Reducibility of cocycles under a Brjuno-Rüssmann arithmetical condition

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Abstract: The arithmetics of the frequency and of the rotation number play a fundamental role in the study of reducibility of analytic quasi-periodic cocycles which are sufficiently close to a constant. In this paper we show how to generalize previous works by L.H.Eliasson which deal with the diophantine case so as to implement a Brjuno-Russmann arithmetical condition both on the frequency and on the rotation number. Our approach adapts the Poschel-Russmann KAM method, which was previously used in the problem of linearization of vector fields, to the problem of reducing cocycles.

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1 Introduction

Cocycles are the fundamental solutions of quasi-periodic linear systems

$$\forall(\theta,t)\in\mathbb{T}^d\times\mathbb{R},\ X'(\theta,t)=A(\theta+t\omega)X(\theta,t) \tag{1}$$

where A is continuous on a torus $\mathbb{T}^d$, matrix-valued and ω is a rationally independent vector of some space $\mathbb{R}^d$ (the space of frequencies). Although the dynamics of such a system can be quite complicated, they are easily studied in case the system is reducible, i.e when there is a map Z, continuous on the double torus $2\mathbb{T}^d=\mathbb{R}^d/2\mathbb{Z}^d$, taking its values in the group of invertible matrices and such that

$$\forall\theta\in 2\mathbb{T}^d, \frac{d}{dt}Z(\theta+t\omega)_{|t=0}=A(\theta)Z(\theta)-Z(\theta)B$$

for some matrix B not depending on θ. Since smoothness is an issue, given a class of functions $\mathcal{C}$, we will say that the cocycle is reducible in $\mathcal{C}$ if Z is in $\mathcal{C}$. Here we will focus on the case in which A takes its values in $sl(2,\mathbb{R})$, which is sufficient, for instance, for the study of the one-dimensional quasi-periodic Schrödinger equation.

The arithmetics of ω seem fundamental in the study of reducibility, as well as the arithmetics of the system's rotation number ρ. At least in the perturbative case, arithmetical conditions of diophantine type have long been used to obtain reducibility, which can be seen as the convergence of a certain sequence of analytic functions: a diophantine condition can be used to control small divisors and make sure that the sequence converges.

In this article, we will give a reducibility result for analytic cocycles under a weaker arithmetical condition than the diophantine one. Thus, we will consider solutions of (1) with $A\in C^\omega_r$, i.e having a holomorphic extension on $\{(z_1,\dots,z_d)\in\mathbb{C}^d,\ \forall j\ \mid \operatorname{Im} z_j\mid< r\}$, with a "weighted norm" $|\cdot|_r$ (see section 2.1). In order to obtain an analytic reducibility result, we will have to pick a frequency and a rotation number with good approximation properties, in the sense of Rüssmann ([7]): ω will have to satisfy a strong irrationality condition controlled by an approximation function G, namely

$$\forall m\in\mathbb{Z}^d\setminus\{0\}, \mid\langle m,\omega\rangle\mid\geq\frac{\kappa}{G(m)}$$

for some positive κ (section 2.1), and ρ will have to satisfy a further arithmetical condition: its approximations by means of linear combinations of the frequencies are controlled by an approximation function g, i.e

$$\forall m\in\mathbb{Z}^d\setminus\{0\}, \mid\rho-\langle m,\omega\rangle\mid\geq\frac{\kappa'}{g(m)}$$

(we will say for short that ρ has g as approximation function with respect to ω with constant κ') with g,G satisfying some extra assumptions. The main result is as follows:

Theorem 1.1 *Let $\kappa>0$ and let G,g be positive increasing functions such that*

- $G(1)\geq 1, g(1)\geq 1,$
- $$\int_1^{+\infty}\frac{\log G(t)+\log g(t)}{t^2}dt<+\infty,$$

- *the map* $t \mapsto \frac{g(t^2)}{G(t)}$ *is bounded.*

Suppose ω *has* G *as an approximation function with constant* κ. *Let* $A \in sl(2,\mathbb{R})$, $r > 0$, $F \in C^\omega_r(\mathbb{T}^d, sl(2,\mathbb{R}))$. *Let* $n_0 \in \mathbb{N}$. *There exist* ϵ_0 *depending only on* g, κ, G, n_0, r *such that if*

1. $$| F |_r \leq \epsilon_0,$$

2. $\rho(A+F)$ *has* g *as an approximation function with respect to* ω *with constant* $\kappa' > \kappa \sup_{t \geq n_0} \frac{g(t^2)}{G(t)}$,

then there exists $r' \in (0,r)$ *such that* $A+F$ *is reducible in* $C^\omega_{r'}(2\mathbb{T}^d, sl(2,\mathbb{R}))$.

A discussion on the dependence of ϵ_0 on g, G and the other parameters is given in subsection 3.4.
The situation is already well-known when $G(t) = t^\tau, \tau > 2(d+1)$ and $g(t) = t^{\tau'}, \tau' \in (d+1, \frac{1}{2}\tau)$: it is included in Eliasson's theorem in [3] and can be stated as follows:

Theorem 1.2 *Assume* ω *is diophantine with constant* $\kappa > 0$ *and exponent* $\tau > 2(d+1)$. *Let* $A \in sl(2,\mathbb{R})$, $r > 0$, $F \in C^\omega_r(\mathbb{T}^d, sl(2,\mathbb{R}))$. *Assume* $\rho(A+F)$ *is diophantine with constant* $\kappa' > 0$ *and exponent* $\tau' \in (d+1, \frac{\tau}{2})$. *There exist* ϵ_0 *depending only on* $\tau, \kappa, \tau', \kappa', r$ *such that if*

$$| F |_r \leq \epsilon_0,$$

then there exists $r' \in (0,r)$ *such that* $A+F$ *is reducible in* $C^\omega_{r'}(2\mathbb{T}^d, sl(2,\mathbb{R}))$.

A new application, however, is when g and G look like exponentials (section 3.4):

Theorem 1.3 *Let* $\kappa > 0, \kappa' > 0$ *and let* $G(t) = e^{\frac{t}{(\log t)^\delta}}, g(t) = e^{t^\alpha}, \delta > 1, \alpha < 1$. *Suppose* ω *has* G *as an approximation function with constant* κ. *Let* $A \in sl(2,\mathbb{R})$, $r > 0$, $F \in C^\omega_r(\mathbb{T}^d, sl(2,\mathbb{R}))$. *There exist* ϵ_0 *depending only on* $\alpha, \kappa, \delta, \kappa', r$ *such that if*

1. $$| F |_r \leq \epsilon_0,$$

2. $\rho(A+F)$ *has* g *as an approximation function with respect to* ω *with constant* κ',

then there exists $r' \in (0,r)$ *such that* $A+F$ *is reducible in* $C^\omega_{r'}(2\mathbb{T}^d, sl(2,\mathbb{R}))$.

Although no perturbative results on cocycles have been obtained without diophantine conditions so far, Brjuno-Rüssmann conditions are already known to be central in the study of the linearization of vector fields (see e.g. [5], [6] and references therein).
Our aim is to adapt the Pöschel-Rüssmann method (see [7] and [6]), which was used in the problem of linearization for vector fields, to the problem of reducing cocycles. It is a KAM-type method in which the speed of convergence is linear.

