Revisiting special relativity: A natural algebraic alternative to Minkowski spacetime

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Minkowski famously introduced the concept of a space-time continuum in 1908, merging three dimensional space with an imaginary time dimension represented by ict, a framework which naturally produced the correct spacetime interval $x^2 - c^2t^2$, and the results of Einstein's theory of special relativity. As an alternative to Minkowski space-time, we replace the unit imaginary $i = \sqrt{-1}$, with the Clifford bivector $\iota = e_1e_2$ for the plane, which also has the property of squaring to minus one, but which can be included without the addition of an extra dimension, as it is a natural part of Clifford's real Cartesian-type plane with the orthonormal basis e_1 and e_2. We find that with the ansatz of spacetime represented by a Clifford multivector, the spacetime metric and the Lorentz transformations, follow immediately as properties of the algebra. Based on the structure of the multivector, a simple semi-classical model is also produced for representing massive particles, giving a new efficient derivation for Compton's scattering formula. We also find a new perspective on the nature of time, now appearing as the bivector of the plane.

Keywords: Special Relativity, Geometric algebra, Clifford algebra, Lorentz transformation, Minkowski

1. Introduction

It has been well established experimentally that the Lorentz transformations, provide the correct translation of space and time measurements from one inertial frame of reference to another. They were initially developed by Lorentz [1904] and previously by Voigt [1887] [Ernst and Hsu, 2001], to explain the null result of the Michelson-Morley experiment, by proposing a length contraction of a laboratory frame of reference moving with respect to a hypothetical aether. Einstein [1905] however, rederived the transformations on the basis of two fundamental postulates, of the invariance of the laws of physics and the invariance of the speed of light, between inertial observers, thus eliminating the need for an aether. Minkowski in 1908, also derived the Lorentz transformations from a different perspective, by postulating a spacetime continuum, from which the results of special relativity also naturally followed [Sexl and Urbantke, 2001]. From an alternative perspective, Zeeman [1964] showed that preserving causality was sufficient to ensure that the coordinate transformations are the Lorentz transformations, along with an invariant maximum speed.

Though Einstein is credited with the definitive explanation of the Lorentz transformations via his two postulates, Minkowski's alternative approach had far-reaching impact [Goenner et al., 1999], as it provided a general structure for spacetime within which the laws of physics could be described. To achieve this he firstly introduced the concept of a uniform four-dimensional space time continuum with the expected Euclidean distance measure $(\Delta s)^2 = (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 + (\Delta x_4)^2$, but where $x_4 = ict$, borrowing the idea of imaginary time proposed by Poincaré, where $i = \sqrt{-1}$ is the unit imaginary, which thus allowed one to view space time as a conventional Euclidean space, while still recovering the required distance measure $(\Delta s)^2 = (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 - c^2(\Delta t)^2$. This idea was received favorably by Einstein, and by the wider scientific community at the time [Einstein and Lawson, 1921], but more recently, with the desire to remain consistent with the real metric of general relativity [Taylor and Wheeler, 1966, Misner et al., 1973], the unit imaginary has been replaced with a four dimensional metric signature $(+,+,+,-)$, because it is more easily extended to a general real metric for curved four-dimensional space.

In this paper, we follow Minkowski's approach, but we postulate an alternate spacetime framework, which is provided by the multivector of a two-dimensional Clifford algebra. Clifford algebra has been used previously to describe spacetime [Hestenes, 1999, 2003, Pavsic, 2004], however these approaches retain a four-dimensional spacetime framework with an associated metric structure, whereas our approach requires a minimal two dimensional Euclidean space, without the need for a metric function, as it arises naturally from the properties of the multivector. In this approach, specifically, we replace the unit imaginary of Minkowski, with the Clifford bivector e_1e_2 of the plane defined by the orthonormal vectors e_1 and e_2, which also has the property of squaring to minus one. The bivector however has several advantages over the unit imaginary, in that, firstly, it is a composite algebraic component of the plane, and so an extra Euclidean dimension is not required, and secondly, the bivector is an algebraic element embedded in a strictly real space, and hence consistent with the real space of general relativity. We find that we are able to adopt Clifford's geometric algebra of two-dimensions as a suitable algebraic framework, because the Lorentz transforms of special relativity act on just the parallel and perpendicular components of vectors relative to a boost direction, thus defining a two-dimensional space.

Clifford's geometric algebra was first published in 1873, extending the work of Grassman and Hamilton, creating a single unified real mathematical framework over Cartesian coordinates, which naturally included the algebraic properties of scalars, complex numbers, quaternions and vectors into a single entity, called the multivector [Doran and Lasenby, 2003]. We find that this general algebraic entity, as part of a real two-dimensional algebra, provides a natural alternative to Minkowski spacetime.

(*a*) *Clifford's algebra of the plane*

In order to represent the plane, Clifford defined two algebraic elements e_1 and e_2, with

$$e_1^2 = e_2^2 = 1, \text{ and } e_1e_2 = -e_2e_1, \tag{1.1}$$

where we note that the composite element $\iota = e_1e_2$ is defined as anticommuting [Doran and Lasenby, 2003], and therefore squares to minus one, that is, $\iota^2 = (e_1e_2)^2 = e_1e_2e_1e_2 = -e_1e_1e_2e_2 = -1$, and can be used to replace, the unit imaginary. A general Clifford multivector for the plane can be written by combining the algebraic elements, as

$$a + x_1e_1 + x_2e_2 + \iota b, \tag{1.2}$$

where a and b are real scalars, $\mathbf{x} = x_1e_1 + x_2e_2$ represents a Cartesian vector, with x_1, x_2 real scalars, and ι is the bivector. We notice, that the multivector, encapsulates a complex-like number $a + \iota b$, but also includes the vector $\mathbf{x}$, thus producing a generalization of a complex number. Thus we have defined an associative but non-commuting algebra in order to describe the plane.

