

Affine.m – *Mathematica* package for computations in representation theory of finite-dimensional and affine Lie algebras

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Abstract

In this paper we present **Affine.m** – program for computations in representation theory of finite-dimensional and affine Lie algebras and describe implemented algorithms. Algorithms are based upon the properties of weights and Weyl symmetry. The most important problems for us are the ones, concerning computation of weight multiplicities in irreducible and Verma modules, branching of representations and tensor product decomposition. These problems have numerous applications in physics and we provide some examples of these applications. The program is implemented in popular computer algebra system *Mathematica* and works with finite-dimensional and affine Lie algebras.

Keywords:

Mathematica; Lie algebra; affine Lie algebra; Kac-Moody algebra; root system; weights; irreducible modules, CFT, Integrable systems

PROGRAM SUMMARY

Manuscript Title: **Affine.m** – *Mathematica* package for computations in representation theory of finite-dimensional and affine Lie algebras

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Program Title: Affine.m

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Programming language: Mathematica

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Operating system: Linux, Windows, MacOS, Solaris

RAM: 5-500 Mb

Keywords: Mathematica; Lie algebra; affine Lie algebra; Kac-Moody algebra; root system; weights; irreducible modules, CFT, Integrable systems

Classification: 5 Computer Algebra, 4.2 Other algebras and groups

Nature of problem:

Representation theory of finite-dimensional Lie algebras has many applications in different branches of physics, including elementary particle physics, molecular physics, nuclear physics. Representations of affine Lie algebras appear in string theories and two-dimensional conformal field theory used for the description of critical phenomena in two-dimensional systems. Also Lie symmetries play major role in study of quantum integrable systems.

Solution method:

We work with weights and roots of finite-dimensional and affine Lie algebras and use Weyl symmetry extensively. Central problems which are the computations of weight multiplicities, branching and fusion coefficients are solved using one general recurrent algorithm based on generalization of Weyl character formula. We also offer alternative implementation based on Freudenthal multiplicity formula which can be faster in some cases.

Restrictions:

Computational complexity grows fast with the rank of algebra, so computations for algebra of rank greater than 8 are non practical.

Unusual features:

We offer the possibility to use traditional mathematical notation for objects in representation theory of Lie algebras in computations if **Affine.m** is used in *Mathematica* notebook interface.

Running time:

From seconds to days depending on rank of algebra and complexity of representations.

1. Introduction

Representation theory of Lie algebras is of central importance for different areas of physics and mathematics. In physics Lie algebras are used for the

description of symmetries of quantum and classical systems. Computational methods in representation theory have a long history [1], there exist numerous software packages for computations related to Lie algebras [2], [3], [4, 5], [6], [7].

Most popular programs [2], [3], [6], [5] are created to study representation theory of simple finite-dimensional Lie algebras. The main computational problems are the following:

1. Construction of root system which determine the properties of Lie algebra including its commutation relations.
2. Weyl group traversal which is important due to Weyl symmetry of root system and characters of representations.
3. Calculation of weight multiplicities, branching and fusion coefficients, which are essential for construction and study of representations.

There are well-known algorithms for these tasks [8], [9], [1], [10]. The third problem is most computation-intensive. There are two different recurrent algorithms which are based on Weyl character formula and Freudenthal multiplicity formula. In this paper we analyze them.

Infinite-dimensional Lie algebras also have growing number of applications in physics for example in conformal field theory and study of integrable systems. But infinite-dimensional algebras are much harder to study and number of available computer programs is much smaller.

Affine Lie algebras [11] constitute important and tractable class of infinite-dimensional Lie algebras. They are constructed as the central extensions of loop algebras of (semi-simple) finite-dimensional Lie algebras and appear naturally in the study of Wess-Zumino-Witten and coset models of conformal field theory [12], [13], [14], [15].

The structure of affine Lie algebras allows to extend computational algorithms created for finite-dimensional Lie algebras [7], [16], [17]. The book [17] with the tables of multiplicities and other computed characteristics of affine Lie algebras and representations was published in 1990. But we are not aware of software packages for popular computer algebra systems which can be used to extend these results. We address this issue and present **Affine.m** – *Mathematica* package for computations in representation theory of affine and finite-dimensional Lie algebras. We describe the features and limitations of the package in present paper. We also provide representation-theoretical background of implemented algorithms and present examples of computations relevant to physics.

The paper starts with an overview of Lie algebras and their representation theory (Sec. 2). Then we describe datastructures of **Affine.m** used to present different objects related to Lie algebras and representations (Sec. 3) and discuss implemented algorithms (Sec. 4). Next section consists of physically interesting examples (Sec. 5). The paper is concluded with the discussion of possible extensions and refinements (Sec. 6).

2. Theoretical background

In this section we remind necessary definitions and present formulae used in computations.

2.1. Lie algebras of finite and affine types

Lie algebra \mathfrak{g} is a vector space with bilinear operation $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$, which is called *Lie bracket*. If we choose the basis X_i in \mathfrak{g} we can specify commutation relations by the *structure constants* C_{ijk} :

$$[X^i, X^j] = \sum_k C_k^{ij} X^k \quad (1)$$

Lie algebra is *simple* if it contains no non-trivial ideals with respect to commutator. *Semisimple* Lie algebra is a direct sum of simple Lie algebras. In present paper we treat simple and semisimple Lie algebras.

Maximal commutative subalgebra (*Cartan subalgebra*) of \mathfrak{g} is denoted by $\mathfrak{h}_{\mathfrak{g}}$. We denote the elements of basis of $\mathfrak{h}_{\mathfrak{g}}$ by H^i .

Killing form on \mathfrak{g} gives a non-degenerate bilinear form (\cdot, \cdot) on $\mathfrak{h}_{\mathfrak{g}}$ which can be used to identify $\mathfrak{h}_{\mathfrak{g}}$ with the subspace of the dual space $\mathfrak{h}_{\mathfrak{g}}^*$ of linear functionals on $\mathfrak{h}_{\mathfrak{g}}$. *Weights* are the elements of $\mathfrak{h}_{\mathfrak{g}}^*$ and are denoted by Greek letters $\mu, \nu, \omega, \lambda \dots$

Special choice of basis gives compact description of commutation relations (1). This basis can be encoded by the root system which is discussed in Section 2.2 (See also [18, 19]).

Loop algebra $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$, corresponding to semisimple Lie algebra \mathfrak{g} , has commutation relations

$$[X^i t^n, X^j t^m] = t^{n+m} \sum_k C_k^{ij} X^k \quad (2)$$

Central extension leads to the appearance of additional term

$$[X^i t^n + \alpha c, X^j t^m + \beta c] = t^{n+m} \sum_k C_k^{ij} X^k + (X^i, X^j) n \delta_{n+m,0} c \quad (3)$$

This algebra $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ is called non-twisted *affine Lie algebra* [11], [20, 21], [17].

We do not treat twisted affine Lie algebras in present paper.

2.2. Modules, weights and roots

Let \mathfrak{g} be finite-dimensional or affine Lie algebra.

Then \mathfrak{g} -module is a vector space V together with a bilinear map $\mathfrak{g} \times V \rightarrow V$ such that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v), \quad \text{for } x, y \in \mathfrak{g}, v \in V \quad (4)$$

Representation of algebra \mathfrak{g} on a vector space V is the homomorphism $\mathfrak{g} \rightarrow gl(V)$ from \mathfrak{g} to Lie algebra of endomorphisms on vector space V with the commutator as the bracket.

For an arbitrary representation it is possible to diagonalize the operators corresponding to Cartan generators H^i simultaneously by special choice of basis $\{v_j\}$ in V :

$$H^i \cdot v_j = \nu_j^i v_j \quad (5)$$

Eigenvalues ν_j^i of Cartan generators on element of basis v_j determine weight $\nu_j \in \mathfrak{h}_{\mathfrak{g}}^*$ such that $\nu_j(H^i) = \nu_j^i$. Vector $v \in V$ is called weight vector of weight λ if $Hv = \lambda_j(H)v$, $\forall H \in \mathfrak{h}$. Weight subspace consists of all weight vectors $V_\lambda = \{v \in V : Hv = \lambda_j(H)v, \forall H \in \mathfrak{h}\}$. Weight multiplicity $m_\lambda = \text{mult}(\lambda) = \dim V_\lambda$ is the dimension of weight subspace.