First of all, we will build a setup in which a system $A+F$ with A constant and F small is conjugated to another system which is arbitrarily close to a constant, in an analytic class which, however, cannot be well controlled: this follows the technique used in [4] and is obtained by iterating (as in subsection 3.1) infinitely many steps (described in section 2) in which one conjugates a system $A_n + F_n$ to a system $A_{n+1} + F_{n+1}$ where A_n, A_{n+1} are constant and $\mid F_{n+1} \mid_{r_n+1} \leq C \mid F_n \mid_{r_n}$, with $C < 1$ being independent of n and r_n being a decreasing sequence controlling how analytic a function is. Thus, if r_n tends to a non zero limit, we have analytic reducibility.
At each step, in order to proceed, the constant part has to be non resonant, and if it is resonant, then we will have to remove the resonances, as explained in subsection 2.2.
Our setup makes sure that r_n tends to a non zero limit whenever there is only a small enough number of steps at which one has to remove resonances in the constant part. The Brjuno-Rüssmann condition on the frequency and on the rotation number of the cocycle is required exactly at this stage.

Acknowledgments The first author would like to thank the Centro Ennio de Giorgi for its financial support. Both authors would like to express their gratitude to Håkan Eliasson for useful discussions which allowed them to substantially improve the paper.

2 The basic step

In this section, we will prove the iterative step, which consists in conjugating a system to another one with a smaller non constant part, whether the constant part be resonant or not.

2.1 Definitions and notations

We will adopt the following conventions:

Definition: Let $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$. We denote by $\mid m \mid$ its modulus:
$\mid m \mid = \sum_{j=1}^{d} \mid m_j \mid$.

Definition: Let $F \in C^0(\mathbb{T}^d)$ and $r > 0$; we say that $F \in C_r^\omega(\mathbb{T}^d)$ if there exists an analytic continuation of F on a product of strips $\{(z_1, \ldots, z_d) \in \mathbb{C}^d, \ \forall j \ \mid \mathrm{Im}\, z_j \mid < r\}$.

On $C_r^\omega(\mathbb{T}^d)$, we shall use the weighted norms:

$$\mid F \mid_r = \sum_{k \in \mathbb{Z}^d} \mid\mid \hat{F}(k) \mid\mid e^{2\pi |k| r} \tag{2}$$

where $\mid\mid . \mid\mid$ is the relevant norm for $\hat{F}(k)$ (for matrices, we use the operator norm).

Notation: For $F \in C_r^\omega(\mathbb{T}^d)$, we denote its truncation by

$$F^N(\theta) = \sum_{|m| \leq N} \hat{F}(m) e^{2i\pi \langle m, \theta \rangle}.$$

Remark: The weighted norms are particularly convenient since they satisfy, for any integer N,

$$| F - F^N |_r = \sum_{k\in\mathbb{Z}^d, |k|>N} || \hat{F}(k) || \, e^{2\pi|k|r} = | F |_r - | F^N |_r \, . \qquad (3)$$

Moreover, they are equivalent to the usual sup norms since it is easy to see that

$$\sup_{|\mathrm{Im}\,\theta|<r} || F(\theta) || \leq | F |_r \, .$$

The following notations give two extensions of the diophantine condition.

Notation: Let

- $\kappa > 0$
- $\omega \in \mathbb{R}^d$
- $G \in C^0(\mathbb{R}^{*+})$ a positive increasing function.

We say that

$$\omega \in NR(\kappa, G)$$

if for all $m \in \mathbb{Z}^d \setminus \{0\}$,

$$| \langle m, \omega \rangle | \geq \frac{\kappa}{G(| m |)}.$$

Notation: Let

- $\alpha \in \mathbb{C}$,
- $\omega \in \mathbb{R}^d$,
- $\kappa' > 0$,
- $N \in \mathbb{N} \setminus \{0\}$,
- $g \in C^0(\mathbb{R}^{*+})$ increasing with positive values;

we say

$$\alpha \in \mathrm{NR}_\omega^N(\kappa', g)$$

if for all $m \in \mathbb{Z}^d \setminus \{0\}$ such that $0 < | m | \leq N$,

$$| \alpha - i\pi\langle m, \omega \rangle | \geq \frac{\kappa'}{g(| m |)} \qquad (4)$$

and we say that

$$\alpha \in \mathrm{NR}_\omega(\kappa', g)$$

if (4) holds for all $m \in \mathbb{Z}^d \setminus \{0\}$.

Remark: If $g(t) = t^\tau$ for some $\tau > 1$, this is a diophantine condition.

2.2 Elimination of resonances

Now we will fix $\omega \in \mathbb{R}^d$ rationally independent (i.e such that for all non zero $m \in \mathbb{Z}^d, \langle m,\omega\rangle \neq 0$): it will be the frequency of the cocycles we will consider.

Remark: There exists a positive increasing and unbounded function $G \in C^0(\mathbb{R}^{*+})$ with $G(1) \geq 1$ and $\kappa > 0$ such that $\omega \in \mathrm{NR}(\kappa, G)$.
Indeed, one can take $\kappa = \min_i \mid \omega_i \mid$ and $G(N) = \max_{|m|\leq N} \frac{\kappa}{|\langle m,\omega\rangle|}$.

As noticed by H. Rüssmann ([7]), a condition $\mathrm{NR}(\kappa, G)$ with G such that

$$\int_1^{\infty} \frac{\log G(t)}{t^2} dt < \infty \tag{5}$$

is fulfilled by all Bruno vectors (see [1]), i.e vectors satisfying:

$$\sum_{k\geq 1} \frac{\mid \log \alpha_{2^k-1} \mid}{2^k} < +\infty \tag{6}$$

where

$$\alpha_k = \min_{l\leq k} \min_{j=1,\dots,d} \min_{|m|=l+1} \mid \langle m,\omega\rangle - \omega_j \mid . \tag{7}$$

In [5] it is shown that condition (6) is equivalent to

$$\sum_{k\geq 1} \frac{\mid \log \alpha_k \mid}{k(k+1)} < +\infty \tag{8}$$

and condition (5) implies condition (8), so that (5), (6) and (8) are equivalent.

Notation: From now on, g will be a fixed positive increasing map defined on $[1, +\infty[$ and such that $g(1) \geq 1$.

Now we shall prove the uniqueness of a resonance when it exists.

Lemma 2.1 *Let $\alpha \in \mathbb{C}$. Let $N \in \mathbb{N} \setminus \{0\}$. There exists $m \in \mathbb{Z}^d$ such that $\mid m \mid \leq N$ and $\alpha - i\pi\langle m,\omega\rangle \in \mathrm{NR}_{\omega}^{N}(\frac{\kappa}{4G(N)}, g)$; if m is non zero, then*

$$\mid \alpha - i\pi\langle m,\omega\rangle \mid < \frac{\kappa}{4G(N)g(\mid m \mid)}$$

and $\alpha - i\pi\langle m,\omega\rangle \in \mathrm{NR}_{\omega}^{N}(\frac{\kappa}{G(N)}, g)$.

Proof: Suppose α is not in $\mathrm{NR}_{\omega}^{N}(\frac{\kappa}{4G(N)}, g)$, i.e there exists $m \in \mathbb{Z}^d, 0 <\mid m \mid \leq N$, such that

$$\mid \alpha - i\pi\langle m,\omega\rangle \mid < \frac{\kappa}{4G(N)g(\mid m \mid)}.$$

Then for all $m' \in \mathbb{Z}^d$ with $0 <\mid m' \mid \leq N$,

$$| \alpha - i\pi\langle m+m',\omega\rangle | \geq | \pi\langle m',\omega\rangle | - | \alpha - i\pi\langle m,\omega\rangle | \geq \frac{\kappa}{G(| m' |)} - \frac{\kappa}{4G(N)g(| m |)} \tag{9}$$

so

$$| \alpha - i\pi\langle m+m',\omega\rangle | \geq \frac{\kappa}{G(N)g(| m' |)} \tag{10}$$

and so $\alpha - i\pi\langle m,\omega\rangle \in \mathrm{NR}_\omega^N(\frac{\kappa}{G(N)}, g)$. $\square$

The following proposition explains how to eliminate resonances in the spectrum of a trace zero matrix.

