Connecting orbits for families of Tonelli Hamiltonians

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Abstract

We investigate the existence of Arnold diffusion-type orbits for systems obtained by iterating in any order the flows of a family of Tonelli Hamiltonians. Our approach is close to the one of Bernard in [3]. When specialized to families of twist maps, our results are similar to those of Moeckel [20] and Le Calvez [15], and generalize the connecting results of Mather for a single twist map in [18].

Résumé

Nous étudions l'existence d'orbites du type "diffusion d'Arnold" pour des systèmes obtenus en itérant dans un ordre quelconque les flots d'une famille d'Hamiltoniens Tonelli. Notre approche au problème est inspirée par celle de Bernard dans [3]. Dans le cas d'une famille d'applications du cylindre déviant la verticale, nos résultats sont similaires à ceux de Moeckel [20] et Le Calvez [15], et généralisent les résultats de Mather pour une seule de ces applications dans [18].

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1 Introduction

Much work has been carried out in order to understand the instability properties of Hamiltonian systems, especially for Hamiltonians which are convex in the momenta variables p. The basic case of a periodic Hamiltonian defined on the cotangent space $T^*\mathbb{T} \cong \mathbb{T} \times \mathbb{R}$ of the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ corresponds to exact-symplectic twist maps on the cylinder, see [21]. Quite a lot is known in this case, thanks for instance to the original works of Birkhoff [4, 5] and to the KAM and Aubry-Mather theories for twist maps. In particular, a general principle is that the non-contractible invariant circles are the unique obstruction to instability phenomena, such as the drift in the p-variable.

The situation becomes more complicated when generalizing to higher dimension, namely to Hamiltonians defined on $T^*\mathbb{T}^d$, $d \in \mathbb{N}$, or, more generally, on the cotangent space T^*M of a d-dimensional manifold M. In this setting, among others the variational approach of Mather and Fathi's weak KAM theory has been fruitful, especially in the framework of the so-called Tonelli Hamiltonians. The Mather, Aubry and Mañé sets introduced by Mather and Fathi generalize the invariant circles and the Aubry-Mather sets for twist maps, and provide at the same time both an obstruction and a dynamical skeleton for the instability phenomena. This has allowed a better comprehension of the mechanisms underlying the phenomenon of Arnold diffusion which was firstly exhibited in the seminal paper [1] on a concrete example.

Some studies have also been devoted to the following different generalization: one keeps the dimension one, and consider instead a family of several twist maps at once, which can be iterated in any order. Following [16], we shall call such a system a *polysystem*, and *polyorbits* its (discrete-time) trajectories, see Definition 1.1 for more rigour. Of course, the trajectories of a map in the family are also trajectories for the polysystem, thus the polysystem presents at least the same unstable behaviors as the single maps in the family. Nevertheless, one expects new kinds of unstable behavior possibly to be created: some obstructions for a map may be circumvented by non-trivial iterations of other maps in the family. Moeckel [20], Le Calvez [15] and Jaulent [14] have studied this problem, extending some results for single twist maps to the polysystem case. In particular, the general emerging principle is that the unique obstructions to instability phenomena, such as the drift in the p-variable, are the common non-contractible invariant circles.

In this paper, we try to merge both generalizations, i.e. we deal with a family of several Hamiltonians in arbitrary dimension and we investigate the presence of unstable polyorbits (often we will call them "diffusion polyorbits" or "connecting polyorbits"). More precisely, we will consider the polysystem associated to a family $\mathcal{F}$ of one-periodic Tonelli Hamiltonians[1] defined on the cotangent space of a compact d-dimensional manifold M without boundary. Just as in the one-dimensional twist map case, one expects that some new unstable behavior may be created by non-trivial iterations of the time-one maps of the family. On the other hand, unlike the single-Hamiltonian case, there is not a definition of Mather, Aubry and Mañé sets for polysystems, hence one may expect the obstructions to come expressed in terms of some more complicated objects.

Our discussion will be in the framework of weak Kam theory, for which we refer to [12]. The ideas will be close to those in Bernard's paper [3], of which the present work may be seen as a generalization to the polysystem case (especially of Section 8 in that paper). We call our method for the construction of unstable polyorbits "Mather mechanism", after the paper [19] which introduced some of the basic ideas of the construction. In [3] a slightly different "Arnold mechanism" is also presented, more reminiscent of the aforementioned paper [1].

The results which we obtain are rather abstract in nature: essentially, they give sufficient conditions in order for the diffusion orbits to occur between two cohomology classes (in the sense of Proposition 1.2). The conditions are encoded, locally around a cohomology class c, in a subspace $R(c)$ of "allowed cohomological directions for diffusion" (Theorem 1.3). This subspace is in turn defined (cf. (5.1) and Proposition 5.5) in terms of some sort of generalized Aubry-Mather sets for the polysystem (the sets $\mathcal{I}_\Phi(\mathcal{G})$ defined in Remark 4.4(i)), which may be in principle quite difficult to decipher. Maybe some further study may lead to more transparent conditions, at least in presence of additional hypotheses. However, in the twist map case we are able to recover "concrete" and "optimal" results (see Corollary 1.4), similar to those already proved with different methods by Le Calvez and Moeckel, and extending some other results of Mather for a single twist map in [18]. On the negative side, using a result of Cui [10] we show that, if the Hamiltonians in the family commute, our mechanism does not give rise to new instability phenomena, which is somehow expected.

As for the interest in studying Hamiltonian polysystems, let us mention that a motivation lies in the fact that the behavior of some complex single-Hamiltonian systems may be to some extent reduced to the analysis of simpler polysystems. We

[1] we recall that a one-periodic Tonelli Hamiltonian is a C^2 function $H(x,p,t)$ defined on $T^*M \times \mathbb{T}$ which is strictly convex and superlinear in p, and whose Hamiltonian flow is complete.

are aware for instance of a recent work of Bounemoura and Pannamen [6], and some works of Marco therein cited.

1.1 Main results

Before introducing our results, let us review the kind of statements which we want to generalize.

For an exact-symplectic twist map F on the cylinder $\mathbb{T} \times \mathbb{R}$, the archetypal instability result is the following: if, for $A < B$, the annulus $\mathbb{T} \times [A, B] \subset \mathbb{T} \times \mathbb{R}$ does not contain any non-contractible invariant circle, then there exists an orbit $(x_n, p_n)_{n\in\mathbb{Z}}$ such that $p_0 < A$ and $p_N > B$ for some $N \in \mathbb{N}$. This dates back to Birkhoff [4, 5], and has been improved in various ways. Two improvements in the framework of Aubry-Mather theory for twist maps will be relevant to us. The first states that if M_{w_1} and M_{w_2} are two Aubry-Mather sets for F of rotation number w_1 and w_2 respectively, such that there is no non-contractible invariant circle between them, then there exists an orbit $\{z_n = (x_n, p_n)\}_{n\in\mathbb{Z}} \subset \mathbb{T} \times \mathbb{R}$ such that

$$\alpha\text{ - lim } z_n \subseteq M_{w_1} \qquad \text{and} \qquad \omega\text{ - lim } z_n \subseteq M_{w_2}.$$

The second states that if $(w_i)_{i\in\mathbb{Z}}$ are rotation numbers such that, for any i, there is no non-contractible invariant circle between the Aubry-Mather sets M_{w_i} and $M_{w_{i+1}}$, then for every sequence $(\varepsilon_i)_i$ of positive number there exists an orbit which visits in turn the ε_i-neighborhood of M_{w_i}. Both these results are due to Mather, we refer to [18] for precise statements.

Of course, for a twist map, non-contractible invariant circles do represent obstructions to the drift in the p-variable, because they disconnect the cylinder, hence the previous statements are optimal. Therefore the principle stemming from these results is that non-contractible invariant circles are the only obstruction to this kind of instability.

For a family of exact-symplectic twist maps on the cylinder, the generalization of the Birkhoff result above obtained by replacing in the statement “non-contractible invariant circle” with “common non-contractible invariant circle” is true. This and other stronger results have been proved by Moeckel, Le Calvez and Jaulent [20, 15, 14]. Again, a common non-contractible invariant circle obviously is a real obstruction to the drift in the p-variable, whence the optimality of these results and the principle that, for a polysystem of exact twist maps, the common non-contractible invariant circles are the only obstruction to this kind of instability.

For the case of a single Hamiltonian in higher dimension, usually only sufficient conditions for the existence of unstable orbits can be proved. A great amount of work has been devoted to this topic. Our approach is close to the one of Mather in [19] and of Bernard in [3] (see also [2, 8, 9]). Their results are better expressed in terms of cohomology classes rather than rotation vectors: in their papers, the authors define equivalence relations in $H^1(M, \mathbb{R})$ such that equivalence between classes implies existence of diffusing orbits between the corresponding Aubry sets. The obstruction for the equivalence is represented, roughly speaking, by the size of the Mañé sets. Notice however that, unlike the one-dimensional case, the obstructions for the equivalence may not always correspond to real obstructions for the dynamics. Nevertheless, if $d = 1$ the obstructions to the equivalence turn out to be exactly the non-contractible invariant circles. Therefore, the results on twist maps mentioned above are recovered, and the equivalence relation is then optimal in this case.

The present paper has the same structure: we define (in terms of pseudographs and of the flows of the Hamiltonians in the family $\mathcal{F}$, see Sections 2 and 3) an equivalence relation $\dashv\vdash_{\mathcal{F}}$ between cohomology classes, which is a natural adaptation to the polysystem case of the relation $\dashv\vdash$ introduced in [3]. We then prove that the occurrence of such a relation implies the existence of diffusing polyorbits, in the sense of Proposition 1.2. We find sufficient conditions (in terms of the "homological size" of some sort of generalized Aubry sets) which ensure, locally around a given class c, the occurrence of the relation. If $d = 1$, this conditions turn out to be also necessary, hence the relation is optimal in this case. For $\mathcal{F}$ composed by a single Hamiltonian, our results exactly reduce to the one in Section 8 of [3].

More precisely, let $\mathcal{F}$ be a family of one-periodic Tonelli Hamiltonians on T^*M, where M is a d-dimensional compact manifold without boundary. For $H \in \mathcal{F}$, we denote by

$$\phi_H \colon T^*M \to T^*M$$

the time-one map of the Hamiltonian flow of H. Let us first rigorously define what we mean by polyorbit.

Definition 1.1 ($\mathcal{F}$-polyorbit)**.** *A bi-infinite sequence $\{z_n\}_{n\in\mathbb{Z}} \subseteq T^*M$ is an $\mathcal{F}$-polyorbit if for every $n \in \mathbb{Z}$ there exists $H \in \mathcal{F}$ such that $\phi_H(z_n) = z_{n+1}$.*

We have (Section 3):

Proposition 1.2. *There exists an equivalence relation $\dashv\vdash_{\mathcal{F}}$ on $H^1(M,\mathbb{R})$ such that:*

- *if $c \dashv\vdash_{\mathcal{F}} c'$ then for every $H, H' \in \mathcal{F}$ there exists a polyorbit which is α-asymptotic to the Aubry set $\tilde{\mathcal{A}}_H(c)$ and ω-asymptotic to $\tilde{\mathcal{A}}_{H'}(c')$;*

- *if $c \dashv\vdash_{\mathcal{F}} c'$ and if η, η' are one-forms of cohomology c, c' respectively, then there exists a polyorbit $(z_n)_{n\in\mathbb{Z}} \subset T^*M$ such that $z_0 \in \mathrm{Graph}\,(\eta)$ and $z_N \in \mathrm{Graph}\,(\eta')$ for some $N \in \mathbb{N}$;*

- *let $(c_i, H_i, \varepsilon_i)_{i\in\mathbb{Z}} \subset H^1(M,\mathbb{R}) \times \mathcal{F} \times]0, +\infty[$ such that $c_i \dashv\vdash_{\mathcal{F}} c_{i+1}$ for every i. Then there exists a polyorbit visiting in turn the ε_i-neighborhoods of the Mather sets $\tilde{\mathcal{M}}_{H_i}(c_i)$. Moreover, if $(c_i, H_i) = (\bar{c}, \bar{H})$ for i small enough (resp. i big enough), then the polyorbit can be taken α-asymptotic to $\tilde{\mathcal{A}}_{\bar{H}}(\bar{c})$ (resp. ω-asymptotic to $\tilde{\mathcal{A}}_{\bar{H}}(\bar{c})$).*

The main result is Theorem 5.7. Let us state it here for finite $\mathcal{F}$, even if it will hold under a weaker assumption.

Theorem 1.3. *Assume $\mathcal{F}$ is finite. Then for every $c \in H^1(M,\mathbb{R})$ there exist a vector subspace $R(c) \subseteq H^1(M,\mathbb{R})$, a neighborhood W of c and $\varepsilon > 0$ such that*

$$c' \ \dashv\vdash_{\mathcal{F}} \ c' + B_\varepsilon R(c) \qquad \forall\, c' \in W.$$

Of course one needs to have information on the subspace $R(c)$ for the result to be interesting. The definition of $R(c)$ is rather abstract and not too easy to handle (cf. the definition given in (5.1) and some equivalent expressions given in Proposition 5.5).

Nevertheless, we are able to prove (Proposition 5.9) that if there exists a $C^{1,1}$ weak Kam solution of cohomology c which is common to all the Hamiltonians in $\mathcal{F}$, then $R(c) = \{0\}$. In addition, if $d = 1$, the viceversa is true: if $R(c) = \{0\}$ then there exists a $C^{1,1}$ weak Kam solution of cohomology c common to all Hamiltonians in $\mathcal{F}$, i.e. a common non-contractible invariant circle.

This fact, together with Theorem 1.3 and Proposition 1.2 yields the following result for families of twist maps (no additional assumptions on $\mathcal{F}$ will be eventually needed):

Corollary 1.4. *Let us consider the polysystem associated to an arbitrary family $\mathcal{F}$ of one-periodic Tonelli Hamiltonians on $\mathbb{T} \times \mathbb{R}$. Let us make the identification $H^1(\mathbb{T},\mathbb{R}) \cong \mathbb{R}$. If, for some $A < B \in \mathbb{R}$, the family $\mathcal{F}$ does not admit an invariant common circle with cohomology in $[A,B]$, then:*

(i) there exists an $\mathcal{F}$-polyorbit $(x_n,p_n)_{n\in\mathbb{Z}}$ satisfying $p_0 = A$ and $p_N = B$ for some $N \in \mathbb{N}$;

(ii) for every $H, H' \in \mathcal{F}$ and every $c, c' \in [A,B]$ there exists an $\mathcal{F}$-polyorbit α-asymptotic to the Aubry set $\tilde{\mathcal{A}}_H(c)$ and ω-asymptotic to $\tilde{\mathcal{A}}_{H'}(c')$;

(iii) for every sequence $(c_i,H_i,\varepsilon_i)_{i\in\mathbb{Z}} \subset [A,B] \times \mathcal{F} \times \,]0,+\infty[$ there exists an $\mathcal{F}$-polyorbit which visits in turn the ε_i-neighborhoods of the Mather sets $\tilde{\mathcal{M}}_{H_i}(c_i)$.

When $d > 1$ some information can still be extracted from the subspace $R(c)$. A sample of what can be obtained will be presented in Proposition 5.11. Very roughly speaking, among the obstructions which prevent $R(c)$ from being large, we find:

- for every finite string $H_1,\dots,H_n$ of elements of $\mathcal{F}$, the invariant sets for the map
$$\phi = \phi_{H_n} \circ \cdots \circ \phi_{H_1};$$
- for every pair H_1, H_2 of elements of $\mathcal{F}$, for every c-weak Kam solution u_1 for H_1 and dual c-weak Kam solution u_2 for H_2, the set
$$\mathrm{Graph}\,(du_1) \cap \mathrm{Graph}\,(du_2).$$

However, unlike the twist map case, such obstructions must be intended in a “negative” way: their smallness is a sufficient condition for $R(c)$ to be large, the converse being not necessarily true.

1.2 Structure of the paper

The paper is organized as follows. In Section 2 we establish some notation and recall some facts about pseudographs and semiconcave functions.

In Section 3 we define the forcing relation $\vdash_{\mathcal{F}}$ and the mutual forcing relation $\dashv\vdash_{\mathcal{F}}$, and we show, like in [3], how the occurrence of such relations implies the diffusion for the polysystem (Proposition 3.2).

In Section 4 we present the objects needed later to put in place what we call the Mather mechanism: Lagrangian action, Lax-Oleinik operators, operations on costs (minimum, composition) and families of costs. Eventually we build the semigroup Σ_c^∞ which acts on the space of pseudographs and encodes informations on the underlying polysystem dynamics. The Subsection 4.3 gathers some needed results in weak Kam theory, rephrased in the language of pseudographs.

In Section 5 the Mather mechanism for the construction of diffusion polyorbits is put in place. The basic step of the mechanism is proved in Subsection 5.1. Then we heuristically show the application to the twist map case in Subsection 5.2. Finally, in Subsection 5.3 we define the subspace $R(c)$ and we prove the general abstract result (Theorem 5.7) which gives sufficient conditions for the occurrence of the relation $\dashv\vdash_{\mathcal{F}}$ in terms of $R(c)$. After the theorem we investigate its application to special cases (such as twist maps and commuting Hamiltonians), and we discuss the properties of $R(c)$ in relation with the dynamics of the polysystem.

2 Notation. The space of pseudographs

In this section we recall from [3] some facts about pseudographs. We refer to that article for a more detailed introduction.

Let M be a d-dimensional compact connected Riemannian manifold without boundary. We denote by Ω the set of smooth closed one-forms on M and by π the projection from the cotangent space T^*M to M. If $\eta \in \Omega$ we denote by $[\eta] \in H^1(M, \mathbb{R})$ its cohomology class and, for $S \subseteq \Omega$, $[S] = \{[\eta] : \eta \in S\}$.

If $u\colon M \to \mathbb{R}$ is a Lipschitz function and $\eta \in \Omega$, then the *pseudograph* $\mathcal{G}_{\eta,u} \subset T^*M$ is defined by

$$\mathcal{G}_{\eta,u} = \{(x, \eta_x + du_x) : x \in M \text{ and } du_x \text{ exists}\}.$$

Given a subset $N \subset M$ and a pseudograph $\mathcal{G}$, the symbol $\mathcal{G}_{|N}$ denotes the restriction of $\mathcal{G}$ above N, that is $\mathcal{G} \cap \pi^{-1}(N)$.