The structure of module is determined by the set of weights since the action of generators E^α on a weight vectors is

$$E^\alpha \cdot v_\lambda \propto v_{\lambda+\alpha} \quad (6)$$

Module structure can be encoded by the formal character of module

$$\text{ch}V = \sum_{\lambda} m_{\lambda} e^{\lambda} \quad (7)$$

Character $\text{ch}V \in \mathcal{E}$ is an element of algebra \mathcal{E} generated by formal exponents of weights. Character can be specialized by taking its value on some element ξ of \mathfrak{h} .

Lie algebra is its own module. The action of generators is called adjoint and it is given by the bracket $ad_X Y = [X, Y]$. *Roots* are weights of adjoint representation of \mathfrak{g} . They encode the commutation relations of algebra in the following way. Denote by Δ the set of roots. For each $\alpha \in \Delta$ there exist a root $-\alpha \in \Delta$ and generators $E^\alpha, E^{-\alpha}$ such that

$$[H^i, E^\alpha] = \alpha^i E^\alpha \quad (8)$$

$$[E^\alpha, E^\beta] = \begin{cases} N_{\alpha, \beta} E^{\alpha+\beta}, & \text{if } \alpha + \beta \in \Delta \\ \frac{2}{(\alpha, \alpha)} \sum_i \alpha^i H^i, & \text{if } \alpha = -\beta \\ 0, & \text{otherwise} \end{cases} \quad (9)$$

Given a root system Δ we can choose a set of positive roots. This is a subset $\Delta^+ \subset \Delta$ such that for each root $\alpha \in \Delta$ exactly one of the roots $\alpha, -\alpha$ is contained in Δ^+ and for any two distinct positive roots $\alpha, \beta \in \Delta^+$ such that $\alpha + \beta \in \Delta$ their sum is also positive $\alpha + \beta \in \Delta^+$. Elements of $-\Delta^+$ are called negative roots.

A positive root is *simple* if it cannot be written as the sum of positive roots. The set of simple roots $\Phi = \{\alpha_i\}$ is a basis of $\mathfrak{h}_{\mathfrak{g}}$ and each root can be written as $\alpha = \sum_i n_i \alpha_i$ with all n_i non-negative or non-positive. In case of finite-dimensional Lie algebra \mathfrak{g} simple roots are numbered from 1 to rank of the algebra $i = 1, \dots, r, \quad r = \text{rank}(\mathfrak{g})$. By numbering simple roots with an index i we introduce lexicographic ordering on the root system Δ . Highest root with respect to this ordering is denoted by $\theta = \sum_{i=1, \dots, r} a_i \alpha_i$, coefficients a_i are called *marks*. It is also highest weight (See section 2.3) of adjoint module. *Comarcs* are equal to $a_i^\vee = \frac{(\alpha_i, \alpha_i)}{2} a_i$.

Although the full set of roots Δ is infinite for affine Lie algebra $\hat{\mathfrak{g}}$ the set of simple roots Φ is finite and its elements are denoted by $\alpha_0, \dots, \alpha_r$ where $r = \text{rank}(\mathfrak{g})$. The roots $\alpha_1, \dots, \alpha_r$ are the roots of the underlying finite-dimensional Lie algebra $\overset{\circ}{\mathfrak{g}}$. The root $\alpha_0 = \delta - \theta$ is the difference of *imaginary root* δ and θ – highest root of the algebra $\overset{\circ}{\mathfrak{g}}$. Note that root multiplicity $\text{mult}(\alpha)$ for affine Lie algebra can be greater than one.

Subalgebra $\mathfrak{b}_+ \subset \mathfrak{g}$ spanned by the generators H^i, E^α for positive roots $\alpha \in \Delta^+$ is called Borel subalgebra.

Parabolic subalgebra $\mathfrak{p}_I \supset \mathfrak{b}_+$ contains Borel subalgebra and is generated by some subset of simple roots $\{\alpha_j : j \in I, I \subset \{1 \dots r\}\}$. It is spanned by the subset of generators $\{H^i\} \cup \{E^\alpha : \alpha \in \Delta^+\} \cup \{E^{-\alpha} : \alpha \in \Delta^+, \alpha = \sum_{j \in I} n_j \alpha_j\}$.

Regular subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is determined by the root system $\Delta_{\mathfrak{a}}$ with the

set of simple roots $\{\beta_i, i = 1, \dots, r_a\}$ which is a subset of set of simple roots $\{\alpha_1, \dots, \alpha_r\} \cup \{\Theta\}$ with the addition of highest root.

The *Weyl group* $W_{\mathfrak{g}}$ is generated by reflections $\{s_i : \mathfrak{h}_{\mathfrak{g}}^* \rightarrow \mathfrak{h}_{\mathfrak{g}}^*\}$ corresponding to simple roots $\{\alpha_i\}$:

$$s_i \cdot \lambda = \lambda - \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \alpha_i \quad (10)$$

Root system and characters of representation are invariant with respect to the action of Weyl group. Root system can be reconstructed from the set of simple roots with the action of Weyl group.

Weyl groups are finite for finite-dimensional Lie algebras and finitely-generated for affine Lie algebras.

Consider an action of element $s_{\alpha} s_{\alpha+\delta}$ of Weyl group of affine Lie algebra $\hat{\mathfrak{g}}$ for α simple root of underlying finite-dimensional Lie algebra \mathfrak{g} . Using the definition (10) it is easy to see that $s_{\alpha} s_{\alpha+\delta} \cdot \lambda = \lambda + \frac{2}{(\alpha, \alpha)} \alpha + \left(\frac{(\alpha, \alpha)}{2(\lambda, \delta)} + (\lambda, \alpha) \right) \delta$. So Weyl group can be presented as semidirect product of Weyl group $W_{\mathfrak{g}}$ of \mathfrak{g} and translations corresponding to the roots of \mathfrak{g} .

Weyl group element can be presented as the product of elementary reflections in multiple ways. Number of elementary reflections in shortest sequence representing element $w \in W_{\mathfrak{g}}$ is called *length* of w and denoted $l(w)$. We also use the notation $\epsilon(w) = (-1)^{l(w)}$ for parity of number of Weyl reflections generating w .

Fundamental domain \bar{C} for the action of Weyl group $W_{\mathfrak{g}}$ on $\mathfrak{h}_{\mathfrak{g}}^*$ is determined by the requirement $\xi \in \bar{C} \Leftrightarrow (\xi, \alpha_i) \geq 0$ for all simple roots α_i . It is called *main Weyl chamber*.

The *Cartan matrix* A is defined by products of simple roots

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \quad (11)$$

and can be used for compact description of Lie algebra commutation relations in Chevalley basis [18], [22], [23].

The form (11) induces the basis dual to the simple roots basis. It is called the *fundamental weights basis*. We denote its elements by ω_i :

$$\langle \omega_i, \alpha_j \rangle = \frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \quad (12)$$

For finite-dimensional Lie algebra there are r fundamental weights, $i = 1, \dots, r$. For affine Lie algebra we have additional fundamental weight $\omega_0 = \lambda$, $(\lambda, \delta) = 1$, $(\lambda, \lambda) = (\delta, \delta) = 0$. Other fundamental weights are equal to $\omega_i = a_i^v \lambda_0 + \overset{\circ}{\omega}_i$, where $\overset{\circ}{\omega}_i$ is the fundamental weight of finite-dimensional Lie algebra \mathfrak{g} .

Weyl vector is equal to the sum of fundamental weights $\rho = \sum_i \omega_i$. It is of importance for the construction of algebra modules.

2.3. Highest weight modules

We consider finitely-generated \mathfrak{g} -modules V such that $V = \bigoplus_{\xi \in \mathfrak{h}_{\mathfrak{g}}^*} V_{\xi}$, where each V_{ξ} is finite-dimensional and there exists finite set of weights $\lambda_1, \dots, \lambda_s$ which generates weight system of V , i.e. if $\dim V_{\xi} \neq 0$ then $\xi = \lambda_i - \sum_{k=1, \dots, r} n_k \alpha_k$ where $n_k \in \mathbb{Z}_+$ (See [24], [25]).

Highest weight module V^{μ} contains one highest weight μ , all other weights are obtained by the subtraction of linear combination of simple roots $\lambda = \mu - n_1 \alpha_1 - \dots - n_r \alpha_r$, $n_k \in \mathbb{Z}_+$.

