THE LOG-CONVEX DENSITY CONJECTURE AND VERTICAL SURFACE AREA IN WARPED PRODUCTS.

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Abstract. We examine the vertical component of surface area in the warped product of a Euclidean interval and a fiber manifold with product density. We determine general conditions under which vertical fibers minimize vertical surface area among regions bounding the same volume and use these results to conclude that in many such spaces vertical fibers are isoperimetric. Our main hypothesis is that the surface area of a fiber be a convex function of the volume it bounds. We apply our results in the specific case of $\mathbb{R}^n - \{0\}$ realized as the warped product $(0, \infty) \times_r S^{n-1}$, providing many new examples of densities where spheres about the origin are isoperimetric, including simple densities with finite volume, simple densities that at the origin are neither log-convex nor smooth, and non-simple densities. We also generalize the results of Kolesnikov and Zhdanov on large balls in $\mathbb{R}^n$ with increasing strictly log-convex simple density. We situate our work in relation to the Log-Convex Density Conjecture of Rosales *et al.* and the recent work by Morgan, Ritoré, and others on formulating a generalized log-convex density/stable spheres conjecture.

Contents

1. Introduction 1
2. Minimization of vertical surface area in warped products 4
3. $\mathbb{R}^n - \{0\}$ with radial density 11
References 16

1. Introduction

In this paper we consider the isoperimetric problem in manifolds with density:

Problem. In a Riemannian manifold M equipped with a positive function Ψ_S weighting surface area and a positive function Ψ_V weighting volume, which region has the least weighted surface area among all regions of weighted volume V_0?

In this paper a hypersurface is always rectifiable and a region always has rectifiable boundary. A region is called isoperimetric if it has minimal weighted surface area among all regions of the same weighted volume. A hypersurface is called isoperimetric if it bounds an isoperimetric region. We note that there is no a priori guarantee that an isoperimetric region exists, and indeed there are simple examples of spaces with density where isoperimetric regions do not exist (see, e.g., [5, Prop. 7.3]). In the rest of this paper we will omit the term "weighted" before surface area and volume and refer to regular surface area and volume as "unweighted." For a general reference on manifolds with density see [13] or better, [14, Ch. 18].

The function Ψ_S from above will be referred to as the *surface density* or *perimeter density* and the function Ψ_V as the *volume density* (or, sometimes on 2-dimensional manifolds, *area density*). By a conformal change of metric one can always take $\Psi_S = \Psi_V$ (see Proposition 2.1), which is referred to as *simple density*, however it is often more convenient to vary the density than to vary the metric and so we allow Ψ_S and Ψ_V to differ. Other interesting and sometimes useful special cases are $\Psi_V = 1$ (*surface density* or *perimeter density*, see e.g. [1], [5, Thm. 7.4]) and $\Psi_S = 1$ (*volume density*, see e.g. [5, proof of Thm. 4.8], [15]).

Most work in manifolds with density has focused on $\mathbb{R}^n$ with simple density (see e.g. [14, Ch. 18], [8], [5],[21],[2], [10], [4], [22], [3], [6]). Of particular interest is radial simple density (density a function of the radius), where interest has centered around the following conjecture of Rosales *et al.* [22, Conj. 3.12]:

Conjecture 1.1 (Log-Convex Density Conjecture [22, Conj. 3.12])**.** *In $\mathbb{R}^n$, $n \geq 2$ with radial log-convex simple density, balls about the origin are isoperimetric for every volume.*

For a simple density, log-convexity is equivalent to the stability of balls about the origin. As pointed out by Morgan [15, 18], a further regularity condition is necessary to avoid examples such as $\mathbb{R}^n$ with simple density e^{r^2-2r+2} where for small volumes isoperimetric regions are approximate balls centered on the unit sphere, or $\mathbb{R}^n$ with simple density r^{-p}, $0 < p \leq n$ where isoperimetric regions do not exist (see [5, Prop. 7.3]). Morgan [15, 16, 18] also discusses the equivalent conjecture in $\mathbb{R}^n$ with general density – that is, that if spheres about the origin are stable then under further regularity conditions at the origin they are isoperimetric, and computes the stability condition on the densities that plays the role of log-convexity in this case. While it is still unclear what the most general statement of the conjecture should be, our Theorem 3.1, which states that spheres are isoperimetric in $\mathbb{R}^n - \{0\}$ with any radial density such that the surface area of spheres is a convex function of the volume they bound and satisfying some additional minor hypotheses, allows us to give several interesting new examples of densities on $\mathbb{R}^n$ (allowing singularity at the origin) for which spheres about the origin are isoperimetric. These examples include simple densities with finite volume, simple densities that are neither log-convex nor smooth at the origin, and non-simple densities (see Example 3.5). In particular, the existence of large families of densities on $\mathbb{R}^n$ which are neither log-convex nor smooth at the origin for which stable spheres are nonetheless isoperimetric complicates the formulation suggested in [15, 18]. In Theorem 2.9, Corollary 2.10, and Corollary 3.7 we generalize a result of Kolesnikov and Zhdanov [8, Prop. 4.7] on large balls about the origin in $\mathbb{R}^n$ with strictly log-convex increasing simple density. We use this generalization to show that for a large family of densities on $\mathbb{R}^n$ for which spheres are stable they are also isoperimetric for large volumes, and we provide several examples of densities where these results apply, including the above-mentioned example of $\mathbb{R}^n$ with simple density e^{r^2-2r+2}, thus giving an example of a space where all spheres are stable but only certain spheres are isoperimetric (Example 3.8).

We obtain our results by analysing the component of surface area tangential to spheres about the origin. This is a special case of the concept of vertical surface area, which we define for a rectifiable hypersurface in the warped product of a real interval and a Riemannian manifold:

Definition 1.2. Let L be a Riemannian manifold of dimension $n-1$ with metric dl^2 and let Z be the interval $(A,B),\ A<B\in\mathbb{R}\cup\pm\{\infty\}$ with the usual metric dr^2. Consider the warped product $Z\times_g L$ with continuous warp factor g giving metric $dr^2+g(r)^2df^2$ and continuous surface density Φ_S and volume density Φ_V. The *vertical surface area* of a rectifiable hypersurface H in the warped product with density $Z\times_g F$ is

$$|H|_{Vert}=\int_H|\vec{n}\cdot\vec{r}|\Phi_S d\mathcal{H}^{n-1}$$

where at any point $\vec{r}$ is the positively oriented unit vector perpendicular to L and $d\mathcal{H}^{n-1}$ is the $n-1$ dimensional Hausdorff measure on H inherited from the warped product $Z\times_g L$ without density. Locally where H is a graph over L,

$$|H|_{Vert}=\int\Phi_S(r(l),l)g(r(l))^{n-1}dL$$

Note that where H is parallel to horizontal fibers the contribution to vertical surface area is 0 and so we can always calculate vertical surface area using only the local formula. Further, the definition of vertical surface area gives trivially the expected inequality

$$|H|_{Vert}\leq|H|,$$

where $|H|$ is the surface area of H, with equality only when H is a union of vertical fibers.