Let us notice that $\mathcal{G}_{\eta,u} = \mathcal{G}_{\eta+df,u-f+\alpha}$ for any smooth f and $\alpha \in \mathbb{R}$. Viceversa, if $\mathcal{G}_{\eta,u} = \mathcal{G}_{\eta',u'}$ then $\eta' = \eta + df$ and $u' = u - f$ for some smooth f determined up to an additive constant. In particular, every pseudograph $\mathcal{G}$ has a well-defined cohomology, which we denote $c(\mathcal{G}) \in H^1(M, \mathbb{R})$. The space E of pseudographs is then a quotient of $\Omega \times \mathrm{Lip}(M)$ and inherits from it the structure of a vector space. This vector space is isomorphic to $H^1(M, \mathbb{R}) \times \big(\mathrm{Lip}(M)/\sim\big)$ where the relation $\sim$ means up to the addition of constants. Given a linear section $S\colon H^1(M, \mathbb{R}) \to \Omega$ (i.e. $[S(c)] = c$), an isomorphism is given by $(c, u) \mapsto \mathcal{G}_{S(c),u}$. The space E can be given a norm via the formula

$$\|\mathcal{G}_{S(c),u}\| = \|c\|_{H^1} + |u|,$$

where $|u|$ denotes half the oscillation of u, i.e. $|u| = (\max u - \min u)/2$.

Changing the section S or the norm $\|\cdot\|_{H^1}$ gives rise to an equivalent norm. In the rest of the paper, S and $\|\cdot\|_{H^1}$ will be considered as fixed. Everything will be well-defined regardless of our choice of S. Sometimes with a little abuse of language we will write c in place of $S(c)$, for instance $\mathcal{G}_{c,u}$ in place of $\mathcal{G}_{S(c),u}$.

We will be mostly concerned with a proper subset of E, namely

$$\mathbb{P} = \left\{\mathcal{G}_{c,u} : c \in H^1(M,\mathbb{R}), u\colon M \to \mathbb{R} \text{ semiconcave}\right\}.$$

Some basic properties of semiconcave functions are quickly reviewed in Subsection 2.0.1. Every $\mathcal{G} \in \mathbb{P}$ is called an *overlapping pseudograph* (the motivation for this terminology is given in [3, Section 2.9]). The set $\mathbb{P}$ is closed under sum and multiplication by a positive scalar, but not under difference or multiplication by a negative scalar. In fact, the dual set $\breve{\mathbb{P}}$ of *anti-overlapping pseudographs* is defined as

$$\breve{\mathbb{P}} = -\mathbb{P},$$

or, in other words, $\mathcal{G}_{\eta,u} \in \breve{\mathbb{P}}$ if and only if u is semiconvex. If $c \in H^1(M,\mathbb{R})$ and $C \subseteq H^1(M,\mathbb{R})$, the symbols $\mathbb{P}_c$ and $\mathbb{P}_C$ stand for

$$\mathbb{P}_c = \{\mathcal{G} \in \mathbb{P} : c(\mathcal{G}) = c\}, \qquad \mathbb{P}_C = \bigcup_{c\in C} \mathbb{P}_c.$$

and analogously for $\breve{\mathbb{P}}_c$ and $\breve{\mathbb{P}}_C$.

Given $\mathcal{G} = \mathcal{G}_{c,u} \in \mathbb{P}_c$ and $\breve{\mathcal{G}} = \mathcal{G}_{c,v} \in \breve{\mathbb{P}}_c$, the set

$$\mathcal{G} \wedge \breve{\mathcal{G}} \subseteq M$$

is defined as the set of the points of minimum of the difference $u - v$. This is a non empty compact set. Moreover, du_x and dv_x exist for every x in $\mathcal{G}\wedge\breve{\mathcal{G}}$ by semiconcavity, and they coincide. For this reason, the following definition yields a non-empty subset of T^*M:

$$\mathcal{G}\tilde{\wedge}\breve{\mathcal{G}} := \mathcal{G}_{|\mathcal{G}\wedge\breve{\mathcal{G}}} = \breve{\mathcal{G}}_{|\mathcal{G}\wedge\breve{\mathcal{G}}} = \mathcal{G} \cap \breve{\mathcal{G}} \cap \pi^{-1}(\mathcal{G} \wedge \breve{\mathcal{G}}) \subseteq \mathcal{G} \cap \breve{\mathcal{G}}$$

and the last inclusion may be strict in general. The set $\mathcal{G}\tilde{\wedge}\breve{\mathcal{G}}$ is compact and is a Lipschitz graph over its projection $\mathcal{G} \wedge \breve{\mathcal{G}}$, by properties of semiconcave functions. Observe that, if $\mathcal{G}$ and $\mathcal{G}'$ are two arbitrary pseudographs in E with the same cohomology class, the definition of $\mathcal{G} \wedge \mathcal{G}'$ is still meaningful, but the set $\mathcal{G}\tilde{\wedge}\mathcal{G}'$ could be empty in general.

Let us also notice that Ω can be naturally regarded as a subset of both $\mathbb{P}$ and $\breve{\mathbb{P}}$. The inclusion is given by $\eta \mapsto \mathcal{G}_{\eta,0} = \text{Graph}\,(\eta)$.

2.0.1 Semiconcave functions

Let us make a brief digression about semiconcave functions. We refer to [7] for a comprehensive exposition in the Euclidean case. On a manifold, the notion of semiconcavity is still meaningful, but the one of semiconcavity constant is chart-dependent. Nevertheless, this difficulty can be bypassed, for instance by taking a finite atlas as shown in [3, Appendix 1]. In this way it is still possible to talk about the best semiconcavity constant of a function u on M (or $M \times M$). It will depend on the particular finite atlas, but this choice will not affect the final results. We shall denote it by $sc(u)$. It satisfies

$$sc\,(\inf_{\lambda}\{u_\lambda\}) \leq \sup_{\lambda}\{sc(u_\lambda)\}, \tag{2.1}$$

for any family of functions $\{u_\lambda\}_\lambda$, provided that the infimum is finite. Moreover, if u_n converges uniformly to u, then

$$sc(u) \leq \liminf sc(u_n). \tag{2.2}$$

A semiconcave function is differentiable at every point of local minimum (and the differential is 0).

A family of functions $\{u_\lambda\}_\lambda$ is called equi-semiconcave if and only if $sc(u_\lambda) \leq C$ for some constant C independent of λ. We will use a lot the following fact: a family of equi-semiconcave functions is equi-Lipschitz (see for instance [7, Theorem 2.1.7]).

Finally, the set of semiconcave functions is closed under sum and multiplication by a positive scalar. A function u such that $-u$ is semiconcave is called semiconvex. A function is both semiconcave and semiconvex if and only if it is $C^{1,1}$.

3 The forcing relation and diffusion polyorbits

Let $\mathcal{F}$ be an arbitrary family of one-periodic Tonelli Hamiltonians on M. In the sequel we will denote with the same symbol $\mathcal{F}$ also the family of Tonelli Lagrangians associated to the Hamiltonians in $\mathcal{F}$ via the Fenchel-Legendre transform. The context will avoid any confusion.

Our goal is to prove existence of diffusion polyorbits, in the sense discussed in the Introduction. In this section, we first adapt to the polysystem framework the notion of forcing relation which was introduced in [3] for the case of a single Hamiltonian. Then, we show (Proposition 3.2) how this relation implies the diffusion: roughly speaking, if the cohomology class c forces the class c', then there will exist diffusion polyorbits from cohomology c to cohomology c', in a sense which will be made precise in the proposition. The aim of the later sections will then be to give sufficient conditions for the forcing relation to occur between two cohomology classes.

Let us recall that we denote by ϕ_H the time-one map of an Hamiltonian H. We define $\phi_{\mathcal{F}}$ of a subset $S \subseteq T^*M$ as follows:

$$\phi_{\mathcal{F}}(S) = \bigcup_{H \in \mathcal{F}} \phi_H(S),$$

so that, for instance $\phi^2_{\mathcal{F}}(S) = \phi_{\mathcal{F}}(\phi_{\mathcal{F}}(S))$. Given two arbitrary subsets S and S' of T^*M, we write

$$S \vdash_{N,\mathcal{F}} S' \quad \overset{\text{def}}{\Longleftrightarrow} \quad S' \subseteq \bigcup_{n=0}^{N} \phi^n_{\mathcal{F}}(S).$$

Notice that $z_1 \in T^*M$ is joined to z_2 by a finite $\mathcal{F}$-polyorbit of length less or equal to N if and only if $\{z_1\} \vdash_{N,\mathcal{F}} \{z_2\}$. We write $S \vdash_{\mathcal{F}} S'$, and we say that S forces S', if $S \vdash_{N,\mathcal{F}} S'$ for some $N \in \mathbb{N}$. We will mainly interested to the case in which S and S' are two pseudographs in $\mathbb{P}$.

We now extend the definition of $\vdash_{\mathcal{F}}$ to cohomology classes. If $\mathbb{W}$ and $\mathbb{W}'$ are two subsets of $\mathbb{P}$, we write

$$\mathbb{W} \ \vdash_{N,\mathcal{F}} \ \mathbb{W}' \quad \overset{\text{def}}{\Longleftrightarrow} \quad \forall \mathcal{G} \in \mathbb{W} \quad \exists \mathcal{G}' \in \mathbb{W}' : \mathcal{G} \vdash_{N,\mathcal{F}} \mathcal{G}'.$$

We write $\mathbb{W} \vdash_{\mathcal{F}} \mathbb{W}'$, and we say that $\mathbb{W}$ forces $\mathbb{W}'$, if $\mathbb{W} \vdash_{N,\mathcal{F}} \mathbb{W}'$ for some $N \in \mathbb{N}$. If $\mathbb{W} = \mathbb{P}_c$ or $\mathbb{W} = \mathbb{P}_C$, for some $c \in H^1(M,\mathbb{R})$ or $C \subseteq H^1(M,\mathbb{R})$, we simply write c or C in place of $\mathbb{P}_c$ or $\mathbb{P}_C$. Similarly for $\mathbb{W}' = \mathbb{P}_c$. So, for instance, if c and c' are two cohomology classes, the relation

$$c \vdash_{N,\mathcal{F}} c'$$

means that for every $\mathcal{G} \in \mathbb{P}_c$ there exists $\mathcal{G}' \in \mathbb{P}_{c'}$ such that $\mathcal{G} \vdash_{N,\mathcal{F}} \mathcal{G}'$.

The relation $\vdash_{\mathcal{F}}$ is reflexive and transitive (between subsets as well as between cohomology classes). In the sequel, it will be useful to consider the symmetrized relation

$$c \dashv\vdash_{\mathcal{F}} c',$$

which means that $c \vdash_{\mathcal{F}} c'$ and $c' \vdash_{\mathcal{F}} c$. We say that c and c' *force each other*. The following fact directly follows from the definitions.

Proposition 3.1. *The relation $\dashv\vdash_{\mathcal{F}}$ is an equivalence relation on $H^1(M,\mathbb{R})$.*

We can now restate and prove Proposition 1.2 about the existence of diffusion polyorbits. The proof is essentially the same as in [3, Proposition 5.3].

Let us recall that a $\mathcal{F}$-polyorbit (or simply a polyorbit) is a bi-infinite sequence $(z_n)_{n\in\mathbb{Z}}$ such that for every n there exists $H_n \in \mathcal{F}$ satisfying $\phi_{H_n}(z_n) = z_{n+1}$. A finite polyorbit is a finite segment of a polyorbit.

Proposition 3.2.

1. *Let $c \vdash_{\mathcal{F}} c'$. Let $H, H' \in \mathcal{F}$ and η, η' be two smooth closed one-forms of cohomology c and c' respectively. Then:*

   (i) there exists a polyorbit which is α-asymptotic to $\tilde{\mathcal{A}}_H(c)$ and ω-asymptotic to $\tilde{\mathcal{A}}_{H'}(c')$;

   (ii) there exists a polyorbit $(z_n)_{n\in\mathbb{Z}}$ which satisfies $z_0 \in \mathrm{Graph}\,(\eta)$ and $z_N \in \mathrm{Graph}\,(\eta')$ for some $N \in \mathbb{N}$;

(iii) there exists a polyorbit $(z_n)_{n\in\mathbb{Z}}$ *which satisfies* $z_0 \in \mathrm{Graph}\,(\eta)$ *and is* ω*-asymptotic to* $\tilde{\mathcal{A}}_{H'}(c')$*;*

(iv) there exists a polyorbit $(z_n)_{n\in\mathbb{Z}}$ *which is* α*-asymptotic to* $\tilde{\mathcal{A}}_H(c)$ *and satisfies* $z_0 \in \mathrm{Graph}\,(\eta')$*.*

2. Let

$$(c_i, H_i, \varepsilon_i)_{i\in\mathbb{Z}} \;\subseteq\; H^1(M,\mathbb{R}) \times \mathcal{F} \times \,]0,+\infty[$$

such that $c_i \vdash_{\mathcal{F}} c_{i+1}$*. Then there exists a polyorbit* $(z_n)_{n\in\mathbb{Z}}$ *which visits in turn the* ε_i*-neighborhoods of the Mather sets* $\tilde{\mathcal{M}}_{H_i}(c_i)$*. Moreover, if* $(c_i, H_i) = (\bar{c}, \bar{H})$ *for* i *small enough (resp.* i *big enough), then the polyorbit can be taken* α*-asymptotic to* $\tilde{\mathcal{A}}_{\bar{H}}(\bar{c})$ *(resp.* ω*-asymptotic to* $\tilde{\mathcal{A}}_{\bar{H}}(\bar{c})$*).*

Proof of 1. The proof of any one of the four statements relies on the following fact: given $\mathcal{G} \in \mathbb{P}_c$ and $\mathcal{G}' \in \breve{\mathbb{P}}_{c'}$, there exists a finite polyorbit joining $\mathcal{G}$ to $\mathcal{G}'$. Indeed, since $c \vdash_{\mathcal{F}} c'$, for any $\mathcal{G} \in \mathbb{P}_c$ there exists $\mathcal{G}'' \in \mathbb{P}_{c'}$ such that $\mathcal{G}'' \subseteq \cup_{n=0}^{N} \phi_{\mathcal{F}}^{n}(\mathcal{G})$, for some N. Hence every point in $\mathcal{G}' \cap \mathcal{G}''$ (we know from Section 2 that this intersection is not empty) is joined by a finite polyorbit to $\mathcal{G}$.

The first statement now follows by taking $\mathcal{G} = \mathcal{G}_{c,u}$ with u a c-weak Kam solution for H and $\mathcal{G}' = \mathcal{G}_{c',u'}$ with u' a dual c'-weak Kam solution for H'. Every point in $\mathcal{G}$ is α-asymptotic for the flow of H to $\tilde{\mathcal{A}}_H(c)$ and every point in $\mathcal{G}'$ is ω-asymptotic for the flow of H' to $\tilde{\mathcal{A}}_{H'}(c')$. This is a general property of weak Kam solutions, which will be recalled in Proposition 4.10. Hence, every finite polyorbit joining $\mathcal{G}$ to $\mathcal{G}'$ can be extended to a bi-infinite polyorbit as in statement (i).

The second statement is obtained in a similar way by taking $\mathcal{G} = \mathcal{G}_{\eta,0}$ and $\mathcal{G}' = \mathcal{G}_{\eta',0}$. The remaining two statements are similar.

Proof of 2. It is a natural adaptation of the proof in [3, Proposition 5.3 (ii)]. □

4 Lagrangian action and Lax-Oleinik operators

In this section we introduce the objects needed to put in place, in the next section, the Mather mechanism for the construction of diffusion polyorbits. We start by summarizing in Proposition 4.1 those aspects of the time-one action of a Tonelli Lagrangian which will be useful in the sequel. It is standard to associate to these actions (or, more generally, to any cost, i.e. any continuous function on $M \times M$) a Lax-Oleinik operator, which can be interpreted also as an operator on pseudographs (formula (4.4)). We describe how the properties of the actions have nice counterparts in the corresponding Lax-Oleinik operators and in the consequent dynamics on pseudographs (Remark 4.4). These nice features are not lost under some operations on costs such as minimums and compositions (Proposition 4.6). In this way, starting from the "basic bricks" of the time-one actions of the Lagrangians in $\mathcal{F}$, we will eventually be able to build, for every cohomology c, a large semigroup Σ_c^∞ of Lax-Oleinik operators (Subsection 4.4).

In the case of just one Tonelli Lagrangian (i.e. $\mathcal{F}$ singleton), this language allows to concisely rephrase some aspects of the weak Kam theory (Subsection 4.3). In the case of general $\mathcal{F}$, the dynamics of Σ_c^∞ on $\mathbb{P}$ is related to the dynamics on T^*M of the semigroup generated by the time-one maps ϕ_H, $H \in \mathcal{F}$. Crucially, the semigroup Σ_c^∞ will contain, after passing to the limit, the operators associated to the Peierls barriers, along with their successive compositions. This aspect, together with the possibility of "shadowing" these operators with "finite-time" ones, will be at the hearth of the Mather mechanism in the next section.

4.1 Lagrangian action

Given a one-periodic Tonelli Lagrangian L on M and a closed smooth one-form η, the time-one action $A_{L,\eta}\colon M \times M \to \mathbb{R}$ is defined by

$$A_{L,\eta}(y,x) = \min_{\gamma(0)=y,\gamma(1)=x} \int_0^1 L\big(\gamma(t),\dot\gamma(t),t\big) - \eta_{\gamma(t)}(\dot\gamma(t))\,dt \tag{4.1}$$

where the minimum is taken over absolutely continuous curves γ. It is well-known that minimizers exist. The following important properties of A_L are also well-known.

Proposition 4.1.

(i) $A_{L,\eta+df}(y,x) = A_{L,\eta}(y,x) + f(y) - f(x)$*; this is immediate from the definition.*

(ii) $\eta \mapsto A_{L,\eta}$ *is continuous if* Ω *is endowed with the topology induced from the space of pseudographs* E *introduced in Section 2 (for a proof see [3, Appendix B.6]).*

In view of (i) above, this is equivalent to the continuity of $c \mapsto A_{L,S(c)}$.

