–Jones isomorphism conjecture in algebraic K-theory and the Baum–Connes conjecture with trivial coefficients).

Observe that $\pi_*(\mathbf{K}_{\mathbb{K}})$ is Bott 2-periodic (due to its identification with topological K-theory). In fact, $\pi_*(\mathbf{K}_{\mathbb{K}})$ is $\mathbb{Z}$ if $*$ is even, and $\{0\}$ if $*$ is odd. The same conclusion holds for the algebraic K-theory of $\mathcal{S}$ [19] and an easy inspection reveals that the canonical inclusion $\mathcal{S}\to\mathbb{K}$ induces a weak homotopy equivalence $\mathbf{K}_{\mathcal{S}}\to\mathbf{K}_{\mathbb{K}}$. It is also shown in ibid. that $\mathcal{S}$ is K-regular and H-unital (whence the Loday assembly map can be defined with coefficients in $\mathcal{S}$, see Remark 2.2). As before, for any discrete and torsion free group G, we identify the Davis–Lück assembly map in K-theory with the Loday assembly map $(\mu^{\mathrm{L}}_{\mathcal{S}})_* : \mathrm{H}_*(BG;\mathbf{K}_{\mathcal{S}})\to\mathrm{K}_*(\mathcal{S}[G])$. Using the naturality of the Loday assembly map, once again we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}_*(BG;\mathbf{K}_{\mathcal{S}}) & \longrightarrow & \mathrm{K}_*(\mathcal{S}[G]) \\
\cong\downarrow & & \downarrow \\
\mathrm{H}_*(BG;\mathbf{K}_{\mathbb{K}}) & \longrightarrow & \mathrm{K}_*(\mathbb{K}[G]),
\end{array}$$

where the vertical arrows are induced by $\mathcal{S}\to\mathbb{K}$. Since the map $\mathbf{K}_{\mathcal{S}}\to\mathbf{K}_{\mathbb{K}}$ is a weak homotopy equivalence the left vertical arrow is an isomorphism. As a consequence of Theorem 6.4 we have proven

Theorem 8.4. *Let G be a discrete and torsion free group. If G satisfies the (split) Kasparov–Novikov conjecture with trivial coefficients, then G and $\mathcal{S}$ satisfy the integral* K*-theoretic (split) Hsiang–Novikov conjecture, i.e., the Loday assembly map $(\mu^{\mathrm{L}}_{\mathcal{S}})_* : \mathrm{H}_*(BG;\mathbf{K}_{\mathcal{S}})\to\mathrm{K}_*(\mathcal{S}[G])$ is (split) injective.*

Remark 8.5. *The above Theorem is an integral* K*-theoretic statement, whereas the main result of Yu in* [61] *proves the rational injectivity of* $(\mu_{\mathcal{S}}^{\mathrm{L}})_*$ *for all discrete groups with no further assumptions.*

Motivated by these observations, we arrive at the following variant of Yu's conjecture:

Conjecture 8.6 (Variant of Yu)**.** *Let G be any discrete group. Let A be any stable C^*-algebra, equipped with the trivial G-action, such that A is* KK*-equivalent to $\mathbb{K}$ (evidently $A \in \mathcal{N}_G$). Then the canonical algebra homomorphism $A[G] \to C_r^*(G, A)$ induces a weak equivalence between their nonconnective algebraic* K*-theory spectra.*

Concluding remark: It is plausible that, building upon the arguments of this article, one can generalize the coefficient C^*-algebra in Theorem 6.4 from $\mathbb{K}$ to any stable G-C^*-algebra A with trivial G-action, such that A belongs to the bootstrap class and $\mathrm{K}_*^{\mathrm{top}}(A)$ is finitely generated and torsion free.

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