We note that general warped products with density have already appeared in [19].

Example 1.3. We can realize $\mathbb{R}^n-\{0\}$ with Euclidean metric as the warped product $(0,\infty)\times_r S^{n-1}$. In this context, we often refer to vertical surface area as *tangential surface area* because it is the component of surface area tangential to spheres about the origin. Analysis of tangential surface area was used in [3, Prop. 4.3] and [5, Thm. 7.4] to prove that in $\mathbb{R}^n-\{0\}$ with density $r^p,\ p<-n$, spheres about the origin minimize tangential surface area and are thus isoperimetric (bounding volume at infinity), and similarly in [5, Prop. 7.5] to give a new proof of the result of Betta, et al. [1, Thm. 4.3] that in $\mathbb{R}^n$ with certain surface densities spheres about the origin are isoperimetric. Section 2 generalizes and refines these ideas.

Remark 1.4. One can also study the weaker inequality $\int_H(\vec{n}\cdot\vec{r})\cdot f\cdot\Phi_S d\mathcal{H}^{n-1}<|H|$ where f is a function on the real interval with $|f|\leq 1$. This is the approach taken by Kolesnikov and Zhdanov [8] in their Proposition 6.7 and the surrounding discussion in the setting of $\mathbb{R}^n$ with increasing simple radial density. Using this formula and the divergence theorem they show that for density $e^{\phi(r)}$ with ϕ convex, radially symmetric and superlinear (e.g. $e^{r^\alpha},\ \alpha>1$), large balls about the origin are isoperimetric. In Corollaries 2.10 and 3.7 we generalize this result using Theorem 2.9 which gives conditions on when a single vertical fiber in a warped product with density minimizes vertical surface area, and in Example 3.8 we give several specific densities where our result applies. Our proof turns on the use of comparison spaces with different surface densities as developed in Section 2. The weighting factor used by Kolesnikov and Zhdanov is similar, however, it is not exactly analogous – their weighting factor can be negative valued whereas our comparison spaces always have positive surface densities. The difference results primarily from the absence of an absolute value around the term $\vec{n} \cdot \vec{r}$ in their approach which thus gives a weaker inequality but allows the application of the divergence theorem.

The most closely related results are those of Kolesnikov and Zhdanov [8, Sec. 6] on large balls in $\mathbb{R}^n$ with increasing strictly log-convex density and those of Diaz et al. [5, Sec. 7] on $\mathbb{R}^n$ with density r^p, $p < 0$, both of which are generalized by this work. We note also that both Montiel [11, 12] and Rafalski [20] have obtained related results for graphs over horizontal regions in warped products.

In Section 2 we prove the most general versions of our theorems in the context of warped products with density. In Section 3 we apply these result to the most interesting case of $\mathbb{R}^n$ with radial density and give many specific examples. Although the results of Section 3 are stated only for radial densities, they apply equally with product surface and volume densities $\Psi_S(r)\Phi(\Theta)$, $\Psi_V(r)\Phi(\Theta)$, and even more generally to product densities of this form on any warped product $(0,\infty)\times_r K$ with K a compact Riemannian manifold (see the beginning of Section 3).

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2. Minimization of vertical surface area in warped products

Proposition 2.1 is a well-known result.

Proposition 2.1. *Let M be an $n-$dimensional riemannian manifold equipped with metric dm^2, continuous surface density Φ_S, and continuous volume density Φ_V. There exists a continuous conformal change of metric on M, $d\tilde{m}^2$, and a positive continuous function, Ψ, such that the volume and surface area of a region in M with metric dm^2 and densities Φ_V and Φ_S is the same as the volume and surface area of the same region in M with metric $d\tilde{m}^2$ and simple density Ψ.*

Proof. Take $d\tilde{m}^2 = \left[\left(\frac{\Phi_V}{\Phi_S}\right) dm\right]^2$ and $\Psi = \frac{\Phi_S^n}{\Phi_V^{n-1}}$. □

One consequence of Proposition 2.1 is that when considering the isoperimetric problem in a manifolds with density one can always reduce to the case of simple density. One can also always make a similar change of coordinates in order to work with volume density or surface density. This reduction, however, does not in general preserve the structure of a warped product, as clarified by the following example:

Example 2.2. Let X be a Riemannian manifold with metric dx^2 and Y a Riemannian manifold with metric dy^2, and let g be a positive continuous function on X. We examine Proposition 2.1 in the case where M is the warped product $X \times_g Y$ with metric $dm^2 = dx^2 + (g(x)dy)^2$ and continuous product densities $\Phi_S^X \Phi_S^Y$ and $\Phi_V^X \Phi_V^Y$. The new metric is given by

$$d\tilde{m}^2 = \left(\frac{\Phi_V^X \Phi_V^Y}{\Phi_S^X \Phi_S^Y}\right)^2 \left(dx^2 + (g(x)dy)^2\right)$$

and thus M with metric $d\tilde{m}^2$ is no longer necessarily a warped product of X and Y even after conformal changes of metric in X and Y. However, if instead we only move the X component of the densities into the new metric on M then we can absorb this factor into the metric on X and the warp factor $g(x)$ in order to obtain a new space given by a warped product $\tilde{X} \times_{\tilde{g}} Y$ (the tilde denoting the change of metric on X and the change of the warp factor g) with surface and volume densities differing only in their Y component and such that surface areas and volumes of regions are the same as in the original warped product M. In particular, if $\Phi_S^Y = \Phi_V^Y = 1$ then this new space is a warped product with simple density.