(iii) $A_{L,\eta}$ *is semiconcave. Even more, if* $C \subset H^1(M,\mathbb{R})$ *is compact, then* $\{A_{L,S(c)}\}_{c\in C}$ *is equi-semiconcave (for a proof see [3, Appendix B.7]).*

(iv) $\partial_x A_{L,\eta}(y,x)$ *exists if and only if* $\partial_y A_{L,\eta}(y,x)$ *exists and in that case we have*

$$\big(x, \eta_x + \partial_x A_{L,\eta}(y,x)\big) = \phi_H\big(y, \eta_y - \partial_y A_{L,\eta}(y,x)\big),$$

where H *is the Hamiltonian associated to* L.

The time-n action A_L^n is defined by letting $A_L^1 = A_L$ and by induction

$$A_{L,\eta}^{n+1}(y,x) = \min_{z\in M}\big\{A_{L,\eta}^n(y,z) + A_{L,\eta}^1(z,x)\big\}$$

or, equivalently,

$$A_{L,\eta}^n(y,x) = \min_{\gamma(0)=y,\gamma(n)=x} \int_0^n L\big(\gamma(t),\dot\gamma(t),t\big) - \eta_{\gamma(t)}(\dot\gamma(t))\,dt,$$

the minimum being over absolutely continuous curves.

It is well-known that, given L, there exists an unique function $\alpha\colon H^1(M,\mathbb{R}) \to \mathbb{R}$ such that the function

$$h_{L,\eta}(y,x) = \liminf_{n\to\infty}\ A^n_{L,\eta}(y,x) + n\alpha([\eta])$$

is real-valued for every η; the family $h_L \equiv \{h_{L,\eta}\}_\eta$ is called the Peierls barrier of L. It clearly satisfies the property (i) of Proposition 4.1; it also satisfies the property (iii), this will be proved in detail in Subsection 4.3.

4.2 Lax-Oleinik operators

For any compact space X, the set of continuous functions $C(X)$ will be endowed with the standard sup-norm $\|\cdot\|_\infty$.

Any continuous function $A \in C(M\times M)$ will be called a cost. To any cost A, it is possible to associate the Lax-Oleinik operator $T_A\colon C(M) \to C(M)$ defined by

$$T_A u(x) = \min_{y\in M} \big\{u(y) + A(y,x)\big\}, \qquad u \in C(M)$$

and the dual Lax-Oleinik operator $\breve{T}_A\colon C(M) \to C(M)$

$$\breve{T}_A u(y) = \max_{x\in M} \big\{u(x) - A(y,x)\big\}, \qquad u \in C(M).$$

We call $\mathcal{I}_A(u) \subseteq M$ the set of points y such that $T_A u(x) = u(y) + A(y,x)$ for some x. Let us now list without proof some basic properties of these objects. We recall that $|\cdot|$ indicates half the oscillation of a function.

Proposition 4.2.

(i) *The minimum and the maximum in the above formulas are actually achieved; $T_A u$ and $\breve{T}_A u$ actually belong to $C(M)$ if $u \in C(M)$;*

(ii) *if A' is another cost and u' another continuous function, then*

$$\begin{aligned} \|T_{A'}u' - T_A u\|_\infty &\le \|A' - A\|_\infty + \|u' - u\|_\infty, \\ |T_{A'}u' - T_A u| &\le |A' - A| + |u' - u| \end{aligned} \tag{4.2}$$

(iii) *$\mathcal{I}_A(u)$ is compact and non-empty;*

(iv) *the set-valued function $(A,u) \mapsto \mathcal{I}_A(u)$ is upper-semicontinuos;*

(v) *for every A and u, we have $\breve{T}_A T_A u \le u$ and*

$$\mathcal{I}_A(u) = \{y \in M : \breve{T}_A T_A u(y) = u(y)\} = \arg\min \big\{u - \breve{T}_A T_A u\big\}.$$

(vi) *for every A and u, we have*

$$T_A \breve{T}_A T_A\, u = T_A\, u \qquad \textit{and} \qquad \breve{T}_A T_A \breve{T}_A\, u = \breve{T}_A\, u;$$

(vii) if A is semiconcave, then $T_A u$ is semiconcave for any u, and $sc(u) \le sc(A)$.

We are going to consider families of costs indexed by closed smooth one-forms. Let us give some definitions.

Definition 4.3. *Let $A \equiv \{A_\eta\}_{\eta\in\Omega}$ be a family of costs indexed by the closed smooth one-forms. We say that A is:*

(i) geometric *if A_η is Lipschitz for every η and*

$$A_{\eta+df}(y,x) = A_\eta(y,x) + f(y) - f(x) \qquad \forall\, f \textit{ smooth}; \tag{4.3}$$

(ii) continuous *if*

$$\Omega \ni \eta \mapsto \; A_\eta \qquad \textit{is continuous}$$

where Ω is endowed with the topology induced from E, see Section 2. Notice that if a family A is geometric, the continuity of $c \mapsto A_{S(c)}$ is sufficient in order to have the continuity of $\eta \mapsto A_\eta$; here S is the linear section chosen in Section 2.

(iii) locally equi-semiconcave *if, for any compact $C \subset H^1(M,\mathbb{R})$, the family $\{A_{S(c)}\}_{c\in C}$ is equi-semiconcave;*

(iv) of $\mathcal{F}$-flow-type *if there exists $N \in \mathbb{N}$ such that the following holds:*

$$\begin{aligned}&\exists\ \partial_x A_\eta(y,x),\ \exists\ \partial_y A_\eta(y,x)\\ &\qquad\qquad \Rightarrow \big(y, \eta_y - \partial_y A_\eta(y,x)\big) \vdash_{N,\mathcal{F}} \big(x, \eta_x + \partial_x A_\eta(y,x)\big);\end{aligned}$$

We say that A is of $N,\mathcal{F}$-flow-type if we want to specify the N.

If all the above conditions are satisfied, we say for short that A is a $\mathcal{F}$-family.

Observe that the Proposition 4.1 says that the time-one actions $\{A_{L,\eta}\}_\eta$, $L \in \mathcal{F}$, are $\mathcal{F}$-families. In the next subsection we are going to introduce some operations on costs which will preserve the property of being an $\mathcal{F}$-family. This will allow to use the Lagrangian time-one actions as "basic bricks" to build lots of $\mathcal{F}$-families of costs.

The utility of a $\mathcal{F}$-family comes from the following remark.

Remark 4.4.

(i) If $A \equiv \{A_\eta\}_{\eta\in\Omega}$ is a geometric family of costs then

$$T_{A_{\eta+df}}(u - f) = T_{A_\eta}(u) - f.$$

Hence, an induced operator on pseudographs $\Phi_A \colon E \to E$ is well-defined by

$$\Phi_A(\mathcal{G}_{\eta,u}) = \mathcal{G}_{\eta,T_{A_\eta}u} \tag{4.4}$$

along with the dual counterpart

$$\breve{\Phi}_A(\mathcal{G}_{\eta,u}) = \mathcal{G}_{\eta,\breve{T}_{A_\eta} u}.$$

Notice that $c(\Phi_A(\mathcal{G})) = c(\breve{\Phi}_A(\mathcal{G})) = c(\mathcal{G})$.

If A' is another geometric family of costs, and if $\mathcal{G} = \mathcal{G}_{c,u}, \mathcal{G}' = \mathcal{G}_{c',u'} \in E$ are two pseudographs, we have the following estimate:

$$\begin{aligned} \|\Phi_A(\mathcal{G}) - \Phi_{A'}(\mathcal{G}')\|_E &= \|c - c'\|_{H^1} + |T_{A_c} u - T_{A'_{c'}} u'| \\ &\leq \|\mathcal{G} - \mathcal{G}'\|_E + |A_c - A'_{c'}| \end{aligned} \tag{4.5}$$

where in the second line we used (4.2).

In the same spirit, $\mathcal{I}_{A_{\eta+df}}(u-f) = \mathcal{I}_{A_\eta}(u)$, therefore the set $\mathcal{I}_{A_\eta}(u)$ is also well-defined on pseudographs, and we will denote it by $\mathcal{I}_A(\mathcal{G})$ or $\mathcal{I}_{\Phi_A}(\mathcal{G})$. Items (v) and (vi) in Proposition 4.2 translate respectively into

$$\mathcal{I}_{\Phi_A}(\mathcal{G}) = \mathcal{G} \wedge \breve{\Phi}_A \Phi_A(\mathcal{G}). \tag{4.6}$$

and

$$\Phi_A \breve{\Phi}_A \Phi_A = \Phi_A, \qquad \breve{\Phi}_A \Phi_A \breve{\Phi}_A = \breve{\Phi}_A. \tag{4.7}$$

(ii) If $\{A_\eta\}_\eta$ is a continuous geometric family, then Φ_A is continuous thanks to the estimate (4.5). Moreover, $\mathcal{I}_A(\mathcal{G})$ is upper-semicontinuous viewed as a (set-valued) function from E to M. Indeed, the composition

$$(\eta, u) \mapsto (A_\eta, u) \mapsto \mathcal{I}_{A_\eta}(u)$$

is upper-semicontinuous (thanks to Proposition 4.2(iv)), and this remains true when passing to the quotient space of pseudographs.

(iii) If $\{A_\eta\}_{\eta\in\Omega}$ is a locally equi-semiconcave geometric family, then $\Phi_A(\mathbb{P}) \subseteq \mathbb{P}$ and $\Phi_A(\mathbb{P}_C)$ is relatively compact for all compact $C \subset H^1(M, \mathbb{R})$. This is a consequence of Proposition 4.2(vii) and the Ascoli-Arzelà Theorem (we recall that equi-semiconcave implies equi-Lipschitz). The analogous result holds true for $\breve{\Phi}_A$.

(iv) If $\{A_\eta\}_\eta$ is a $N, \mathcal{F}$-flow-type, locally equi-semiconcave and geometric family of costs, then

$$\mathcal{G}_{|\mathcal{I}_A(\mathcal{G})} \ \vdash_{N,\mathcal{F}} \ \Phi_A(\mathcal{G}) \qquad \forall\, \mathcal{G} \in \mathbb{P}. \tag{4.8}$$

This important fact is obtained by writing $\mathcal{G} = \mathcal{G}_{\eta,u}$ and then applying Proposition 4.5. The dual statement is also true and is proved analogously. It can be expressed as

$$\mathcal{G}_{|\breve{\mathcal{I}}_A(\mathcal{G})} \ \vdash_{N,-\mathcal{F}} \ \breve{\Phi}_A(\mathcal{G}) \qquad \forall\, \mathcal{G} \in \breve{\mathbb{P}}.$$

Here we have denoted by $-\mathcal{F}$ the family $\{-H : H \in \mathcal{F}\}$; its elements are not Tonelli Hamiltonians but the relation $\vdash_{-\mathcal{F}}$ is still meaningful. We have also denoted by $\breve{\mathcal{I}}_A(\mathcal{G})$ the set of points $x \in M$ such that $\breve{T}_{A_\eta} u(y) = u(x) - A_\eta(y, x)$ for some y (and η and u are such that $\mathcal{G} = \mathcal{G}_{\eta,u}$).

Proposition 4.5. *Suppose the family of costs* $\{A_\eta\}_\eta$ *satisfies the assumptions in Remark 4.4(iv). Let* $u\colon M \to \mathbb{R}$ *be semiconcave and* $v = T_{A_\eta} u$*. Then, for every* x *such that* dv_x *exists and for every* y *such that* $v(x) = u(y) + A_\eta(y, x)$*, we have*

$$\exists\, du_y \quad \textit{and} \quad (y, \eta_y + du_y) \vdash_{N,\mathcal{F}} (x, \eta_x + dv_x).$$

Proof. The proof is essentially the same as in [3, Proposition 2.7] but we report it for completeness. Let x be such that dv_x exists, and let y be such that $v(x) = u(y) + A_\eta(y, x)$. From the definition of T_{A_η}, one gets that the function $y' \mapsto u(y') + A_\eta(y', x)$ has a minimum at y. Being the sum of two semiconcave functions, both of them have to be differentiable at y and

$$du_y + \partial_y A_\eta(y, x) = 0.$$

Similarly, the function $x' \mapsto v(x') - A_\eta(y, x')$ has a maximum at x. Since dv_x exists and $-A_\eta$ is semiconvex, we get that

$$dv_x - \partial_x A_\eta(y, x) = 0.$$

Hence by the $N, \mathcal{F}$-flow-type property we get the desired result:

$$(x, \eta_y + du_y) = \big(x, \eta_y - \partial_y A_\eta(y, x)\big) \vdash_{N,\mathcal{F}} \big(x, \eta_x + \partial_x A_\eta(y, x)\big) = (x, \eta_x + dv_x).$$

□

4.2.1 Operations on costs and families of costs

There are three quite natural operations on costs. For A, A' two costs and $\lambda \in \mathbb{R}$, they are defined as follows:

$$\begin{aligned}
&(A, \lambda) \mapsto A + \lambda && \textit{(addition of constant)}\\
&(A, A') \mapsto \min\{A, A'\} && \textit{(minimum)}\\
&(A, A') \mapsto A' \circ A(y, x) = \min_{z \in M} \big\{A(y, z) + A'(z, x)\big\} && \textit{(composition).}
\end{aligned}$$

It is easily checked that the three of them are continuous in their arguments and that the Lax-Oleinik operators well-behave in the following sense: for $u \in C(M)$, we have

$$\begin{aligned}
T_{A+\lambda} u &= T_A u + \lambda\\
T_{\min\{A,A'\}} &= \min\{T_A u, T_{A'} u\}\\
T_{A' \circ A} u &= T_{A'} \circ T_A u.
\end{aligned} \tag{4.9}$$

We can define the same operations on families of costs in the obvious way: for $A \equiv \{A_\eta\}_\eta$ and $A' \equiv \{A'_\eta\}_\eta$,

$$(A + \lambda)_\eta = A_\eta + \lambda([\eta]), \qquad \min\{A, A'\}_\eta = \min\{A_\eta, A'_\eta\}, \qquad (A' \circ A)_\eta = A'_\eta \circ A_\eta,$$

where we suppose that λ is a function from $H^1(M, \mathbb{R})$ to $\mathbb{R}$.

The following proposition shows that these operations preserve the properties expressed in the Remark 4.4.

Proposition 4.6. *Let A, A' be two $\mathcal{F}$-families of costs, and $\lambda\colon H^1(M,\mathbb{R}) \to \mathbb{R}$ be a continuous function. Then $A+\lambda$, $\min\{A, A'\}$ and $A' \circ A$ are $\mathcal{F}$-families as well. Moreover, we control the semiconcavity constants as follows (we omit the subscript η):*

$$sc(A+\lambda) = sc(A) \qquad sc(\min\{A, A'\}) \le \max\{sc(A), sc(A')\}$$
$$sc(A' \circ A) \le \max\{sc(A), sc(A')\}.$$

Proof. We have to verify the four conditions in Definition 4.3. Conditions (i) and (ii) are easy. The condition (iii), i.e. the local equi-semiconcavity, is obvious for the family $A+\lambda$ and is true for the family $\min\{A, A'\}$ because of (2.1). For $A' \circ A$, we have, at fixed z, that $(x, y) \mapsto A(y, z) + A'(z, x)$ is $\max\{sc(A), sc(A')\}$-semiconcave, hence $(x, y) \mapsto A' \circ A(y, x)$ is $\max\{sc(A), sc(A')\}$-semiconcave too, again by (2.1).

As for the condition (iv), i.e. the $\mathcal{F}$-flow-type property, it is obvious for $A+\lambda$. For $\min\{A, A'\}$, we notice that, by semiconcavity, $\partial_x \min\{A, A'\}_\eta(y, x)$ exists if and only if both $\partial_x A_\eta(y, x)$ and $\partial_x A'_\eta(y, x)$ exist and coincide, and in this case

$$\partial_x \min\{A, A'\}_\eta(y, x) = \partial_x A_\eta(y, x) = \partial_x A'\eta(y, x).$$

The same happens with ∂_y. Hence, we can use the $\mathcal{F}$-flow-type property of A to obtain

$$\begin{aligned}\big(y, \eta_y - \partial_y \min\{A, A'\}_\eta(y, x)\big) &= \big(y, \eta_y - \partial_y A_\eta(y, x)\big) \vdash_{N,\mathcal{F}} \big(x, \eta_x + \partial_x A_\eta(y, x)\big)\\ &= \big(x, \eta_x + \partial_x \min\{A, A'\}_\eta(y, x)\big).\end{aligned}$$

It remains to prove the $\mathcal{F}$-flow-type property for $A' \circ A$. Let y, x be such that $\partial_y (A' \circ A)_\eta(y, x)$ and $\partial_x (A' \circ A)_\eta(y, x)$ exist, and let z be a point of minimum in the expression

$$\big(A' \circ A\big)_\eta(y, x) = \min_{z \in M} \{A_\eta(y, z) + A'_\eta(z, x)\}.$$

Considering here y as a fixed parameter, we can apply Proposition 4.5. Using the $\mathcal{F}$-flow-type property for A', we obtain that

$$(z, \eta_z + \partial_x A_\eta(y, z)) \vdash_{N',\mathcal{F}} (x, \eta_x + \partial_x (A' \circ A)_\eta(y, x)).$$

Similarly, we can consider x as a fixed parameter and obtain

$$(y, \eta_y + \partial_y A_\eta(y, z)) \vdash_{N,\mathcal{F}} (z, \eta_z + \partial_x A_\eta(y, z)).$$

This implies the conclusion. $\square$

From (4.9) we also deduce that

$$\Phi_{A+\lambda} = \Phi_A, \qquad \Phi_{A' \circ A} = \Phi_{A'} \circ \Phi_A$$
$$\mathcal{I}_{A+\lambda}(\mathcal{G}) = \mathcal{I}_A(\mathcal{G}), \quad \mathcal{I}_{\min\{A, A'\}}(\mathcal{G}) \subseteq \mathcal{I}_A(\mathcal{G}) \cup \mathcal{I}_{A'}(\mathcal{G}), \quad \mathcal{I}_{A' \circ A}(\mathcal{G}) \subseteq \mathcal{I}_A(\mathcal{G}). \quad (4.10)$$

Instead, we are not able to find an analogous formula for $\Phi_{\min\{A,A'\}}$. Let us notice that even if $\Phi_{A+\lambda} = \Phi_A$, the operation of adding a constant is not completely immaterial: it has a role for operators associated to costs such as $\min\{A+\lambda, A'+\lambda'\}$. If $\lambda'-\lambda$ is sufficiently big, then the corresponding operator will be Φ_A, and if $\lambda-\lambda'$ is sufficiently big, the operator will be $\Phi_{A'}$. Intermediate values of $\lambda'-\lambda$ will correspond to intermediate situations.