The most simple type of highest weight modules is the Verma module M^{μ} who's space can be defined as the space

$$M^{\mu} = U(\mathfrak{g}) \underset{U(\mathfrak{b}_+)}{\otimes} D^{\mu}(\mathfrak{b}_+), \quad (13)$$

with respect to the multiplication in $U(\mathfrak{g})$, where $\underset{U(\mathfrak{b}_+)}{\otimes}$ means that the action of elements of $U(\mathfrak{b}_+)$ “falls through” the left part of tensor product onto the right part. Here \mathfrak{b}_+ is Borel subalgebra, $D^{\mu}(\mathfrak{b}_+)$ is a representation of \mathfrak{b}_+ such that $D(E^{\alpha}) = 0$, $D(H) = \mu(H)$ for any positive root α . Elements of \mathfrak{g} act from the left and we should commute all the elements of \mathfrak{b}_+ to the right, so that they can act on the space $D_{\lambda}(\mathfrak{b}_+)$.

Weight multiplicities in Verma module can be found from the Weyl character formula

$$\text{ch} M^{\mu} = \frac{e^{\mu}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\text{mult}(\alpha)}} = \frac{e^{\mu}}{\sum_{w \in W} \epsilon(w) e^{w\rho - \rho}} \quad (14)$$

Here we have used the Weyl denominator identity

$$R := \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\text{mult}(\alpha)} = \sum_{w \in W} \epsilon(w) e^{w\rho - \rho}, \quad (15)$$

and $\epsilon(w) := \det(w)$ is equal to the parity of number of Weyl reflections generating w .

Verma module M^μ has the unique maximal submodule and the unique nontrivial simple quotient L^μ which is an *irreducible highest weight module*.

Irreducible highest weight modules have no non-trivial submodules. Weyl character formula for irreducible highest weight modules is

$$\text{ch}L^\mu = \frac{\sum_{w \in W} \epsilon(w) e^{w(\mu+\rho)-\rho}}{\sum_{w \in W} \epsilon(w) e^{w\rho-\rho}} = \sum_{w \in W} \epsilon(w) \text{ch}M^{w(\mu+\rho)-\rho} \quad (16)$$

Thus the character of an irreducible highest weight module can be seen as the combination of characters of Verma modules. (This fact is a consequence of the Bernstein-Gelfand-Gelfand resolution ([26, 27], see also [24]).)

Construction of generalized Verma modules is analogous to (13), but representation of Borel subalgebra is substituted by the representation of parabolic subalgebra $\mathfrak{p}_I \supset \mathfrak{b}_+$ generated by some subset $\{\alpha_I\}$ of simple roots $I \subset \{1, \dots, r\}$:

$$M_I^\mu = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} L_{\mathfrak{p}_I}^\mu.$$

Denote a formal element $R_I := \prod_{\alpha \in \Delta^+ \setminus \Delta_{\mathfrak{p}_I}^+} (1 - e^{-\alpha})^{\text{mult}(\alpha)}$. Then the character of generalized Verma module can be written as

$$\text{ch}M_I^\mu = \frac{1}{R_I} \text{ch}L_{\mathfrak{p}_I}^\mu. \quad (17)$$

We can use Weyl character formula to obtain recurrent relation on weight multiplicities which can be used for calculations [28, 29].

For irreducible highest-weight modules recurrent relation has form

$$m_\xi = - \sum_{w \in W \setminus e} \epsilon(w) m_{\xi - (w(\rho) - \rho)} + \sum_{w \in W} \epsilon(w) \delta_{(w(\mu+\rho) - \rho), \xi}. \quad (18)$$

Formulae for Verma and generalized Verma modules differs only in second term on the right-hand side. In case of Verma module it is just $\delta_{\xi, \mu}$. For generalized Verma module summation in second term on the right-hand side (18) is over the Weyl subgroup generated by reflections corresponding to roots $\{\alpha_I\}$.

Other recurrent formula can be obtained from the study of Casimir elements action on irreducible highest weight modules [18]:

$$m_\lambda = \frac{2}{(\mu + \rho)^2 - (\lambda + \rho)^2} \sum_{\alpha \in \Delta^+} \sum_{k \geq 1} (\lambda + k\alpha, \alpha) m_{\lambda + k\alpha}. \quad (19)$$

It is called the Freudenthal multiplicity formula. Note that it is applicable only to irreducible modules.

We discuss the use of formulae (18) and (19) for computations in section 4.

Now consider an algebra \mathfrak{g} and a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$. Simple roots β_i of the subalgebra \mathfrak{a} can be presented as linear combinations of algebra roots α_j : $\beta_i = \sum_{j=1, \dots, r_{\mathfrak{g}}} k_j \alpha_j$, $j = 1, \dots, r_{\mathfrak{a}}$.

Each \mathfrak{g} -module is also an \mathfrak{a} -module, although $L_{\mathfrak{g}}^{\mu}$ is not irreducible as \mathfrak{a} -module. It can be decomposed into the sum of irreducible \mathfrak{a} -modules:

$$L_{\mathfrak{g}}^{\mu} = \bigoplus_{\nu} b_{\nu}^{\mu} L_{\mathfrak{a}}^{\nu} \quad (20)$$

Coefficients in this decomposition are called the branching coefficients.

It is possible to calculate branching coefficients by the construction and repeated subtraction of characters of modules $L_{\mathfrak{a}}^{\nu}$. This traditional approach has serious limitations especially in case of affine Lie algebras. We discuss them in the end of section 4.

Now we describe alternative approach which is based upon recurrent relation on branching coefficients. But before we proceed to this recurrent relation we need several definitions.

For a subalgebra $\mathfrak{a} \subset \mathfrak{g}$ we introduce a subalgebra \mathfrak{a}_{\perp} . Consider the root subspace $\mathfrak{h}_{\perp \mathfrak{a}}^*$ orthogonal to \mathfrak{a} ,

$$\mathfrak{h}_{\perp \mathfrak{a}}^* := \{\eta \in \mathfrak{h}^* | \forall h \in \mathfrak{h}_{\mathfrak{a}}; \eta(h) = 0\},$$

and the roots (correspondingly – positive roots) of \mathfrak{g} orthogonal to \mathfrak{a} ,

$$\begin{aligned} \Delta_{\mathfrak{a}_{\perp}} & : = \{\beta \in \Delta_{\mathfrak{g}} | \forall h \in \mathfrak{h}_{\mathfrak{a}}; \beta(h) = 0\}, \\ \Delta_{\mathfrak{a}_{\perp}}^+ & : = \{\beta^+ \in \Delta_{\mathfrak{g}}^+ | \forall h \in \mathfrak{h}_{\mathfrak{a}}; \beta^+(h) = 0\}. \end{aligned} \quad (21)$$

Let $W_{\mathfrak{a}_{\perp}}$ be the subgroup of W generated by the reflections w_{β} with the roots $\beta \in \Delta_{\mathfrak{a}_{\perp}}^+$. The subsystem $\Delta_{\mathfrak{a}_{\perp}}$ determines the subalgebra \mathfrak{a}_{\perp} with the Cartan subalgebra $\mathfrak{h}_{\mathfrak{a}_{\perp}}$.

Cartan subalgebra can be decomposed in the following way: $\mathfrak{h} = \mathfrak{h}_{\mathfrak{a}} \oplus \mathfrak{h}_{\mathfrak{a}_{\perp}} \oplus \mathfrak{h}_{\perp}$

We also introduce notation

$$\widetilde{\mathfrak{a}}_{\perp} := \mathfrak{a}_{\perp} \oplus \mathfrak{h}_{\perp} \quad \widetilde{\mathfrak{a}} := \mathfrak{a} \oplus \mathfrak{h}_{\perp}. \quad (22)$$

For \mathfrak{a} and \mathfrak{a}_\perp we consider the corresponding Weyl vectors, $\rho_{\mathfrak{a}}$ and $\rho_{\mathfrak{a}_\perp}$ and compose the so called "defects" $\mathcal{D}_{\mathfrak{a}}$ and $\mathcal{D}_{\mathfrak{a}_\perp}$ of the injection:

$$\mathcal{D}_{\mathfrak{a}} := \rho_{\mathfrak{a}} - \pi_{\mathfrak{a}}\rho, \quad \mathcal{D}_{\mathfrak{a}_\perp} := \rho_{\mathfrak{a}_\perp} - \pi_{\mathfrak{a}_\perp}\rho. \quad (23)$$

For $\mu \in P^+$ consider the linked weights $\{(w(\mu + \rho) - \rho) | w \in W\}$ and their projections to $h_{\mathfrak{a}_\perp}^*$ additionally shifted by the defect $-\mathcal{D}_{\mathfrak{a}_\perp}$:

$$\mu_{\mathfrak{a}_\perp}(w) := \pi_{\mathfrak{a}_\perp}[w(\mu + \rho) - \rho] - \mathcal{D}_{\mathfrak{a}_\perp}, \quad w \in W.$$