If we are only interested in the vertical surface area, however, then we can always obtain a simple comparison space in the form of a simple product with surface density:

Proposition 2.3. *Let L be a Riemannian manifold of dimension $n-1$ with metric dl^2 and let Z be the interval $(A, B),\ A < B \in \mathbb{R} \cup \pm\{\infty\}$ with the usual metric dr^2. Consider the warped product $Z \times_g L$ with continuous warp factor g and continuous product densities $\Phi_S^Z \Phi_S^L$ and $\Phi_V^Z \Phi_V^L$. For a fixed $a \in (A, B)$, let $s(r)$ be the function*

$$s(r) = \int_a^r \Phi_V^Z(t) g(t)^{n-1} dt,$$

and let $\tilde{Z}$ be the interval $(s(A), s(B))$. Let $\Psi(s) = \Phi_S^Z(r(s)) g(r(s))^{n-1}$ and let $\tilde{L}$ be the space L after conformal change of metric $d\tilde{l} = (\left[\Phi_V^L\right]^{1/(n-1)} dl)$. Then, the map $(r, l) \mapsto (s(r), l)$ from $Z \times_g L$ with densities $\Phi_S^Z \Phi_S^L$ and $\Phi_V^Z \Phi_V^L$ to the product $\tilde{Z} \times \tilde{F}$ with surface density $\Psi(s) \frac{\Phi_S^L(l)}{\Phi_V^L(l)}$ is a C^1 diffeomorphism that preserves volume and vertical surface area.

Remark. If L with density Φ_V^L has finite volume then the coordinate $s(r)$ of Proposition 2.3 is a constant times the signed volume of the region $(a, r) \times L$, and if fibers $\{r\} \times L$ have finite surface area then $\Psi(s)$ is a constant times the surface area of the fiber $\{r(s)\} \times L$.

Proof. We examine the local elements of volume in these two spaces:

$$dV_{Z \times_g L} = \Phi_V^Z(r) \Phi_V^L(l) g(r)^{n-1} dr dL = ds d\tilde{L} = dV_{\tilde{Z} \times \tilde{L}}$$

thus volume is preserved. For the local elements of vertical surface area, we observe

$$\Phi_S^Z(r) \Phi_S^L(l) g(r)^{n-1} dL = \Psi(s) \frac{\Phi_S^L(l)}{\Phi_V^L(l)} d\tilde{L},$$

which completes the proof. □

In Lemmas 2.4 and 2.6 we give sufficient conditions for vertical fibers to minimize vertical surface area in an important family of these model spaces.

Lemma 2.4. *Let L be a Riemannian manifold of dimension $n-1$ with metric dl^2 giving finite total $n-1$ dimensional measure and let Z be the interval $(A, B),\ A, B \in \mathbb{R} \cup \pm\{\infty\}$ with the usual metric dr^2. In the product $Z \times L$ with volume density 1 and convex surface density $\Psi(r)$, for any rectifiable hypersurfaces H such that $H - \{r_0\} \times L$ is the boundary of a signed oriented region of net volume 0 that is bounded away from A and B in almost every horizontal fiber $Z \times \{l\}$, $|H|_{Vert} \geq |\{r\} \times L|$.*

If Ψ does not approach 0 at B (resp. A) the condition that the region be bounded away from B (resp. A) in almost every fiber can be weakened to the region not containing an open interval about B (resp. A) in almost every fiber.

Proof. Suppose H is such that $H - \{r_0\} \times L$ is the boundary of a signed oriented region of net volume 0 that is bounded away from A and B in almost every horizontal fiber $Z \times \{l\}$. By translation of the interval Z, we can assume $r_0 = 0$.

Let $R = R^+ - R^-$ be the signed oriented region bounded by $H - \{0\} \times L$ of net volume 0 and bounded away from the origin and infinity in almost every fiber $Z \times \{l\}$. Then,

$$0 = \int_{R^+} dV - \int_{R^-} dV$$

and by Fubini,

$$0 = \int_L h(l)dl$$

where

$$h(l) = h^+(l) + h^-(l),$$

$$h^+(l) = \int_{Z\times\{l\}\cap R^+} dr, \quad h^-(l) = -\int_{Z\times\{l\}\cap R^-} dr,$$

and h, h^+, and h^- are defined almost everywhere. Furthermore, we claim that almost everywhere $h(l)$ is contained between the smallest and largest r-coordinates of points in $Z \times \{l\} \cap H$ (which is non-empty almost everywhere because for almost all l, $Z \times \{l\} \cap R$ is bounded away from A and B): Because $Z \times \{l\} \cap R^+$ is bounded away from B and all the r coordinates in R^+ are positive, $h^+(l)$ is between 0 and the largest r-coordinate in $Z \times \{l\} \cap R^+$ which, if non-zero, is the largest r-coordinate in $Z \times \{l\} \cap H$. Similarly, because $Z \times \{l\} \cap R^-$ is bounded away from A, $h^-(\Theta)$ is between the smallest r-coordinate in $Z \times \{l\} \cap R^-$ and 0, and the smallest r-coordinate in $Z \times \{l\} \cap R^-$ is the smallest r-coordinate in $Z \times \{l\} \cap H$ if it is non-zero. Thus, if both the intersections $Z \times \{l\} \cap R^+$ and $Z \times \{l\} \cap R^-$ contain more than the point with r-coordinate 0 then $h(l)$ is contained between the smallest and largest r-coordinates of points in $Z \times \{l\} \cap H$. If $Z \times \{l\} \cap R^+$ contains only the r-coordinate 0 then it must be that $Z \times \{l\} \cap R^-$ contains the interval $(r_{max}, 0)$ where r_{max} is the largest r-coordinate in $Z \times \{l\} \cap R^-$ which in this case is the largest r-coordinate in $Z \times \{l\} \cap H$, and in particular $h^-(l) \leq r_{max}$ so that we again reach the conclusion. We argue similarly if $Z \times \{l\} \cap R^-$ contains only the r-coordinate 0.

Since H is rectifiable its intersections with the rays $Z \times \{l\}$ are L-almost everywhere transversal (H differs from a countable union of C^1 hypersurfaces by a set of $n-1$ dimensional Hausdorff measure 0 and this holds for these hypersurfaces), and thus

$$|H|_{Vert} = \int_L \left(\sum_{r \in Z\times\{l\}\cap H} \Psi(r) \right) dL$$

and since Ψ is convex and h is between the minimum and maximum r-value in each fiber

$$|H|_{Vert} \geq \int_L \Psi(h(l))dL,$$

and by Jensen's inequality (applied to the normalization of the measure dL which has finite total measure),

$$|H|_{Vert} \geq \int_L \Psi(0)dL,$$

and the quantity on the right is the surface area of $\{0\} \times L$.