4.3 Weak Kam theory

In this subsection we consider the special case $\mathcal{F} = \{L\}$ and we rephrase in the language of pseudographs some standard results in weak Kam theory. Some of them have already been used in Proposition 3.2, and some others will be used in Section 5.

An important role in the theory is played by the so-called weak Kam solutions. There are several equivalent definitions for them. The one which we are going to use is: given a Tonelli Lagrangian L and a cohomology class c, a c-weak Kam solution for L is a solution $u \in C(M)$ of the equation

$$u = T_{A_{L,c}} u + \alpha_L(c),$$

where $\alpha_L \colon H^1(M,\mathbb{R}) \to \mathbb{R}$ is Mather's α-function appeared in Subsection 4.1. A dual weak Kam solution is defined as a solution $u \in C(M)$ of the equation

$$u = \breve{T}_{A_{L,c}} u - \alpha_L(c)$$

In fact, $\alpha_L(c)$ is the unique constant such that the above equations admit a solution. We say that u is a weak Kam solution (resp. dual weak Kam solution) if it is a c-weak Kam solution (resp. dual weak Kam solution) for some c.

It is no surprise, in view of the definition of Φ_{A_L} in (4.4), that the language of pseudographs allows to concisely reformulate these concepts. From that definition it is indeed immediate that:

$$u \text{ is a } c\text{-weak Kam solution for } L \;\Leftrightarrow\; \mathcal{G}_{c,u} \text{ is a fixed point of } \Phi_{A_L}. \tag{4.11}$$

In view of this, we shall call weak Kam solutions as well the fixed points of Φ_{A_L}, and c-weak Kam solutions the fixed points in $\mathbb{P}_c$. Analogously for dual weak Kam solutions, with $\breve{\Phi}_A$ in place of Φ_A. Notice that two c-weak Kam solutions u and u' differing by a constant correspond to the same weak Kam solution $\mathcal{G}_{c,u} = \mathcal{G}_{c,u'}$.

Another important object in weak Kam theory is the Peierls barrier h_L, introduced in Subsection 4.1. Let us point out that

$$h_{L,c} = \lim_{n\to\infty} \lim_{m\to\infty} \min\{A^n_{L,c} + n\alpha, A^{n+1}_{L,c} + (n+1)\alpha, \dots, A^m_{L,c} + m\alpha\}, \tag{4.12}$$

and that, by Proposition 4.6, the families of costs appearing in the right-hand side are locally equi-semiconcave in the sense of Definition 4.3, with a local (in c) common bound for their semiconcavity constants. Hence, they have a local (in c) common bound for their Lipschitz constants. By the Ascoli-Arzelà theorem, this implies that the two limits are uniform (for any fixed c). Since uniform limits preserve semiconcavity constants, we get that the family of costs h_L is locally equi-semiconcave in the sense of Definition 4.3. Remark 4.4(iii) thus applies.

The next proposition reformulates in our language the well-known identities

$$\min_{z\in M}\big\{h_{L,c}(y,z)+A_{L,c}(z,x)+\alpha_L(c)\big\}=h_{L,c}(y,x),$$
$$\min_{z\in M}\big\{A_{L,c}(y,z)+h_{L,c}(z,x)+\alpha_L(c)\big\}=h_{L,c}(y,x),$$
$$\min_{z\in M}\big\{h_{L,c}(y,z)+h_{L,c}(z,x)\big\}=h_{L,c}(y,x)\qquad\forall\, y,x\in M.$$

Proposition 4.7. *Let $h_L\equiv\{h_{L,c}\}_c$ be the family of costs associated to the Peierls barrier of L. The following identities hold true:*

$$\Phi_{A_L}\circ\Phi_{h_L}=\Phi_{h_L}$$
$$\Phi_{h_L}\circ\Phi_{A_L}=\Phi_{h_L}$$
$$\Phi_{h_L}\circ\Phi_{h_L}=\Phi_{h_L}$$

This proposition has important consequences. Indeed, it implies the following characterizations of weak Kam solutions.

Proposition 4.8. *Let L be a Tonelli Lagrangian, $c\in H^1(M,\mathbb{R})$ and $u\colon M\to\mathbb{R}$ be a continuous function. The following are equivalent:*

(i) u is a c-weak Kam solution for L;

(ii) $\mathcal{G}_{c,u}$ is a fixed point of Φ_{A_L};

(iii) $\mathcal{G}_{c,u}$ is a fixed point of Φ_{h_L};

(iv) $\mathcal{G}_{c,u}$ belongs to $\Phi_{h_L}(E)$.

The dual statement obtained by replacing 'c-weak Kam solution' with 'dual c-weak Kam solution', Φ with $\breve{\Phi}$ and $\mathbb{P}$ with $\breve{\mathbb{P}}$ is also true.

Proof.
$(i)\Leftrightarrow(ii)$ has been already pointed out in (4.11);
$(iii)\Rightarrow(ii)$: let $\mathcal{G}$ be such that $\Phi_{h_L}(\mathcal{G})=\mathcal{G}$. We then have, by Proposition 4.7,

$$\Phi_{A_L}(\mathcal{G})=\Phi_{A_L}\Phi_{h_L}(\mathcal{G})=\Phi_{h_L}(\mathcal{G})=\mathcal{G};$$

$(iii)\Rightarrow(iv)$ is obvious;
$(iv)\Rightarrow(iii)$: let $\mathcal{G}\in\Phi_{h_L}(E)$; then there exists $\mathcal{G}'\in E$ such that $\Phi_{h_L}(\mathcal{G}')=\mathcal{G}$. By Proposition 4.7,

$$\Phi_{h_L}(\mathcal{G})=\Phi_{h_L}\Phi_{h_L}(\mathcal{G}')=\Phi_{h_L}(\mathcal{G}')=\mathcal{G};$$

$(ii)\Rightarrow(iii)$: for a given $\mathcal{G}\in\mathbb{P}$, the set of costs A such that $\Phi_A(\mathcal{G})=\mathcal{G}$ is closed under addition of constants, finite minimums, compositions and uniform limits. From $\Phi_{A_L}(\mathcal{G})=\mathcal{G}$ and expression (4.12) we thus get $\Phi_{h_L}(\mathcal{G})=\mathcal{G}$.

The dual statement is proved analogously. $\square$

It is clear from the previous Proposition that weak Kam solutions belong to $\mathbb{P}$ and dual weak Kam solutions belong to $\breve{\mathbb{P}}$. In $d = 1$, it is known that the non-contractible invariant circles are exactly the pseudographs which are both weak Kam solutions and dual weak Kam solutions. The following proposition will be crucial in the proof of Proposition 5.9. As usual, H denotes the Tonelli Hamiltonian associated to L via the Fenchel-Legendre transform.

Proposition 4.9.

(i) A weak Kam solution $\mathcal{G} \subset T^*M$ is invariant for ϕ_H^{-1}. A dual weak Kam solution is invariant for ϕ_H^1;

(ii) if $\mathcal{G}$ is a weak Kam solution belonging to $\breve{\mathbb{P}}$, then automatically $\mathcal{G}$ is a dual weak Kam solution. Analogously, a dual weak Kam solution belonging to $\mathbb{P}$ is a weak Kam solution;

(iii) if $\mathcal{G}$ is both a weak Kam solution and a dual weak Kam solution, then $\mathcal{G}$ is a Lipschitz ϕ_H-invariant graph over M;

Proof.

(i) Let $\mathcal{G}$ be a weak Kam solution. From Remark 4.4(iv) we know that

$$\mathcal{G}_{|\mathcal{I}_{A_L}(\mathcal{G})} \vdash_{1,\{L\}} \Phi_{A_L}(\mathcal{G}).$$

that is, using the fact that $\Phi_A(\mathcal{G}) = \mathcal{G}$,

$$\mathcal{G} \subseteq \phi_H^1\big(\mathcal{G}_{|\mathcal{I}_{A_L}(\mathcal{G})}\big).$$

This proves $\phi_H^{-1}(\mathcal{G}) \subseteq \mathcal{G}$, that is the first claim. The dual claim is obtained analogously, starting from the dual version of Remark 4.4(iv).

(ii) Let $\mathcal{G}$ be a weak Kam solution belonging to $\breve{\mathbb{P}}$. From the dual version of Remark 4.4(iv) and part (i), we deduce

$$\breve{\Phi}_{A_L}(\mathcal{G}) \subseteq \phi_H^{-1}(\mathcal{G}) \subseteq \mathcal{G}.$$

It is easy to verify that if an anti-overlapping pseudograph is contained in an overlapping one, the two must coincide. Thus $\breve{\Phi}_{A_L}(\mathcal{G}) = \mathcal{G}$, that is $\mathcal{G}$ is a dual weak Kam solution. The dual statement is analogous.

(iii) Let $\mathcal{G}$ be both a weak Kam and a dual weak Kam solution. It is immediate from part (i) that $\mathcal{G}$ is invariant both in the past and in the future. Moreover, $\mathcal{G}$ has to belong to $\mathbb{P} \cap \breve{\mathbb{P}}$, hence it is a Lipschitz graph over M (recall that a function both semiconcave and semiconvex is $C^{1,1}$). $\square$

We have just seen that a weak Kam solution $\mathcal{G}$ is invariant for ϕ_H^{-1}. Hence the sequence $\phi_H^{-n}(\mathcal{G})$ is decreasing in n. Moreover, one may prove (see [3], or Proposition 4.16 in which we are going to prove some analogous statements in more general situations) that its intersection is a compact invariant set in both past and future, and

$$\bigcap_{n\in\mathbb{N}} \phi_H^{-n}(\mathcal{G}) = \mathcal{G}_{|\mathcal{I}_{h_L}(\mathcal{G})}$$

In view of (4.6), this set coincides with $\mathcal{G}\tilde{\wedge}\breve{\Phi}_{h_L}(\mathcal{G})$.

We now introduce the c-Aubry set of L, denoted by $\tilde{\mathcal{A}}_L(c)$, which appears in Proposition 3.2. One of the possible definitions is the following:

$$\tilde{\mathcal{A}}_L(c) = \bigcap \big\{\mathcal{G}_{|\mathcal{I}_{h_L}(\mathcal{G})} : \mathcal{G} \text{ is a } c\text{-weak Kam solution}\big\} \subseteq T^*M.$$

For a weak Kam solution $\mathcal{G}$, the set $\mathcal{I}_{h_L}(\mathcal{G}) = \mathcal{G} \wedge \breve{\Phi}_{h_L}(\mathcal{G})$ is also called the Aubry set of $\mathcal{G}$.

If $\mathcal{G} \in \mathbb{P}_c$ and $\mathcal{G}' \in \breve{\mathbb{P}}_c$, it is always true (see Section 2) that $\mathcal{G}\tilde{\wedge}\mathcal{G}' \subseteq T^*M$ is a compact set which is a Lipschitz graph over its projection $\mathcal{G} \wedge \mathcal{G}' \subseteq M$, hence it is clear that $\tilde{\mathcal{A}}_L(c)$ is a compact Lipschitz graph over its projection too. It is invariant, being the intersection of invariant sets. It is less obvious from this description, but true, that $\tilde{\mathcal{A}}_L(c)$ is non-empty.

Let us denote by $\mathbb{V}_L$ and $\breve{\mathbb{V}}_L$ respectively the sets of weak Kam solutions and dual weak solutions for L. The function Φ_{h_L} and $\breve{\Phi}_{h_L}$ are inverse to each other when restricted to these sets. More precisely,

$$\Phi_{h_L} \circ \breve{\Phi}_{h_L|\mathbb{V}_L} = \mathrm{id}, \qquad \breve{\Phi}_{h_L} \circ \Phi_{h_L|\breve{\mathbb{V}}_L} = \mathrm{id}.$$

This is due to the formulas (4.7). A pair of the type $(\mathcal{G}, \breve{\Phi}_{h_L}(\mathcal{G})) \in \mathbb{V} \times \breve{\mathbb{V}}$ is, up to a constant, a conjugate weak Kam pair in the sense of Fathi (see [12]). Indeed, we see from Proposition 4.2(v) that if u and $\breve{u}$ are such that $(\mathcal{G}_{c,u}, \mathcal{G}_{c,\breve{u}}) \in \mathbb{V} \times \breve{\mathbb{V}}$ and $\mathcal{G}_{c,\breve{u}} = \breve{\Phi}_{h_L}(\mathcal{G}_{c,u})$, then $u - \breve{u}$ is constant on the Aubry set of $\mathcal{G}$ (this constant is zero if we choose $\breve{u} = \breve{T}_{h_L} T_{h_L} u$).

The following property (which has been used in the proof of Proposition 3.2) tells us that weak Kam solutions may be seen as a sort of unstable manifolds of the Aubry set of L, and dual weak Kam solutions as stable manifolds. For the proof we refer to [3, Proposition 4.3].

Proposition 4.10. *For every c-weak kam solution $\mathcal{G}$ and every $z \in \mathcal{G}$, the α-limit of z for ϕ_H^1 is contained in $\tilde{\mathcal{A}}(c)$. Analogously, every point in a dual c-weak Kam solution is ω-asymptotic to $\tilde{\mathcal{A}}(c)$.*

Let us now give one of the possible definitions of the Mather set $\tilde{\mathcal{M}}_L(c)$: it is the union of the supports of the invariant measures for ϕ_H^1 which are contained in $\tilde{\mathcal{A}}(c)$. It is a compact invariant set. Finally, the following is one of the possible definitions of the Mañé set $\tilde{\mathcal{N}}_L(c)$:

$$\tilde{\mathcal{N}}_L(c) = \bigcup \big\{\mathcal{G}_{|\mathcal{I}_{h_L}(\mathcal{G})} : \mathcal{G} \text{ is a } c\text{-weak Kam solution}\big\}.$$

This also can be proved to be a compact invariant set. We have

$$\tilde{\mathcal{M}}_L(c) \subseteq \tilde{\mathcal{A}}_L(c) \subseteq \tilde{\mathcal{N}}_L(c) \subseteq T^*M.$$

We refer to [12], [2] or [17] for a detailed analysis.

4.4 The semigroup Σ_c^∞

In this section we somehow generalize the previous subsection to the case of more than one Tonelli Hamiltonian. Let us recall that our final aim is to get informations about the forcing relation $\vdash_{\mathcal{F}}$, in order to apply Proposition 3.2. We notice that the properties in Proposition 4.1 of the time-one action of a Tonelli Lagrangian, and in particular (*iv*), point in that direction. Proposition 4.6 tells us that these nice properties are preserved by addition of constants, minimums and compositions. Let us then call σ the class of all those families of costs which can be obtained starting from the time-one actions $A_L, L \in \mathcal{F}$, through a finite number of these operations; for instance, if $L_1, L_2 \in \mathcal{F}$, then σ contains $A^n_{L_1}$, $\min\{A^n_{L_1} : N \leq n \leq N'\}$, $A^{n_2}_{L_2} \circ A^{n_1}_{L_1}$, $\min\{A_{L_1} + \lambda_1, A_{L_2} + \lambda_2\}$, and so on, for all possible choices of parameters $n, N, \ldots$. Of course, σ contains $\{A_L : L \in \mathcal{F}\}$ and is closed under addition of constants, minimums and compositions. By Proposition 4.6 we immediately deduce:

Proposition 4.11. *Every family $A \in \sigma$ is a $\mathcal{F}$-family according to Definition 4.3. Hence, all the conclusions of Remark 4.4 apply to A, and in particular we have*

$$\mathcal{G}_{|\mathcal{I}_A(\mathcal{G})} \ \vdash_{\mathcal{F}} \ \Phi_A(\mathcal{G}) \qquad \forall\, \mathcal{G} \in \mathbb{P}.$$

We define $\Sigma = \{\Phi_A : A \in \sigma\}$, where Φ_A is the operator defined by the formula (4.4). By the formula $\Phi_{A'} \circ \Phi_A = \Phi_{A' \circ A}$, Σ is a semigroup with respect to the composition.

For a given $c \in H^1(M, \mathbb{R})$, we define $\sigma_c = \{A_c : A \in \sigma\}$. Let σ_c^∞ be the closure of σ_c in $C(M \times M)$. Observe that the elements of σ_c^∞ are costs and not families of costs. It is clear that σ_c^∞ is the smallest class containing $\{A_{L,c} : L \in \mathcal{F}\}$ and closed under addition of constants, minimums, compositions and uniform limits. Let us point out the important fact that the Peierls barriers $h_{L,c}$ belong to σ_c^∞ as well, since the limits involved in the definition are uniform (see the discussion after relation (4.12)).

In order to have good compactness properties, we will often make the assumption that $\mathcal{F}$ is equi-semiconcave, according to the following definition:

Definition 4.12. *We say that the family $\mathcal{F}$ is* equi-semiconcave *if, for every fixed c, the time-one actions $\{A_{L,c} : L \in \mathcal{F}\}$ form an equi-semiconcave set of functions on $M \times M$.*

Of course, a finite family $\mathcal{F}$ composed by Tonelli Hamiltonians is equi-semiconcave.

Let us now assume $\mathcal{F}$ equi-semiconcave. The elements of σ_c^∞ are then equi-semiconcave by the estimates in Proposition 4.6, hence equi-Lipschitz. By Arzeli-Ascolà theorem, σ_c^∞ is then closed under pointwise limits too. Being closed under minimums, it is also closed under countable inf and liminf, provided that they are finite. Being a separable space, it is actually closed under arbitrary inf and liminf, provided that they are finite.

