The Asymptotic Mandelbrot Law of Some Evolution Networks

Li Li

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Abstract

In this letter, we study some evolution networks that grow with linear preferential attachment. Based upon some recent results on the quotient Gamma function, we give a rigorous proof of the asymptotic Mandelbrot law for the degree distribution $p_k \propto (k+c)^{-\gamma}$ in certain conditions. We also analytically derive the best fitting values for the scaling exponent γ and the shifting coefficient c.

Complex networks are now the joint focus of many branches of research[1−3]. Particularly, the scale-free property of some networks attracts continuous interests, due to their importance and pervasiveness[4−6]. In short, this property means that the degree distribution of a network obeys a power law $P(k) \propto k^{-\gamma}$, where k is the degree and $P(k)$ is the corresponding probability density, and the scaling exponent γ is a constant. A pioneering model that generates power-law degree distribution was presented by Barabási and Albert (BA)[4].

In recent studies, it was found that in some complex networks, e.g. transportation networks[7] and social collaboration networks[8], the degree distribution follows the so-called "shifted power law"[9] $P(k) \propto (k+c)^{-\gamma}$, where the shifting coefficient c is another constant. This property is also called "Mandelbrot law"[10].

To understand the origins of such Mandelbrot law, Ren, Yang and Wang[11] proposed a interesting growing network that is generated with linear preferential attachment. In such networks, there exits a recursive dependence relationship between every two consecutive degrees

$$p(k)\left[k+\frac{2+2m\beta}{1-\beta}\right]=p(k-1)\left[k+\frac{2m\beta}{1-\beta}-1\right] \quad (1)$$

where where $k = 2, ..., n$, n is the number of nodes. m is a positive integer constant and $\beta \in [0, 1]$ is another constant.

Defining $a = \frac{2m\beta}{1-\beta} - 1$, $b = \frac{2+2m\beta}{1-\beta}$, we can abbreviate Eq.(1) as

$$p_k\,[k+b] = p_{k-1}\,[k+a] \quad (2)$$

To derive the asymptotic of the degree distribution, Ren, Yang and Wang[11] studied the following three kinds of approximations:

I) forward-difference approximation, assuming

$$\frac{dp(k)}{dk} \approx p(k) - p(k-1) = p(k) - \frac{k+b}{k+a}p(k) = \frac{a-b}{k+a}p(k) \tag{3}$$

we have an estimation of the power-law as

$$p(k) \propto (k+a)^{-(b-a)} \tag{4}$$

II) backward-difference approximation, assuming

$$\frac{dp(k)}{dk} \approx p(k+1) - p(k) = \frac{k+1+a}{k+1+b}p(k) - p(k) = \frac{a-b}{k+1+b}p(k) \tag{5}$$

we have another estimation of the power-law as

$$p(k) \propto (k+b+1)^{-(b-a)} \tag{6}$$

III) Suppose we must have a Mandelbrot law $p(k) \propto (k+c)^{-\gamma}$. As a result, we have $p(k-1) \propto (k-1+c)^{-\gamma}$. Substitute these two approximations in the logarithm type of Eq.(2), we have

$$\ln\frac{k+a}{k+b} = \ln\frac{p(k)}{p(k-1)} = -\gamma\ln(k+c) + \gamma\ln(k-1+c) \tag{7}$$

Rewrite Eq.(7) as

$$\ln\frac{1+a\frac{1}{k}}{1+b\frac{1}{k}} = \gamma\ln\frac{1+(c-1)\frac{1}{k}}{1+c\frac{1}{k}} \tag{8}$$

and apply the second order Taylor expansion of $\frac{1}{k}$ in Eq.(8), we have

$$p(k) \propto \left(k + \frac{b+a+1}{2}\right)^{-(b-a)} \tag{9}$$

All these three estimations indicates that the scaling exponent of the degree distribution should be $-(b-a)$. Simulation results[11] show that Eq.(9) gives the best approximation accuracy of the empirical distributions. However, we still need a rigorous proof of this interesting finding.