To prove the last statement of the Lemma we observe that in a fiber where the region is neither bounded away from B nor contains an open interval about B, H must intersect the fiber transversely in infinitely many points in any neighborhood of B. Since Ψ is convex and does not approach 0 at B, it has a positive minimum in a neighborhood of B bounded away from A, and thus if there is a set of positive measure of such fibers then H has infinite vertical surface area. Since we can assume that H has finite vertical surface area, we conclude the set of such fibers is of measure 0. Since by hypothesis the set of fibers where the region contains an open interval about B has measure 0, almost everywhere the region must be bounded away from B. □

Remark 2.5. In the setting of Lemma 2.4, if Ψ is not convex then in general vertical fibers do not minimize vertical surface area: consider the cylinder $\mathbb{R} \times S^1$ with non-convex smooth perimeter density Ψ so that there exists $r_0 < r_1$ such that $\Psi(\frac{r_0+r_1}{2}) > \frac{\Psi(r_0)+\Psi(r_1)}{2}$. Then the curve given by the arcs $\{r_0\} \times [0, \pi]$, $\{r_1\} \times [\pi, 2\pi]$ and the radial segments $[r_0, r_1] \times \{\pi\}$ and $[r_0, r_1] \times \{0\}$ bounds net area 0 with the circle $\left\{\frac{r_o+r_1}{2}\right\} \times S^1$ and has less vertical perimeter. This example can be generalized to any interval and any type of vertical fiber – just take two half spaces of the vertical fiber at radii as chosen above and join them along the set of horizontal fibers over their border.

Lemma 2.6. *Let L be a Riemannian manifold of dimension $n-1$ with metric dl^2 giving finite total measure on L and let Z be the interval (A, B), $A, B \in \mathbb{R} \cup \pm\{\infty\}$ with the usual metric dr^2. Consider the product $Z \times L$ with volume density 1 and continuous surface density $\Psi(r)$ with Ψ convex.*

(1) *If $Z = (0, \infty)$ and $\lim_{r\to 0} \Psi(r) = 0$ then fibers $\{r\} \times L$ minimize vertical surface area among hypersurfaces bounding the same volume.*
(2) *If $Z = (-1, 1)$, $\lim_{r\to -1} \Psi(r) = 0$ and $\int_0^1 \Psi(r) = \infty$, then fibers $\{r\} \times L$ for $r \leq 0$ minimize vertical surface area among hypersurfaces of finite surface area bounding the same volume.*
(3) *If $Z = (-\infty, \infty)$ and $\lim_{r\to-\infty} \Psi(r) > 0$ and $\lim_{r\to\infty} \Psi(r) > 0$ then fibers $\{r\} \times L$ minimize vertical surface area among hypersurfaces with which they bound net volume zero.*

Proof. (1):

Let R be a region with rectifiable boundary with $|R| = |(0, r_0) \times L|$ and suppose $|\partial R|_{Vert} < |\{r_0\} \times L|$. Because there is infinite volume at infinity and R has finite volume, almost every fiber of R does not contain an interval around infinity. We have that $\lim_{r\to 0} |\{r\} \times L| = 0$ (from finite surface area of fibers and the limit of Ψ), that $\lim_{r\to 0} |(0, r) \times L| = 0$, and that the surface area $|\{r\} \times L|$ is a continuous function of the volume $|(0, r) \times L|$, and thus we can take r small enough so that $|\partial(R \cup (0, r] \times L)|_{Vert} < |\{r'\} \times L|$ where r' is such that $|R \cup (0, r] \times L| = |(0, r') \times L|$. However, the convexity condition combined with the limit at 0 implies that Ψ is non-decreasing and thus does not approach 0 at ∞ so we can apply Lemma 2.4 to $\partial(R \cup (0, r] \times L)$ to obtain a contradiction.

(2):

We will show that for each region R with rectifiable boundary and finite surface area there exists a vertical fiber bounding the same volume and having vertical surface area less than or equal to that of R. The result will then follow because the convexity combined with the limit and positivity of Ψ imply that Ψ is increasing and thus the vertical fibers for $r \leq 0$ have vertical surface area strictly less than that of the fibers for $r > 0$.

Let R be a region with rectifiable boundary and finite surface area. Suppose that for both R and R^C the sets of points such that the fibers contain open intervals at infinity has positive measure. Then for r sufficiently large, the vertical fibers $\{r\} \times L \cap R$ have $n-1$ dimensional L-measure contained in a compact real interval bounded away from 0 and the maximum measure of L. Since L has finite total measure, in this space isoperimetric regions exist for all volumes, and since the isoperimetric profile is a continuous function, there is an isoperimetric inequality on L that implies for r sufficiently large, say $r > 1 - \delta$, $\partial(\{r\} \times L \cap R)$ has $n-2$ dimensional surface area greater than a fixed $\epsilon > 0$ viewed as a surface in L. In particular, by analyzing the horizontal component of surface area, we conclude that $|\partial R| \geq \epsilon \int_{1-\delta}^{1} \Psi(r)dr > \infty$.

Thus for either R or R^C almost every fiber does not contain an open interval at infinity. If it is R, we can proceed as in case *(1)*. If it is R^C then since the space has finite total volume, R^C also has finite volume and we can proceed as in case *(1)* with the region R^C. However, the same fiber that bounds volume $|R^C|$ at -1 bounds volume $|R|$ at 1, and since R and R^C have the same boundary, we obtain the conclusion for R.

(3):

Let H be a rectifiable hypersurface such that $H - \{r\} \times L$ bounds the region $R = R^+ - R^-$ with signed volume 0. In particular, R^+ and R^- both have finite volume, and since there is infinite volume at both plus and minus infinity, as in (1) we obtain that for almost every horizontal fiber R does not contain an open interval at $-\infty$ or ∞. Thus, we can apply Lemma 2.4 to obtain the result. □

Remark 2.7. In Lemma 2.6 case *(1)*, the condition on the limit of the surface density is necessary – otherwise, fibers bounding small volumes will have a surface area bounded below by some constant so that for sufficiently small volume they cannot be isoperimetric. In case *(2)*, the condition on the integral of the surface density is necessary to avoid regions bounded by horizontal surfaces $Z \times M$ for some $n-2$ dimensional submanifold $M \subset L$ which have zero vertical surface area - it guarantees that these regions have infinite surface area (this is also why the extra hypothesis on finite surface area is necessary in the statement). The conditions on the limits in case *(3)* is used in the proof to rule out certain regions but may or may not be necessary.