Let us fix $c \in H^1(M,\mathbb{R})$. Every element $A \in \sigma_c^\infty$ is a cost and not a family of costs, hence in general it will not be associated to an operator from $\mathbb{P}$ to $\mathbb{P}$. Nevertheless, we can still define the operator $\Phi_A \colon \mathbb{P}_c \to \mathbb{P}_c$ by

$$\Phi_A(\mathcal{G}_{c,u}) = \mathcal{G}_{c,T_A u}.$$

Since the costs in σ_c^∞ are semiconcave, the image $\Phi_A(\mathbb{P}_c)$ is really contained in $\mathbb{P}_c$ (cf. Remark. 4.4(iii)). We define

$$\Sigma_c^\infty = \{\Phi_A \colon \mathbb{P}_c \to \mathbb{P}_c : A \in \sigma_c^\infty\}.$$

This is a semigroup. The action of Σ_c^∞ gives a dynamics on $\mathbb{P}_c$, which encodes informations about the dynamics on T^*M generated by the time-one maps ϕ_H, $H \in \mathcal{F}$. Moreover, we can identify Σ_c^∞ with σ_c^∞ modulo addition of constants. It then becomes a subset of a normed space by

$$\|\Phi_A\| = |A| = \frac{\max A - \min A}{2}.$$

Let us state some other properties. Item *(iv)* in the following Proposition is a sort of "shadowing" property.

Proposition 4.13. *Let $\mathcal{F}$ be equi-semiconcave.*

(i) The composition

$$\begin{aligned} \circ \colon \Sigma_c^\infty \times \Sigma_c^\infty &\to \Sigma_c^\infty \\ (\Phi', \Phi) &\mapsto \Phi' \circ \Phi \end{aligned}$$

is continuous.

(ii) for every $\Phi, \Phi' \in \Sigma_c^\infty$ and $\mathcal{G}, \mathcal{G}' \in \mathbb{P}_c$, it holds

$$\|\Phi(\mathcal{G}) - \Phi'(\mathcal{G}')\|_{\mathbb{P}} \leq \|\mathcal{G} - \mathcal{G}'\|_{\mathbb{P}} + \|\Phi - \Phi'\| \tag{4.13}$$

and in particular every Φ is 1-Lipschitz.

(iii) The function $\mathcal{I}_\Phi(\mathcal{G})$ is upper-semicontinuous in both $\Phi \in \Sigma_c^\infty$ and $\mathcal{G} \in \mathbb{P}$.

(iv) For all $\Phi \in \Sigma_c^\infty, \mathcal{G} \in \mathbb{P}_c$ and $\mathbb{U}$ neighborhood of $\Phi(\mathcal{G})$ in $\mathbb{P}_c$ there exists $\Phi' \in \Sigma$ such that $\Phi'(\mathcal{G}) \in \mathbb{U}$ (in particular $\mathcal{G} \vdash_{\mathcal{F}} \mathbb{U}$ by Proposition 4.11).

Proof. Item *(i)* is a direct consequence of the continuity of the composition between costs. Item *(ii)* is analogous to the estimate (4.5). Item *(iii)* is an easy consequence of Proposition 4.2 (iv). For item *(iv)*, it suffices to choose $A \in \sigma_c$ such that Φ_A is close enough to Φ. The conclusion follows then from item *(ii)*. □

Proposition 4.14. *If $\mathcal{F}$ is an equi-semiconcave family, then σ_c^∞ is an equi-semiconcave set of functions and Σ_c^∞ is compact.*

Proof. We already pointed out the equi-semiconcavity of σ_c^∞. This implies equi-Lipschitzianity. Since everything is up to additive constants, the compactness follows from Ascoli-Arzelà Theorem. $\square$

In the next proposition we gather some properties of the minimal subsets of the dynamical system $(\mathbb{P}_c, \Sigma_c^\infty)$ which will be needed in the next section. We recall that a minimal subset is a compact subset of $\mathbb{P}_c$ which is stable by the semigroup Σ_c^∞ and which does not contain any proper subset with the same properties. For compact spaces the existence of minimal subsets is a standard Zorn's Lemma argument (and actually, for compact metric spaces the Zorn's Lemma is not needed, see the proof in [13]), but our space $\mathbb{P}_c$ is not compact, so this does not apply. Nevertheless, the argument can be easily adapted when Σ_c^∞ is compact, as we are going to show.

The existence of minimal components is the unique point in the mechanism where the equi-semiconcavity of $\mathcal{F}$ seems crucial. Anyway, it is perhaps possible to drop this assumption, at the cost of slightly complicating the construction in the next section. In Corollary 5.10 the assumption will be eventually dropped for the case $d = 1$. See also Remark 5.12.

Proposition 4.15. *Assume $\mathcal{F}$ is equi-semiconcave (so that Σ_c^∞ is compact by Proposition 4.14). Then:*

(i) there exists a minimal set; more precisely, for every $\mathcal{G} \in \mathbb{P}_c$ its orbit $\{\Phi(\mathcal{G}) : \Phi \in \Sigma^\infty\}$ contains a minimal set;

(ii) for every $\mathcal{G} \in \mathbb{P}_c$ there exists $\Phi \in \Sigma_c^\infty$ such that $\Phi(\mathcal{G})$ belongs to a minimal set;

(iii) $\mathcal{G} \in \mathbb{P}_c$ belongs to a minimal component $\mathbb{M}$ if and only if for every $\Phi \in \Sigma_c^\infty$ there exists $\Phi' \in \Sigma_c^\infty$ such that $\Phi'\Phi(\mathcal{G}) = \mathcal{G}$; in this case, $\mathbb{M}$ coincides with the orbit of $\mathcal{G}$; in particular, every minimal component $\mathbb{M}$ is transitive: for every $\mathcal{G}, \mathcal{G}' \in \mathbb{M}$ there exists $\Phi \in \Sigma_c^\infty$ such that $\Phi(\mathcal{G}) = \mathcal{G}'$.

Proof.

(i) Given $\mathcal{G} \in \mathbb{P}_c$, its orbit is invariant and compact because Σ_c^∞ is a compact semigroup. By a general result in topological dynamics, it contains a minimal set;

(ii) this is immediate from (i);

(iii) let $\mathcal{G} \in \mathbb{M}$ with $\mathbb{M}$ minimal, and consider $\Phi \in \Sigma_c^\infty$. The orbit of $\Phi(\mathcal{G})$ contains a minimal component by the proof of (i), and is contained in $\mathbb{M}$ because $\mathbb{M}$ is invariant. By minimality of $\mathbb{M}$, the orbit has to coincide with $\mathbb{M}$. Viceversa, suppose that for every $\Phi \in \Sigma_c^\infty$ there exists $\Phi' \in \Sigma_c^\infty$ such that $\Phi'\Phi(\mathcal{G}) = \mathcal{G}$. We know that the orbit of $\mathcal{G}$ contains a minimal set $\mathbb{M}$ by (i). The assumption says that every invariant set contained in the orbit of $\mathcal{G}$ must contain $\mathcal{G}$ as well. We deduce that $\mathbb{M}$ coincides with the orbit of $\mathcal{G}$. $\square$

In order to have a better understanding of the operators in Σ_c^∞ and the minimal components of $\mathbb{P}_c$, let us now further investigate about these objects in some special cases.

- *Case* $\mathcal{F} = \{L\}$. This was the case of the previous subsection. In addition to what already said there, one can show that Σ_c^∞ is commutative in this case, and that $\Phi\Phi_{h_{L,c}} = \Phi_{h_{L,c}}\Phi = \Phi_{h_{L,c}}$ for every $\Phi \in \Sigma_c^\infty$. It is then easy to verify that $\mathbb{M}$ is a minimal component if and only if $\mathbb{M} = \{\mathcal{G}\}$ for some c-weak Kam solution $\mathcal{G}$.

- *Commuting Hamiltonians.* If the Hamiltonians in $\mathcal{F}$ commute with each other, i.e. their Poisson bracket satisfies

$$\{H, G\} + \partial_t H - \partial_t G = 0 \qquad \forall\, H, G \in \mathcal{F},$$

  then it is known (see [10] for the time-periodic case and [11, 22] for the autonomous case) that the associated Lax-Oleinik semigroups commute and that the Hamiltonians in the family share the same weak Kam solutions and the same Peierls barrier $\{h_c\}_c$. Thus Σ_c^∞ is commutative and $\Phi\Phi_{h_c} = \Phi_{h_c}\Phi = \Phi_{h_c}$ for every $\Phi \in \Sigma_c^\infty$. In particular, the minimal components are exactly the c-weak Kam solutions of one (hence all) Hamiltonian in $\mathcal{F}$.

- *General case.* For every $\Phi_A \in \Sigma_c^\infty$ it is possible to define an analogous of the Peierls barrier. Indeed, arguing as for the case $A = A_{L,c}$, one can show (see [23]) that there exists a unique real number α_A such that the liminf

$$h_A = \liminf_{n\to+\infty} A^n + n\alpha_A \tag{4.14}$$

  is real-valued. Exactly as for the Peierls barrier, we have $\Phi_{h_A} \in \Sigma_c^\infty$, and analogous statements to Propositions 4.7 and 4.8*(ii)-(iii)-(iv)* hold. In particular the image of Φ_{h_A} coincides with its fixed points and with the fixed points of Φ_A.

  Even if the general operator in Σ_c^∞ seems to be quite obscure, something can be said for particular ones. We discuss the properties of two of them in the next propositions. Let us pick two Hamiltonians H_1 and H_2 in $\mathcal{F}$, and call A_1, A_2 their c-time-one actions and h_1, h_2 their c-Peierls barriers (we omit the subscript c). The two operators in Σ_c^∞ which we discuss in the propositions are those associated to the costs $A_2 \circ A_1$ and $h_2 \circ h_1$.

  As for the minimal components, they are also quite obscure. It is easy to verify that if $\mathcal{G}$ is a weak Kam solution common to every Hamiltonian in $\mathcal{F}$, then $\mathcal{G}$ is fixed by the whole Σ_c^∞ and is thus a minimal component. Another easy property is that every minimal component $\mathbb{M}$ must contain a fixed point of every $\Phi_A \in \Sigma_c^\infty$: indeed, by invariance we have $\Phi_{h_A}(\mathbb{M}) \subseteq \mathbb{M}$, and, by the properties of h_A, the image of Φ_{h_A} consists of the fixed points of Φ_A.

Proposition 4.16. *Let $H_1, H_2 \in \mathcal{F}$, and call A_1, A_2 their time-one actions and ϕ_1, ϕ_2 their time-one maps. Let us also denote $A = A_2 \circ A_1$ and $\phi = \phi_2 \circ \phi_1$. Let us consider the operator Φ_A. The following hold true:*

(i) for every $\mathcal{G} \in \mathbb{P}_c$ and every $n \in \mathbb{N}$ it holds

$$\phi^{-n}\big(\Phi_{A^n}(\mathcal{G})\big) \subseteq \mathcal{G}_{|\mathcal{I}_{A^n}(\mathcal{G})};$$

(ii) the fixed points $\mathcal{G}$ of Φ_A are invariant in the past with respect to ϕ; more precisely, they satisfy

$$\phi^{-n}(\mathcal{G}) \subseteq \mathcal{G}_{|\mathcal{I}_{A^n}(\mathcal{G})}.$$

(iii) for every fixed point $\mathcal{G}$ of Φ_A, the set $\mathcal{G}_{|\mathcal{I}_{h_A}(\mathcal{G})}$ is invariant in the past and in the future with respect to ϕ;

(iv) every point in $\mathcal{G}$ is α-asymptotic to $\mathcal{G}_{|\mathcal{I}_{h_A}(\mathcal{G})}$ with respect to ϕ.

Proof.

(i) This is a more precise version of the relation

$$\mathcal{G}_{|\mathcal{I}_{A^n}(\mathcal{G})} \;\vdash_{\mathcal{F}}\; \Phi_{A^n}(\mathcal{G})$$

of Remark 4.4 (*iv*). It follows by a refinement of the proof of Proposition 4.5, using property (*iv*) in Proposition 4.1;

(ii) it is immediate from item (*i*) since by definition a fixed point satisfies $\Phi_A(\mathcal{G}) = \mathcal{G}$;

(iii) let $\mathcal{G}$ be a fixed point of Φ_A. It follows from item (*ii*) that the set $\cap_n \phi^{-n}(\mathcal{G})$, if non-empty, is invariant both in the past and in the future. Hence it suffices to show that this intersection is equal to $\mathcal{G}_{|\mathcal{I}_{h_A}(\mathcal{G})}$. For this aim, let us first notice that

$$\mathcal{I}_{h_A}(\mathcal{G}) = \bigcap_n \mathcal{I}_{A^n}(\mathcal{G}) \tag{4.15}$$

Indeed, from $h_A \circ A^n = h_A$ and relations 4.10, it follows that the first set is smaller than the second. For the reverse inclusion, write $\mathcal{G} = \mathcal{G}_{c,u}$, consider $\bar{y}$ belonging to the intersection in the right-hand side and let $x_n \in M$ be such that $u(x_n) = u(\bar{y}) + A^n(\bar{y}, x_n) + n\alpha_A$. Then by definition of h_A every accumulation point x of the sequence x_n satisfies

$$u(x) \geq u(\bar{y}) + h_A(\bar{y}, x)$$

and since $u(x) = \min_y\{u(y) + h_A(y,x)\}$, we get that the minimum is achieved in $\bar{y}$, thus $\bar{y} \in \mathcal{I}_{h_A}(\mathcal{G})$. This proves (4.15). In order to conclude the proof of item (*iii*), it suffices to prove that

$$\bigcap_n \phi^{-n}(\mathcal{G}) = \mathcal{G}_{|\bigcap_n \mathcal{I}_{A^n}(\mathcal{G})}.$$

The first set is smaller than the second by item (*ii*). The reverse inclusion follows from the fact that, if $z \in \mathcal{G}_{|\mathcal{I}_{A^{n+1}}(\mathcal{G})}$, then $\phi^n(z) \in \mathcal{G}$. This follows from property (*iv*) in Proposition 4.1 and a refinement of the proof of Proposition 4.5.

(iv) Let $z \in \mathcal{G}$. By item (*ii*), for every $N \in \mathbb{N}$ the sequence $\phi^{-n}(z)$ lies in $\mathcal{G}_{|\mathcal{I}_{A^N}(\mathcal{G})}$ for n big enough. Since this is a closed set, every α-limit of the sequence stay in it. Taking the intersection over N gives the result, by equation 4.15. $\square$

Proposition 4.17. *Let $H_1, H_2 \in \mathcal{F}$, and call h_1, h_2 their Peierls barriers. Let us also denote $A_c = h_{2,c} \circ h_{1,c}$, for a fixed c. Let us consider the operator $\Phi_{A,c} \in \Sigma_c^\infty$ and the subsets $\mathbb{V}_{1,c}, \mathbb{V}_{2,c}, \mathbb{V}_{A,c} \subset \mathbb{P}_c$ of the fixed points of $\Phi_{h_{1,c}}, \Phi_{h_{2,c}}$ and $\Phi_{A,c}$ respectively. Then $\mathbb{V}_{A,c}$ is contained in $\mathbb{V}_{2,c}$ and is isometric to a subset of $\mathbb{V}_{1,c}$.*

Proof. Obviously $\mathbb{V}_{A,c}$ is contained in the image of $\Phi_{A,c}$, which is contained in the image of $\Phi_{h_{2,c}}$, that is $\mathbb{V}_{2,c}$. Moreover, since $\Phi_{A,c} = \Phi_{h_{2,c}} \circ \Phi_{h_{1,c}}$, we get that $\Phi_{h_{2,c}}$ is a left inverse for $\Phi_{h_{1,c}}$ on $\mathbb{V}_{A,c}$. Since both of them are 1-Lipschitz (cf. Proposition 4.13 (ii)), $\Phi_{h_{1,c}}$ is an isometry between $\mathbb{V}_{A,c}$ and $\Phi_{h_{1,c}}(\mathbb{V}_{A,c})$, which is a subset of $\mathbb{V}_{1,c}$. $\square$

Let us point out that, if $d = 1$, the whole of Σ_c^∞ would not be needed for the purposes of this article. Indeed, the heuristic discussion in Section 5.2 as well as the proof of Proposition 5.9 show that the Peierls barrier operators $\Phi_{h_L}, L \in \mathcal{F}$, would suffice to get optimal results. Nevertheless, if $d > 1$, considering the whole of Σ_c^∞ should in principle give strictly stronger (even if more abstract) results.

5 The Mather mechanism

We are now going to describe the Mather mechanism for the construction of diffusion polyorbits. This section is organized as follows: we first prove the technical results which are at the core of the Mather mechanism. Then we heuristically show how they can be applied to polysystems of twist maps. Finally we prove a general theorem and discuss some consequences and applications.

Loosely speaking, the mechanism works in the following way: we will be able to associate to every $c \in H^1(M, \mathbb{R})$ a subspace $R(c) \subseteq H^1(M, \mathbb{R})$ of "allowed cohomological directions" for the forcing relation $\dashv\vdash_{\mathcal{F}}$ (and hence for the diffusion, in view of Proposition 3.2). The obstruction for this subspace to be large will be, roughly, the homological size of the sets $\mathcal{I}_\Phi(\mathcal{G})$, for $\mathcal{G} \in \mathbb{P}_c$ and $\Phi \in \Sigma_c^\infty$.

Throughout the whole section, the family $\mathcal{F}$ is assumed equi-semiconcave in the sense of Definition 4.12, unless otherwise stated. For a subset $S \subseteq M$, we call $S^\perp \subseteq \Omega$ the vector subspace of the smooth closed one-forms whose support is disjoint from S and $[S^\perp]$ its projection on $H^1(M, \mathbb{R})$. It follows from the finite dimensionality of $H^1(M, \mathbb{R})$ that there always exists an open set $U \supseteq S$ such that $[U^\perp] = [S^\perp]$. Such a U will be called an *adapted neighborhood* of S. Let us point out that, if $M = \mathbb{T}$, we have $[S^\perp] = \{0\}$ if and only if $S = \mathbb{T}$, and otherwise $[S^\perp] = H^1(\mathbb{T}, \mathbb{R}) \cong \mathbb{R}$.