Proposition 2.2 *Let $A \in sl(2,\mathbb{R})$ with eigenvalues $\pm\alpha$. Let $N \in \mathbb{N}$. Suppose that α is not in $\mathrm{NR}_\omega^N(\frac{\kappa}{4G(N)}, g)$. There exists $\Phi \in \cap_{r'\geq 0} C^\omega_{r'}(2\mathbb{T}^d, SL(2,\mathbb{R}))$ and a numerical constant C' such that*

$$\forall r' \geq 0, \ | \Phi |_{r'} \leq C' e^{\pi N r'}; \ | \Phi^{-1} |_{r'} \leq C' e^{\pi N r'} \tag{11}$$

and if $\tilde{A}$ with eigenvalues $\pm\tilde{\alpha}$ is such that

$$\partial_\omega \Phi = A\Phi - \Phi\tilde{A} \tag{12}$$

then $\tilde{\alpha} \in \mathrm{NR}_\omega^N(\frac{\kappa}{G(N)}, g)$.
Moreover,

$$| \tilde{\alpha} | < \frac{\kappa}{4G(N)}.$$

Proof: Lemma 2.1 gives a number $m, 0 < | m | \leq N$, such that letting

$$\tilde{\alpha} = \alpha - i\pi\langle m,\omega\rangle$$

then $\tilde{\alpha} \in \mathrm{NR}_\omega^N(\frac{\kappa}{G(N)}, g)$. Let P be such that $P^{-1}AP$ is diagonal and $|| P || = 1$. We define

$$\Phi(\theta) = P^{-1} \begin{pmatrix} e^{i\pi\langle m,\theta\rangle} & 0 \\ 0 & e^{-i\pi\langle m,\theta\rangle} \end{pmatrix} P.$$

Relation (12) gives

$$\tilde{A} = P^{-1} \begin{pmatrix} \tilde{\alpha} & 0 \\ 0 & -\tilde{\alpha} \end{pmatrix} P.$$

To obtain the estimate (11), we use an estimate shown for instance in [4], Lemma A:

$$|| P^{-1} || \leq \max\left(1, \left(\frac{C. || A ||}{2 | \alpha |}\right)^6\right)$$

where C is a numerical constant, and since A is diagonalizable whenever its eigenvalues are non zero,

$$\| P^{-1} \| \leq \max\left(1, \left(\frac{C. \mid \alpha \mid}{2 \mid \alpha \mid}\right)^6\right) \leq C'$$

where C' is a numerical constant, which gives (11). □

2.3 Solution of the linearized homological equation

Our aim is to solve an equation of the form

$$\partial_\omega Z = (A+F)Z - Z(A'+F')$$

where A and F are known, $A \in sl(2,\mathbb{R})$ and F is analytic with values in $sl(2,\mathbb{R})$. If A is non-resonant, we first solve

$$\partial_\omega \tilde{X} = [A, \tilde{X}] + aF^N - a\hat{F}(0)$$

where F^N is some truncation of F and a is close enough to 1; then we define $A' = A + a\hat{F}(0)$ and F' by

$$\partial_\omega e^{\tilde{X}} = (A+F)e^{\tilde{X}} - e^{\tilde{X}}(A'+F')$$

and then we estimate F' to get $\mid F' \mid_{r'} \leq \sqrt{1-a} \mid F \mid_r$. If A is resonant, we conjugate $A+F$ to a system $\tilde{A}+\tilde{F}$ where $\tilde{A}$ is non-resonant and we proceed in the same way as in the non-resonant case. So, from now on, to simplify the notations, we will assume that $A, \tilde{A}, A' \in sl(2,\mathbb{R})$, that $F, \tilde{F}, \tilde{X}, F'$ have values in $sl(2,\mathbb{R})$ and Z, Φ have values in $SL(2,\mathbb{R})$.

Proposition 2.3 *Let*

- $N \in \mathbb{N}$,
- $r, r' > 0$.

Let $\tilde{A}$ with eigenvalues $\pm\tilde{\alpha} \in \mathrm{NR}_\omega^N(\frac{\kappa}{4G(N)}, g)$. Let $\tilde{F} \in C_r^\omega(\mathbb{T}^d)$. Then equation

$$\forall \theta \in \mathbb{T}^d,\ \partial_\omega \tilde{X}(\theta) = [\tilde{A}, \tilde{X}(\theta)] + \tilde{F}^N(\theta) - \hat{\tilde{F}}(0);\ \hat{\tilde{X}}(0) = 0 \tag{13}$$

has a unique solution $\tilde{X} \in C_{r'}^\omega(\mathbb{T}^d)$ such that

$$\mid \tilde{X} \mid_{r'} \leq \frac{4}{\kappa} G(N) g(N) \mid \tilde{F}^N \mid_{r'} \tag{14}$$

Proof: In Fourier series, equation (13) can be written:

$$\begin{aligned} \forall m \in \mathbb{Z}^d, 0 < \mid m \mid \leq N \Rightarrow 2i\pi\langle m, \omega\rangle \hat{\tilde{X}}(m) = [\tilde{A}, \hat{\tilde{X}}(m)] + \hat{\tilde{F}}(m); \\ \mid m \mid \in \{0\} \cup [N+1, +\infty[\Rightarrow 2i\pi\langle m, \omega\rangle \hat{\tilde{X}}(m) = [\tilde{A}, \hat{\tilde{X}}(m)]. \end{aligned} \tag{15}$$

So for $\mid m \mid \in \{0\} \cup [N+1, +\infty[$, $\hat{\tilde{X}}(m) = 0$ is a solution (not necessarily unique).

For $0 <| m |\leq N$, the solution is formally written as

$$\hat{\tilde{X}}(m) = \mathcal{L}_m^{-1}\hat{\tilde{F}}(m) \tag{16}$$

where $\mathcal{L}_m$ is the operator

$$\mathcal{L}_m : sl(2,\mathbb{R}) \to sl(2,\mathbb{R}),\ M \mapsto 2i\pi\langle m,\omega\rangle M - [\tilde{A}, M].$$

Its spectrum is $\{2i\pi\langle m,\omega\rangle - 2\tilde{\alpha}, 2i\pi\langle m,\omega\rangle + 2\tilde{\alpha}, 2i\pi\langle m,\omega\rangle\}$.
Since $\omega \in \mathrm{NR}(\kappa, G)$ and $\tilde{\alpha} \in \mathrm{NR}_\omega^N(\frac{\kappa}{4G(N)}, g)$, $\mathcal{L}_m$ is invertible and we have for all $0 <| m |\leq N$,

$$|| \mathcal{L}_m^{-1} ||\leq \max\{\frac{G(| m |)}{\kappa}, \frac{4G(N)g(| m |)}{\kappa}\} = \frac{4G(N)g(| m |)}{\kappa}$$

therefore for all $m \in \mathbb{Z}^d$ such that $0 <| m |\leq N$,

$$|| \hat{\tilde{X}}(m) ||\leq 4G(N)\frac{g(| m |)}{\kappa} || \hat{\tilde{F}}(m) || \tag{17}$$

therefore

$$\begin{aligned} | \tilde{X} |_{r'} &\leq 4G(N) \sum_{m\in\mathbb{Z}^d\setminus\{0\},|m|\leq N} \frac{g(| m |)}{\kappa} || \hat{\tilde{F}}(m) || e^{2\pi|m|r'} \\ &\leq 4\frac{G(N)g(N)}{\kappa} \sum_{m\in\mathbb{Z}^d\setminus\{0\},|m|\leq N} || \hat{\tilde{F}}(m) || e^{2\pi|m|r'} \\ &\leq 4\frac{G(N)g(N)}{\kappa} | \tilde{F}^N |_{r'} . \Box \end{aligned} \tag{18}$$

2.4 Solution of the full homological equation without resonances

This section explains the basic step in case the constant part is non resonant.