Using Proposition 2.3, we can lift Lemma 2.6 to a much more general setting:

Theorem 2.8. *Let L be a Riemannian manifold of dimension $n-1$ with metric dl^2 and let Z be the interval (A, B), $A < 0 < B$ with standard metric dr^2. Consider a warped product $Z \times_g L$ (g continuous) with metric $dr^2 + g(r)^2 dl^2$ and continuous product surface density $\Psi_S(r)\Phi(l)$ and volume density $\Psi_V(r)\Phi(l)$. Suppose that the surface area of fibers $\{r\} \times L$ and the signed volume of any annulus $[0, r] \times L$, $A < r < B$ is finite and that the surface area of fibers $\{r\} \times L$ is a convex function of the signed volume of the annulus $[0, r] \times L$.*

(1) *If there is infinite total volume, $(A, 0] \times L$ has finite volume, and*

$$\lim_{r \to A} |\{r\} \times L| = 0$$

then fibers $\{r\} \times L$ minimize vertical surface area among hypersurfaces bounding the same volume and thus are uniquely isoperimetric for all volumes.

(2) *If there is finite total volume V_0,*

$$\lim_{r \to A} |\{r\} \times L| = 0,$$

and

$$\int_0^B \Psi_S(r) g(r)^{n-2} dr = \infty$$

then fibers $\{r\} \times L$ such that $|(A, r) \times L| \leq V/2$ minimize vertical surface area among hypersurfaces bounding the same volume and having finite surface area and thus are uniquely isoperimetric for all volumes.

(3) *If both $(A, 0] \times L$ and $[0, B) \times L$ have infinite volume and both*

$$\lim_{r \to A} |\{r\} \times L| > 0$$

and

$$\lim_{r \to B} |\{r\} \times L| > 0$$

then fibers $\{r\} \times L$ minimize vertical surface area among rectifiable hypersurfaces with which they bound net volume 0 and thus uniquely minimize surface area among such surfaces.

Remark. By reflecting the interval we see statements 1 and 2 also hold reversing the roles of A and B.

Proof. After applying Proposition 2.3 and possibly shifting or scaling the interval, we reduce to the corresponding cases of Lemma 2.6: The component of the density depending on l is merged completely into the metric and, following the remark after Proposition 2.3, the convexity of the surface density on the resulting space is equivalent to the convexity of surface area of fibers as a function of the volume of annuli.

For *(1)* and *(3)*, the conditions on the limit translate directly.

For *(2)*, the condition on the limit translates directly, however, the condition on the integrals is not the same. This reflects the fact that while vertical surface area is preserved in Proposition 2.3, surface area is not, and so the regions of finite surface area are not necessarily the same. However, the condition given allows us to deduce by essentially the same argument as in the proof of Lemma 2.6-*(2)* that regions of finite surface area contain open intervals about B only in a set of horizontal fibers of measure 0 where the measure on L is induced by the volume element on L, $\Phi(l)dL$, which is the same as the measure on $\tilde{L}$ induced by $\tilde{dl}$. After this, the rest of the proof goes through unchanged.

That vertical fibers are uniquely isoperimetric follows because surface area is greater than vertical surface area with equality only when the surface is a union of vertical fibers. In cases *(1)* and *(2)* the surface area of fibers is increasing so that multiple fibers are always worse than a fiber of small radius bounding the same volume, and in case *(3)* multiple fibers are always worse by convexity because for multiple fibers to bound net volume zero with a given fiber when there is infinite volume at both the origin and infinity some of them must lie on both sides of the given fiber. □

It is possible to generalize Theorem 2.8 to the case where only certain fibers minimize vertical surface area:

Theorem 2.9. *Let L be a Riemannian manifold of dimension $n-1$ with metric dl^2 and let Z be the interval (A, B), $A < 0 < B$ with standard metric dr^2. Consider a warped product $Z \times_g L$ (g continuous) with metric $dr^2 + g(r)^2 dl^2$ and continuous product surface density $\Psi_S(r)\Phi(l)$ and volume density $\Psi_V(r)\Phi(l)$. Suppose that the surface area of fibers $\{r\} \times L$ and the volume of annuli $[0, r] \times L$, $A < r < B$ are finite.*

1. *Suppose there is infinite total volume and $(A, 0] \times L$ has finite volume. Let F be the function that sends a volume V to the surface area of the unique vertical fiber $\{r\} \times L$ such that $|(A, r] \times L| = V$. If there is a function $\tilde{F}$ and a V_0 such that $\tilde{F} \le F$, $\tilde{F}$ is convex, $\lim_{V \to 0} \tilde{F}(V) = 0$ and $\tilde{F}(V_0) = F(V_0)$, then the fiber $\{r\} \times L$ such that $|(A, r) \times L| = V_0$ minimizes vertical surface area among surfaces bounding the same volume and thus is uniquely isoperimetric for volume V_0.*
2. *Suppose both $(A, 0] \times L$ and $[0, B) \times L$ have infinite volume. Let F be the function that sends a signed volume V to the surface area of the unique vertical fiber $\{r\} \times L$ such that $|[0, r] \times L| = V$. If there is a function $\tilde{F}$ and a V_0 such that $\tilde{F} \le F$, $\tilde{F}$ is convex, $\lim_{V \to -|(A,0] \times L|} \tilde{F}(V) > 0$, $\lim_{V \to |[0,B) \times L|} \tilde{F}(V) > 0$, and $\tilde{F}(V_0) = F(V_0)$ then the fiber $\{r\} \times L$ such that $|(0, r) \times L| = V_0$ minimizes vertical surface area among rectifiable hypersurfaces with which it bounds net volume 0 and thus uniquely minimizes surface area among such surfaces.*

Remark. It is possible to give a version of Theorem 2.9 in the finite volume case, however the condition on the surface area density makes the hypothesis more complicated to state and we have found no interesting examples where it applies, and thus we omit it here.