5.1 The basic step

Let us introduce some notations: for $c \in H^1(M,\mathbb{R})$, $\mathcal{G} \in \mathbb{P}_c, \Phi \in \Sigma_c^\infty$, we define

$$R_\Phi(\mathcal{G}) = [\mathcal{I}_\Phi(\mathcal{G})^\perp] = [\mathcal{G} \wedge \breve{\Phi}\Phi(\mathcal{G})^\perp] \subseteq H^1(M,\mathbb{R}).$$

Here the second equality follows from 4.6. More generally, for $\Phi_1, \ldots, \Phi_n \in \Sigma_c^\infty$, we define

$$R_{\Phi_1,\ldots,\Phi_n}(\mathcal{G}) = \Big[\mathcal{I}_{\Phi_n\circ\cdots\circ\Phi_1}(\mathcal{G})^\perp + \mathcal{I}_{\Phi_n\circ\cdots\circ\Phi_2}(\Phi_1(\mathcal{G}))^\perp + \cdots + \mathcal{I}_{\Phi_n}\big(\Phi_{n-1}\circ\cdots\circ\Phi_1(\mathcal{G})\big)^\perp\Big].$$

We will see in Proposition 5.3 that the subspace $R_{\Phi_1,\ldots,\Phi_n}(\mathcal{G})$ should be intended as a subspace of "allowed cohomological directions for the forcing relation, through the composition $\Phi_n \circ \cdots \circ \Phi_1$, starting from $\mathcal{G}$". By taking the union over all finite strings $(\Phi_1, \ldots, \Phi_n)$, one should get a space of "allowed cohomological directions for the forcing relation starting from $\mathcal{G}$". Afterward, by intersecting over all $\mathcal{G}$ in $\mathbb{P}_c$, one should get a space of "allowed cohomological directions for the forcing relation starting from c", which is basically what we are looking for in order to apply Proposition 3.2. This motivates the following definitions:

$$\begin{aligned} R(\mathcal{G}) &= \bigcup_{\substack{\Phi_1,\ldots,\Phi_n \in \Sigma_c^\infty \\ n\in\mathbb{N}}} R_{\Phi_1,\ldots,\Phi_n}(\mathcal{G}) \\ R(c) &= \bigcap_{\mathcal{G}\in\mathbb{P}_c} R(\mathcal{G}) \end{aligned} \tag{5.1}$$

Notice that at this stage it is not clear weather $R(\mathcal{G})$ or $R(c)$ are vector subspaces. In Proposition 5.5 several equivalent expressions for $R(c)$ will be given. They will imply that $R(c)$ is indeed a vector subspace, and $R(\mathcal{G})$ is a vector subspace for every $\mathcal{G}$ in a minimal component of $\mathbb{P}_c$.

We shall write $R_\mathcal{F}(c)$ when we want to emphasize the dependence on the family $\mathcal{F}$. For a vector subspace $V \subseteq H^1(M,\mathbb{R})$, we denote the ε-radius ball centered at the origin by $B_\varepsilon(V)$.

The following lemma is the basic key step in the accomplishment of the Mather mechanism. Indeed, given a family of costs $A \in \sigma$, the lemma shows how a pseudograph $\mathcal{G}$ may force nearby cohomologies, with the set $\mathcal{I}_A(\mathcal{G})$ acting as an obstruction to this phenomenon. Furthermore, the semicontinuity in $\mathcal{G}$ of $\mathcal{I}_A(\mathcal{G})$ allows to extend the conclusion to a whole neighborhood of $\mathcal{G}$.

Lemma 5.1. *Let A be a $\mathcal{F}$-family of costs according to the Definition 4.3 (in particular, $A \in \sigma$ will work). Let Φ_A be the associated operator on pseudographs. Then, for every $\mathcal{G} \in \mathbb{P}$ and for every neighborhood $\mathbb{U}$ of $\Phi_A(\mathcal{G})$ in $\mathbb{P}$ there exist $N \in \mathbb{N}$, a neighborhood $\mathbb{W}$ of $\mathcal{G}$ and an $\varepsilon > 0$ such that:*

$$\forall\, \mathcal{G}' \in \mathbb{W}, \quad \forall\, c \in c(\mathcal{G}') + B_\varepsilon R_{\Phi_A}(\mathcal{G}) \qquad \exists\, \mathcal{G}'' \ \textit{ such that}$$
$$\mathcal{G}'' \in \mathbb{U}, \qquad \mathcal{G}' \vdash_{N,\mathcal{F}} \mathcal{G}'', \qquad c(\mathcal{G}'') = c.$$

Proof. Let us fix $\mathcal{G}$, $\mathbb{U}$ and an adapted neighborhood U of $\mathcal{I}_A(\mathcal{G})$. The set function $\mathcal{G} \mapsto \mathcal{I}_A(\mathcal{G})$ is upper semicontinuous by Remark 4.4 (ii), so that there exists a neighborhood $\mathbb{W}'$ of $\mathcal{G}$ such that $\mathcal{I}_A(\mathcal{G}') \subseteq U$ for all $\mathcal{G}' \in \mathbb{W}'$. Moreover, by continuity of Φ, we can suppose that $\Phi(\mathbb{W}') \subseteq \mathbb{U}$. The function

$$\mathbb{P} \times U^{\perp} \ni (\mathcal{G}, \nu) \mapsto \mathcal{G} + \mathcal{G}_{\nu,0}$$

is continuous, hence there exists a neighborhood $\mathbb{W}$ of $\mathcal{G}$ and a neighborhood W of 0 in $U^{\perp}$ such that $\mathbb{W} + \mathcal{G}_{W,0} \subseteq \mathbb{W}'$. Projections are open maps, so the projection of W on the cohomology contains a ball $B_\varepsilon[U^{\perp}]$ centered at 0. With these choices of $\mathbb{W}$ and ε, let $\mathcal{G}' \in \mathbb{W}$ and $c \in c(\mathcal{G}') + B_\varepsilon[U^{\perp}]$. We can then take as $\mathcal{G}''$ the pseudograph $\Phi_A(\mathcal{G}' + \mathcal{G}_{\nu,0})$ where $\nu \in W$ satisfies $[\nu] = c - c(\mathcal{G}')$. Indeed, by Remark 4.4(iv) we find N such that

$$\mathcal{G}' \ \vdash_{0,\mathcal{F}} \ \mathcal{G}'_{|U} = \big(\mathcal{G}' + \mathcal{G}_{\nu,0}\big)_{|U} \ \vdash_{N,\mathcal{F}} \ \Phi_A\big(\mathcal{G}' + \mathcal{G}_{\nu,0}\big) = \mathcal{G}''. \qquad \square$$

The Lemma 5.1 easily extends to operators in Σ_c^∞.

Proposition 5.2. *Let $\Phi \in \Sigma_c^\infty$. Then, for every $\mathcal{G} \in \mathbb{P}_c$ and for every neighborhood $\mathbb{U}$ of $\Phi(\mathcal{G})$ in $\mathbb{P}$ there exist $N \in \mathbb{N}$, a neighborhood $\mathbb{W}$ of $\mathcal{G}$ and an $\varepsilon > 0$ such that:*

$$\begin{gathered} \forall\, \mathcal{G}' \in \mathbb{W}\,, \ c \in c(\mathcal{G}') + B_\varepsilon R_\Phi(\mathcal{G}) \qquad \exists\, \mathcal{G}'' \quad \textit{such that} \\ \mathcal{G}'' \in \mathbb{U}, \qquad \mathcal{G}' \ \vdash_{N,\mathcal{F}} \ \mathcal{G}'', \qquad c(\mathcal{G}'') = c. \end{gathered}$$

Proof. Let us fix $\mathcal{G}$ and $\mathbb{U}$, and let us consider $\mathcal{I}_\Phi(\mathcal{G})$ and one of its adapted neighborhoods U. By Proposition 4.13 there exists $A \in \sigma$ such that $\Phi_A(\mathcal{G}) \in \mathbb{U}$ and $\mathcal{I}_A(\mathcal{G}) \subseteq U$. This implies

$$R_{\Phi_A}(\mathcal{G}) = [\mathcal{I}_A(\mathcal{G})^{\perp}] \supseteq [U^{\perp}] = [\mathcal{I}_\Phi(\mathcal{G})^{\perp}] = R_\Phi(\mathcal{G}).$$

We apply the previous proposition and we get the result. $\square$

In the following proposition we prove two similar results which show how Proposition 5.2 has a good behavior under composition. The second version is in principle stronger but, at least for an equi-semiconcave family $\mathcal{F}$, the first version would eventually lead to the same results. Therefore *a posteriori* the second version is not strictly needed here.

The main point in both results is that, if we compose several operators in Σ_c^∞, the set of allowed directions which we get is greater than just the union of the allowed directions obtained by applying separately Proposition 5.2 to each operator. In fact, we obtain the vector subspace generated by this union.

Proposition 5.3. *Let $\Phi_1, \dots, \Phi_n \in \Sigma_c^\infty$. Then: for every $\mathcal{G} \in \mathbb{P}$ and for every neighborhood $\mathbb{U}$ of $\Phi_n \circ \cdots \circ \Phi_1(\mathcal{G})$ in $\mathbb{P}$ there exist $N \in \mathbb{N}$, a neighborhood $\mathbb{W}$ of $\mathcal{G}$ and an $\varepsilon > 0$ such that:*

$$\begin{gathered} \forall\, \mathcal{G}' \in \mathbb{W}, \ \forall\, c \in c(\mathcal{G}') + B_\varepsilon\Big(R_{\Phi_1}(\mathcal{G}) + R_{\Phi_2}(\Phi_1(\mathcal{G})) + \cdots + R_{\Phi_n}\big(\Phi_{n-1} \circ \cdots \circ \Phi_1(\mathcal{G})\big)\Big) \\ \exists\, \mathcal{G}'' : \qquad \mathcal{G}'' \in \mathbb{U}, \qquad \mathcal{G}' \ \vdash_{N,\mathcal{F}} \ \mathcal{G}'', \qquad c(\mathcal{G}'') = c. \end{gathered}$$

Stronger version. *Under the same assumptions,*

$$\forall\, \mathcal{G}' \in \mathbb{W}, \quad \forall\ c \in \ \ c(\mathcal{G}') + B_\varepsilon R_{\Phi_1,\dots,\Phi_n}(\mathcal{G})$$
$$\exists\, \mathcal{G}'' : \quad \mathcal{G}'' \in \mathbb{U}, \qquad \mathcal{G}' \ \vdash_{N,\mathcal{F}}\ \mathcal{G}'', \qquad c(\mathcal{G}'') = c.$$

Proof. We suppose for simplicity $n = 2$. The result is obtained by applying two times the Proposition 5.2 and by noticing that if Z', Z'' are linear subspaces of a normed space and $\varepsilon', \varepsilon'' > 0$, then $B_{\varepsilon'} Z' + B_{\varepsilon''} Z''$ contains $B_\varepsilon(Z' + Z'')$ for some $\varepsilon > 0$. □

Proof of the stronger version. Let us suppose $n = 2$ for simplicity. Let us consider $\mathcal{G} \in \mathbb{P}$ and a neighborhood $\mathbb{U}$ of $\Phi_2\Phi_1(\mathcal{G})$. Let U_1 and U_2 be adapted neighborhoods in M of $\mathcal{I}_{\Phi_2\Phi_1}(\mathcal{G})$ and $\mathcal{I}_{\Phi_2}(\Phi_1(\mathcal{G}))$ respectively. By Proposition 4.13, there exist $A_1, A_2 \in \sigma$ and a neighborhood $\mathbb{W}'$ of $\mathcal{G}$ in $\mathbb{P}$ such that

$$\Phi_{A_2}\Phi_{A_1}(\mathcal{G}') \in \mathbb{U}, \quad \mathcal{I}_{A_2 \circ A_1}(\mathcal{G}') \subseteq U_1 \quad \text{and} \quad \mathcal{I}_{A_2}(\Phi_{A_1}(\mathcal{G}')) \subseteq U_2 \qquad \forall\, \mathcal{G}' \in \mathbb{W}'.$$

Let us now consider $\eta_1 \in U_1^\perp$ and $\eta_2 \in U_2^\perp$. Given $\mathcal{G}' = \mathcal{G}_{\eta,u} \in \mathbb{W}'$, we have

$$\Phi_2\big(\Phi_1(\mathcal{G}' + \mathcal{G}_{\eta_1,0}) + \mathcal{G}_{\eta_2,0}\big) = \mathcal{G}_{\eta+\eta_1+\eta_2,v}$$

with

$$v := T_{A_{2,\eta+\eta_1+\eta_2}} w \qquad w := T_{A_{1,\eta+\eta_1}} u.$$

Let $x \in M$ be a point such that dv_x exists. By Proposition 4.5, if z is a point which realizes the minimum in the formula for $v(x)$, then dw_z exists and

$$dw_z + \eta_z + \eta_{1,z} + \eta_{2,z} \quad \vdash_{N_2,\mathcal{F}} \quad dv_x + \eta_x + \eta_{1,x} + \eta_{2,x}$$

for some $N_2 \in \mathbb{N}$. In the same way, if y realizes the minimum in the formula for $w(z)$, then

$$du_y + \eta_y + \eta_{1,y} \quad \vdash_{N_1,\mathcal{F}} \quad dw_z + \eta_z + \eta_{1,z}$$

for some $N_1 \in \mathbb{N}$.

A generalization of the upper-semicontinuity result in 4.4 (*ii*) shows that if $[\eta_1 + \eta_2] \in B_\varepsilon[U_1^\perp + U_2^\perp]$ with ε small enough, then $y \in U_1$ and $z \in U_2$. We thus have $\eta_{1,y} = 0$ and $\eta_{2,z} = 0$ and therefore

$$du_y + \eta_y \quad \vdash_{N_1+N_2,\mathcal{F}} \quad dv_x + \eta_x + \eta_{1,x} + \eta_{2,x}$$

which is to say

$$\mathcal{G}' = \mathcal{G}_{\eta,u} \quad \vdash_{N_1+N_2,\mathcal{F}} \quad \mathcal{G}_{\eta+\eta_1+\eta_2,v}.$$

The proof is now completed with $\mathcal{G}'' = \mathcal{G}_{\eta+\eta_1+\eta_2,v}$, up to choosing $\mathcal{G}'$ in a smaller neighborhood $\mathbb{W} \subseteq \mathbb{W}'$ in such a way that ε can be fixed independently of $\mathcal{G}'$. □

Notice that, trivially, Lemma 5.1 is a particular case of Proposition 5.2, which in turn is a particular case of Proposition 5.3.

5.2 Heuristic application to twist maps

Even without the main general theorem 5.7 of the next subsection, it is possible at this stage, using just the Proposition 5.3, to derive some results about diffusion in polysystems of exact-symplectic twist maps on the cylinder. The discussion in this subsection will just be an heuristic one, even if everything could be made rigorous. The corresponding rigorous results will be proven in greater generality in the next subsection (Proposition 5.9 and Corollary 5.10).

Let $\mathcal{F}$ be a family of Tonelli Hamiltonians on $\mathbb{T}\times\mathbb{R}$. For simplicity we assume $\mathcal{F} = \{H_1, H_2\}$, and call L_1, L_2 the corresponding Lagrangians. Let us fix $c \in H^1(M, \mathbb{R})$. We now show that either there exists a circle of cohomology c which is invariant for both H_1 and H_2 (which obviously provides an obstruction to diffusion), or c forces a whole neighborhood of cohomology classes (and thus there exists diffusion in the sense of Proposition 3.2).

Indeed, suppose that such an invariant common circle exists. It is standard that it can be identified with a pseudograph $\mathcal{G}$ which is invariant for both ϕ_{H_1} and ϕ_{H_2}. In particular, by the very definition of forcing relation in Section 3, $\mathcal{G}$ is the only pseudograph forced by $\mathcal{G}$, and thus c is the only cohomology class forced by c.

Vice versa, let us suppose that there does not exist such a common invariant circle. Let us consider $\mathcal{G} \in \mathbb{P}_c$, and let us apply $\Phi_{h_2} \circ \Phi_{h_1}$ to it (h_1 and h_2 are the Peierls barrier of H_1, H_2). By the first version of Proposition 5.3, we get

$$\mathcal{G} \ \vdash_{\mathcal{F}} \ c + B_\varepsilon\big(R_{\Phi_{h_1}}(\mathcal{G}) + R_{\Phi_{h_2}}(\Phi_{h_1}(\mathcal{G}))\big) \qquad \forall\, \mathcal{G} \in \mathbb{P}_c. \tag{5.2}$$

Recall that the image of Φ_{h_1} is contained in $\mathbb{P}$ and consists precisely of the weak Kam solutions for H_1, while the image of $\breve{\Phi}_{h_2}$ is contained in $\breve{\mathbb{P}}$ and consists of the dual weak Kam solutions for H_2. By assumption there do not exist common invariant circles, hence, in view of Proposition 4.9,

$$\Phi_{h_1}(\mathcal{G}) \neq \breve{\Phi}_{h_2}\Phi_{h_2}\Phi_{h_1}(\mathcal{G}).$$

This implies (due to $d = 1$) that $R_{\Phi_{h_2}}(\Phi_{h_1}(\mathcal{G})) = H^1(\mathbb{T}, \mathbb{R})$. Thus the formula (5.2) implies that every $\mathcal{G} \in \mathbb{P}_c$ forces a whole neighborhood of cohomology classes. In that formula, ε depends in principle on $\mathcal{G}$, but one can show that by compactness it is possible to choose it uniformly in $\mathcal{G}$. Therefore c forces a whole neighborhood of cohomology classes, as claimed.

Notice how in this one-dimensional case our construction is optimal, in the following sense: the obstructions to the mechanism (i.e. the "homological size" of the sets $\mathcal{I}_\Phi(\mathcal{G})$) are real obstructions to the diffusion (i.e. the common invariant circles). On the contrary, in $d > 1$ the construction likely gives just sufficient conditions for the diffusion: the obstructions to this mechanism may be circumvented by a different diffusion mechanism.

5.3 A general theorem and some applications

We can summarize the argument used in Subsection 5.2 for the one-dimensional case by saying that we have applied Proposition 5.3 to $\Phi_{h_{L_2}} \circ \Phi_{h_{L_1}}$, and the result turned out to be optimal (so that there were no need to consider any other $\Phi \in \Sigma_c^\infty$). Moreover, switching the order and considering $\Phi_{h_{L_1}} \circ \Phi_{h_{L_2}}$ would have led to the same result. The generalization of this argument to an arbitrary dimension d is not completely straightforward: the choice of the operator could in principle make a difference, and it is much less clear if the allowed directions which we can obtain are optimal or not.