Proposition 2.4 *Let $0 < r' \leq r$, $a' \in (0,1]$, $N \in \mathbb{N}$, $\tilde{F} \in C_r^\omega(\mathbb{T}^d)$, $\tilde{A} \in sl(2,\mathbb{R})$. If $\sigma(\tilde{A}) = \{\pm\alpha\}, \alpha \in \mathrm{NR}_\omega^N(\frac{\kappa}{4G(N)}, g)$, then there exists*

- $\tilde{X}, F' \in C_{r'}^\omega(\mathbb{T}^d)$,
- $A' \in sl(2,\mathbb{R})$

such that

$$| A' - \tilde{A} |\leq|| \hat{\tilde{F}}(0) || \tag{19}$$

$$\partial_\omega e^{\tilde{X}} = (\tilde{A} + \tilde{F})e^{\tilde{X}} - e^{\tilde{X}}(A' + F') \tag{20}$$

$$| \tilde{X} |_{r'}\leq 4a'\frac{G(N)g(N)}{\kappa} | \tilde{F}^N |_{r'} \tag{21}$$

and

$$\begin{aligned} | F' |_{r'} &\leq e^{|\tilde{X}|_{r'}}(1-a') \mid \tilde{F} |_{r'} + e^{|\tilde{X}|_{r'}} a' \mid \tilde{F} - \tilde{F}^N |_{r'} \\ &+ e^{|\tilde{X}|_{r'}} \mid \tilde{F} |_{r'} | \tilde{X} |_{r'} (e^{|\tilde{X}|_{r'}} + a' + a' e^{|\tilde{X}|_{r'}}). \end{aligned} \tag{22}$$

Proof: Let $\tilde{X}$ be a solution of

$$\forall \theta \in \mathbb{T}^d,\ \partial_\omega \tilde{X}(\theta) = [\tilde{A}, \tilde{X}(\theta)] + a' \tilde{F}^N(\theta) - a' \hat{\tilde{F}}(0);\ \hat{\tilde{X}}(0) = 0 \tag{23}$$

as given by Proposition 2.3 (so it satisfies (21)). Let $A' = \tilde{A} + \hat{\tilde{F}}(0)$ so that (19) holds, and let F' be defined by

$$\partial_\omega e^{\tilde{X}} = (\tilde{A} + \tilde{F}) e^{\tilde{X}} - e^{\tilde{X}}(A' + F').$$

We have

$$F' = e^{-\tilde{X}}(\tilde{F} - a'\tilde{F}^N) + e^{-\tilde{X}}\tilde{F}(e^{\tilde{X}} - Id) + a'(e^{-\tilde{X}} - Id)\hat{\tilde{F}}(0) - e^{-\tilde{X}} \sum_{k\geq 2} \frac{1}{k!} \sum_{l=0}^{k-1} \tilde{X}^l (a'\tilde{F}^N - a'\hat{\tilde{F}}(0)) \tilde{X}^{k-1-l}. \tag{24}$$

Now

$$| \tilde{F} - a'\tilde{F}^N |_{r'} \leq a' \mid \tilde{F} - \tilde{F}^N |_{r'} + (1-a') \mid \tilde{F} |_{r'} . \tag{25}$$

Thus

$$\begin{aligned} | F' |_{r'} &\leq e^{|\tilde{X}|_{r'}}(1-a') \mid \tilde{F} |_{r'} + e^{|\tilde{X}|_{r'}} a' \mid \tilde{F} - \tilde{F}^N |_{r'} \\ &+ e^{|\tilde{X}|_{r'}} \mid \tilde{F} |_{r'} | \tilde{X} |_{r'} (e^{|\tilde{X}|_{r'}} + a' + a' e^{|\tilde{X}|_{r'}}). \ \Box \end{aligned} \tag{26}$$

Remark: Denote $\epsilon = | \tilde{F} |_r$. Suppose

$$2G(N)g(N)\epsilon \leq \frac{\kappa(1-a')}{2}. \tag{27}$$

Then (21) implies

$$| \tilde{X} |_{r'} \leq a'(1-a')$$

which implies that $e^{|\tilde{X}|_{r'}} \leq 2$, and therefore, by (22), if one assumes moreover that

$$e^{-2\pi N(r-r')} \leq 1 - a' \tag{28}$$

with $r' > 0$, then

$$| \tilde{F} - \tilde{F}^N |_{r'} \leq (1-a') \mid \tilde{F} - \tilde{F}^N |_r$$

therefore

$$| F' |_{r'} \leq 2(1-a')\epsilon + 2a'(1-a')\epsilon + 2\epsilon a'(1-a')(2+3a'). \tag{29}$$

Thus, if a' is close enough to 1 (i.e larger than $1-\frac{1}{14^2}$),

$$| F' |_{r'} \leq (1-a')^{\frac{1}{2}}\epsilon. \tag{30}$$

2.5 Solution of the full homological equation with resonances

This section presents the basic step when there are resonances in the constant part.

Proposition 2.5 *Let $a \in (0,1), c_0 > 0$, C' as in Proposition 2.2, $N \in \mathbb{N}$, $r > \frac{2\log(g(N)G(N))}{\pi N}$, $F \in C^\omega_r(\mathbb{T}^d)$, $| F |_r = \epsilon$, $A \in sl(2,\mathbb{R})$. Suppose the eigenvalues $\pm\alpha$ of A are not in $\mathrm{NR}^N_\omega(\frac{\kappa}{4G(N)}, g)$. If*

$$2G(N)^2 g(N)^2 \epsilon \leq \frac{(1-a)^2}{2}\kappa^2 \tag{31}$$

and

$$eC'(G\cdot g)(N+1)^{-c_0} \leq 1-a \tag{32}$$

then letting $r' = \frac{r}{2} - c_0 \frac{\log(G\cdot g)(N+1)}{4\pi N}$, there exists

- $F' \in C^\omega_{r'}(\mathbb{T}^d)$,
- $A' \in sl(2,\mathbb{R})$,
- $Z \in C^\omega_{r'}(2\mathbb{T}^d)$,

such that

$$\partial_\omega Z = (A+F)Z - Z(A'+F') \tag{33}$$

and

$$| F' |_{r'} \leq (1-a)\epsilon. \tag{34}$$

Proof: One first applies Proposition 2.2 on A. Let $\Phi, \tilde{A}$ be as in Proposition 2.2 so that $\sigma(\tilde{A}) = \pm\tilde{\alpha}$, $\tilde{\alpha} \in \mathrm{NR}^N_\omega(\frac{\kappa}{G(N)}, g)$ and $| \tilde{\alpha} | \leq \frac{\kappa}{4G(N)}$; let $\tilde{F} = \Phi F \Phi^{-1}$. Notice that, by construction of Φ, the map $\tilde{F}$ remains continuous on $\mathbb{T}^d$. Apply Proposition 2.4 with $a' = 1$ and with $r' = \frac{r}{2} - c_0\frac{\log(G\cdot g)(N+1)}{4\pi N}$ to get $\tilde{X} \in C^\omega_{r'}(\mathbb{T}^d), A', F' \in C^\omega_{r'}(\mathbb{T}^d)$ such that (20) and (22) hold as well as

$$| \tilde{X} |_{r'} \leq 4\frac{G(N)g(N)}{\kappa} | \tilde{F}^N |_{r'} \tag{35}$$

and let $Z = \Phi e^{\tilde{X}} \in C^\omega_{r'}(2\mathbb{T}^d)$ so that Z satisfies (33). Condition (31) implies that

$$(G\cdot g)(N) | \tilde{X} |_{r'} \leq \frac{(1-a)^2 | \tilde{F}^N |_{r'}}{\epsilon}$$

so (22) with $a' = 1$ gives

$$| F' |_{r'} \leq eC' | F - F^N |_r e^{-2\pi N(r-2r')} + eC' | F^N |_{r'} e^{2\pi N r'} \frac{(1-a)^2}{(G\cdot g)(N)}(2e+1) \qquad (36)$$

and by the choice of r',

$$| F' |_{r'} \leq eC' | F |_r (G\cdot g)(N+1)^{-c_0}. \qquad (37)$$

This implies, by assumption (32), that

$$| F' |_{r'} \leq (1-a) | F |_r . \square \qquad (38)$$

3 Iteration, reducibility and arithmetical conditions

3.1 Iteration

Here comes the actual Brjuno-Rüssmann condition:

Assumption 1 *The functions g and G satisfy*

$$\int_1^\infty \frac{\log[g(t)G(t)]}{t^2} dt < \infty. \qquad (39)$$

In order to iterate the basic step, we will now fix the parameters as follows: let C' be as in Proposition 2.2.

- Let $r_0 > 0$.
- Let $n_0 \in \mathbb{N}$.
- Let $c_0 = \frac{r_0}{4^{n_0+3}(\sup_{t\in[1,n_0]} \frac{\log(G\cdot g)(t+1)}{t}+1)}$.
- Let $a \in [1 - \frac{1}{14^2}, 1)$ such that

$$1 - a \leq \frac{1}{(G\cdot g)(2)^2}.$$

- Let ϵ_0 such that

$$\int_{(G\cdot g)^{-1}\left(\frac{\kappa}{2(1-a)^{\frac{n_0-5}{4}}\epsilon_0^{\frac{1}{2}}}\right)}^\infty \frac{\log(G\cdot g)(t)}{t^2} dt \leq \frac{r_0}{4^{n_0+2}} \qquad (40)$$

  and such that

$$eC'\epsilon_0^{\frac{c_0}{4}} \leq (1-a)^2\kappa^2. \qquad (41)$$

- For all $n \in \mathbb{N}$, let $\epsilon_n = (1-a)^{\frac{n}{2}}\epsilon_0$;
- For all $n \in \mathbb{N}$, let N_n be the biggest integer such that

$$(G \cdot g)(N_n)^2 \leq \frac{(1-a)^2}{4\epsilon_n}\kappa^2$$

(N_n exists since $\epsilon_n \leq \frac{(1-a)^2\kappa^2}{4e(G\cdot g)(1)^2}$). In this way, we have

$$\frac{\kappa^2}{4(1-a)^{\frac{n_0-5}{2}}\epsilon_0} = \frac{(1-a)^2}{4\epsilon_{n_0-1}}\kappa^2 \leq (G\cdot g)(N_{n_0})^2$$

therefore

$$(G\cdot g)^{-1}\left(\frac{\kappa}{2(1-a)^{\frac{n_0-5}{4}}\sqrt{\epsilon_0}}\right) \leq N_{n_0}$$

and so

$$\int_{N_{n_0}}^{\infty} \frac{\log(G\cdot g)(t)}{t^2}dt \leq \frac{r_0}{4^{n_0+2}}. \tag{42}$$

Remark: : The number ϵ_0 will then only depend on a, κ, g, G, n_0 and r_0 (the higher n_0 and the smaller r_0 are, the smaller ϵ_0 will be).

To simplify the notations, from now on the functions A_n are understood to be in $sl(2,\mathbb{R})$, while the F_n have their values in $sl(2,\mathbb{R})$ and Z'_n, Z_n have their values in $SL(2,\mathbb{R})$.

Proposition 3.1 *Let*

- $A \in sl(2,\mathbb{R})$,
- $F \in C^{\omega}_{r_0}(\mathbb{T}^d)$.

If $\mid F \mid_{r_0} \leq \epsilon_0$, *then there exist sequences*

- $(r_n)_{n\in\mathbb{N}}, r_n > 0$,
- $Z_n, \in C^{\omega}_{r_n}(2\mathbb{T}^d)$,
- A_n *with spectrum* $\pm\alpha_n$,
- $F_n \in C^{\omega}_{r_n}(2\mathbb{T}^d)$,
- $m_n \in \mathbb{Z}^d$,

such that

1. *if all* m_n *are zero when* $n \geq n_0$, *then* r_n *has a positive limit;*

2. *if* $m_n \neq 0$ *then* $\mid \alpha_n - \pi\langle m_n, \omega\rangle \mid \leq \frac{\kappa}{4G(N_n)}$*;*

3. m_n *has modulus less than* N_n*,*

4. $\mid F_n \mid_{r_n} \leq \epsilon_n$*;*

5. $\partial_\omega Z_n = (A+F)Z_n - Z_n(A_n + F_n)$*;*

6. $\mid \alpha_{n-1} - i\pi\langle m_{n-1}, \omega\rangle - \alpha_n \mid \leq \sqrt{\epsilon_{n-1}}$*.*

Remark: Proposition 3.1 implies that $A+F$ is reducible in $C^\omega_{r'}$ for some $r' > 0$ if all m_n are zero for $n \geq n_0$.

Proof:

This proposition is shown by recurrence. Suppose these sequences are defined up to some $n \in \mathbb{N}$ and suppose that for all $n' \leq \min(n-1, n_0)$, $r_{n'+1} \geq \frac{r_{n'}}{4}$.

First case: Suppose that $\alpha_n \in NR^{N_n}_\omega(\frac{\kappa}{4G(N_n)}, g)$. Let $r_{n+1} = r_n - c_0\frac{|\log(1-a)|}{2\pi N_n}$, so that $r_{n+1} \geq \frac{r_n}{2}$ if $n \leq n_0$. One can apply Proposition 2.4 with

- $r = r_n$
- $r' = r_{n+1}$
- $N = N_n$
- $\tilde{F} = F_n$
- $\tilde{A} = A_n$

and obtain $Z'_n = e^{\tilde{X}_n} \in C^\omega_{r_{n+1}}(\mathbb{T}^d)$, $F_{n+1} \in C^\omega_{r_{n+1}}(\mathbb{T}^d)$, A_{n+1} such that

$$\partial_\omega Z'_n = (A_n + F_n)Z'_n - Z'_n(A_{n+1} + F_{n+1}) \tag{43}$$

and

$$\mid F_{n+1} \mid_{r_{n+1}} \leq (1-a)^{\frac{1}{2}}\epsilon_n = \epsilon_{n+1}. \tag{44}$$

One then takes $Z_{n+1} = Z_n Z'_n$.