Among the weights $\{\mu_{\mathfrak{a}_\perp}(w) | w \in W\}$ one can always choose those located in the fundamental chamber $\overline{C_{\mathfrak{a}_\perp}}$. Let U be the set of representatives u for the classes $W/W_{\mathfrak{a}_\perp}$ such that

$$U := \{u \in W \mid \mu_{\mathfrak{a}_\perp}(u) \in \overline{C_{\mathfrak{a}_\perp}}\}. \quad (24)$$

Thus we can form the subsets:

$$\mu_{\bar{\mathfrak{a}}}(u) := \pi_{\bar{\mathfrak{a}}}[u(\mu + \rho) - \rho] + \mathcal{D}_{\mathfrak{a}_\perp}, \quad u \in U, \quad (25)$$

and

$$\mu_{\mathfrak{a}_\perp}(u) := \pi_{\mathfrak{a}_\perp}[u(\mu + \rho) - \rho] - \mathcal{D}_{\mathfrak{a}_\perp}, \quad u \in U. \quad (26)$$

Notice that the subalgebra \mathfrak{a}_\perp is regular by definition since it is built on a subset of roots of the algebra \mathfrak{g} .

Denote by $k_\xi^{(\mu)}$ signed branching coefficients. If $\xi \in \bar{C}_{\mathfrak{a}}$ is in main Weyl chamber $k_\xi^{(\mu)} = b_\xi^{(\mu)}$ otherwise $k_\xi^{(\mu)} = \epsilon(w)b_{w(\xi+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}^{(\mu)}$ where $w \in W_{\mathfrak{a}}$ is such that $w(\xi + \rho_{\mathfrak{a}}) - \rho_{\mathfrak{a}} \in W_{\mathfrak{a}}$.

Now we can use the Weyl character formula to write a recurrent relation [30] for signed branching coefficients $k_\xi^{(\mu)}$ corresponding to an injection $\mathfrak{a} \hookrightarrow \mathfrak{g}$:

$$k_\xi^{(\mu)} = -\frac{1}{s(\gamma_0)} \left(\sum_{u \in U} \epsilon(u) \dim \left(L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)} \right) \delta_{\xi - \gamma_0, \pi_{\bar{\mathfrak{a}}}(u(\mu + \rho) - \rho)} + \sum_{\gamma \in \Gamma_{\bar{\mathfrak{a}} \rightarrow \mathfrak{g}}} s(\gamma + \gamma_0) k_{\xi + \gamma}^{(\mu)} \right). \quad (27)$$

The recursion is governed by the set $\Gamma_{\mathfrak{a} \rightarrow \mathfrak{g}}$ called the injection fan. The latter is defined by the carrier set $\{\xi\}_{\mathfrak{a} \rightarrow \mathfrak{g}}$ for the coefficient function $s(\xi)$

$$\{\xi\}_{\bar{\mathfrak{a}} \rightarrow \mathfrak{g}} := \{\xi \in P_{\bar{\mathfrak{a}}} \mid s(\xi) \neq 0\}$$

appearing in the expansion

$$\prod_{\alpha \in \Delta^+ \setminus \Delta_{\mathfrak{a}_\perp}^+} (1 - e^{-\pi_{\bar{\mathfrak{a}}}\alpha})^{\text{mult}(\alpha) - \text{mult}_{\mathfrak{a}}(\pi_{\bar{\mathfrak{a}}}\alpha)} = - \sum_{\gamma \in P_{\bar{\mathfrak{a}}}} s(\gamma) e^{-\gamma}; \quad (28)$$

The weights in $\{\xi\}_{\tilde{\mathfrak{a}} \rightarrow \mathfrak{g}}$ are to be shifted by γ_0 – the lowest vector in $\{\xi\}$ – and the zero element is to be eliminated:

$$\Gamma_{\mathfrak{a} \rightarrow \mathfrak{g}} = \{\xi - \gamma_0 \mid \xi \in \{\xi\}\} \setminus \{0\}. \quad (29)$$

The formula (18) is a particular case of recurrent relation for branching coefficients (27) in the case of Cartan subalgebra $\mathfrak{a} = \mathfrak{h}_{\mathfrak{g}}$.

If the root system of \mathfrak{a}_{\perp} is generated by some subset of \mathfrak{g} simple roots $\alpha_1, \dots, \alpha_r$, then the recurrent relation (27) is connected with the generalized Bernstein-Bernstein-Gelfand resolution for parabolic Verma modules [31].

Another particular case of this formula is connected with tensor product decomposition. Consider the tensor product of two irreducible \mathfrak{g} -modules $L^{\mu} \otimes L^{\nu}$. It is also a \mathfrak{g} -module but not irreducible in general. So

$$L^{\mu} \otimes L^{\nu} = \bigoplus_{\gamma} f_{\gamma}^{\mu\nu} L^{\gamma} \quad (30)$$

The coefficients $f_{\gamma}^{\mu\nu}$ are called fusion coefficients. The problem of computation of fusion coefficients is equivalent to branching problem for the diagonal subalgebra $\mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$ (see [32]). So our implementation of recurrent algorithm can be used to decompose tensor products (See Section 5.1).

In the case of affine Lie algebras $\mathfrak{g}, \mathfrak{a}$ the multiplicities m_{ν} and the corresponding branching coefficients b_{ν} can be regarded as the coefficients in power series decomposition of string and branching functions correspondingly:

$$\sigma_{\nu}(q) = \sum_{n=0}^{\infty} m_{\nu-n\delta} q^n, \quad \nu = \sum_j c_j \omega_j, \quad c_j \geq 0 \quad (31)$$

$$b_{\nu}(q) = \sum_{n=0}^{\infty} b_{\nu-n\delta} q^n, \quad \nu = \sum_j c_j \omega_j, \quad c_j \geq 0 \quad (32)$$

String and branching functions have modular and analytic properties which are important for conformal field theory, especially in coset models and the study of CFT on higher genus surfaces [33], [13], [12], [34].

Affine.m calculates weight multiplicities and branching coefficients for affine Lie algebras up to some finite grade. We present examples of computations in Sections 5.3, 5.4. Now we proceed to the description of datastructures and algorithms implemented in **Affine.m**.

3. Core datastructures

Having introduced necessary mathematical objects, problems and relations we now describe the related datastructures of **Affine.m**. Although *Mathematica* is untyped language it is possible to create structured objects and do type checks with patterns [35], [36].

3.1. Weights

Weights are represented by two data-structures: `finiteWeight` for finite-dimensional Lie algebras and `affineWeight` for affine.

Internally the finite weight is a `List` with the `Head` `finiteWeight`, its components are the coordinates of the weight in the orthogonal basis of Bourbaki [23].

Affine weight is an extension of a finite weight by supplying it with level and grade coordinates. There is a set of functions defined for finite and affine weights. The complete list can be found in online help of the package. The most important are the definitions of addition, multiplication by number and scalar product (bilinear form) for weights. These definitions allow to use traditional notation when working with **Affine.m**:

```
w=makeFiniteWeight[{1,0,3}];
v=makeFiniteWeight[{3,2,1}];
2*w+v==makeFiniteWeight[{5,2,7}]
w.v==6
```

The use of orthogonal basis in the internal structure of weights allows us to work with weights without complete specification of root system which is useful for study of branching, since roots of the subalgebra can be specified by hand.

3.2. Root systems

To specify an algebra of finite or affine type it is enough to fix its root system. Root systems are represented by two datatypes `finiteRootSystem` and `affineRootSystem`. The latter is an extension of the former. We offer several different constructors for these datastructures. It is possible to specify the set of simple roots explicitly, for example to study the subalgebra $B_2 \subset B_4$ we can use the definition

```
b2b4=makeFiniteRootSystem[ { {1,-1,0,0}, {0,1,0,0} } ]
```

There are constructors for root systems of simple finite-dimensional Lie algebras:

```
b2=makeSimpleRootSystem[B,2]
```

We use typographic features of *Mathematica* frontend to offer mathematical notation for simple Lie algebras:

```
B2 == makeFiniteRootSystem[ { {1, -1}, {0, 1} }]
```

Non-twisted affine root systems can be created as affine extensions of finite root systems, e.g.

```
b2affine = makeAffineExtension[b2]
```

In notebook interface this can be written simply as \hat{B}_2 .