In order to overcome these difficulties, we will adopt a slightly more abstract approach. This will give stronger conclusions, at the cost of a certain difficulty to interpret the obstructions which we will found.

We start with a "raw" result which follows immediately from Proposition 5.3.

Proposition 5.4. *Let c be fixed. For any $\mathcal{G} \in \mathbb{P}_c$ and any finite string $s = (\Phi_1, \dots, \Phi_n)$ of elements of Σ_c^∞, there exist $\varepsilon(\mathcal{G}, s) > 0$ such that*

$$c \vdash_{\mathcal{F}} c + \bigcap_{\mathcal{G} \in \mathbb{P}_c} \bigcup_{\substack{s=(\Phi_1,\dots,\Phi_n) \\ n \in \mathbb{N}}} B_{\varepsilon(\mathcal{G},s)}\big(R_{\Phi_1,\dots,\Phi_n}(\mathcal{G})\big). \tag{5.3}$$

Proof. Recall that $c \vdash_{\mathcal{F}} c'$ if and only if $\mathcal{G} \vdash_{\mathcal{F}} c'$ for all $\mathcal{G} \in \mathbb{P}_c$. The result is then a consequence of Proposition 5.3. □

The general theorem 5.7 will consist in a refined (but at the same time simplified) version of this raw result. Roughly speaking, it will be possible to replace the intersection over $\mathcal{G} \in \mathbb{P}_c$ with an intersection over a smaller set, to replace the union with a sum of vector subspaces and to choose ε uniformly in $\mathcal{G}, \Phi$. This will simplify the right-hand side, and will lead in the end to a unique subspace $R(c) \subseteq H^1(M, \mathbb{R})$ encoding all the information. Moreover, exploiting some semicontinuity, the result will be proved to hold for c' close enough to c; it will also be possible to replace the forcing relation $\vdash_{\mathcal{F}}$ with the mutual forcing relation $\dashv\vdash_{\mathcal{F}}$, and to have a locally uniform control on the N appearing in its definition.

In order to motivate what follows, let us observe that the map $\mathcal{G} \mapsto R(\mathcal{G})$ is non-increasing along the action of elements of Σ_c^∞. More precisely,

$$R(\Phi(\mathcal{G})) \subseteq R(\mathcal{G}) \qquad \forall\, \mathcal{G} \in \mathbb{P}_c, \Phi \in \Sigma_c^\infty. \tag{5.4}$$

This can be interpreted by saying that this map is a sort of multi-valued Lyapunov function for the dynamics in $(\mathbb{P}_c, \Sigma_c^\infty)$. Since we are interested in the set $R(c)$, which is the intersection of all the sets $R(\mathcal{G})$, it is natural to look at the minimal components of the dynamics, whose properties have been analysed in Proposition 4.15.

For a minimal component $\mathbb{M}$ of $(\mathbb{P}_c, \Sigma_c^\infty)$ let us define

$$R(\mathbb{M}) = \bigcap_{\mathcal{G} \in \mathbb{M}} R(\mathcal{G}).$$

Proposition 5.5 (Equivalent expressions for $R(c)$)**.**

(i) We have:

$$R(c) = \bigcap_{\mathbb{M} \text{ minimal}} R(\mathbb{M}).$$

(ii) We have the following equivalent expressions for $R(\mathbb{M})$*:*

$$\begin{aligned} R(\mathbb{M}) = R(\mathcal{G}) &= \sum_{\substack{\Phi_1,\dots,\Phi_n \in \Sigma_c^\infty \\ n \in \mathbb{N}}} R_{\Phi_1,\dots,\Phi_n}(\mathcal{G}) \qquad \text{for any fixed } \mathcal{G} \in \mathbb{M} \\ &= R_{\Phi_1,\dots,\Phi_n}(\mathcal{G}) \qquad \text{for some } \Phi_1,\dots,\Phi_n \text{ depending on } \mathcal{G} \\ &= \sum_{\substack{\mathcal{G} \in \mathbb{M} \\ \Phi \in \Sigma_c^\infty}} R_\Phi(\mathcal{G}). \end{aligned}$$

In particular, $R(\mathbb{M})$ *is a vector subspace for every* $\mathbb{M}$*, and the same holds for* $R(c)$*.*

Proof. Let us prove item (i). By the definition of $R(c)$ and $R(\mathbb{M})$, it is clear that $R(c) \subseteq R(\mathbb{M})$ for every minimal component $\mathbb{M}$, hence $R(c) \subseteq \cap_{\mathbb{M}} R(\mathbb{M})$. For the reverse inclusion, let us notice that, since $\mathcal{F}$ is equi-semiconcave, by Proposition 4.15(ii) for every $\mathcal{G} \in \mathbb{P}_c$ there exists $\Phi \in \Sigma_c^\infty$ such that $\Phi(\mathcal{G})$ belongs to a minimal component. By (5.4),

$$R(\mathcal{G}) \supseteq R(\Phi(\mathcal{G})) \supseteq \bigcap_{\mathbb{M}} R(\mathbb{M}).$$

By taking the intersection over all $\mathcal{G} \in \mathbb{P}_c$, one gets the desired inclusion.

Let us now prove item (ii). Thanks to relation (5.4) and the transitivity of minimal components (Proposition 4.15(iii)), we gets that the function $\mathcal{G} \mapsto R(\mathcal{G})$ is constant on every minimal component. This proves that $R(\mathbb{M}) = R(\mathcal{G})$ for any $\mathcal{G} \in \mathbb{M}$. Moreover, again by transitivity, for every two strings $(\Phi_1, \dots, \Phi_n)$ and $(\Phi'_1, \dots, \Phi'_{n'})$ it holds

$$R_{\Phi_1,\dots,\Phi_n}(\mathcal{G}) + R_{\Phi'_1,\dots,\Phi'_{n'}}(\mathcal{G}) = R_{\Phi_1,\dots,\Phi_n,\Phi,\Phi'_1,\dots,\Phi'_{n'}}(\mathcal{G}) \subseteq R(\mathcal{G})$$

where Φ is any operator in Σ_c^∞ such that $\Phi \circ \Phi_n \circ \cdots \circ \Phi_1(\mathcal{G}) = \mathcal{G}$. This proves that

$$R(\mathcal{G}) \supseteq \sum_{\substack{\Phi_1,\dots,\Phi_n \in \Sigma_c^\infty \\ n \in \mathbb{N}}} R_{\Phi_1,\dots,\Phi_n}(\mathcal{G}) \qquad \forall\, \mathcal{G} \in \mathbb{M}$$

and the opposite inclusion is easy from the definitions. Moreover, since the dimension of $H^1(M, \mathbb{R})$ is finite, this also prove that, for some $n \in \mathbb{N}$,

$$R(\mathcal{G}) = R_{\Phi_1,\dots,\Phi_n}(\mathcal{G}) \qquad \text{for some} \quad \Phi_1, \dots, \Phi_n \in \Sigma_c^\infty.$$

The equality $R(\mathbb{M}) = \sum_{\mathcal{G} \in \mathbb{M}} \sum_{\Phi \in \Sigma_c^\infty} R_\Phi(\mathcal{G})$ follows by similar arguments. □

Let us mention that, starting from the last expression for $R(\mathbb{M})$ above, one can show that considering just the weaker version of Proposition 5.3 would eventually lead to the same results.

Remark 5.6. The function $\mathcal{F} \mapsto R_{\mathcal{F}}(c)$ is increasing. This is natural in view of the interpretation of $R_{\mathcal{F}}(c)$ as a set of allowed directions for diffusion, and follows by an inspection of the definitions (in fact, the map $\mathcal{F} \mapsto \Sigma_c^\infty(\mathcal{F})$ is also increasing). In particular, let us point out that, since $R_{\mathcal{F}}(c)$ is a vector subspace,

$$R_{\mathcal{F}}(c) \supseteq \bigcup_{H \in \mathcal{F}} R_{\{H\}}(c).$$

In general though the inclusion can be strict: we will see that this will be the case for two twist maps with non-common non-contractible invariant circles of cohomology c.

We can now restate and prove Theorem 1.3 of the Introduction, which is a generalization of Theorem 0.11 in [3] to the polysystem case.

Theorem 5.7. *Let $\mathcal{F}$ be a family of one-periodic Tonelli Hamiltonians defined on the cotangent space of a boundaryless compact manifold M. Assume that $\mathcal{F}$ is equi-semiconcave in the sense of Definition 4.12. Let $c \in H^1(M,\mathbb{R})$. Then there exist a neighborhood W of c in $H^1(M,\mathbb{R})$, $\varepsilon > 0$ and $N \in \mathbb{N}$ such that*

$$c' \ \dashv\vdash_{N,\mathcal{F}} \ c' + B_\varepsilon R(c) \qquad \forall\, c' \in W.$$

Proof. We subdivide the proof into four steps.

Step 1. For every $\mathbb{M} \subset \mathbb{P}_c$ minimal and every $\mathcal{G} \in \mathbb{M}$ there exist a neighborhood $\mathbb{W}_{\mathcal{G}}$ of $\mathcal{G}$ in $\mathbb{P}$, a natural number $N_{\mathcal{G}}$ and $\varepsilon_{\mathcal{G}} > 0$ such that

$$\mathcal{G}' \ \vdash_{N_{\mathcal{G}},\mathcal{F}} \ c(\mathcal{G}') + B_{\varepsilon_{\mathcal{G}}} R(c) \qquad \forall\, \mathcal{G}' \in \mathbb{W}_{\mathcal{G}}.$$

Let $\mathbb{M}$ be minimal and $\mathcal{G} \in \mathbb{M}$. Let $\Phi_1, \dots, \Phi_n \in \Sigma_c^\infty$ such that $R_{\Phi_1,\dots,\Phi_n}(\mathcal{G}) = R(\mathbb{M})$. This is possible thanks to Proposition 5.5. Let us then apply Proposition 5.3 to $\mathcal{G}$ and to the composition $\Phi_n \circ \cdots \circ \Phi_1$. Call $\mathbb{W}_{\mathcal{G}}$, $N_{\mathcal{G}}$ and $\varepsilon_{\mathcal{G}}$ the objects yielded by that Proposition. Since $R(c) \subseteq R(\mathbb{M})$, we have

$$\mathcal{G}' \ \vdash_{N_{\mathcal{G}},\mathcal{F}} \ c(\mathcal{G}') + B_{\varepsilon_{\mathcal{G}}} R(c) \qquad \forall\, \mathcal{G}' \in \mathbb{W}_{\mathcal{G}}, \tag{5.5}$$

as desired.

Step 2. For every $\mathcal{G} \in \mathbb{P}_c$ there exist a neighborhood $\mathbb{W}_{\mathcal{G}}$ of $\mathcal{G}$ in $\mathbb{P}$, a natural number $N_{\mathcal{G}}$ and $\varepsilon_{\mathcal{G}} > 0$ such that

$$\mathcal{G}' \ \vdash_{N_{\mathcal{G}},\mathcal{F}} \ c(\mathcal{G}') + B_{\varepsilon_{\mathcal{G}}} R(c) \qquad \forall\, \mathcal{G}' \in \mathbb{W}_{\mathcal{G}}.$$

Let $\mathcal{G} \in \mathbb{P}_c$. By Proposition 4.15, there exists $\Phi \in \Sigma_c^\infty$ such that $\Phi(\mathcal{G})$ is in a minimal component. Moreover, by Proposition 4.13 (iv) there exists $A \in \sigma$ such that $\Phi_A(\mathcal{G}) \in \mathbb{W}_{\Phi(\mathcal{G})}$. By continuity, $\Phi_A(\mathcal{G}') \in \mathbb{W}_{\Phi(\mathcal{G})}$ if $\mathcal{G}'$ is in a small enough neighborhood $\mathbb{W}_{\mathcal{G}}$ of $\mathcal{G}$. By Proposition 4.11 and by Step 1, there exists $N_A \in \mathbb{N}$ such that

$$\mathcal{G}' \quad \vdash_{N_A,\mathcal{F}} \quad \Phi_A(\mathcal{G}') \quad \vdash_{N_{\Phi(\mathcal{G})},\mathcal{F}} \quad c(\mathcal{G}') + B_{\varepsilon_{\Phi(\mathcal{G})}} R(c) \qquad\qquad \forall\, \mathcal{G}' \in \mathbb{W}_{\mathcal{G}}.$$

Thus we can take $N_{\mathcal{G}} = N_A + N_{\Phi(\mathcal{G})}$ and $\varepsilon_{\mathcal{G}} = \varepsilon_{\Phi(\mathcal{G})}$.

Step 3. There exist a neighborhood W' of c in $H^1(M,\mathbb{R})$, a natural number N and $\varepsilon' > 0$ such that

$$c' \quad \vdash_{N,\mathcal{F}} \quad c' + B_{\varepsilon'} R(c) \qquad \forall\, c' \in W'.$$

Let us choose A_0 in σ (no matter which one, for instance $A_0 = A^1_L$ with $L \in \mathcal{F}$ will work). The closure of $\Phi_{A_0}(\mathbb{P}_c)$ is compact, thus we can extract a finite subfamily $\{\mathcal{G}_j\}_j \subseteq \overline{\Phi_{A_0}(\mathbb{P}_c)}$ such that $\mathbb{W} = \cup_j \mathbb{W}_{\mathcal{G}_j}$ covers $\overline{\Phi_{A_0}(\mathbb{P}_c)}$. Moreover, it is true that $\mathbb{W}$ also covers $\Phi_{A_0}(\mathbb{P}_{W'})$ for a sufficiently small neighborhood W' of c. Indeed, consider an arbitrary neighborhood W'' of c. The function $\mathcal{G} \mapsto c(\mathcal{G})$ is continuous on the compact set $\overline{\Phi_{A_0}(\mathbb{P}_{W''})} \setminus \mathbb{W}$, hence its image is compact too. Since c does not belong to this image, we can take as W' the intersection of W'' with the complementary of the image.

In other words, for any $\mathcal{G}' \in \mathbb{P}_{W'}$ there exists $\bar{j}$ such that $\Phi_{A_0}(\mathcal{G}') \in \mathbb{W}_{\mathcal{G}_{\bar{j}}}$. Hence we obtain

$$\mathcal{G}' \quad \vdash_{N_{A_0},\mathcal{F}} \quad \Phi_{A_0}(\mathcal{G}') \quad \vdash_{\max_j N_{\mathcal{G}_j},\mathcal{F}} \quad c(\mathcal{G}') + B_{\min_j \varepsilon_{\mathcal{G}_j}} R(c) \qquad \forall\, \mathcal{G}' \in \mathbb{P}_{W'}.$$

Thus we can take $N = N_{A_0} + \max_j N_{\mathcal{G}_j}$ and $\varepsilon' = \min_j \varepsilon_{\mathcal{G}_j}$, and the Step 3 is proved.

Step 4. There exist a neighborhood W of c in $H^1(M,\mathbb{R})$, a natural number N and $\varepsilon > 0$ such that

$$c' \quad \dashv\vdash_{N,\mathcal{F}} \quad c' + B_{\varepsilon} R(c) \qquad \forall\, c' \in W.$$

In order to obtain the mutual forcing relation starting from the one-side forcing relation of Step 3, it suffices to take $W \subseteq W'$ and $\varepsilon \leq \varepsilon'$ small enough in such a way that $W + B_\varepsilon R(c) \subset W'$. This makes possible to apply the one-side forcing in the opposite direction. This concludes the proof of Step 4 (we keep the same N as in the Step 3) and of the Theorem. □

Remark 5.8. A careful analysis of the proof of the theorem shows that the multi-valued function $c \mapsto R(c)$ is lower-semicontinuous: for any c there exists a neighborhood Z such that $R(c) \subseteq R(c')$ for every $c' \in Z$. Nevertheless, the statement of the theorem is somehow stronger, because in some sense yields semicontinuity also on N and ε.

In the sequel we draw some relations between the subspace $R(c)$ and the underlying Hamiltonian polysystem dynamics.

Proposition 5.9. *Assume $\mathcal{F}$ equi-semiconcave. If there exists a $C^{1,1}$ c-weak Kam solution which is common to all $H \in \mathcal{F}$, then $R(c) = \{0\}$. If $d = 1$ the viceversa holds: if $R(c) = \{0\}$ then all the Hamiltonians in $\mathcal{F}$ have an invariant circle in common.*

Proof. If there exists such a weak kam solution as in the statement, we can identify it with a pseudograph $\mathcal{G} \in \mathbb{P}_c \cap \breve{\mathbb{P}}_c$. It is easy to verify that every $\Phi \in \Sigma_c^\infty$ must then satisfies $\Phi(\mathcal{G}) = \breve{\Phi}(\mathcal{G}) = \mathcal{G}$. The singleton $\{\mathcal{G}\}$ is thus a minimal set for $\mathbb{P}_c$ and, in view of formula (4.6), it satisfies $R(\{\mathcal{G}\}) = \{0\}$.

On the other hand, if $d = 1$ and $R(c) = \{0\}$, then there exists a minimal set $\mathbb{M}$ such that

$$\sum_{\mathcal{G}\in\mathbb{M},\Phi\in\Sigma_c^\infty} [\mathcal{I}_\Phi(\mathcal{G})^\perp] = \{0\},$$

which means, thanks once more to (4.6), that $\mathcal{G} = \breve{\Phi}\Phi(\mathcal{G})$ for every $\mathcal{G} \in \mathbb{M}$ and $\Phi \in \Sigma_c^\infty$. Let us apply this to the operator $\Phi = \Phi_{h_{L,c}}$ associated to the Peierls barrier $h_{L,c}$ of a Lagrangian $L \in \mathcal{F}$. We get

$$\mathcal{G} = \breve{\Phi}_{h_{L,c}}\Phi_{h_{L,c}}(\mathcal{G}) \in \mathrm{Im}(\breve{\Phi}_{h_{L,c}}) \qquad \forall\, L \in \mathcal{F},$$

hence $\mathcal{G}$ is a dual weak kam solution for every $L \in \mathcal{F}$, which in addition belongs to $\mathbb{P}$. This implies the result, by Proposition 4.9. □

We now can restate and prove the Corollary 1.4 about families of exact twist maps. The condition of equi-semiconcavity on $\mathcal{F}$ is dropped.