(i) *Geometric product*

A key property of Clifford's algebra, is given by the product of two vectors. Given the vectors $\mathbf{u} = u_1e_1 + u_2e_2$ and $\mathbf{v} = v_1e_1 + v_2e_2$, then using the distributive law for multiplication over addition, as assumed for an algebraic field, we find

$$\mathbf{uv} = (u_1e_1 + u_2e_2)(v_1e_1 + v_2e_2) = u_1v_1 + u_2v_2 + (u_1v_2 - v_1u_2)e_1e_2, \tag{1.3}$$

using the properties defined in Eq. (1.1). We identify $u_1v_1 + u_2v_2$ as the dot product and $(u_1v_2 - v_1u_2)e_1e_2$ as the wedge product, giving

$$\mathbf{uv} = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v}. \tag{1.4}$$

Hence the algebraic product of two vectors produces a sum of the dot and wedge products, with the significant advantage that this algebraic product now has an inverse operation. For $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ unit vectors, giving $\hat{\mathbf{u}}\cdot\hat{\mathbf{v}} = \cos\theta$ and $\hat{\mathbf{u}}\wedge\hat{\mathbf{v}} = \imath\sin\theta$, we therefore have $\hat{\mathbf{u}}\hat{\mathbf{v}} = \cos\theta + \imath\sin\theta$, where θ is the angle between the two vectors.

We can see from Eq. (1.4), that for the case of a vector multiplied by itself, that the wedge product will be zero and hence the square of a vector $\mathbf{v}^2 = \mathbf{v}\cdot\mathbf{v} = v_1^2 + v_2^2$, becomes a scalar quantity. Hence the Pythagorean length of a vector is simply $\sqrt{\mathbf{v}^2}$, and so we can find the inverse vector

$$\mathbf{v}^{-1} = \frac{\mathbf{v}}{\mathbf{v}^2}. \tag{1.5}$$

That is, the reciprocal of a Clifford vector is simply a vector with the same direction, and the inverse length.

2. Clifford multivectors as a framework for space and time

After inspecting Minkowski's definition of spacetime coordinates and Eq. (1.2), we are therefore led to describe spacetime events as the multivector difference

$$\Delta X = \Delta x_1e_1 + \Delta x_2e_2 + \imath c\Delta t = \Delta\mathbf{x} + \imath c\Delta t, \tag{2.1}$$

with $\Delta\mathbf{x}$ representing the change in position vector in the plane and Δt represents the change in observer time, where we restrict our analysis to two-dimensional space. This is without loss of generality, however, as we can always re-orientate our plane, to lie in the plane of the relative velocity vector between the frames. The interpretation of the coordinate change in Eq. (2.1), is the same as conventionally interpreted [Taylor and Wheeler, 1966], representing an observer moving through a preconfigured coordinate system, which at each point has a properly synchronized clock, from which the moving observer can read off the other frames local time t and position $\mathbf{x}$. An example on the use of the multivector in Eq. (2.1) is applied to π^+-meson decay in Section 3. We then find the spacetime interval to be

$$(\Delta X)^2 = (\Delta\mathbf{x} + \imath c\Delta t)(\Delta\mathbf{x} + \imath c\Delta t) = (\Delta\mathbf{x})^2 - c^2(\Delta t)^2 + c\Delta t\Delta\mathbf{x}\imath + c\Delta t\imath\Delta\mathbf{x} = (\Delta\mathbf{x})^2 - c^2(\Delta t)^2, \tag{2.2}$$

using the fact that $\Delta\mathbf{x}$ and $\imath$ anticommute, because $\imath$ anticommutes with e_1 and e_2, and $\imath^2 = -1$, giving the correct spacetime interval. We notice here an immediate simplification through use of the multivector, in that we are not required to define the dot product in order to calculate the metric, but it is produced directly by simply squaring the multivector. For the rest frame of the particle, that is, not moving with respect to the chosen frame, we have $(\Delta X_0)^2 = -c^2(\Delta\tau)^2$, where we define in this case t to represent the proper time τ of the particle. We have assumed that the speed c is the same in the rest and the moving frame, as required by Einstein's second postulate. Now, if the spacetime interval defined in Eq. (2.2) is invariant, which we demonstrate in a later section using the transformations defined by Eq. (2.17), then we can equate the rest frame interval to the moving frame interval, giving

$$c^2(\Delta\tau)^2 = c^2(\Delta t)^2 - (\Delta\mathbf{x})^2 = c^2(\Delta t)^2 - \mathbf{v}^2(\Delta t)^2 = c^2(\Delta t)^2\left(1 - \frac{\mathbf{v}^2}{c^2}\right), \tag{2.3}$$

with $\Delta x = \mathbf{v}\Delta t$, and hence, taking the square root, we find the time dilatation formula $\Delta t = \gamma\Delta\tau$ where

$$\gamma = \frac{1}{\sqrt{1-\mathbf{v}^2/c^2}}. \tag{2.4}$$

From Eq. (2.1), we can now calculate the proper velocity through the differential form of Eq. (2.1) with respect to the proper time difference, giving the velocity multivector

$$U = \frac{dX}{d\tau} = \frac{d\mathbf{x}}{dt}\frac{dt}{d\tau} + \iota c\frac{dt}{d\tau} = \gamma\mathbf{v} + \gamma\iota c, \tag{2.5}$$

where we use $\frac{dt}{d\tau} = \gamma$ and $\mathbf{v} = \frac{d\mathbf{x}}{dt}$. We then find

$$U^2 = (\gamma\mathbf{v} + \gamma\iota c)^2 = \left(\frac{1}{1-\mathbf{v}^2/c^2}\right)(\mathbf{v}^2 - c^2) = -c^2. \tag{2.6}$$

We define the momentum multivector

$$P = mU = \gamma m\mathbf{v} + \iota\gamma mc = \mathbf{p} + \iota\frac{E}{c}, \tag{2.7}$$

with the relativistic momentum $\mathbf{p} = \gamma m\mathbf{v}$ and the total energy $E = \gamma mc^2$.

Now, as $U^2 = -c^2$, then $P^2 = -m^2c^2$ is an invariant describing the conservation of momentum and energy, which gives

$$P^2c^2 = \mathbf{p}^2c^2 - E^2 = -m^2c^4, \tag{2.8}$$

or $E^2 = m^2c^4 + \mathbf{p}^2c^2$, giving the standard relativistic expression for the conservation of momentum-energy. Employing the de Broglie relations $\mathbf{p} = \hbar\mathbf{k}$ and $E = \hbar w$, we find using Eq. (2.7), the wave multivector

$$K = \frac{P}{\hbar} = \mathbf{k} + \iota\frac{w}{c}. \tag{2.9}$$

Similarly then we have

$$K^2 = \mathbf{k}^2 - \frac{w^2}{c^2} = -\frac{1}{\lambda_c^2}, \tag{2.10}$$

giving the correct dispersion relation for a wave which is relativistically invariant, where $\lambda_c = \frac{\hbar}{mc}$ is the reduced Compton wave length. That is, we have a phase velocity $v_p = \frac{w}{|\mathbf{k}|} = c\sqrt{1+\frac{1}{\lambda_c^2k^2}}$ and the group velocity, $v_g = \frac{dw}{dk} = \frac{c^2}{v_p}$. We now find the dot product of the wave and spacetime multivectors $K\cdot X = \mathbf{k}\cdot\mathbf{x} - wt$, giving the phase of a traveling wave.