Semisimple Lie algebras can be created as sums of simple:

```
A1 @ A1 == finiteRootSystem[2, 2, {finiteWeight[2, {1, 0}], finiteWeight[2, {0, 1}]]
```

Predicate `rootSystemQ` checks if the object is root systems of finite or affine type.

List of simple roots is the property of root system so it is accessed as `rs[simpleRoots]`.

We have implemented several functions to get major properties of root systems. Weyl vector is given by the function `rho[rs_?rootSystemQ]`:

```
In[1] = rho[b2]
Out[1] = finiteWeight[2, {3/2, 1/2}]
```

Positive roots can be constructed with `positiveRoots[rs_?rootSystemQ]`. For affine Lie algebra this and related functions return the list up to fixed grade. This grade limit is set through the value of `rs[gradeLimit]` which is equal to 10 by default. List of roots (up to `gradeLimit`) is returned by `roots[rs]`. Cartan matrix and fundamental weights are calculated by the functions `cartanMatrix` and `fundamentalWeights` correspondingly.

It is possible to specify a weight of Lie algebra by its Dynkin label

```
weight[b2][1,2,3] == makeFiniteWeight[{2, 1}]
```

The function `dynkinLabels[rs_?rootSystemQ][wg_?weightQ]` returns Dynkin labels of weight `wg` in the root system `rs`.

Elements of Weyl group are specified by the set of indices of reflections, so element $w = s_1 s_2 s_1$ of Weyl group of algebra B_2 is constructed with `weylGroupElement[b2][1,2,1]`. Then it can be applied to the weights:

```
w = weylGroupElement[b2][1,2,1];
w @ makeFiniteWeight[{1,0}] == makeFiniteWeight[{-1,0}]
```

Computation of lexicographically minimal form [10, 37] for Weyl group elements can be conveniently implemented through the use of pattern-matching in *Mathematica*. In [38] rewrite rules for simple finite dimensional and affine Lie algebras are presented as *Mathematica* patterns. Our presentation of Weyl group elements is compatible with the code of [38]

```

In [1] = <<A3reduce;
        reduce[s[1,2,1,2,1,3,2,1,1]]
Out [1] = s[2, 3, 2]

In [2] = (weylGroupElement[A3] @@ reduce[s[1,2,1,2,1,3,2,1,1]]) @ weight[A3][-1,-2,-1]
Out [2] = finiteWeight[4, {-2, 2, 1, -1}]

In [3] = dynkinLabels[A3][Out[2]]
Out [3] = {-4, 1, 2}

```

3.3. Formal elements

We represent formal characters of representations by special structure `formalElement`. This structure is a hash-table implemented with `DownValues`. The keys are weights at the exponents and the values are corresponding multiplicities. `makeFormalElement[{\gamma_1, \dots, \gamma_n}, {m_1, \dots, m_n}]` creates data-structure which represents the element $\sum_{i=1}^n m_i e^{\gamma_i}$ of formal algebra \mathcal{E} . The operations of formal algebra \mathcal{E} are implemented for `formalElement` data-type: formal elements can be added, multiplied by number or exponent of weight. There is also multiplication of formal elements but no division.

```

In [1] = makeFormalElement[{makeFiniteWeight[{1,1}],makeFiniteWeight[{0,0}],{1,2}] *
        (2 * Exp[makeFiniteWeight[{1,0}]] *
        makeFormalElement[{makeFiniteWeight[{1,1}],makeFiniteWeight[{0,0}],{1,2}]);
In [2] = In[1][weights]
Out [2] = {finiteWeight[2, {1, 0}], finiteWeight[2, {2, 1}], finiteWeight[2, {3, 2}]}

In [3] = In[1][multiplicities]
Out [3] = {8, 8, 2}

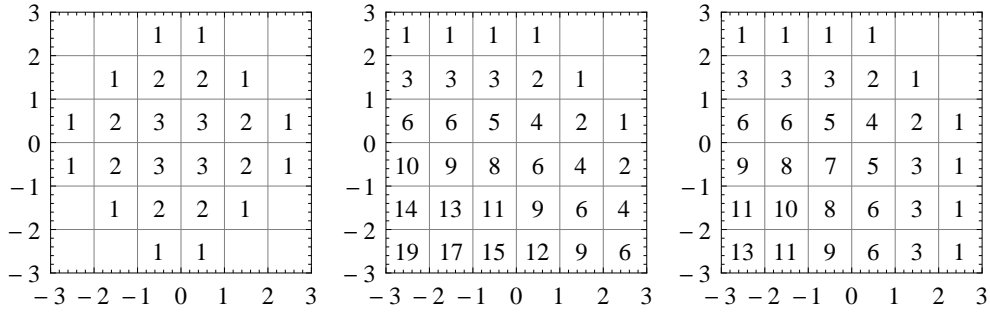
```

3.4. Modules

Affine.m can be used to study different kinds of modules, i.e. Verma modules, irreducible modules and parabolic Verma modules. We need datastructure `module` to represent generic module of Lie algebra \mathfrak{g} . Module properties can be deduced from its set of singular weights using Weyl character formulae (14),(15),(17),(16). Set of singular weights can have Weyl symmetry. It can be symmetry with respect to Weyl group $W_{\mathfrak{g}}$ or with respect to some subalgebra $W_{\mathfrak{a}}$ as in the case of parabolic Verma modules. Then it is possible to study only main Weyl chamber $C_{\mathfrak{a}}$. To use this symmetry generic constructor for `module` datastructure accepts several parameters `makeModule[rs_?rootSystemQ][singWeights_formalElement,subs_?rootSystemQ|emptyRootSystem[],limit:10]`. Here `rs` is root system of Lie algebra \mathfrak{g} , `singWeights` is the set of singular weights, `subs` is root system corresponding to Weyl group $W_{\mathfrak{a}}$ which is the symmetry of

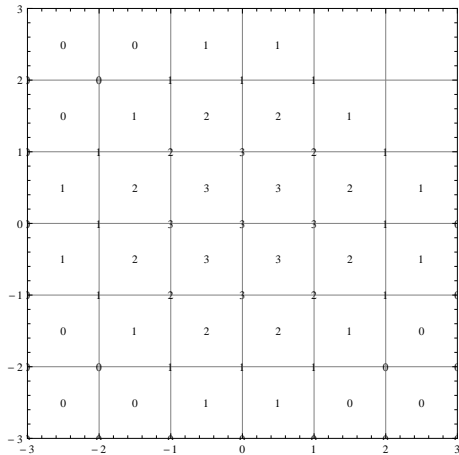
the set of singular weights. Parameter `limit` limits computation for infinite-dimensional modules such as Verma or parabolic Verma. There are several specialized constructors for different types of highest weight modules:

```
vm=makeVermaModule[B2][{2,1}];
pm=makeParabolicVermaModule[B2][weight[B2][2,1],{1}];
im=makeIrreducibleModule[B2][2,1];
GraphicsRow[textPlot/@{im,vm,pm}]
```

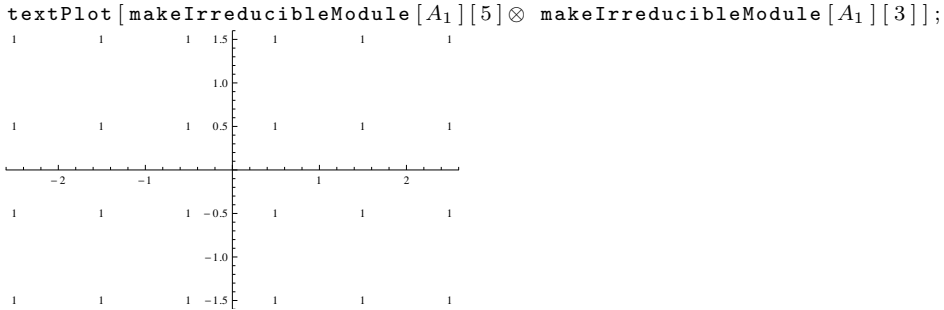


As we already stated properties of the module are encoded by its singular element. Function `singularElement[m_module]` returns singular element of module as `formalElement` datastructure. Character (up to `limit` for (parabolic) Verma modules) is returned by function `character[m_module]`. Direct sum of modules is module and we use natural notation

```
im1=makeIrreducibleModule[B2][weight[B2][2,1]];
im2=makeIrreducibleModule[B2][weight[B2][1,2]];
textPlot[im1⊕ im2]
```



Tensor product is also implemented but only for finite-dimensional Lie algebras, since tensor product of affine Lie algebra modules leads to rich new structures [39, 40, 41] which are out of the scope of present paper.