Proof. We can replace the r-component of the surface density with a new r-component so that the surface area of vertical fibers is now given by $\tilde{F}$. By our hypothesis on $\tilde{F}$ we can apply Theorem 2.8 to this new space. Because $\tilde{F} \le F$, the new surface density is everywhere less than or equal to the original surface density and so the surface area (resp. vertical surface area) of any region considered as a region in this new space is less than or equal to its surface area (resp. vertical surface area) as a region in the original space. Because $\tilde{F}(V_0) = F(V_0)$, the fiber in question has the same surface area and vertical surface area in both spaces. Volume is the same for all regions in both spaces, and thus this fiber also satisfies the conclusions of Theorem 2.8 in the original space. □

We deduce the following useful corollary:

Corollary 2.10. *Let L be a Riemannian manifold of dimension $n-1$ with metric dl^2 and let Z be the interval $(0, B)$ with standard metric dr^2. Consider a warped product $Z \times_g L$ (g continuous) with metric $dr^2 + g(r)^2 dl^2$ and continuous product surface density $\Psi_S(r)\Phi(l)$ and volume density $\Psi_V(r)\Phi(l)$. Suppose that the surface area of fibers $\{r\} \times L$ and the volume of annuli $(0, r) \times L$, $0 < r < B$ are finite, and that there is infinite total volume. Let F be the function that sends a volume V to the surface area of the unique vertical fiber $\{r\} \times L$ such that $|(0, r] \times L| = V$. If F is eventually convex, F is bounded below by a line through the origin of positive* *slope, and* $\lim_{V\to\infty} F'(V) = \infty$ *then for sufficiently large* r*, vertical fibers* $\{r\} \times L$ *are isoperimetric.*

Proof. In order to apply Theorem 2.9, it suffices to find a convex function $\tilde{F}$ such that $\tilde{F} \leq F$, $\tilde{F}(V) = F(V)$ for V sufficiently large, and $\lim_{V\to 0} \tilde{F}(V) = 0$. Because F is eventually convex and $\lim_{V\to\infty} F'(V) = \infty$, there is eventually a tangent line bounding F from below with a non-negative V-intercept. Denote by l_0 such a line, tangent to the graph of F at V_0. Denote by l_{-1} a line through the origin bounding F from below and let V_{-1} be the V-coordinate of the intersection of l_0 and l_{-1} (forcibly $V_{-1} \leq V_0$). Then we define

$$\tilde{F}(V) = \begin{cases} l_{-1}(V) & 0 < V \leq V_{-1} \\ l_0(V) & V_{-1} \leq V \leq V_0 \\ F(V) & V_0 \leq V \end{cases},$$

which has the desired properties. □

3. $\mathbb{R}^n - \{0\}$ with radial density

We apply the results of Section 2 in the motivating case of $\mathbb{R}^n - \{0\}$ considered as the warped product $(0,\infty) \times_r S^{n-1}$ with radial density. In this context, we refer to vertical surface area as tangential surface area, following [5]. We note that the results of this section extend to $(0,\infty) \times_r S^{n-1}$ with continuous product surface density $\Psi_S(r)\Phi(\Theta)$ and volume density $\Psi_V(r)\Phi(\Theta)$ (note that Φ must be the same for both the surface and volume density), as well as to warped products $(0,\infty)\times_r K$ for any compact Riemannian manifold K with densities of the same form.

Theorem 3.1. *Consider* $\mathbb{R}^n - \{0\}$ *with continuous radial surface density* Ψ_S *and continuous radial volume density* Ψ_V *such that the surface area of a sphere of radius* r *is a convex function of the signed volume of the annulus* $[1, r] \times S^{n-1}$.

(1) *If there is infinite total volume and either*
   (a) *there is finite volume at the origin and* $\lim_{r\to 0} |\partial B_r| = 0$ *or*
   (b) *there is finite volume at infinity and* $\lim_{r\to\infty} |\partial B_r| = 0$*,*

   then spheres about the origin minimize tangential surface area among hypersurfaces bounding the same volume and thus are uniquely isoperimetric for all volumes.

(2) *If there is finite total volume and*

$$\lim_{r\to 0} |\partial B_r| = 0 \text{ (resp. } \lim_{r\to\infty} |\partial B_r| = 0\text{)},$$

   then spheres about the origin bounding volume less than or equal to half the total volume of the space at the origin (resp. at infinity) minimize tangential surface area among hypersurfaces of finite surface area bounding the same volume and thus are uniquely isoperimetric for all volumes.

(3) *If there infinite volume at both the origin and infinity and both*

$$\lim_{r\to 0} |\partial B_r| > 0 \text{ and } \lim_{r\to\infty} |\partial B_r| > 0,$$

   then any sphere about the origin S *minimizes vertical surface area and thus uniquely minimizes surface area among rectifiable hypersurfaces* H *bounding net volume zero with* S.

Proof. This is just Theorem 2.8 applied in this setting. To obtain, for example, *(1)-(b)*, we consider the warped product with the interval reflected about 0. The only case that needs extra verification is *(2),* for which we must verify that $\int_1^\infty \Phi_S(r) r^{n-2} dr = \infty$ (the other case following similarly). However, by the convexity condition and the fact that $\lim_{r\to 0} |\partial B_r| = 0$, $|\partial B_r| = c \cdot r^{n-1}\Psi_S(R)$ must be increasing, and thus

$$\int_1^\infty r^{n-2}\Psi_S(r) \geq \int_1^\infty \frac{1}{r}\Psi_S(1) = \infty.$$

□

We obtain the following simplified statement in the case of simple density:

Theorem 3.2. *Consider $\mathbb{R}^n - \{0\}$ with continous simple radial density e^ϕ, where $|\partial B_r|$ is a log-convex function of r.*

1. *If there is infinite total volume and finite volume at infinity then spheres about the origin minimize tangential surface area among hypersurfaces bounding the same volume and thus complements of balls about the origin are uniquely isoperimetric for all volumes.*
2. *If there is finite total volume then spheres about the origin bounding volume greater than half the volume of the space at the origin minimize tangential surface area among hypersurfaces bounding the same volume and thus are uniquely isoperimetric for all volumes.*
3. *If there infinite volume at both the origin and infinity then any sphere about the origin S minimizes vertical surface area and thus uniquely minimizes surface area among rectifiable hypersurfaces H bounding net volume zero with S.*

Remark. For a smooth simple density the log-convexity of $|\partial B_r|$ is equivalent to $\phi'' \geq \frac{n-1}{r^2}$. Thus, for a smooth simple density the cases of Theorem 3.1 where $\lim_{r\to 0} |\partial B_r| = 0$ cannot occur because this gives a contradiction.