(*a*) *Rotations in space*

Euler's formula for complex numbers, carries over unchanged for the bivector ι, with which we define a rotor

$$R = \cos\theta + \iota\sin\theta = \mathrm{e}^{\iota\theta}, \tag{2.11}$$

which produces a rotation by θ on the e_1e_2 plane, in the same way as rotations on the Argand diagram. For example, for a unit vector $\mathbf{v} = e_1$ along the e_1 axis, acting with the rotor from the right we find $\mathbf{v}R = e_1(\cos\theta + \iota\sin\theta) = \cos\theta e_1 + e_2\sin\theta$, thus describing an anti-clockwise rotation by θ. If we alternatively act from the left with the rotor, we will find a clockwise rotation by θ.

However, we now show, that a rotation can be described more generally as a sequence of two reflections. Given a vector $\mathbf{n}_1$ normal to a reflecting surface, with an incident ray given by $\mathbf{I}$, then we find the reflected ray [Doran and Lasenby, 2003]

$$\mathbf{r} = -\mathbf{n}_1\mathbf{I}\mathbf{n}_1. \tag{2.12}$$

If we apply a second reflection, with a unit normal $\mathbf{n}_2$, then we have

$$\mathbf{r} = \mathbf{n}_2\mathbf{n}_1\mathbf{I}\mathbf{n}_1\mathbf{n}_2 = (\cos\theta - \iota\sin\theta)\mathbf{I}(\cos\theta + \iota\sin\theta) = \mathrm{e}^{-\iota\theta}\mathbf{I}\mathrm{e}^{\iota\theta}, \tag{2.13}$$

using Eq. (1.4) for two unit vectors. If the two normals $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel, then no rotation is produced. In fact the rotation produced is twice the angle between the two normals.

Hence rotations are naturally produced by conjugation, where if we seek to rotate a vector $\mathbf{v}$ by an angle θ, we calculate

$$\mathbf{v}' = \mathrm{e}^{-\iota\theta/2}\mathbf{v}\mathrm{e}^{\iota\theta/2}, \tag{2.14}$$

which rotates in an anticlockwise direction. The rotation formula in Eq. (2.14) above, can in two-space, be simplified to a single right acting operator $\mathbf{v}' = \mathbf{v}\mathrm{e}^{\iota\theta}$. However this simplification is only possible in two-dimensions for the special case of rotations on vectors, and will not work on other algebraic elements or in higher dimensions, and hence Eq. (2.14) is the preferred way to apply operators such as rotors on vectors and multivectors.

(*b*) *The Lorentz Group*

We found that the exponential of the bivector $\mathrm{e}^{\iota\theta}$, describes rotations in the plane, as shown in Eq. (2.14), however, more generally, we can define the exponential of a full multivector M defined as in Eq. (1.2), by constructing the Taylor series

$$\mathrm{e}^M = 1 + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \dots \tag{2.15}$$

which is absolutely convergent for all multivectors M [Hestenes, 1999]. Also, for multivectors M, N, we have $\mathrm{e}^M\mathrm{e}^N = \mathrm{e}^{M+N}$, *if and only if* $MN = NM$, and because of the closure of multivectors under addition and multiplication, we see that the exponential of a multivector, must also produce another multivector, and we find, in fact, a unique multivector $N = \mathrm{e}^M$, for each multivector M [Hestenes, 1999]. The inverse operation, that is, finding the logarithm of N, is not always defined, for example, for N a pure vector, then an exponential form does not exist. However as noted in Eq. (2.12), acting on a vector by conjugation with a vector produces a reflection, which is not part of the homogeneous Lorentz group.

To simplify notation [French, 1987] we will dispense with the notation ΔX describing a change in observer coordinates, and simply assume that the coordinate systems coincide at $t = t' = 0$ with $\mathbf{x} = \mathbf{x}' = 0$, so that we can write this change as simply X. So, selecting transformations that leave the spacetime interval given by $X^2 \equiv (\Delta X)^2$, defined in Eq. (2.2), invariant, defines the homogeneous Lorentz group. For the multivector $M = a + \phi\hat{\mathbf{v}} + \iota\theta$, we define the dagger operation $M^\dagger = a - \phi\hat{\mathbf{v}} - \iota\theta$, we firstly find for a general rotation e^M, that $\mathrm{e}^M\mathrm{e}^{M^\dagger} = \mathrm{e}^{a+\phi\hat{\mathbf{v}}+\iota\theta}\mathrm{e}^{a-\phi\hat{\mathbf{v}}-\iota\theta} = \mathrm{e}^{2a}\mathrm{e}^{\phi\hat{\mathbf{v}}+\iota\theta}\mathrm{e}^{-\phi\hat{\mathbf{v}}-\iota\theta} = \mathrm{e}^{2a}$ a scalar, but in order to not rescale the space we require $a = 0$. Hence looking at all transformations of the form $\mathrm{e}^M = \mathrm{e}^{\phi\hat{\mathbf{v}}+\iota\theta}$, acting by conjugation, we find

$$X'^2 = \mathrm{e}^M X \mathrm{e}^{M^\dagger}\mathrm{e}^M X \mathrm{e}^{M^\dagger} = \mathrm{e}^M X^2 \mathrm{e}^{M^\dagger} = X^2, \tag{2.16}$$

using associativity and the fact that $\mathrm{e}^M\mathrm{e}^{M^\dagger} = 1$, and that X^2 is a scalar as shown in Eq. (2.2), and so unaffected by boosts and rotations. Hence all transformations of the form

$$\mathrm{e}^{\phi\hat{\mathbf{v}}+\iota\theta}, \tag{2.17}$$

will leave the spacetime interval invariant, and so defines the homogeneous Lorentz group. Hence we see that the operator defined by Eq. (2.17), will leave the spacetime interval unchanged, confirming that $(\Delta\mathbf{x})^2 - c^2(\Delta t)^2$ is an invariant, as assumed in Eq. (2.3).