4. Computational algorithms

As we have already stated in section 2.3 there exist two recurrent relations which can be used to calculate weight multiplicities in irreducible modules. Both algorithms proceed in the following way to calculate weight multiplicities:

1. Create the list of weights in main Weyl chamber by subtraction of all possible combinations of simple roots from the highest weight (e.g. for finite-dimensional algebra subtract α_1 from μ while inside \bar{C} , then subtract α_2 from all the weights already obtained etc).
2. Sort the list of weights by their product with Weyl vector.
3. Use a recurrent formula. If the weight required for recurrent computation is outside the main chamber use Weyl symmetry.

The difference in performance of algorithms is in the number of previous values required to get the multiplicity of weight under consideration. For Weyl formula-based recurrent relation (18) it is constant and equal to the number of elements in Weyl group (if we are far from the boundary of representation diagram). When Freudenthal formula (19) is used number of previous values grows with the distance from the external border of representation. So Freudenthal formula is faster if the weight is close to the border or the rank of the algebra and the size of Weyl group is large [8]. Note that Freudenthal formula is valid for irreducible modules only, so it can not be used to study (generalized) Verma modules.

We have made some experiments with our implementations of Freudenthal formula and formula (19) and have got Figure 1, which depicts dependence of computation time on number of weights in module.

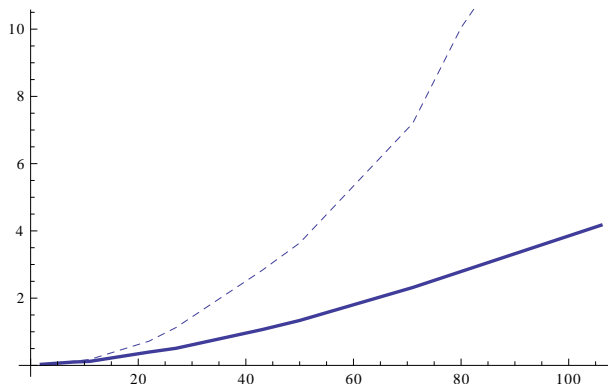


Figure 1: Running time of algorithms based on Freudenthal formula (19) (dashed) and recurrent relation (18) (solid) with the number of weights in \bar{C} for calculation of multiplicities in representations of B_2 .

In the calculation of branching coefficients use of Freudenthal formula requires full construction of formal characters of algebra representation and all the representations of subalgebra. It is impractical when the rank of algebra and subalgebra are big, for example for maximal subalgebra.

Alternative algorithm which was presented in the paper [30] proceeds as follows: It contains the following steps:

1. Construct the root system $\Delta_{\mathfrak{a}}$ for the embedding $\mathfrak{a} \rightarrow \mathfrak{g}$.
2. Select all positive roots $\alpha \in \Delta^+$ orthogonal to \mathfrak{a} , i.e. form the set $\Delta_{\mathfrak{a}_{\perp}}^+$.
3. Construct the set $\Gamma_{\mathfrak{a} \rightarrow \mathfrak{g}}$. Relation (2) defines the sign function $s(\gamma)$ and the set $\Phi_{\mathfrak{a} \subset \mathfrak{g}}$ where the lowest weight γ_0 is to be subtracted to get the fan: $\Gamma_{\mathfrak{a} \rightarrow \mathfrak{g}} = \{\xi - \gamma_0 | \xi \in \Phi_{\mathfrak{a} \subset \mathfrak{g}}\} \setminus \{0\}$.
4. Construct the set $\widehat{\Psi}^{(\mu)} = \{w(\mu + \rho) - \rho; w \in W\}$ of singular weights for the \mathfrak{g} -module $L^{(\mu)}$.
5. Select the weights $\{\mu_{\widetilde{\mathfrak{a}_{\perp}}} (w) = \pi_{\widetilde{\mathfrak{a}_{\perp}}} [w(\mu + \rho) - \rho] - \mathcal{D}_{\mathfrak{a}_{\perp}} \in \overline{C_{\widetilde{\mathfrak{a}_{\perp}}}}\}$. Since the set $\Delta_{\mathfrak{a}_{\perp}}^+$ is fixed we can easily check whether the weight $\mu_{\widetilde{\mathfrak{a}_{\perp}}} (w)$ belongs to the main Weyl chamber $\overline{C_{\widetilde{\mathfrak{a}_{\perp}}}}$ (by computing its scalar product with the fundamental weights of \mathfrak{a}_{\perp}^+).
6. For the weights $\mu_{\widetilde{\mathfrak{a}_{\perp}}} (w)$ calculate dimensions of the corresponding modules, $\dim \left(L_{\widetilde{\mathfrak{a}_{\perp}}}^{\mu_{\widetilde{\mathfrak{a}_{\perp}}} (w)} \right)$, using the Weyl dimension formula and construct the singular element $\Psi_{(\mathfrak{a}, \mathfrak{a}_{\perp})}^{(\mu)}$.

7. Calculate the anomalous branching coefficients using the recurrent relation (27) and select among them those corresponding to the weights in the main Weyl chamber \overline{C}_α .

We can speed up the algorithm by one-time computation of the representatives of the conjugate classes W/W_{α_\perp} .

Consider the regular embedding $B_2 \subset B_4$. In this case fan consists of 24 elements. In order to decompose B_4 module we need to construct the subset of singular weights of the module which projects to the main Weyl chamber of subalgebra B_2 . Full set of singular weights consists of 384 elements. Required subset contains at most 48 elements. So time of the construction of required subset is negligible if the number of branching coefficients is greater than that. We may estimate the total number of required operations for the computation of branching coefficients as the product of number of elements in main Weyl chamber of subalgebra with non-zero branching coefficients and number of elements in fan. In the case of direct algorithm we need to compute the multiplicities for each module in the decomposition, so the number of operations grows faster than square of number of elements in main Weyl chamber of subalgebra with non-zero branching coefficients.

To further illustrate this performance issue we include the Figure 2 where we show the time required for computations of branching coefficients for $B_3 \subset B_4$.

5. Examples

In this section we present some examples of computations available with **Affine.m** with the code required to produce these results.

5.1. Tensor product decomposition for finite-dimensional Lie algebras

Computation of fusion coefficients for the decomposition of tensor product of highest-weight modules to the direct sum of irreducible modules has numerous applications in physics. For example, we can consider spin of composite system such as atom. Another interesting example is integrable spin chain consisting of N particles with the spins living in some representation L of Lie algebra \mathfrak{g} with \mathfrak{g} -invariant Hamiltonian H , describing nearest-neighbour spin-spin interaction. In order to solve such system, i.e. find eigenstates of Hamiltonian, we need to decompose $L^{\otimes N}$ into the direct sum of irreducible \mathfrak{g} -modules of lower dimension and diagonalize the Hamiltonian on these modules.

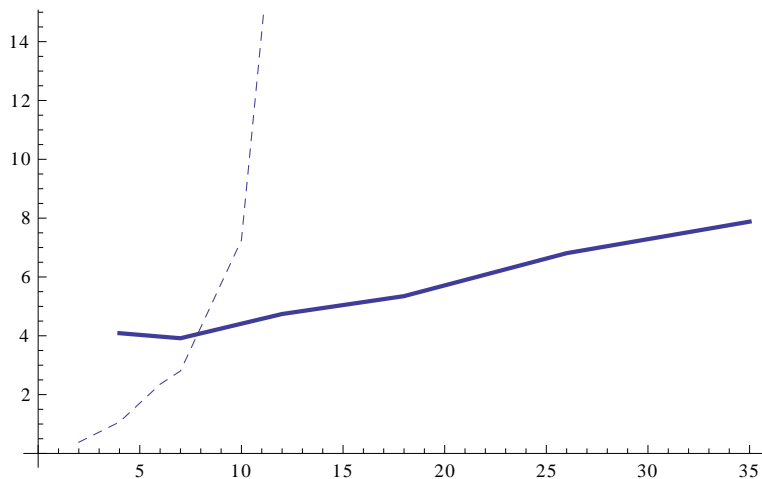


Figure 2: Running time of algorithms based on Freudenthal formula (19) (dashed) and recurrent relation (27) (solid) with the number of weights in \bar{C} for calculation of branching coefficients for $B_3 \subset B_4$.

For fundamental representations of simple Lie algebras it is sometimes possible to get analytic result for the dependence of decomposition coefficients on N (See [32]). Our code give numerical values and can be used to check this analytic results.