Indeed, further assuming $\sum_{k=1}^{n} p(k) = 1$, we have the following matrix equation

$$\begin{bmatrix} 2+a & -(2+b) & 0 & ... & 0 \\ 0 & 3+a & -(3+b) & ... & 0 \\ & & ... & & \\ 0 & 0 & ... & n+a & -(n+b) \\ 1 & 1 & ... & 1 & 1 \end{bmatrix} \begin{bmatrix} p(1) \\ p(2) \\ ... \\ p(n-1) \\ p(n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ ... \\ 0 \\ 1 \end{bmatrix} \tag{10}$$

Using Gaussian elimination algorithm, we can directly solve $p(n)$ from Eq.(10) as

$$\begin{aligned} p(n) &= \left[1+\frac{n+b}{n+a}+...+\prod_{j=2}^{n}\frac{j+b}{j+a}\right]^{-1} \\ &= \left[1+\sum_{i=2}^{n}\prod_{j=i}^{n}\frac{j+b}{j+a}\right]^{-1} \end{aligned} \tag{11}$$

Based on the recursive relationship Eq.(2), for a given n, we have

$$\begin{aligned} p(k) &= p(n)\left(\prod_{j=k+1}^{n}\frac{j+b}{j+a}\right) = p(n)\left(\frac{\prod_{j=1}^{n}\frac{j+b}{j+a}}{\prod_{j=1}^{k}\frac{j+b}{j+a}}\right) \\ &= p(n)\left(\prod_{j=1}^{n}\frac{j+a}{j+b}\right)\left(\prod_{j=1}^{k}\frac{j+a}{j+b}\right) \end{aligned} \tag{12}$$

where $k = 1, ..., n-1$.

It is well known that for Gamma function $\Gamma(z)$, we have $\Gamma(z+1) = z\Gamma(z)$. So, we get

$$(j+b) = \frac{\Gamma(j+1+b)}{\Gamma(j+b)}, \quad (j+a) = \frac{\Gamma(j+1+a)}{\Gamma(j+a)} \tag{13}$$

where $j = 1, ..., n-1$.

From Eq.(12), we have

$$\begin{aligned} p(k) &= p(n)\left(\prod_{j=1}^{n}\frac{j+a}{j+b}\right)\left(\prod_{j=1}^{k}\frac{\Gamma(j+1+a)}{\Gamma(j+a)}\right)\left(\prod_{j=1}^{k}\frac{\Gamma(j+b)}{\Gamma(j+1+b)}\right) \\ &= p(n)\left(\prod_{j=1}^{n}\frac{j+a}{j+b}\right)\frac{\Gamma(k+1+a)}{\Gamma(1+a)}\frac{\Gamma(1+b)}{\Gamma(k+1+b)} \\ &= \lambda\cdot\frac{\Gamma(k+1+a)}{\Gamma(k+1+b)} \end{aligned} \tag{14}$$

where $\lambda = p(n)\left(\prod_{j=1}^{n}\frac{j+a}{j+b}\right)\frac{\Gamma(1+b)}{\Gamma(1+a)}$ is a constant.

Eq.(14) indicates that $p(k)$ has the same asymptotic behavior of $\frac{\Gamma(k+1+a)}{\Gamma(k+1+b)}$. Actually, the quotient of two Gamma functions is a difficult problem that received consistent attentions[12−15]. There are numbers of approximation formulas which are not accurate enough for the above applications. Fortunately, an important results had been obtained very recently[15] as

Lemma 1[15] Given two constants s and t, when $x \to \infty$, we have

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{\frac{1}{t-s}} \sim \sum_{k=0}^{\infty} F_k(t,s)x^{-n+1} \tag{15}$$

where $F_k(t,s)$ are the polynomials of order n defined by

$$F_0(t,s) = 1 \tag{16}$$

$$F_n(t,s) = \frac{1}{n}\sum_{k=1}^{n}(-1)^{k+1}\frac{B_{k+1}(t) - B_{k+1}(s)}{(k+1)(t-s)}F_{n-k}(t,s) \tag{17}$$

where $n \geq 1$, $B_k(t)$ is the Bernoulli polynomials (page 40 of [16]) for t.

Based on **Lemma 1**, from Eq.(14), we can have an accurate expansion of the degree distribution as follows

$$\left[\frac{p(k)}{\lambda}\right]^{\frac{1}{a-b}} \sim k + \frac{a+b+1}{2} + \frac{1-(a-b)^2}{24}k^{-1} + ... \tag{18}$$

As $k \to \infty$, we have $\left[\frac{p(k)}{\lambda}\right]^{\frac{1}{a-b}} \approx k + \frac{a+b+1}{2}$. Thus, we reach the following conclusion rigorously.

Theorem 1 The degree distribution follows an asymptotic Mandelbrot law Eq.(9) for some complex networks that grow with linear preferential attachment depicted by Eq.(2).

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