Proof. We apply Theorem 3.1. The convexity conditions are easily checked to be equivalent. Thus, the only thing left to verify is that in the first two cases spheres of large radius have surface area approaching zero. However, finite volume at infinity gives us that $|\partial B_r|$ is decreasing with $\lim_{r\to\infty} |\partial B_r| = 0$ – indeed, by the log-convexity condition, if $|\partial B_r|$ is ever non-decreasing then it is always non-decreasing and thus there is infinite volume at infinity, and if it is decreasing but not decreasing to 0 then there is also infinite volume at infinity. □

Diaz *et al.* [5, Thm 7.4] used the technique of tangential surface area to recover a result of Betta et al. on when spheres about the origin are isoperimetric in the case of surface density. We restate it here, noting it follows again as a corollary of the more general Theorem 3.1. The modified convexity condition is obtained by subtracting off the value at the origin and noting that spheres are isoperimetric in $\mathbb{R}^n$ with constant density.

Theorem 3.3 (Surface density, [1, Thm 4.3])**.** *In $\mathbb{R}^n$ with non-decreasing radial surface density Ψ such that*

$$(\Psi(r^{1/n}) - \Psi(0))r^{1-1/n}$$

is convex, spheres about the origin are isoperimetric.

Remark 3.4 (Volume density). In $\mathbb{R}^n - \{0\}$ with volume density e^ϕ, the convexity condition of Theorem 3.1 becomes $\phi'(r) \leq -1/r$. However, any decreasing volume density can be handled with simpler arguments (see [15]).

Example 3.5. We present here some applications of Theorems 3.1 and 3.2.

(1) Diaz *et al.* [5, Prop 7.5] used the technique of tangential surface area to prove that in $\mathbb{R}^n - \{0\}$ with simple density r^p, $p < -n$, spheres about the origin are isoperimetric bounding volume at infinity. Using Theorem 3.2 we can extend this family to $r^p e^{\phi(r)}$ with $\phi'' \geq 0$ and either $\phi'(r) \leq 0$ and $p < -n$ or $\phi'(r) \leq -\epsilon < 0$ and $p = -n$. Notably, at the origin these densities are neither log-convex nor smooth. Figure 3.1 *(d)* shows the graph of surface area of spheres as a function of volume for one density in this family.

(2) Theorem 3.2 also allows us to generalize the first example further to $\mathbb{R}^n - \{0\}$ with simple density $r^p e^{\phi(r)}$, $-n < p \leq -(n-1)$, $\phi'' \geq 0$ and $\phi' \leq -\epsilon < 0$ (for example $\phi(r) = -ar + b$, $a > 0$), which gives finite total volume: in these spaces, spheres bounding half or more of the volume of the space at the origin are isoperimetric.

(3) Theorem 3.2 gives a similar result in $\mathbb{R}^n - \{0\}$ with simple density r^{-n}, which has infinite volume at both the origin or infinity: in this space, any sphere about the origin S minimizes surface area among all rectifiable hypersurfaces bounding net volume zero with S.

(4) Consider $\mathbb{R}^n - \{0\}$ with volume density Ψ_V such that there is finite volume at the origin and infinite volume at infinity and surface area density $\Psi_S(r) = G(|B_r|)/r^{n-1}$ where G is convex and approaches zero at the origin. Then the surface area of balls about the origin as a function of their volume is proportional to G and thus convex so we can apply Theorem 3.1 to show balls about the origin are isoperimetric. We can use this method to construct many specific examples, for instance, $\Psi_V = r^m$ and $\Psi_S = r^k$ with $m \geq 0$ and $k \geq m + 1$. This example provides a generalization to higher dimensions of part of the result of Diaz et. al. [5, Thm. 4.17 and Prop. 4.23] on isoperimetric regions in sectors with simple density r^p, which after a conformal change of coordinates is equivalent to $\mathbb{R}^2$ with differing perimeter and area densities.

(5) Instead of density, we can can consider a conformal change of metric (which is equivalent to certain differing densities on surface area and volume): Consider $\mathbb{R}^2 - \{0\}$ with metric $e^\phi ds$ where ds is the standard Euclidean metric and ϕ is a function of r. A calculation shows that the perimeter of circles as a function of their area is convex at the area of the circle of radius r if and only if

$$\phi''(r) \geq \phi'(r)^2 + \frac{\phi'(r)}{r} + \frac{1}{r^2}.$$

In particular, Theorem 3.1-(3) applies with $\phi = -\log r$ which gives the punctured plane with metric $\frac{1}{r} \cdot ds$, i.e. a cylinder (we can also obtain this result directly by applying Theorem 3.1-(3) to $(-\infty, \infty) \times S^1$ with density 1). By adding $\phi'(r)/r$ to each side we can rewrite the convexity condition as

$$\kappa(r) \geq \left(\phi'(r) + \frac{1}{r}\right)^2$$

where κ is the Gaussian curvature at radius r. Howards *et al.* [7, Sec. 9] give other results on the isoperimetric problem in $\mathbb{R}^2$ with non-Euclidean metrics, in particular showing that circles are isoperimetric if the metric is smooth and has curvature that decreases with radius.

Example 3.6. In $\mathbb{R}^n - \{0\}$ with simple density $r^p \ \ -n < p < 0$, isoperimetric regions do not exist [5, Prop. 7.3]. Although the convexity condition holds for $-n < p \leq -(n-1)$, Theorem 3.1 does not apply: there is infinite total volume with finite volume at the origin but the surface area of small spheres does not go to 0. Indeed, this is always the case for a simple density satisfying these volume hypotheses. In particular, from non-existence for $p = -(n-1)$, we see that the condition in Theorem 3.1-(1) that the surface area goes to zero cannot be weakened to the condition that the surface area of small spheres remains bounded (as noted already in remark 2.7 for our model spaces).

3.1. **Large balls in $\mathbb{R}^n - \{0\}$.** Using Corollary 2.10 we generalize Kolesnikov and Zhdanov's [8, Prop. 4.7] results on large balls about the origin in $\mathbb{R}^n$ with simple density. In particular, we obtain that in $\mathbb{R}^n$ with simple density non-singular at the origin and exhibiting a type of eventually strict log-convexity, spheres about the origin bounding large volume are uniquely isoperimetric (Corollary 3.7). In Example 3.8 we demonstrate several densities where Theorem 2.9, Corollary 2.10 or Corollary 3.7 can be applied.