Consider tensor power of B_2 first fundamental representations $(L^{[1,0]})^{\otimes 4}$. Decomposition coefficients are just branching coefficients for tensor product module to the diagonal subalgebra $B_2 \subset B_2 \oplus B_2 \oplus B_2 \oplus B_2$. So following code calculates these coefficients:

```
fm = makeIrreducibleModule[B2][1, 0];
tp = ((fm⊗ ]fm)⊗ fm)⊗ fm;
subs = makeFiniteRootSystem[
  {1/4*{1, -1, 1, -1, 1, -1, 1, -1},
   1/4*{0, 1, 0, 1, 0, 1, 0, 1}}];
bc = branching[tp, subs];
{bc[#], dynkinLabels[subs][#]} & /@ bc[weights]
```

It produces list of highest weights and tensor product decomposition coefficients:

```
{{1, {4, 0}}, {3, {2, 2}}, {0, {3, 0}},
 {2, {0, 4}}, {3, {1, 2}}, {6, {2, 0}},
 {6, {0, 2}}, {1, {1, 0}}, {3, {0, 0}}}
```

Returning to the problem of spin chain Hamiltonian diagonalization we can see that instead of diagonalizing operator in space of dimension 625 we can diagonalize operators in spaces of dimensions 55, 81, 30, 35, 35, 14, 10, 5, 1.

5.2. Branching and parabolic Verma modules

We illustrate generalized BGG-resolution with the diagrams of G_2 parabolic Verma modules which appear in the decomposition of irreducible module $L_{G_2}^{[1,1]}$:

$$\text{ch}(L^\mu) = \sum_{u \in U} e^{\mu_{\bar{\alpha}}(u)} \epsilon(u) \text{ch} M_I^{\mu_{\alpha^\perp}(u)}. \quad (33)$$

Character of $L^{[1,1]}$ is presented in Figure 3, characters of generalized Verma modules in decomposition (33) are shown in Figure 4. Characters in upper row appear in (33) with positive sign and in lower row with negative.

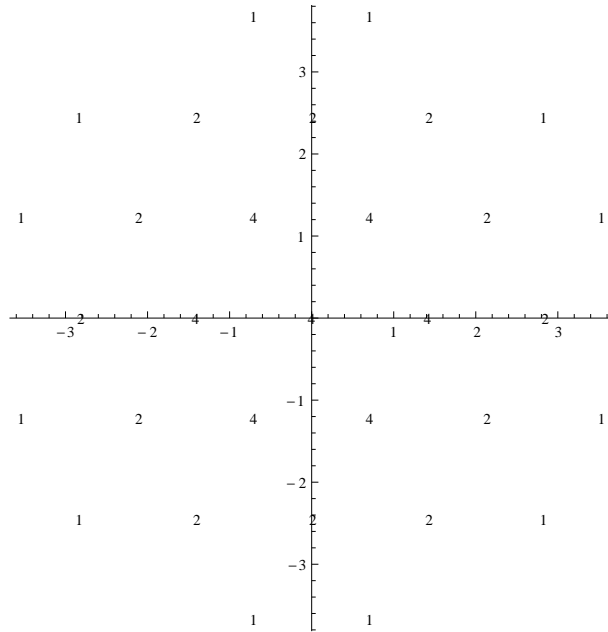


Figure 3: Character of irreducible G_2 -module $L^{[1,1]}$

5.3. String functions of affine Lie algebras and CFT models

String functions can be used to present the formal character of affine Lie algebra highest weight representation. They have interesting analytic and modular properties [11, 33, 42].

Affine.m produces power series decomposition for string functions. Consider affine Lie algebra $sl(\hat{3}) = \hat{A}_2$ highest weight module $L^{(1,0,0)}$. To get string functions we can use the code:

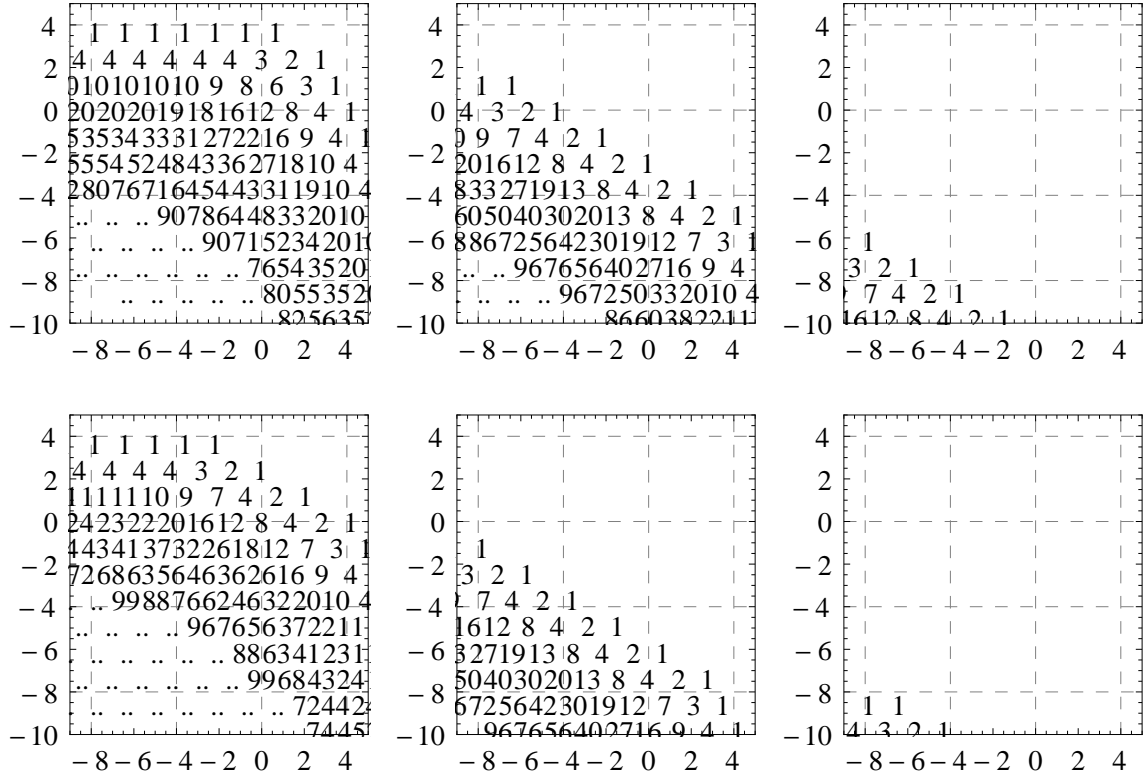


Figure 4: Character of G_2 generalized Verma modules appearing in decomposition of $L^{[1,1]}$. Parabolic Verma modules in upper row appear in decomposition with positive sign, in lower row – with negative.

```

stringFunctions[ $\hat{A}_2, \{1, 1, 2\}$ ]
{{0, 0, 4},
 2q + 10q2 + 40q3 + 133q4 + 398q5 + 1084q6 + 2760q7 + 6632q8 + 15214q9 + 33508q10},
{{0, 3, 1},
 2q + 12q2 + 49q3 + 166q4 + 494q5 + 1340q6 + 3387q7 + 8086q8 + 18415q9 + 40302q10},
{{1, 1, 2},
 1 + 6q + 27q2 + 96q3 + 298q4 + 836q5 + 2173q6 + 5310q7 + 12341q8 + 27486q9 + 59029q10},
{{2, 2, 0},
 1 + 8q + 35q2 + 124q3 + 379q4 + 1052q5 + 2700q6 + 6536q7 + 15047q8 + 33248q9 + 70877q10},
{{3, 0, 1},
 2 + 12q + 49q2 + 166q3 + 494q4 + 1340q5 + 3387q6 + 8086q7 + 18415q8 + 40302q9 + 85226q10}

```

Similarly for affine Lie algebra \hat{G}_2 we get

```

stringFunctions[ $\hat{G}_2, \{1, 1, 0\}$ ]
{{2, 0, 0},
 1 + 8q + 37q2 + 138q3 + 431q4 + 1227q5 + 3208q6 + 7901q7},

```

```

{{0, 0, 1},
 3q + 18q2 + 73q3 + 247q4 + 736q5 + 2000q6 + 5070q7},
{{1, 1, 0},
 1 + 7q + 32q2 + 117q3 + 370q4 + 1055q5 + 2780q6 + 6880q7},
{{0, 2, 0},
 3q + 15q2 + 63q3 + 210q4 + 633q5 + 1725q6 + 4407q7}

```

5.4. Branching functions and coset models of conformal field theory

More complex models of CFT can be obtained from cosets G/A corresponding to the embedding $\mathfrak{a} \subset \mathfrak{g}$. These models can be studied as gauge theories [43, 44].