Corollary 5.10. *Let $M = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let $\mathcal{F}$ be a (non-necessarily equi-semiconcave) family of one-periodic Tonelli Hamiltonians on $T^*M \cong \mathbb{T} \times \mathbb{R}$. Let us make the identification $H^1(\mathbb{T}, \mathbb{R}) \cong \mathbb{R}$. If, for some $A < B \in \mathbb{R}$, the family $\mathcal{F}$ does not admit an invariant common circle with cohomology in $[A, B]$, then:*

(i) there exists an $\mathcal{F}$-polyorbit $(x_n, p_n)_{n\in\mathbb{Z}}$ satisfying $p_0 = A$ and $p_N = B$ for some $N \in \mathbb{N}$;

(ii) for every $H, H' \in \mathcal{F}$ and every $c, c' \in [A, B]$ there exists an $\mathcal{F}$-polyorbit α-asymptotic to the Aubry set $\tilde{\mathcal{A}}_H(c)$ and ω-asymptotic to $\tilde{\mathcal{A}}_{H'}(c')$

(iii) for every sequence $(c_i, H_i, \varepsilon_i)_{i\in\mathbb{Z}} \subset [A, B] \times \mathcal{F} \times \mathbb{R}^+$ there exists an $\mathcal{F}$-polyorbit which visits in turn the ε_i-neighborhoods of the Mather sets $\tilde{\mathcal{M}}_{H_i}(c_i)$.

Proof. If $\mathcal{F}$ is finite, the conclusion is immediate: by Proposition 5.9, $R(c) = \mathbb{R}$ for every $c \in [A, B]$, hence by Theorem 5.7 $[A, B]$ is contained in the same equivalence class for $\dashv\vdash_{\mathcal{F}}$. Therefore Proposition 3.2 applies, and allows to prove the results: for instance, in order to prove item (i) one applies Proposition diffusion (ii) with $\eta \equiv A$ and $\eta' \equiv B$.

If $\mathcal{F}$ is arbitrary, we just reduce to the case of $\mathcal{F}$ finite thanks to the following fact: if the family $\mathcal{F}$ does not admit invariant common circles with cohomology in

$[A, B]$, then there exists a finite subfamily $\mathcal{F}' \subset \mathcal{F}$ with the same property. Indeed, suppose that this is not the case and let us arbitrarily pick H_0 in $\mathcal{F}$: then the set $C(\mathcal{F}')$ defined by

$$\begin{aligned} C(\mathcal{F}') &= \left\{\mathcal{G} \in \mathbb{P}_{[A,B]} : \mathcal{G} \text{ is a } C^{1,1} \text{ weak Kam solution for all } H \in \mathcal{F}' \cup \{H_0\}\right\} \\ &= \bigcap_{H \in \mathcal{F}' \cup \{H_0\}} \Big(\{\mathcal{G} : \Phi_{A_H}(\mathcal{G}) = \mathcal{G}\} \cap \{\mathcal{G} : \breve{\Phi}_{A_H}(\mathcal{G}) = \mathcal{G}\}\Big) \ \cap \ \mathbb{P}_{[A,B]} \end{aligned}$$

is non-empty for all finite $\mathcal{F}' \subseteq \mathcal{F}$. The second line in the above expression tells us that $C(\mathcal{F}')$ is also closed, hence compact, because it is contained in $\Phi_{A_{H_0}}(\mathbb{P}_{[A,B]})$ which is relatively compact by Remark 4.4(iii). Furthermore, the sets $C(\mathcal{F}')$ satisfy the finite intersection property, because

$$C(\mathcal{F}'_1) \cap \cdots \cap C(\mathcal{F}'_n) = C(\mathcal{F}'_1 \cup \cdots \cup \mathcal{F}'_n) \neq \emptyset.$$

By compactness, the whole intersection is non-empty too:

$$\bigcap_{\substack{\mathcal{F}' \subseteq \mathcal{F} \\ \mathcal{F}' \text{finite}}} C(\mathcal{F}') \neq \emptyset.$$

Its elements are the invariant circles common to all the Hamiltonians of the family $\mathcal{F}$. This contradicts the assumptions. $\square$

Let us further discuss about the implications of Theorem 5.7 in some special cases.

- *Case* $\mathcal{F} = \{L\}$. This is the case extensively treated in [3]. In that paper, $R(c)$ was defined as

$$R(c) = \bigcap_{\mathcal{G}\ c\text{-weak Kam solution}} [\mathcal{I}_{\Phi_{h_c}}(\mathcal{G})^{\perp}], \tag{5.6}$$

  Let us check that this definition coincides with the one given here. Indeed, from Section 4.4 we know that the minimal components in $\mathbb{P}_c$ are exactly the c-weak Kam solutions for L, and that $\Phi_{h_c} \circ \Sigma_c^\infty = \Sigma_c^\infty \circ \Phi_{h_c} = \Phi_{h_c}$. Therefore,

$$\mathcal{I}_\Phi(\mathcal{G}) \supseteq \mathcal{I}_{\Phi_{h_c} \circ \Phi}(\mathcal{G}) = \mathcal{I}_{\Phi_{h_c}}(\mathcal{G}), \qquad \forall\, \mathcal{G} \in \mathbb{P}_c, \Phi \in \Sigma_c^\infty$$

  (the first inclusion follows from 4.10). The equality of (5.6) with our definition of $R(c)$ is then easy to verify.

  The obstruction to the diffusion via the Mather mechanism is then the homological size of $\mathcal{I}_{\Phi_{h_c}}(\mathcal{G})$, for every c-weak Kam solution $\mathcal{G}$. This set is also called Aubry set of $\mathcal{G}$ (see Section 4.3), and taking the union over the c-weak Kam solutions $\mathcal{G}$ one gets the projection on M of the Mañé set $\tilde{\mathcal{N}}(c) \subset T^*M$. In fact, a relation between $R(c)$ and the homology (in T^*M) of $\tilde{\mathcal{N}}(c)$ is given in [3, Lemma 8.2].

- *Case $d = 1$.* In this case the mutual forcing relation $\dashv\vdash_{\mathcal{F}}$ is well understood thanks to Proposition 5.9: there exists a closed set in $H^1(M, \mathbb{R})$ which is the set of cohomology classes c for which there exists a common invariant circle of cohomology c. The equivalence classes for $\dashv\vdash_{\mathcal{F}}$ are the elements of this set and the connected components of its complementary.

  In this case the Mather mechanism is optimal in two ways: first, because it allows a complete description of the equivalence classes for $\dashv\vdash_{\mathcal{F}}$; second, because the equivalence relation $\dashv\vdash_{\mathcal{F}}$ completely characterize the possible unstable behaviors of the polysystem (in other words, the obstructions for the mechanism, i.e. the common non-contractible invariant circles, are real obstructions for the dynamics).

- *Commuting hamiltonians.* By the discussion in Section 4.4, we know that there exists an operator $\Phi_{h_c} \in \Sigma_c^\infty$ such that $\Phi_{h_c} \circ \Phi = \Phi \circ \Phi_{h_c} = \Phi_{h_c}$ for every $\Phi \in \Sigma_c^\infty$. We also know that h_c is the common Peierls barrier of all the Hamiltonians in $\mathcal{F}$, and that the minimal components in $\mathbb{P}_c$ are exactly the c-weak Kam solutions for one (hence all) Hamiltonian in $\mathcal{F}$. Arguing as in the case of a single Hamiltonian, one gets

$$R_{\mathcal{F}}(c) = \bigcap_{\mathcal{G}\ c\text{-weak Kam solution}} [\mathcal{I}_{\Phi_{h_c}}(\mathcal{G})^\perp],$$

  hence $R_{\mathcal{F}}(c) = R_{\{H\}}(c)$ for every $H \in \mathcal{F}$: the obstructions are the same than those of every single Hamiltonian in $\mathcal{F}$. Therefore, the polysystem does not present any new kind of instability phenomena with respect to each system regarded separately (at least using the Mather mechanism presented here).

- *General case.* The general situation appears much messier. Nevertheless, some information can still be extracted. For instance, let us suppose that V is a one-dimensional subspace of $H^1(M, \mathbb{R})$ not contained in $R(c)$. Then, there must exists a minimal component $\mathbb{M}$ such that V is not contained in $R(\mathbb{M})$. In particular, by Proposition 5.5 and by invariance of $\mathbb{M}$, we have that

$$V \nsubseteq R_{\Phi_1,\ldots,\Phi_n}(\mathcal{G}) \qquad \forall\, \mathcal{G} \in \mathbb{M},\ \forall\, \Phi_1, \ldots, \Phi_n \in \Sigma_c^\infty. \tag{5.7}$$

  By making different choices of $\mathcal{G}$ and $\Phi_1, \ldots, \Phi_n$, one in principle gets a plethora of conditions, which may become arbitrarily complicated. Two samples of the kind of statements which can be obtained are proved in the next proposition.

  Let us also point out that the condition above can be interpreted, at least for some choices of $\mathcal{G}$ and Φ as a sort of "homologically transverse intersection" between some generalized stable and unstable manifolds. Indeed, by definition of $R_{\Phi_1,\ldots,\Phi_n}(\mathcal{G})$ the obstructions essentially boil down to the various sets $\mathcal{I}_\Phi(\mathcal{G}) = \mathcal{G} \wedge \breve{\Phi}\Phi(\mathcal{G})$. By property (iv) in Proposition 4.16 and its dual version, we see that, at least for some choices of $\mathcal{G}$ and Φ, we can interpret $\mathcal{G}$ as an unstable manifold for some switched flow and $\breve{\Phi}\Phi(\mathcal{G})$ as a stable manifold of another switched flow.

Proposition 5.11. *Suppose that V is a one-dimensional subspace of $H^1(M,\mathbb{R})$ not contained in $R(c)$.*

(i) For every arbitrary finite string $H_1, \dots, H_k$ of Hamiltonians in $\mathcal{F}$, there exists a subset $S \subset T^*M$ such that: it is a Lipschitz graph over its projection on M, it is contained in a pseudograph of cohomology c, it is invariant (both in past and in future) for the switched flow

$$\phi = \phi_{H_k} \circ \dots \circ \phi_{H_1},$$

and its projection $\pi(S) \subseteq M$ satisfies

$$V \nsubseteq [\pi(S)^\perp].$$

(ii) For every pair of Hamiltonians $H_0, H_1 \in \mathcal{F}$ there exists a c-weak Kam solution $\mathcal{G}_0$ for H_0 and a dual c-weak Kam solution $\mathcal{G}_1$ for H_1 such that

$$V \nsubseteq \left[(\mathcal{G}_0 \wedge \mathcal{G}_1)^\perp\right].$$

Moreover, call h_0, h_1 the Peierls barriers of H_0 and H_1. Then without loss of generality we can also suppose that $\Phi_{h_0}\Phi_{h_1}(\mathcal{G}_0) = \mathcal{G}_0$ and $\Phi_{h_1}\Phi_{h_0}(\mathcal{G}_1) = \mathcal{G}_1$.

Proof.

(i) Call $A_1, \dots, A_k$ the time-one actions of $H_1, \dots, H_k$, and consider the composition

$$A = A_k \circ \dots \circ A_1.$$

In (5.7) take $n = 1$, $\Phi_1 = \Phi_{h_{A_c}}$ and $\mathcal{G}$ a fixed point of Φ_1 belonging to $\mathbb{M}$ (let us recall that, by invariance, every minimal component $\mathbb{M}$ contains such a fixed point). Since $V \nsubseteq R(c)$, we have

$$V \nsubseteq [\mathcal{I}_{\Phi_1}(\mathcal{G})^\perp].$$

Set $S = \mathcal{G}_{|\mathcal{I}_{\Phi_1}(\mathcal{G})}$. By a natural generalization of Proposition 4.16, S is invariant for ϕ, thus the conclusion of item (i) is achieved.

(ii) Call h_0, h_1 the Peierls barrier of H_0 and H_1. In (5.7) take $n = 1$, $\Phi_1 = \Phi_{h_{1c}}$ and $\mathcal{G}$ a fixed point of $\Phi_{h_{0c}}$ belonging to $\mathbb{M}$. We have then,

$$V \nsubseteq [\mathcal{I}_{\Phi_1}(\mathcal{G})^\perp] = [\mathcal{G} \wedge \breve{\Phi}_1\Phi_1(\mathcal{G})^\perp].$$

This proves the first part of the result, with $\mathcal{G}_0 = \mathcal{G}$ and $\mathcal{G}_1 = \breve{\Phi}_1\Phi_1(\mathcal{G})$. The second part follows similarly, by taking as $\mathcal{G}$ a fixed point of $\Phi_{h_{0,c}}\Phi_{h_{1,c}}$ (see Proposition 4.17).

$\square$

We end with two remarks.

Remark 5.12 (Case $\mathcal{F}$ not equi-semiconcave). For an arbitrary family $\mathcal{F}$ of Tonelli Hamiltonians we can still define a subspace $R_{\mathcal{F}}(c)$ analogous to the one defined for an equi-semiconcave family. Indeed, we know that if $\mathcal{F}'$ and $\mathcal{F}''$ are equi-semiconcave subfamilies of $\mathcal{F}$ such that $\mathcal{F}' \subseteq \mathcal{F}''$, then $R_{\mathcal{F}'}(c) \subseteq R_{\mathcal{F}''}(c)$. This implies that the union of all $R_{\mathcal{F}'}(c), \mathcal{F}' \subset \mathcal{F}$ finite, is a subspace of H^1. We can then define $R_{\mathcal{F}}(c)$ to be equal to this union. By the finite dimension of H^1, we have $R_{\mathcal{F}}(c) = R_{\mathcal{F}_0}(c)$ for an equi-semiconcave subfamily $\mathcal{F}_0$. Moreover, Theorem 5.7 and Proposition 5.9 remain true, by just replacing $\mathcal{F}$ with $\mathcal{F}_0$ in the proofs. Nevertheless, this does not allow to construct polyorbits for which a non-equi-semiconcave family of Hamiltonians is genuinely needed.

Remark 5.13 (More on the minimal sets $\mathbb{M}$). A further study of the minimal components $\mathbb{M}$ leads to some more equivalent expressions for $R(\mathbb{M})$. Recall from Proposition 5.5 that

$$R(\mathbb{M}) = \sum_{\mathcal{G} \in \mathbb{M}, \Phi \in \Sigma_c^\infty} R_\Phi(\mathcal{G}).$$

A first equivalent expression is

$$R(\mathbb{M}) = \sum_{\substack{A \in \sigma_c^\infty \\ \mathcal{G} \text{ fixed point of } \Phi_A \text{ in } \mathbb{M}}} R_{\Phi_{h_A}}(\mathcal{G}).$$

Indeed, every addend of the first sum is contained in some addend of the second sum (the vice versa being obvious): given $\mathcal{G} \in \mathbb{M}$ and $\Phi \in \Sigma_c^\infty$, let us take Φ' such that $\Phi'\Phi(\mathcal{G}) = \mathcal{G}$. Such a Φ' exists by Proposition 4.15(iii). By definition of Σ_c^∞, there exists $A \in \sigma_c^\infty$ such that $\Phi' \circ \Phi = \Phi_A$. Since $\mathcal{I}_{\Phi' \circ \Phi}(\mathcal{G}) \subset \mathcal{I}_\Phi(\mathcal{G})$, we have

$$R_\Phi(\mathcal{G}) \subseteq R_{\Phi_A}(\mathcal{G}) \subseteq R_{\Phi_{h_A}}(\mathcal{G}),$$

as claimed.

Another equivalent expression follows from the following ideas. Thanks to the property $\mathcal{I}_{\Phi' \circ \Phi}(\mathcal{G}) \subseteq \mathcal{I}_\Phi(\mathcal{G})$, we have that, for a fixed $\mathcal{G}$, the function $\Phi \mapsto \mathcal{I}_\Phi(\mathcal{G})$ is non-increasing along the orbits of the dynamical system

$$\begin{aligned} \Sigma_c^\infty \times \Sigma_c^\infty &\to \Sigma_c^\infty \\ (\Phi', \Phi) &\mapsto \Phi' \circ \Phi. \end{aligned}$$

As in Proposition 4.15, one can show that there exist minimal components Λ for this dynamical system and that for every Φ there exists Φ' belonging to one of this minimal components such that $\mathcal{I}_{\Phi'}(\mathcal{G}) \subseteq \mathcal{I}_\Phi(\mathcal{G})$. Moreover if Φ, Φ' belong to the same minimal component Λ, then $\mathcal{I}_{\Phi'}(\mathcal{G}) = \mathcal{I}_\Phi(\mathcal{G})$, so that we can call it $\mathcal{I}_\Lambda(\mathcal{G})$. We finally deduce that for every $\mathbb{M}$ minimal in $\mathbb{P}_c$

$$R(\mathbb{M}) = \sum_{\Lambda \text{ minimal}} [\mathcal{I}_\Lambda(\mathcal{G})^\perp] \qquad \forall\, \mathcal{G} \in \mathbb{M}.$$

Finally, one last property of minimal sets: if $\mathbb{M}, \mathbb{M}'$ are minimal, then the distance function $\mathcal{G} \mapsto d(\mathcal{G}, \mathbb{M}')$ is constant on $\mathbb{M}$. Indeed, the distance function is 1-Lipschitz and $\mathbb{M}'$ is invariant, therefore $d(\Phi(\mathcal{G}), \mathbb{M}') = d(\Phi(\mathcal{G}), \Phi(\mathbb{M}')) \leq d(\mathcal{G}, \mathbb{M}')$ for all $\mathcal{G} \in \mathbb{P}_c$ and $\Phi \in \Sigma_c^\infty$. We thus have, by transitivity of the dynamics in $\mathbb{M}$,

$$d(\mathcal{G}, \mathbb{M}') = d(\mathcal{G}', \mathbb{M}') \qquad \forall\, \mathcal{G}, \mathcal{G}' \in \mathbb{M}$$

For the same reasons, the set of those couples $(\mathcal{G}, \mathcal{G}') \in \mathbb{M} \times \mathbb{M}'$ such that $d(\mathcal{G}, \mathcal{G}') = d(\mathbb{M}, \mathbb{M}')$ is invariant by Σ_c^∞.

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