Second case: α_n is not in $NR^{N_n}_\omega(\frac{\kappa}{4G(N_n)}, g)$.
Assumption (31) is satisfied by definition of N_n; assumption (32) is also satisfied since, by maximality of N_n,

$$(G\cdot g)(N_n+1)^{-c_0} \leq \left(\frac{2\epsilon_n}{(1-a)^2\kappa^2}\right)^{\frac{c_0}{2}} \leq \left(\frac{2\epsilon_0}{(1-a)^2\kappa^2}\right)^{\frac{c_0}{2}} \tag{45}$$

which, together with (41), implies that

$$(G\cdot g)(N_n+1)^{-c_0} \leq \frac{1-a}{eC'}. \tag{46}$$

Therefore, one can apply Proposition 2.5 with

- $r = r_n$,
- $r' = r_{n+1} = \frac{r_n}{2} - c_0 \frac{\log(G\cdot g)(N_n+1)}{\pi N_n}$, so that $r_{n+1} \geq \frac{r_n}{4}$ if $n \leq n_0$,
- $N = N_n$

to get

- $A_{n+1} \in sl(2,\mathbb{R})$,
- $F_{n+1} \in C^{\omega}_{r_{n+1}}(\mathbb{T}^d)$,
- $Z'_n \in C^{\omega}_{r_{n+1}}(2\mathbb{T}^d)$,

such that

$$\partial_\omega Z'_n = (A_n + F_n)Z'_n - Z'_n(A_{n+1} + F_{n+1})$$

and

$$\mid F_{n+1} \mid_{r_{n+1}} \leq (1-a) \mid F_n \mid_{r_n} \leq \epsilon_{n+1}$$

One then takes $Z_{n+1} = Z_n Z'_n$.

Let us show that $(r_n)_n$ has a positive limit if all m_n are zero for $n \geq N_0$. We have

$$\begin{aligned} \lim_n r_n &= r_{n_0} - \sum_{k=n_0}^{\infty} (r_k - r_{k+1}) \\ &\geq \frac{r_0}{4^{n_0}} - \sum_{k \geq n_0} \frac{\mid \log(1-a) \mid}{2\pi N_k}. \end{aligned} \tag{47}$$

Now, for all n,

$$N_n = E\left((G\cdot g)^{-1} \left(\frac{(1-a)\kappa}{2\sqrt{\epsilon_n}} \right) \right)$$

thus

$$\lim_n r_n \geq \frac{r_0}{4^{n_0}} - \frac{\mid \log(1-a) \mid}{2\pi} \int_{n_0}^{\infty} \left[(G\cdot g)^{-1} \left(\frac{\kappa}{2(1-a)^{\frac{n}{4}-1}} \right) \right]^{-1} dn. \tag{48}$$

Through the change of variables $X = \frac{\kappa}{2(1-a)^{\frac{n}{4}-1}}$,

$$\lim_n r_n \geq \frac{r_0}{4^{n_0}} - \int_{(G\cdot g)(N_{n_0})}^{\infty} \frac{1}{\pi (G\cdot g)^{-1}(X) X} dX. \tag{49}$$

Letting now $Y = (G\cdot g)^{-1}(X)$, the integral becomes

$$\begin{aligned}\int_{(G\cdot g)(N_{n_0})}^{\infty}\frac{1}{\pi(G\cdot g)^{-1}(X)X}dX &= \int_{N_{n_0}}^{\infty}\frac{1}{\pi Y(G\cdot g)(Y)}d(G\cdot g)(Y)\\ &= [\frac{\log(G\cdot g)(Y)}{\pi Y}]_{N_{n_0}}^{\infty}+\int_{N_{n_0}}^{\infty}\frac{\log(G\cdot g)(Y)}{\pi Y^2}dY\\ &= \frac{-\log(G\cdot g)(N_{n_0})}{N_{n_0}}+\int_{N_{n_0}}^{\infty}\frac{\log(G\cdot g)(Y)}{Y^2}dY.\end{aligned} \tag{50}$$

Therefore

$$\lim_n r_n \geq \frac{r_0}{4^{n_0}}+\frac{\log(G\cdot g)(N_{n_0})}{\pi N_{n_0}}-\frac{1}{\pi}\int_{N_{n_0}}^{\infty}\frac{\log(G\cdot g)(Y)}{Y^2}dY \tag{51}$$

so, by (42),

$$\lim_n r_n \geq \frac{r_0}{4^{n_0+1}} \tag{52}$$

which is positive. □

3.2 A link between the Brjuno sum and the allowed perturbation

It is easily seen that the condition on ϵ_0 can also be expressed more conveniently as the following sufficient condition:

$$\epsilon_0 \leq \exp\left(-\frac{r_0}{4^{n_0}}-\mid\log\frac{\kappa}{2(1-a)^{n_0}}\mid -2\int_1^{\infty}\frac{\log(g\cdot G)(t)}{t^2}dt\right) \tag{53}$$

Indeed, we have the following bound:

$$\begin{aligned}\mid\frac{1}{2}\log\epsilon_0+\int_1^{\infty}\frac{\log(g\cdot G)(t)}{t^2}dt)\mid &=\mid\int_1^{\infty}\frac{\log\sqrt{\epsilon_0}}{t^2}dt+\int_1^{\infty}\frac{\log(g\cdot G)(t)}{t^2}dt)\mid\\ &=\mid\int_1^{\infty}\frac{\log(\sqrt{\epsilon_0}g\cdot G)(t)}{t^2}dt)\mid\\ &\leq\frac{r_0}{4^{n_0}}+\int_1^{(g\cdot G)^{-1}(\frac{\kappa}{2(1-a)^{n_0}\sqrt{\epsilon_0}})}\frac{\mid\log\sqrt{\epsilon_0}(g\cdot G)(t)\mid}{t^2}dt\end{aligned} \tag{54}$$

and the conclusion follows easily from the upper bound on t.

3.3 Reducibility theorem

We will now need one more assumption on the approximation functions G and g.

Assumption 2 *The map $t\mapsto\frac{g(t^2)}{G(t)}$ is bounded.*

Now we can prove the main result:

Theorem 3.2 *Let $A \in sl(2,\mathbb{R})$, $r>0$, $F \in C^\omega_r(\mathbb{T}^d)$. Let $n_0 \in \mathbb{N}$. Assume $\rho(A+F) \in \mathrm{NR}_\omega(\kappa', g)$ with $\kappa' > \kappa \sup_{t \geq n_0} \frac{g(t^2)}{G(t)}$.*
Under assumptions 1 and 2 on the approximation functions g and G, there exist $\epsilon_0 > 0$ depending only on g, κ, G, n_0, r such that if

$$\mid F \mid_r \leq \epsilon_0,$$

then there exists $r' \in (0, r)$ such that $A+F$ is reducible in $C^\omega_{r'}(2\mathbb{T}^d)$.

Proof: Let $a \in [\max(1 - \frac{1}{14^2}, 1 - \frac{1}{G \cdot g(2)^2}), 1[$. Let $\epsilon_0 > 0, (\epsilon_n)_{n\in\mathbb{N}}, (N_n)_{n\in\mathbb{N}}$ as defined at the beginning of section 3.1. Let $(r_n), (\alpha_n), (m_n), (A_n), (F_n), (Z_n)$ be the sequences given by Proposition 3.1.
The sequence (A_n) is bounded in $sl(2,\mathbb{R})$ for the operator norm so taking a subsequence (A_{n_k}), we find that A_{n_k} tends to some $A_\infty \in gl(2,\mathbb{R})$. Now $\rho(A_\infty)$ is the limit of $\rho(A_{n_k})$ (see [3], Lemma A.3) which implies that for all n,

$$\rho(A_\infty) = \rho(A_{n+1}) - \lim_{k\to\infty} \sum_{j=n+1}^{n_k - 1} (\rho(A_j) - \rho(A_{j+1})).$$

Moreover,

$$\rho(A+F) = \rho(A_\infty) + \pi \sum_{j\geq 0} \langle m_j, \omega\rangle \tag{55}$$

(see also [3]). Therefore

$$\begin{aligned} \mid \rho(A+F) - \pi \sum_{j\leq n} \langle m_j, \omega \rangle \mid &= \mid \rho(A_\infty) + \pi \sum_{j \geq n+1} \langle m_j, \omega\rangle \mid \\ &\leq \mid \alpha_{n+1} \mid + \sum_{j\geq n+1} \mid \alpha_j - \pi\langle m_j, \omega\rangle - \alpha_{j+1} \mid \\ &\leq \mid \alpha_{n+1} \mid + \sum_{j \geq n+1} \sqrt{\epsilon_j} \end{aligned} \tag{56}$$