Branching functions of the embedding $\mathfrak{a} \subset \mathfrak{g}$ are the partition functions of CFT on the torus (see [13]).

As first example we show computation of branching functions for the embedding $\hat{A}_1 \rightarrow \hat{B}_2$ up to tenth grade:

```

branchingFunctions[\hat{B}_2, makeAffineExtension[makeFiniteRootSystem[{{1, 1}}]], {1, 1, 1}]

{{3, 0},
 2 + 14q + 52q2 + 154q3 + 410q4 + 994q5 + 2248q6 + 4832q7 + 9934q8 + 19680q9 + 37802q10},
{{2, 1},
 4 + 20q + 72q2 + 220q3 + 584q4 + 1424q5 + 3248q6 + 7012q7 + 14488q8 + 28844q9 + 55616q10},
{{0, 3},
 4q + 20q2 + 68q3 + 200q4 + 516q5 + 1224q6 + 2736q7 + 5808q8 + 11820q9 + 23236q10},
{{1, 2},
 2 + 14q + 54q2 + 168q3 + 462q4 + 1148q5 + 2656q6 + 5812q7 + 12130q8 + 24358q9 + 47328q10}

```

Another example is computation of branching functions for the regular embedding $\hat{B}_2 \subset \hat{C}_3$:

```

sub=makeAffineExtension[parabolicSubalgebra[C3][2, 3]];
branchingFunctions[\hat{C}_3, sub, {2, 0, 0, 0}]

{{0, 1, 0},
 2q - 20q3 + 24q4 + 82q5 - 320q6 + 108q7},
{{1, 0, 0},
 1 - q - 8q2 + 19q3 + 16q4 - 156q5 + 205q6 + 640q7},
{{0, 0, 1},
 q - 5q3 + 7q5}

```

6. Conclusion

We have presented the package **Affine.m** for computations in representation theory of finite-dimensional and affine Lie algebras. It can be used to study Weyl groups, roots systems, irreducible, Verma and parabolic Verma modules of finite-dimensional and affine Lie algebras. In present paper we

have also discussed main ideas used for implementation of the package and described most important notions of representation theory required to use **Affine.m**.

We have demonstrated that recurrent approach based upon Weyl character formula is not only useful for calculations but also allows to see connection with (generalized) Bernstein-Bernstein-Gelfand resolution.

Also we have presented examples of computations with this package connected with problems of physics and mathematics.

In future versions of our software we are going to treat twisted affine Lie algebras, extended affine Lie algebras and provide more direct support for tensor product decomposition.

Acknowledgements

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Appendix A. Software package

The package can be freely downloaded from <http://github.com/naa/Affine>. To get the development code use the command

```
git clone git://github.com/naa/Affine.git
```

Contents of the package:

Affine/	root folder
demo/	demonstrations
demo.nb	demo notebook
paper.nb	code for the paper
doc/	documentation folder
figures/	figures in paper
timing.pdf	diagram showing performance
branching-timing.pdf	... for branching coefficients
irrep-sum.pdf	sum of B2 irreps
irrep-verma-pverma.pdf	irrep, Verma, (p)Verma for B2
G2-irrep.pdf	irrep for G2

G2-pverma.pdf	parabolic Verma for G2
tensor-product.pdf	tensor product of A1-modules
bibliography.bib	bibliographic database
paper.pdf	present paper
paper.tex	paper source
TODO.org	list of issues
src/	source folder
affine.m	main software package
tests/	unit tests folder
tests.m	unit tests
README.markdown	installation and usage notes

References

References

- [1] Belinfante, J. and Kolman, B., *A survey of Lie groups and Lie algebras with applications and computational methods*, Society for Industrial Mathematics, 1989.
- [2] Nutma, T., *Simplie*.
- [3] Van Leeuwen, M., *Euromath Bull* **1** (1994) 83.
- [4] Stembridge, J., *Journal of Symbolic Computation* **20** (1995) 755.
- [5] Stembridge, J., *Coxeter/weyl packages for maple*.
- [6] Fischbacher, T., (2002).
- [7] Fuchs, J., Schellekens, A. N., and Schweigert, C., *Nucl. Phys.* **B473** (1996) 323.
- [8] Moody, R. and Patera, J., *AMERICAN MATHEMATICAL SOCIETY* **7** (1982).
- [9] Stembridge, J., *Math. Soc. Japan, Tokyo* (2001) 1.
- [10] Casselman, W., *Inventiones mathematicae* **116** (1994) 95.
- [11] Kac, V., *Infinite dimensional Lie algebras*, Cambridge University Press, 1990.

- [12] Walton, M., (1999).
- [13] Di Francesco, P., Mathieu, P., and Senechal, D., *Conformal field theory*, Springer, 1997.
- [14] Goddard, P., Kent, A., and Olive, D., *Physics Letters B* **152** (1985) 88 .
- [15] Dunbar, D. C. and Joshi, K. G., *Int. J. Mod. Phys.* **A8** (1993) 4103.
- [16] Gannon, T., (2001).
- [17] Kass, S., Moody, R., Patera, J., and Slansky, R., *Affine Lie algebras, weight multiplicities, and branching rules*, SI, 1990.
- [18] Humphreys, J., *Introduction to Lie algebras and representation theory*, Springer, 1997.
- [19] Humphreys, J., *Reflection groups and Coxeter groups*, Cambridge Univ Pr, 1992.
- [20] Wakimoto, M., *Infinite-dimensional Lie algebras*, American Mathematical Society, 2001.
- [21] Wakimoto, M., *Lectures on infinite-dimensional Lie algebra*, World scientific, 2001.
- [22] Fulton, W. and Harris, J., *Representation theory: a first course*, volume 129, Springer Verlag, 1991.
- [23] Bourbaki, N., *Lie groups and Lie algebras*, Springer Verlag, 2002.
- [24] Humphreys, J., *Representations of semisimple Lie algebras in the BGG category O* , Amer Mathematical Society, 2008.
- [25] Carter, R., *Lie algebras of finite and affine type*, Cambridge University Press, 2005.
- [26] Bernstein, J., Gel'fand, I., and Gel'fand, S., *Funktsional'nyi Analiz i ego prilozheniya* **10** (1976) 1.
- [27] Bernstein, I., Gel'fand, I., and Gel'fand, S., *Functional Analysis and Its Applications* **5** (1971) 1.

- [28] Il'in, M., Kulish, P., and Lyakhovsky, V., Zapiski Nauchnykh Seminarov POMI **374** (2010) 197.
- [29] Kulish, P. and Lyakhovsky, V., Symmetry, Integrability and Geometry: Methods and Applications **4**.
- [30] Lyakhovsky, V. and Nazarov, A., Journal of Physics A: Mathematical and Theoretical **44** (2011) 075205.
- [31] Lyakhovsky, V. and Nazarov, A., (2011).
- [32] Kulish, P. P., Lyakhovsky, V. D., and Postnova, O. P., ArXiv e-prints (2011).
- [33] Kac, V. and Wakimoto, M., Advances in mathematics(New York, NY. 1965) **70** (1988) 156.
- [34] Walton, M., Nuclear Physics B **322** (1989) 775.
- [35] Shifrin, L., *Mathematica programming: an advanced introduction*.
- [36] Maeder, R., *Computer science with Mathematica*, Cambridge University Press, 2000.
- [37] Casselman, W., Automata to perform basic calculations in coxeter groups, in *Representations of groups: Canadian Mathematical Society annual seminar, June 15-24, 1994, Banff, Alberta, Canada*, volume 16, page 35, Canadian Mathematical Society, 1995.
- [38] van der Kallen, W., Computing shortlex rewrite rules with mathematica.
- [39] Kazhdan, D. and Lusztig, G., AMERICAN MATHEMATICAL SOCIETY **7** (1994).
- [40] Kazhdan, D. and Lusztig, G., Journal of the American Mathematical Society **6** (1993) 905.
- [41] Kazhdan, D. and Lusztig, G., AMERICAN MATHEMATICAL SOCIETY **6** (1993).
- [42] Kac, V. and Peterson, D., Adv. in Math **53** (1984) 125.
- [43] Hwang, S. and Rhedin, H., Mod. Phys. Lett. **A10** (1995) 823.

[44] Hwang, S. and Rhedin, H., Nuclear Physics B **406** (1993) 165.