(i) *Spacetime boosts*

Using the first component of the homogeneous Lorentz group defined in Eq. (2.17), operators of the form $\mathrm{e}^{\phi\hat{\mathbf{v}}}$, where the vector $\mathbf{v} = v_1 e_1 + v_2 e_2 \mapsto \phi\hat{\mathbf{v}}$, where $\hat{\mathbf{v}}$ is a unit vector, with $\hat{\mathbf{v}}^2 = 1$, we find

$$\mathrm{e}^{\phi\hat{\mathbf{v}}} = 1 + \phi\hat{\mathbf{v}} + \frac{\phi^2}{2!} + \frac{\phi^3\hat{\mathbf{v}}}{3!} + \frac{\phi^4}{4!} + \cdots = \cosh\phi + \hat{\mathbf{v}}\sinh\phi. \tag{2.18}$$

Also, defining an orthogonal vector to $\mathbf{v}$ given by $\mathbf{w} = \iota\mathbf{v}$, then we find $\mathbf{w}^2 = (\iota\mathbf{v})^2 = \mathbf{v}^2$, and therefore $\mathrm{e}^{\phi\iota\hat{\mathbf{v}}} = \cosh\phi + \iota\hat{\mathbf{v}}\sinh\phi$.

So applying the exponential operator to the general spacetime vector $X = \mathbf{x} + \iota ct$, using the transformation

$$X' = \mathrm{e}^{-\iota\hat{\mathbf{v}}\phi/2} X \mathrm{e}^{\iota\hat{\mathbf{v}}\phi/2}, \tag{2.19}$$

then in the $\hat{\mathbf{v}} = e_1$ direction, for example, we find the transformed spacetime coordinates

$$\begin{aligned} X' &= \mathrm{e}^{-\iota\frac{\hat{\mathbf{v}}\phi}{2}}(xe_1 + ye_2 + \iota ct)\mathrm{e}^{\iota\frac{\hat{\mathbf{v}}\phi}{2}} \\ &= \mathrm{e}^{\phi e_2} x_1 e_1 + x_2 e_2 + \iota ct \mathrm{e}^{\phi e_2} \\ &= (\cosh\phi x_1 - ct\sinh\phi)e_1 + x_2 e_2 + \iota(ct\cosh\phi - \sinh\phi x_1), \end{aligned} \tag{2.20}$$

which is the conventional Lorentz boost, in terms of the rapidity ϕ, defined by $\tanh\phi = v/c$, which can be rearranged to give $\cosh\phi = \gamma$ and $\sinh\phi = \gamma v/c$. Substituting these relations we find

$$X' = \gamma(x_1 - vt)e_1 + x_2 e_2 + \iota\gamma\left(ct - \frac{vx_1}{c}\right), \tag{2.21}$$

which thus gives the transformation $x_1' = \gamma(x_1 - vt)$, $x_2' = x_2$ and $ct' = \gamma(ct - \frac{vx_1}{c})$, the correct Lorentz boost of coordinates. The formula in Eq. (2.19) can be simply inverted to give

$X = \mathrm{e}^{\iota\hat{\mathbf{v}}\phi/2} X' \mathrm{e}^{-\iota\hat{\mathbf{v}}\phi/2}$, using the fact that $\mathrm{e}^{\iota\hat{\mathbf{v}}\phi/2}\mathrm{e}^{-\iota\hat{\mathbf{v}}\phi/2} = \mathrm{e}^0 = 1$. The relativity of simultaneity is a fundamental result of special relativity, and from the perspective of the Clifford multivector Eq. (2.1), we see that it stems from the fact that, during a boost operation, the terms for space e_1 and e_2 become mixed, resulting in the bivector term e_1e_2, thus creating a variation in the observers time coordinate. Similarly the momentum multivector, shown in Eq. (2.7), will follow the same coordinate transformation law between frames shown in Eq. (2.19), with $P' = \mathrm{e}^{-\iota\hat{\mathbf{v}}\phi/2} P \mathrm{e}^{\iota\hat{\mathbf{v}}\phi/2}$.

We find that the Lorentz boost of electromagnetic fields, as opposed to coordinates, is similar to Eq. (2.19) above, except that we omit the ι bivector in the exponent, that is we are boosting in a perpendicular direction. Given an electric field as $\mathbf{E} = E_x e_1 + E_y e_2$, then applying the boost according to Eq. (2.14), using as an example the exponentiation of a vector $\mathbf{v} \mapsto \phi e_1$, in the e_1 direction, we find

$$\begin{aligned} \mathrm{e}^{-\frac{\hat{\mathbf{v}}\phi}{2}}\mathbf{E}\mathrm{e}^{\frac{\hat{\mathbf{v}}\phi}{2}} &= \left(\cosh\frac{\phi}{2} - e_1\sinh\frac{\phi}{2}\right)(E_x e_1 + E_y e_2)\left(\cosh\frac{\phi}{2} + e_1\sinh\frac{\phi}{2}\right) \quad (2.22) \\ &= E_x e_1 + E_y e_2(\cosh\phi + e_1\sinh\phi) \\ &= E_x e_1 + \gamma E_y e_2 - e_1 e_2 \frac{E_y\gamma v}{c}, \end{aligned}$$

which are the correct Lorentz transformations for an electromagnetic field. That is, the parallel field is unaffected, the perpendicular field E_y has been increased to γE_y and the term $e_1e_2E_y\gamma v/c$, represents the e_1e_2 plane, also describable with an orthogonal vector e_3 in three-space, hence this term gives the expected induced magnetic field B_z.

Hence the exponential map of a Clifford vector, naturally produces the correct Lorentz transformation of spacetime coordinates and the electromagnetic field in the plane, using the spacetime coordinate multivector given by Eq. (2.1) and the field multivector $F = \mathbf{E} + \iota c B$.

(ii) *Velocity addition rule*

If we apply two consecutive parallel boosts, $\mathbf{v}_1 = v_1\hat{\mathbf{v}} \mapsto \phi_1\hat{\mathbf{v}}$ and $\mathbf{v}_2 = v_2\hat{\mathbf{v}} \mapsto \phi_2\hat{\mathbf{v}}$, where $\tanh\phi = \frac{v}{c}$, we have the combined boost operation

$$\mathrm{e}^{\phi_1\hat{\mathbf{v}}}\mathrm{e}^{\phi_2\hat{\mathbf{v}}} = \mathrm{e}^{(\phi_1+\phi_2)\hat{\mathbf{v}}}. \tag{2.23}$$

Hence we have a combined boost velocity

$$v = c\tanh(\phi_1+\phi_2) = \frac{\tanh\phi_1 + \tanh\phi_2}{1+\tanh\phi_1\tanh\phi_2} = \frac{v_1+v_2}{1+v_1v_2/c^2}, \tag{2.24}$$

the standard relativistic velocity addition formula. By inspection, the velocity addition formula implies that a velocity can never be boosted past the speed c, which confirms c as a speed limit.