Suppose $\rho(A+F)$ satisfies

$$\forall m \in \mathbb{Z}^d,\ \mid \rho(A+F) - \pi\langle m, \omega\rangle \mid \geq \frac{\kappa'}{g(\mid m \mid)}.$$

In particular,

$$\mid \rho(A+F) - \pi \sum_{j\leq n} \langle m_j, \omega\rangle \mid \geq \frac{\kappa'}{g(\mid \sum_{j\leq n} m_j \mid)}$$

and so

$$\frac{\kappa'}{g(\mid \sum_{j\leq n} m_j \mid)} \leq \sum_{j\geq n+1} \sqrt{\epsilon_j} + \mid \alpha_{n+1} \mid. \tag{57}$$

Let $n > n_0$. Assume $m_n \neq 0$. Then we have

$$| \alpha_{n+1} | \leq | \alpha_n - \pi\langle m_n, \omega\rangle | + | \alpha_n - \pi\langle m_n, \omega\rangle - \alpha_{n+1} | \leq \frac{\kappa}{4G(N_n)} + \sqrt{\epsilon_n}$$

so

$$\kappa' \leq \left[\sum_{j\geq n} \sqrt{\epsilon_j} + \frac{\kappa}{4G(N_n)}\right] g(| \sum_{j\leq n} m_j |). \tag{58}$$

Thus

$$\kappa' \leq \left[\sum_{j\geq n} \sqrt{\epsilon_j} + \frac{\kappa}{4G(N_n)}\right] g(\sum_{j\leq n} | m_j |). \tag{59}$$

Now

$$\sum_{j\geq n} \sqrt{\epsilon_j} = \frac{1}{1-(1-a)^{\frac{1}{4}}} \sqrt{\epsilon_n} \leq 2\sqrt{\epsilon_n}$$

and since, by definition of N_n,

$$\epsilon_n \leq \frac{(1-a)^2\kappa^2}{4G(N_n)^2 g(N_n)^2}$$

then

$$\kappa' \leq \frac{\kappa}{G(N_n)} g(\sum_{j\leq n} | m_j |). \tag{60}$$

Now, note that $\sum_{j\leq n} | m_j | \leq N_n^2$. This comes from the fact that, denoting by m_{j_k} the subsequence of non-zero m_j's, then for all k,

$$\begin{aligned} | \langle m_{j_{k+1}}, \omega\rangle | &< | \langle m_{j_{k+1}}, \omega\rangle - \alpha_{j_{k+1}} | + | \langle m_{j_k}, \omega\rangle - \alpha_{j_k} + \alpha_{j_{k+1}} | + | \langle m_{j_k}, \omega\rangle - \alpha_{j_k} | \\ &< \frac{\kappa}{4G(N_{j_{k+1}})} + 2\sqrt{\epsilon_{j_k}} + \frac{\kappa}{4G(N_{j_k})} \\ &\leq \frac{\kappa}{4G(N_{j_{k+1}})} + \frac{\kappa}{2G(N_{j_k})} + \frac{\kappa}{4G(N_{j_k})} \end{aligned} \tag{61}$$

which together with the arithmetic condition on ω implies that $N_{j_{k+1}} > N_{j_k} \geq k$. Therefore

$$\kappa' \leq \frac{\kappa}{G(N_n)} g(N_n^2) \tag{62}$$

and since by assumption $\frac{g(t^2)}{G(t)}$ is bounded,

$$\kappa' \leq \kappa \sup_{t\geq n} \frac{g(t^2)}{G(t)}. \tag{63}$$

In other words, if

$$\kappa' > \kappa \sup_{t \geq n} \frac{g(t^2)}{G(t)} \tag{64}$$

then $m_n = 0$ for all $n \geq n_0$; and so $A + F$ is analytically reducible. □

This proves Theorem 1.1. Here is an easy consequence of the main result:

Corollary 3.3 *Let $A \in sl(2, \mathbb{R})$, $r > 0$, $F \in C_r^\omega(\mathbb{T}^d)$. Assume*

1. *the map $t \mapsto \frac{g(t^2)}{G(t)}$ tends to 0,*
2. *$\rho(A + F) \in \mathrm{NR}_\omega(\kappa', g)$ for some $\kappa' > 0$.*

There exist ϵ_0 depending only on $g, \kappa, G, \rho(A + F), r$ such that if

$$\mid F \mid_r \leq \epsilon_0,$$

then there exists $r' \in (0, r)$ such that $A + F$ is reducible in $C_{r'}^\omega(2\mathbb{T}^d)$.

Proof: Let n_0 be the smallest integer such that $\kappa' > \sup_{t \geq n_0} \frac{g(t^2)}{G(t)}$. Take ϵ_0 as in Theorem 3.2 so that it really depends on $g, \kappa, G, \rho(A + F), r$ and apply Corollary 3.2. □

Theorem 1.2 is a particular case of Corollary 3.3 since we can take $g(t) = t^\mu, G(t) = t^{\mu'}$ with $\mu' \geq \frac{\mu}{2}, \mu \geq 1, \mu' \geq 1$, as we will see in the next section.

3.4 Possible choices of approximation functions

Here we give a few examples of approximation functions to which Theorem 3.2 can be applied.

Verification of Assumption 2 Here are a few examples where Assumption 2 holds, i.e $\frac{g(t^2)}{G(t)}$ is bounded:

1. $g(t) = t^\mu, G(t) = t^{\mu'}$ with $\mu' \geq \frac{\mu}{2}, \mu \geq 1, \mu' \geq 1$;
2. $g(t) = e^{t^\alpha}, G(t) = e^{t^{\alpha'}}$ with $\alpha \leq \frac{\alpha'}{2}, \alpha < 1, \alpha' < 1$;
3. $g(t) = e^{t^\alpha}, G(t) = e^{\frac{t}{(\log t)^\delta}}, \alpha < 1, \delta > 1$.

In the example 1, and if $\mu' > \frac{\mu}{2}$, then, as noted in section 3.4, the condition on ϵ_0 does not depend on n_0 and κ' might be arbitrarily small, which corresponds to Eliasson's full-measure reducibility result in [3].

Smallness conditions We shall make conditions (40) and (41) more explicit for the particular cases that we mentioned before, namely, when $(g \cdot G)(t) = t^{\mu+\mu'}, \mu, \mu' > 2$ (diophantine case), when $(g\cdot G)(t) = e^{t^\alpha + t^{\alpha'}}, \alpha, \alpha' < 1$ and when $(g\cdot G)(t) = e^{\frac{t}{(\log t)^\delta}+t^\alpha}, \delta > 1, \alpha < 1$.
Recall the condition (40):

$$\int_{(g\cdot G)^{-1}\left(\frac{\kappa}{2(1-a)^{\frac{n_0-5}{4}}\sqrt{\epsilon_0}}\right)}^{\infty} \frac{\log(g\cdot G)(t)}{t^2}dt \le \frac{r_0}{4^{n_0+2}} \tag{40}$$

Lemma 3.4 *If $(g \cdot G)(t) = t^{\mu+\mu'}$, condition (40) is satisfied if*

$$(1-a)^{\frac{1}{8(\mu+\mu')}} \le \frac{1}{2} \tag{65}$$

and

$$\epsilon_0 \le (\frac{r_0}{8(\mu+\mu')})^{4\mu}(1-a)^{\frac{3}{2}}\kappa \tag{66}$$

or if

$$\epsilon_0 \le (\frac{r_0}{4^{n_0+3}(\mu+\mu')})^{4(\mu+\mu')}\kappa. \tag{67}$$