Corollary 3.7. *In $\mathbb{R}^n$ with continuous radial density $\Psi = e^{\phi(r)}$, if there exist $r_0 > 0$ and $\epsilon > 0$ such that ϕ is twice continuously differentiable with $\phi''(r) \geq \frac{\epsilon}{r}$ for all $r > r_0$, then large spheres about the origin are uniquely isoperimetric.*

Remark. In fact, it suffices to have ϕ' go to infinity and ϕ'' eventually greater than $\frac{n-1}{r^2}$.

Proof. Let F be the function mapping the volume of a ball about the origin in $\mathbb{R}^n$ with simple density Ψ to its surface area. Because Ψ is continuous and

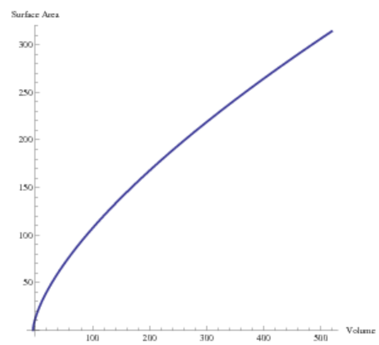
$$0 < \Psi(0) < \infty,$$

there is a positive constant C such that near the origin the surface area of a ball is greater than C/r times its volume. Thus, F has a vertical asymptote at volume 0. Now, for V sufficiently large,

$$F'(V) = \left(\frac{(n-1)}{r} + \phi'(r)\right)$$

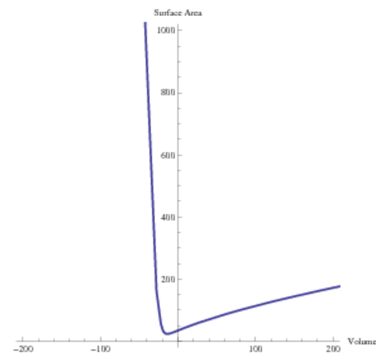
$$F''(V) = \left(\phi''(r) - \frac{n-1}{r^2}\right) \cdot \frac{|S_1|}{|S_r|}$$

where r is the radius of the ball of volume V, and thus F is eventually convex. Furthermore since ϕ' becomes arbitrarily large, $\lim_{V\to\infty} F'(V) = \infty$. Because F is positive outside of zero, continuous, and eventually increasing, it has an infimum greater than zero on any interval bounded away from 0, and since F also has a vertical asymptote at volume 0, there must exist a line of positive slope through the origin bounding F from below. Finally, since $\Psi(0)$ being finite implies that there is finite volume at the origin we can apply Corollary 2.10. □



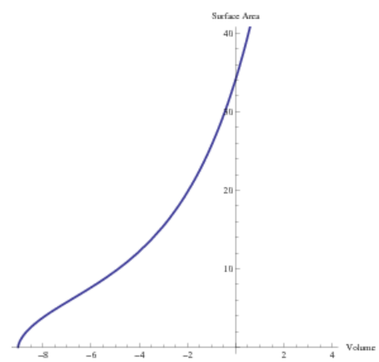
$\Psi_V = \Psi_S = 1.$

(a) The vertical asymptote at the origin is typical of densities bounded away from 0 and infinity at the origin, as in the proof of Corollary 3.7.



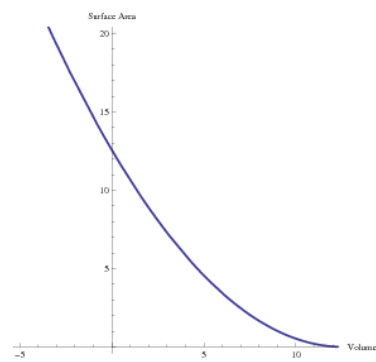
$\Psi_V = \Psi_S = e^{r^{-1}}.$

(b) This space has infinite volume at both the origin and infinity, and one can visually verify from the graph that Theorem 2.9-*(2)* applies for small spheres which thus minimize surface area among surfaces with which they bound net volume 0 (see example 3.8).



$\Psi_V = e^r, \ \Psi_S = e^{r^8}.$

(c) One can visually verify from the graph that for this density the hypotheses of Theorem 2.9 are satisfied and for large spheres which are thus isoperimetric (see Example 3.8).



$\Psi_V = \Psi_S = r^{-4}.$

(d) This density is singular and extremely non-convex at the origin but ball complements are isoperimetric for all volumes (see Example 3.5).

FIGURE 3.1. Graph of the surface area of the sphere of radius r as a function of the signed volume of the annulus $[1, r] \times S^2$ for various densities on $\mathbb{R}^3$.

Example 3.8. We give some applications of Theorem 2.9, Corollary 2.10, and Corollary 3.7:

(1) In $\mathbb{R}^n$ with simple density e^{r^α}, $\alpha > 1$, Corollary 3.7 shows that large spheres about the origin are isoperimetric (originally shown by Kolesnikov and Zhdanov [8, Prop. 4.7]).
(2) In $\mathbb{R}^n$ with simple density $e^{p(r)}$ with p a polynomial of degree greater than or equal to 2 and positive leading coefficient, Corollary 3.7 shows that large spheres about the origin are isoperimetric. This include, for example, $\mathbb{R}^n$ with simple density e^{r^2-2r+2}, where for small volumes isoperimetric regions are approximate balls centered on the unit circle [15, 18]. Thus we obtain examples of spaces where spheres are stable but only isoperimetric for certain volumes.
(3) In $\mathbb{R}^3$ with density $\Psi_V = e^r$, $\Psi_S = e^{r^8}$, large spheres about the origin are isoperimetric: It is easy to see from the graph of surface area as a function of volume (Figure 3.1 *(c)*) that an appropriate convex function exists and thus we can apply Theorem 2.9. Alternatively, one can show directly that Corollary 2.10 applies.
(4) As pointed out by Morgan [17], in $\mathbb{R}^n$ with simple density e^{r^α}, $\alpha < 0$, small spheres about the origin minimize tangential surface area among hypersurfaces with which they bound volume 0 and uniquely minimize surface area among such hypersurfaces. Indeed, to apply Theorem 2.9-*(2)* we can produce an appropriate convex function using the same ideas as in the proof of Corollary 2.10. We can also see this graphically in Figure 3.1 *(b)*.

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