Hence, we have now demonstrated from the ansatz of the spacetime coordinate described by the multivector shown in Eq. (2.1), that we produce the correct Lorentz transformations, where the variable c is indeed found to be an invariant speed limit. Numerically therefore, c can be identified as the speed of light, since this is the only known physical object which travels at a fixed speed and represents a universal speed limit.

3. Applications

(a) π^+-meson decay

A classic example of experimental proof for the special theory of relativity is its application to the decay of π^+-mesons, which are observed to enter the atmosphere at high velocity $\mathbf{v}$ from outer space, having a known decay time at rest of $\tau_\pi = 2.55 \times 10^{-8}$ s, giving a spacetime coordinate of $X = \iota c\tau_\pi$. Boosting these coordinates to the π^+-meson velocity, we have a boost $\mathrm{e}^{\iota \hat{\mathbf{v}}\phi/2}$, where $\tanh\phi = v/c$, so we therefore find

$$X' = RXR^\dagger = \mathrm{e}^{-\iota\hat{\mathbf{v}}\phi/2}\iota c\tau_\pi \mathrm{e}^{\iota\hat{\mathbf{v}}\phi/2} = \iota c\tau_\pi \mathrm{e}^{\iota\hat{\mathbf{v}}\phi} = \iota c\tau_\pi(\cosh\phi + \iota\hat{\mathbf{v}}\sinh\phi) = \gamma\mathbf{v}\tau_\pi + \iota\gamma c\tau_\pi. \tag{3.1}$$

So that we have a decay time in laboratory coordinates of $ct = \gamma c\tau_\pi$, with a track length in the laboratory of $\mathbf{x} = \gamma\mathbf{v}\tau_\pi$, in agreement with experimental determinations [French, 1987].

(b) Thomas rotation

It is known, that a surprising result occurs when we apply two non-parallel boosts, followed by their inverse boosts, in that the velocity of the frame does not return to zero. Furthermore, there is a rotation of the frame, called the Thomas rotation, a result, in fact, not noticed until 1925 [Taylor and Wheeler, 1966].

For the case of two consecutive general boosts given by

$$R = \mathrm{e}^{-\iota\phi_2\hat{\mathbf{v}}_2/2}\mathrm{e}^{-\iota\phi_1\hat{\mathbf{v}}_1/2} = \mathrm{e}^{-\iota\phi_c\hat{\mathbf{v}}_c/2}\mathrm{e}^{-\iota\theta/2}, \tag{3.2}$$

where we use the results from the Appendix A, to write this in terms of a single combined boost $\phi_c\hat{\mathbf{v}}_c$ and a rotation θ, finding, using the results from the Appendix A,

$$\tan\frac{\theta}{2} = \frac{\sin\delta\sinh\frac{\phi_1}{2}\sinh\frac{\phi_2}{2}}{\cos\delta\sinh\frac{\phi_1}{2}\sinh\frac{\phi_2}{2} - \cosh\frac{\phi_1}{2}\cosh\frac{\phi_2}{2}}, \tag{3.3}$$

where δ is the angle between the boost directions, given by $\cos\delta = \hat{\mathbf{v}}_1 \cdot \hat{\mathbf{v}}_2$. Hence we can see that only for parallel boosts, that is $\delta = 0$, will there not in fact be a Thomas rotation θ, of the frame.

We can also write the Thomas rotation as a single exponential of a multivector

$$R = \mathrm{e}^{-\iota\phi_t\hat{\mathbf{v}}_t/2 - \iota\theta_t/2}, \tag{3.4}$$

using the results of the Appendix B. Hence the homogeneous Lorentz group defined by Eq. (2.17), naturally encompasses the rotations, boosts and Thomas rotations of frames.

(c) A circle as seen by a moving observer

We locate a circle on the plane, centered a unit distance along the e_2 axis, seen by a static observer located at the origin as

$$\hat{\mathbf{x}} = \sin\theta e_1 + \cos\theta e_2, \tag{3.5}$$

which forms a position multivector representing the edge

$$X = \mathbf{x} + \iota ct = ct\hat{\mathbf{x}} + \iota ct, \tag{3.6}$$

where we have written the position vector, representing a photon trajectory traveling from the circle as $\mathbf{x} = ct\hat{\mathbf{x}}$, assuming propagation at the speed of c, in the direction $\hat{\mathbf{x}}$. We also have a moving observer at the origin, with a boost $R = \mathrm{e}^{\iota e_1 \phi/2}$, where $\tanh\phi = v/c$.

Hence we find the new position of the circle, from the boosted coordinates as

$$\begin{aligned} X' &= RXR^{\dagger} \\ &= ct\mathrm{e}^{\iota e_1\phi/2}\sin\theta e_1 \mathrm{e}^{-\iota e_1\phi/2} + ct\cos\theta e_2 + ct\mathrm{e}^{\iota e_1\phi/2}\iota\mathrm{e}^{-\iota e_1\phi/2} \\ &= ct(\sin\theta\cosh\phi + \sinh\phi)e_1 + ct\cos\theta e_2 + \iota ct(\sin\theta\sinh\phi + \cosh\phi). \end{aligned} \tag{3.7}$$

For light, we require $x' = ct$ and hence from Eq. (3.7), by equating the e_1 and ι coefficients, we require $|ct(\sin\theta\cosh\phi + \sinh\phi)e_1 + ct\cos\theta e_2| = |\iota ct(\sin\theta\sinh\phi + \cosh\phi)|$, and hence the light rays follow the unit vector

$$\hat{\mathbf{x}}' = \frac{(\sin\theta\cosh\phi + \sinh\phi)e_1 + \cos\theta e_2}{\sin\theta\sinh\phi + \cosh\phi}. \tag{3.8}$$

If we consider the vector $\mathbf{m} = e_2 + \sinh\phi\cos\theta e_1$, then

$$\mathbf{m}\cdot\hat{\mathbf{x}}' = \cosh\alpha\cos\theta, \tag{3.9}$$

which shows that $\mathbf{m}$ points towards the center of the moving circle. If we now normalize $\mathbf{m}$, we find

$$\hat{\mathbf{m}}\cdot\hat{\mathbf{x}}' = \cos\theta' = \frac{\cos\theta\cosh\phi}{\sqrt{1+\sinh^2\phi\cos^2\theta}} \tag{3.10}$$

or $\tan\theta' = \frac{\tan\theta}{\gamma}$. Hence the circle appears to shrink, although still appears as a circle.