Proof: Rewrite the condition as

$$\int_b^\infty \frac{(\mu+\mu')\log t}{t^2}dt \le \frac{r_0}{4^{n_0+2}} \tag{68}$$

where b is short for $\left(\frac{\kappa}{2(1-a)^{\frac{n_0-5}{4}}\sqrt{\epsilon_0}}\right)^{\frac{1}{(\mu+\mu')}}$. Integrating by parts (i.e integrating $\frac{1}{t^2}$ and derivating $\log t$), this is

$$(\mu+\mu')\frac{\log b + 1}{b} \le \frac{r_0}{4^{n_0+2}}. \tag{69}$$

It is enough that

$$2(\mu+\mu')\frac{1}{\sqrt{b}} \le \frac{r_0}{4^{n_0+2}} \tag{70}$$

that is,

$$2(\mu+\mu')\left(\frac{2}{\kappa}(1-a)^{\frac{n_0-5}{4}}\sqrt{\epsilon_0}\right)^{\frac{1}{2(\mu+\mu')}} \le \frac{r_0}{4^{n_0+2}}. \tag{71}$$

which is true if (67) is satisfied. If moreover

$$(1-a)^{\frac{1}{8(\mu+\mu')}} \le \frac{1}{2} \tag{72}$$

then (71) is satisfied as long as

$$\epsilon_0 \leq (\frac{r_0}{8(\mu+\mu')})^{4(\mu+\mu')}(1-a)^{\frac{3}{2}}. \ \square \tag{73}$$

Lemma 3.5 *If* $(g\cdot G)(t) = e^{t^\alpha + t^{\alpha'}}, \alpha' < \alpha < 1$, *then* (40) *holds if*

$$\epsilon_0 \leq \frac{\kappa}{4}\exp\left[-2\left(\frac{2\cdot 4^{n_0+2}}{r_0(1-\alpha)}\right)^{\frac{\alpha}{1-\alpha}}\right]. \tag{74}$$

Proof: Condition (40) holds if

$$\int_{(g\cdot G)^{-1}\left(\frac{\sqrt{\kappa}}{2(1-a)^{\frac{n_0-3}{4}}\sqrt{\epsilon_0}}\right)}^{\infty} \frac{2}{t^{2-\alpha}}dt \leq \frac{r_0}{4^{n_0+2}} \tag{75}$$

i.e

$$(g\cdot G)^{-1}\left(\frac{\sqrt{\kappa}}{2(1-a)^{\frac{n_0-3}{4}}\sqrt{\epsilon_0}}\right) \geq \left(\frac{2\cdot 4^{n_0+2}}{r_0(1-\alpha)}\right)^{\frac{1}{1-\alpha}} \tag{76}$$

and since $g\cdot G$ is increasing, this amounts to

$$\frac{\kappa}{4(1-a)^{\frac{n_0-3}{2}}\epsilon_0} \geq (g\cdot G)\left(\left(\frac{2\cdot 4^{n_0+2}}{r_0(1-\alpha)}\right)^{\frac{1}{1-\alpha}}\right)^2. \tag{77}$$

So condition (40) holds if

$$(1-a)^{\frac{n_0-3}{2}}\epsilon_0 \leq \frac{\kappa}{4}\exp\left[-2\left(\frac{2\cdot 4^{n_0+2}}{r_0(1-\alpha)}\right)^{\frac{\alpha}{1-\alpha}}\right] \tag{78}$$

so, in particular, (74) is a sufficient condition. $\square$

Lemma 3.6 *If* $(g\cdot G)(t) = e^{\frac{t}{(\log t)^\delta} + t^\alpha}, \alpha < 1, \delta > 1$, *then* (40) *holds if*

$$\epsilon_0 \leq \frac{\kappa}{4}\left((g\cdot G)\circ\exp\left[\left(\frac{4^{n_0+3}}{r_0(\delta-1)(1-\alpha)}\right)^{\frac{1}{(\delta-1)(1-\alpha)}}\right]\right)^{-2}. \tag{79}$$

Proof: In this case, (40) can be rewritten

$$\int_b^\infty \frac{1}{t^{2-\alpha}}dt + \int_b^\infty \frac{1}{t(\log t)^\delta}dt \leq \frac{r_0}{4^{n_0+2}} \tag{80}$$

with b short for $(g\cdot G)^{-1}\left(\frac{\kappa}{2(1-a)^{\frac{n_0-5}{4}}\sqrt{\epsilon_0}}\right)$. Integrating by parts (integrating $\frac{1}{t}$ and derivating $\frac{1}{(\log t)^\delta}$), we compute

$$\int_b^\infty \frac{1}{t(\log t)^\delta}dt = [\frac{1}{(\log t)^{\delta-1}}]_b^\infty + \delta\int_b^\infty \frac{1}{t(\log t)^\delta}dt \tag{81}$$

which implies

$$(\delta-1)\int_b^\infty \frac{1}{t(\log t)^\delta}dt = \frac{1}{(\log b)^{\delta-1}} \tag{82}$$

so that (40) is equivalent to

$$\frac{b^{\alpha-1}}{1-\alpha} + \frac{1}{(\delta-1)(\log b)^{\delta-1}} \leq \frac{r_0}{4^{n_0+2}}. \tag{83}$$

Now $\frac{1}{(\delta-1)(\log b)^{\delta-1}} \leq \frac{r_0}{4^{n_0+3}}$ if

$$(g\cdot G)\circ \exp\left[\left(\frac{4^{n_0+3}}{r_0(\delta-1)}\right)^{\frac{1}{\delta-1}}\right]^2 \leq \frac{\kappa}{2(1-a)^{\frac{n_0-5}{4}}\sqrt{\epsilon_0}} \tag{84}$$

and $\frac{b^{\alpha-1}}{1-\alpha} \leq \frac{r_0}{4^{n_0+3}}$ if

$$\epsilon_0 \leq \frac{\kappa}{4}\exp\left[-2\left(\frac{4^{n_0+3}}{r_0(1-\alpha)}\right)^{\frac{\alpha}{1-\alpha}}\right]$$

so that (40) holds if

$$\epsilon_0 \leq \frac{\kappa}{4}\left((g\cdot G)\circ \exp\left[\left(\frac{4^{n_0+3}}{r_0(\delta-1)(1-\alpha)}\right)^{\frac{1}{(\delta-1)(1-\alpha)}}\right]\right)^{-2}. \ \square \tag{85}$$

Now for the condition (41): first note that in these examples, the mention of $\sup_{t\in[1,n_0]}\frac{\log(G\cdot g)(t+1)}{t}$ in condition (41) can be suppressed since it is a non increasing function, and therefore (41) holds if, for instance,

$$eC'\epsilon_0^{\frac{r_0}{4^{n_0+5}}} \leq (1-a)^2\kappa \tag{86}$$

So in example 1, (41) holds for a suitable a if

$$eC'\epsilon_0^{\frac{r_0}{4^{n_0+5}}} \leq \frac{1}{2^{2(\mu+\mu')}}\kappa \tag{87}$$

and in example 2, (41) holds for a suitable a if

$$eC'\epsilon_0^{\frac{r_0}{4^{n_0+5}}} \leq \kappa\exp(-4\cdot 4^{\frac{1}{1-\alpha'}}) \tag{88}$$

while in example 3, it holds under the analogous condition

$$eC'\epsilon_0^{\frac{r_0}{4^{n_0+5}}} \leq \kappa\exp(-4\cdot 4^{\frac{1}{1-\alpha}}). \tag{89}$$

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