For example, the diameter of the moon is approximately $2\theta = 0.5$ degree, but if the earth was moving at half the speed of light relative to the moon, that is $\tanh\phi = 0.5$, then using Eq. (3.10), its diameter would appear to shrink down to $2\theta' = 0.435$ degrees, or to 87% of its original size.

As we are working with a two-dimensional Clifford algebra, we can only make conclusions about observations in the plane, although it was noted by Penrose [1959] that a sphere will also remain a sphere in this situation.

(*d*) *Modeling fundamental particles as multivectors*

The multivector defined in Eq. (2.1), will now be used as a representation for individual particles, from which the results of special relativity, such as Lorentz contraction and time dilatation, also arise.

Using the multivector defined in Eq. (2.9), we can define a particle moving with a velocity $\mathbf{v}$, given by

$$P = \hbar\mathbf{k} + \iota\frac{\gamma\hbar\omega_0}{c}, \tag{3.11}$$

where $\hbar\mathbf{k} = \gamma m\mathbf{v}$. As expected the momentum multivector is invariant between inertial frames with $P^2 = -mc^2$, in agreement with Eq. (2.8). For a particle at rest, we therefore have $P_0 = \iota\frac{E}{c} = \iota\frac{\hbar\omega_0}{c}$, where we use the de Broglie relation between total energy E and the frequency $E = \hbar\omega$, to find

$$\omega_0 = \frac{mc^2}{\hbar}. \tag{3.12}$$

Integrating the momentum multivector with respect to the proper time τ, remembering that $dt = \gamma d\tau$, and dividing by the rest mass m we find

$$X = \mathbf{x} + \iota \frac{\hbar \theta_0}{mc} = \mathbf{x} + \iota \lambda_c \theta_0, \tag{3.13}$$

where $\omega_0 = \frac{d\theta_0}{dt}$ and the constant of integration being zero as we set $\mathbf{x} = 0$ and $\theta_0 = 0$ at $\tau = 0$. Inspecting the bivector component, we find

$$\lambda_c d\theta_0 = \lambda_c w_0 dt = \left(\frac{\hbar}{mc}\right)\left(\frac{mc^2}{\hbar}\right) dt = cdt, \tag{3.14}$$

showing that the local time of the particle can be identified as the phase of the de Broglie wave, with $dt = \left(\frac{\lambda_c}{c}\right) d\theta_0$.

The form of the multivector naturally leads to a model for the electron, analogous to a *zitterbewegung* model, first described by Schrödinger [1930], an effect now extensively verified by experiment, [Wunderlich, 2010, Gerritsma et al., 2010, Zawadzki and Rusin, 2011]. The *zitterbewegung* model assumes that the electron consists of lightlike particle oscillating at the speed of light, with an amplitude equal to the reduced Compton wavelength, where the macroscopically observed velocity $\mathbf{v}$ of the electron now represents the drift velocity of this lightlike particle.

In order to produce a simplified semi-classical model, we assume a circular periodic motion with a radius

$$r_0 = \frac{\hbar}{m_e c} = \lambda_c, \tag{3.15}$$

where $\lambda_c = \frac{\hbar}{m_e c}$ is the reduced Compton wavelength, which then gives the tangential velocity $v = r_0 \omega_0 = \left(\frac{\hbar}{2m_e c}\right)\left(\frac{2m_e c^2}{\hbar}\right) = c$ as required for a *zitterbewegung* model.

We know from inspection of Eq. (3.11), that under a boost, the frequency ω_0 will increase to $\omega = \gamma\omega_0$, with the radius required to shrink to $r = r_0/\gamma$, so that the tangential velocity $v = rw = \left(\frac{r_0}{\gamma}\right)\gamma w_0 = r_0 w_0 = c$, remains at the speed of light. We also found in Section 3 (c), that a circle will shrink by the ratio of γ in agreement with this result. Hence this simplified two-dimensional Bohr-type model in Fig. 1, indicates that under a boost, the de Broglie frequency will increase to $\gamma\omega_0$ implying an energy and hence a mass increase γm_0, the frequency increase also implies time dilatation, with the local time being provided by the phase of the de Broglie wave, and the shrinking radius producing length contraction, thus producing known relativistic effects.

In the footsteps of previous investigations [Schrödinger, 1930, Penrose, 2004, Hestenes, 1990], a future development is to extend this work to three dimensional space.

(*e*) *Scattering processes*

It is established that energy and momentum conservation applies in relativistic dynamics, provided that the rest energy $m_0 c^2$ is now included along with the appropriate relativistic corrections, that is, defining momentum as $\gamma m\mathbf{v}$, and the energy as γmc^2. We now find, however, that the two conservation laws can be bundled into a single momentum multivector defined in Eq. (2.7).

For example, if we are given a set of particles which are involved in an interaction, which then produce another set of particle as output. Then, in order to describe this collision interaction process we firstly include a separate momentum multivector for each

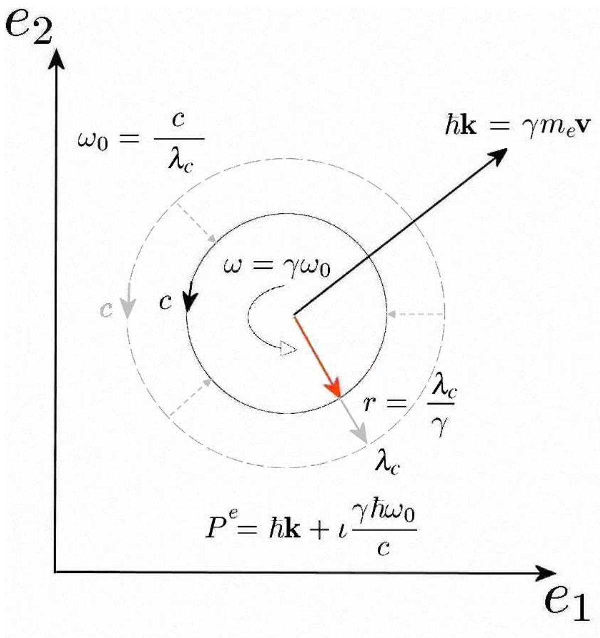

Figure 1. Multivector model for the electron, consisting of a light-like particle orbiting at the de Broglie angular frequency ω_0 at a radius of λ_c in the rest frame, and when in motion described generally by the multivector $P_e = \hbar\mathbf{k} + \iota\frac{\gamma\hbar\omega_0}{2c}$. Under a boost, the de Broglie angular frequency will increase to $\gamma\omega_0$, giving an apparent mass increase and time dilatation, the electron radius will also shrink by γ, implying length contraction, thus naturally producing the key results of special relativity.

particle, and then energy and momentum conservation between the initial and final states is simply given by

$$\sum P_{\text{initial}} = \sum P_{\text{final}}, \tag{3.16}$$

assuming we are dealing with an isolated system. We know $E = |\mathbf{p}|c$ for a massless particle, so using Eq. (2.7) we write the momentum multivector for a photon as $\Gamma = \mathbf{p} + \iota|\mathbf{p}|$, which gives $\Gamma^2 = 0$ and for a massive particle $P^2 = -m_0^2c^2$ as shown in Eq. (2.8).

For Compton scattering, which involves an input photon striking an electron at rest, with the deflected photon and moving electron as products, we can write energy and momentum conservation using the multivectors as simply $\Gamma_i + P_i = \Gamma_f + P_f$, which we can rearrange to

$$(\Gamma_i - \Gamma_f) + P_i = P_f. \tag{3.17}$$

Squaring both sides we find

$$(\Gamma_i - \Gamma_f)^2 + P_i(\Gamma_i - \Gamma_f) + (\Gamma_i - \Gamma_f)P_i + P_i^2 = P_f^2, \tag{3.18}$$

remembering that in general the multivectors do not commute. Now, we have the generic results that $P_i^2 = P_f^2 = -m_0^2c^2$ and $(\Gamma_i - \Gamma_f)^2 = \Gamma_i^2 + \Gamma_f^2 - \Gamma_i\Gamma_f - \Gamma_f\Gamma_i = -2\Gamma_i \cdot \Gamma_f = -2(\mathbf{p}_i \cdot \mathbf{p}_i + |\mathbf{p}_i||\mathbf{p}_f|) = 2|\mathbf{p}_i||\mathbf{p}_f|(1 - \cos\theta)$, using $\Gamma_i^2 = \Gamma_f^2 = 0$. For the following two terms in Eq. (3.18), using $P_i = \iota m_0 c$, we have $m_0c(\iota(\Gamma_i - \Gamma_f) + (\Gamma_i - \Gamma_f)\iota) = -2m_0c(|\mathbf{p}|_i - |\mathbf{p}|_f)$.

We therefore find from Eq. (3.18) that

$$|\mathbf{p}_i||\mathbf{p}_f|(1-\cos\theta) - m_0 c(|\mathbf{p}_i| - |\mathbf{p}_f|) = 0. \tag{3.19}$$

Dividing through by $|\mathbf{p}_i||\mathbf{p}_f|$ and substituting $|\mathbf{p}| = \frac{h}{\lambda}$ we find Compton's well known formula

$$\lambda_f - \lambda_i = \frac{h}{m_0 c}(1-\cos\theta). \tag{3.20}$$

The advantage of the momentum multivector is that energy and momentum conservation can be considered in unison as shown in Eq. (3.17), which also provides a clear solution path, whereas conventional textbook methods rely on manipulating two separate equations describing momentum and energy conservation [French, 1987].

4. Discussion

It is well established that Clifford's geometric algebra, is a natural formalism to study the geometrical operations of the plane, such as reflections and rotations [Doran and Lasenby, 2003]. However, we demonstrate additionally that space time represented as the Clifford multivector, as shown in Eq. (2.1), is a natural alternative to Minkowski spacetime, producing the correct spacetime interval and the required Lorentz transformation, directly from the properties of the algebra. Hence we propose that multivectors are the natural mathematical structure to describe spacetime.

The definition of a spacetime event as a multivector in Eq. (2.1), also provides a new perspective on the nature of time, in that rather than being defined as an extra Euclidean dimension, it becomes instead a composite quantity of space, the bivector e_1e_2. Minkowski's famous quote is therefore particularly apt, *Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality* [Einstein, 1952].

As we have seen in Eq. (2.11), a bivector represents a rotation, and a possible interpretation is that time is defined by the angular rotation of the de Broglie frequency $w = \frac{E}{\hbar}$ related to the frequency of the *zitterbewegung*, as hypothesized by Schrödinger, Penrose and others [Schrödinger, 1930],[Penrose, 2004],[Hestenes, 1990], and encapsulated by a two dimensional model derived from the multivector description in Eq. (3.11), shown in Fig. 1.

Such a view of time as an entity possessing rotational attributes, has also been supported by recent experiments, which have identified a fluctuating electric field at the de Broglie frequency for an electron [Catillon et al., 2008], and the use of the rotating electric field in circularly polarized light as an attosecond clock to probe atomic processes [Pfeiffer et al., 2011, Ueda and Ishikawa, 2011, Eckle et al., 2008, Krausz and Ivanov, 2009]. Hence society's popular description of time, as the 'river of time', based in part on the pronouncement of Newton in the Principia, Book 1 [Newton, 1686], that time . . . *flows equably without relation to anything external* . . . , along with time being promoted by Minkowski as a fourth dimension, may perhaps need to be amended to include a rotational aspect, and adopting a water analogy, time might be viewed descriptively as a whirlpool or an eddy.

The spacetime multivector defined in Eq. (2.1) for the plane, can obviously be rotated into a larger three-dimensional space, without losing its algebraic properties, becoming equivalent to spacetime events conventionally described by the Lorentz four-vectors $x^\mu = (ct, \mathbf{x})$, $\mu \in [0,1,2,3]$, and similarly for the velocity and momentum multivectors defined in Eq. (2.5) and Eq. (2.7) respectively, however the Clifford multivector provides a more natural algebraic setting, without the requirement of matrices or a metric in order to describe the Lorentz transformations. Also, Minkowski spacetime diagrams, consisting of a space axis and a time axis, still apply, though the time axis no longer represents a Euclidean time dimension, but simply shows the algebraic relationship between time as a bivector and space as a vector. The abstract nature of Minkowski diagrams is confirmed by the rotation of the coordinate axes for the moving observer, which are tilted with respect to the original frame when displayed on the Minkowski diagram, a practice which is purely formal and not indicating a real rotation of the space or time axes between the frames [French, 1987]. Also the use of the momentum multivector defined in Eq. (2.7) allows the application of momentum and energy conservation within a single algebraic entity, as shown in the applications in Section 3, for Compton scattering.

In summary, this approach from a pure mathematical perspective based on the ansatz of spacetime represented by a Clifford multivector shown in Eq. (2.1), produces the correct spacetime metric and Lorentz transformations directly from the properties of the algebra, without needing to reference other postulates. This systematic approach, is also shown to be advantageous in describing the Lorentz transformations, in that an exhaustive exploration of the exponential map of a multivector, naturally produced rotations, boosts and the Thomas rotation of frames, and in fact the full homogeneous Lorentz group represented simply as the multivector exponential $\mathrm{e}^{\phi\hat{\mathbf{v}}+\iota\theta}$. The Lorentz group is typically seen as $so(3,1)$, describing three space and one time dimension. However as we have shown, this is larger than required in order to describe the algebra of space and time defined by the Lorentz transformations.

We also see educational benefits with the use of multivectors as a description of spacetime, which allow the Lorentz transformations to arise naturally in a simplified algebraic setting, without any unnecessary mathematical 'overheads', such as matrices, the dot product or metric structures.

5. Appendix

(*a*) *Boost-rotation form of a multivector*

Given a general two-space multivector

$$M = a + w_1 e_1 + w_2 e_2 + \iota b = r\cos\alpha + s\cos\beta e_1 + s\sin\beta e_2 + \iota r\sin\alpha, \tag{5.1}$$

in order to write this in the exponential form

$$\begin{aligned} \rho\mathrm{e}^{\phi\hat{\mathbf{v}}}\mathrm{e}^{\theta\iota} \quad = \quad & \rho\Big(\cosh\phi\cos\theta + \sinh\phi(v_x\cos\theta - v_y\sin\theta)e_1 \\ & + \sinh\phi(v_y\cos\theta + v_x\sin\theta)e_2 + \iota\cosh\phi\sin\theta\Big), \end{aligned} \tag{5.2}$$

consisting of a separate boost and rotation, we require

$$\begin{aligned} \theta &= \alpha = \arctan\left(\frac{b}{a}\right), \ \rho = \sqrt{r^2 - s^2} = \sqrt{a^2 + b^2 - w_1^2 - w_2^2} \\ \phi &= \operatorname{arctanh}\left(\frac{s}{r}\right) = \operatorname{arctanh}\left(\frac{\sqrt{w_1^2 + w_2^2}}{\sqrt{a^2 + b^2}}\right) \\ v_1 &= \cos(\beta - \alpha) = \frac{w_1 a + w_2 b}{\sqrt{w_1^2 + w_2^2}\sqrt{a^2 + b^2}}, \ v_2 = \sin(\beta - \alpha) = \frac{w_2 a - w_1 b}{\sqrt{w_1^2 + w_2^2}\sqrt{a^2 + b^2}}, \end{aligned} \tag{5.3}$$

from which we can determine θ, ϕ, $\mathbf{v}$ and ρ. Hence we can convert from a general multivector M to the exponential form shown in Eq. (5.2). Inspecting Eq. (5.3), we see that this form of a multivector will not be possible if $a^2+b^2=0$, that is if we have a pure vector, but as already noted in Eq. (2.12), this will produce a pure reflection. If $r^2<s^2$ then ρ goes imaginary, however this would imply that we are trying to apply a boost with $\frac{v}{c}>1$, which once again shows that the speed of light is a speed limit. Hence we see that applying boosts through the exponential maps of the velocity vector, it is impossible to boost past the speed of light, and hence this also confirms the speed c as a speed limit for particles.

(*b*) *Exponential of the full multivector*

We found that exponentiating the even subalgebra, that is $a+\iota b$, produces rotations, and exponentiating a vector $\mathbf{v}=v_1e_1+v_2e_2$, produces a Lorentz boost of the field, so we now seek the exponential of a full multivector. This cannot simply be split into two separate operations, as the vector and bivector terms do not commute.

Firstly, we observe that, defining $B=\mathbf{v}+\iota b$, then $B^2=(\mathbf{v}+\iota b)^2=\mathbf{v}^2-b^2+b\mathbf{v}\iota+b\iota\mathbf{v}=\mathbf{v}^2-b^2$, produces a scalar. The B term can also be separated from the scalar a, as they commute, as follows

$$\begin{aligned}
\mathrm{e}^M=\mathrm{e}^{a+\mathbf{v}+\iota b} &= \mathrm{e}^a\mathrm{e}^{\mathbf{v}+\iota b} \\
&= \mathrm{e}^a\left(1+B+\frac{\mathbf{v}^2-b^2}{2!}+\frac{B(\mathbf{v}^2-b^2)}{3!}+\frac{(\mathbf{v}^2-b^2)^2}{4!}+\ldots\right) \\
&= \mathrm{e}^a\Bigg(1+\frac{\mathbf{v}^2-b^2}{2!}+\frac{(\mathbf{v}^2-b^2)^2}{4!}+\ldots \\
&\quad+\frac{B}{\sqrt{\mathbf{v}^2-b^2}}\left(\sqrt{\mathbf{v}^2-b^2}+\frac{\sqrt{\mathbf{v}^2-b^2}(\mathbf{v}^2-b^2)}{3!}+\ldots\right)\Bigg) \\
&= \mathrm{e}^a\left(\cosh|B|+\hat{B}\sinh|B|\right),
\end{aligned} \tag{5.4}$$

where $|B|=\sqrt{\mathbf{v}^2-b^2}$, assuming $\mathbf{v}^2>b^2$, and $\hat{B}=\frac{B}{|B|}=\frac{\mathbf{v}+\iota b}{|B|}$. If $v^2<b^2$ we simply replace the hyperbolic trigonometric functions with trigonometric functions, and if $\mathbf{v}^2=b^2$, then referring to the second line of the above derivation, we see that all terms following B are zero, and so, in this case $\mathrm{e}^M=\mathrm{e}^a(1+B)=\mathrm{e}^a(1+\mathbf{v}+\iota b)$. The reverse process, of finding the exponent for a given multivector, represents taking the log of a multivector.

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