Injective objects and retracts of Fraïssé limits

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Abstract

We present a purely category-theoretic characterization of retracts of Fraïssé limits. For this aim, we consider a natural version of injectivity with respect to a pair of categories (a category and its subcategory). It turns out that retracts of Fraïssé limits are precisely the objects that are injective relatively to such a pair. One of the applications is a characterization of non-expansive retracts of Urysohn's universal metric space.
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Contents

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1 Introduction

Recall that a *Fraïssé class* is a countable class $\mathcal{F}$ of finitely generated models of a fixed first-order language, satisfying the following conditions:

(i) Given $a, b \in \mathcal{F}$ there exists $d \in \mathcal{F}$ such that both a and b embed into d (Joint Embedding Property).

(ii) Given $a, b \in \mathcal{F}$ and embeddings $i\colon c \to a$, $j\colon c \to b$, there exist $w \in \mathcal{F}$ and embeddings $k\colon a \to w$ and $\ell\colon b \to w$ such that $k \circ i = \ell \circ j$ (Amalgamation Property).

(iii) Given $a \in \mathcal{F}$, every substructure of a is isomorphic to an element of $\mathcal{F}$.

The *Fraïssé limit* of $\mathcal{F}$ is a countable model U such that, up to isomorphism,

$$\mathcal{F} = \{a \subseteq U\colon a \text{ is a finitely generated substructure of } U\}$$

and for every isomorphism $h\colon a \to b$ between finitely generated substructures of U there exists an automorphism $H\colon U \to U$ such that $H \supseteq h$. The latter property is called *ultrahomogeneity*. It is a classical theorem of Roland Fraïssé [9] that the Fraïssé limit exists and is unique, up to isomorphism. Uncountable versions of Fraïssé limits were studied by Jónsson [12, 13].

A recent result of Dolinka [5] characterizes, under certain assumptions, countable models that are embeddable as retracts into the Fraïssé limit. Namely, he proves that, under certain conditions on the Fraïssé class, retracts of the Fraïssé limit are precisely the (countable) algebraically closed models.

The aim of this note is to extend Dolinka's characterization to the case of category-theoretic Fraïssé limits, at the same time weakening the assumption on the class of objects. Namely, Dolinka's result assumes that models are finite and for each natural number n there exist only finitely many isomorphic types of models generated by a set of cardinality n. We do not make any such assumptions. Our result relates retracts of Fraïssé limits to a natural variant of injectivity. Among new applications, we characterize non-expansive retracts of the universal metric space of Urysohn. This metric space is formally not a Fraïssé limit, because the category of finite metric spaces is uncountable.

However, it can be "approximated" by Fraïssé limits of countable subcategories (e.g. by considering rational distances only).

Category-theoretic approach to Fraïssé limits comes from the author's paper [14], motivated by a much earlier work of Droste and Göbel [8] and by a recent work of Irwin and Solecki [11] on projective Fraïssé limits. In [14] the key notion is a *Fraïssé sequence* rather than a Fraïssé limit. This turns out to be convenient because it allows to work in one category (corresponding to finitely generated models), forgetting about the existence or non-existence of colimits. In order to speak about retractions, we need to work with a pair of categories, both with the same objects; the first one allows "embeddings" only, while the second one allows all possible homomorphisms.

1.1 Categories of sequences

Fix a category $\mathfrak{K}$. We shall treat $\mathfrak{K}$ as the class of arrows, the class of objects will be denoted by $\mathrm{Ob}(\mathfrak{K})$ and the set of $\mathfrak{K}$-arrows with domain x and codomain y will be denoted by $\mathfrak{K}(x,y)$. A *sequence* in $\mathfrak{K}$ is simply a covariant functor from ω into $\mathfrak{K}$. One can think that the objects of $\mathfrak{K}$ are "small" structures (e.g. finitely generated models of a fixed language). Sequences in $\mathfrak{K}$ form a bigger category of "large" structures. For category-theoretic notions we refer to [15].

We shall use the following convention: Sequences in $\mathfrak{K}$ will be denoted by capital letters $X, Y, Z, \ldots$ and the objects of $\mathfrak{K}$ will be denoted by small letters $x, y, z, \ldots$. Fix a sequence $X\colon \omega \to \mathfrak{K}$. Recall that formally X assigns to each natural number n an object $X(n)$ of $\mathfrak{K}$ and X assigns a $\mathfrak{K}$-arrow $X(n,m)\colon X(n) \to X(m)$ for each pair $\langle n, m\rangle$ of natural numbers such that $n \leqslant m$. We shall always write x_n instead of $X(n)$ and x_n^m instead of $X(n,m)$. Note that being a functor imposes the conditions $x_n^n = \mathrm{id}_{x_n}$ and $x_k^m = x_\ell^m \circ x_k^\ell$ for $k \leqslant \ell \leqslant m$.

An arrow from a sequence X to a sequence Y is, by definition, a natural transformation from the functor X into the functor $Y \circ \psi$, where $\psi\colon \omega \to \omega$ is increasing (i.e. ψ is a covariant functor from ω to ω). We identify some arrows that "potentially converge" to the same limit. More precisely, given natural transformations τ_0 and τ_1 from the sequence X to $Y \circ \psi_0$ and $Y \circ \psi_1$, respectively, we say that τ_0 is *equivalent* to τ_1, if the diagram consisting of both sequences X, Y together with all arrows induced by τ_0 and τ_1 is commutative. This is indeed an equivalence relation and it commutes with the composition, therefore $\sigma\mathfrak{K}$ indeed becomes a category. In order to illustrate this idea, observe that every sequence is isomorphic to its cofinal subsequence. Indeed, if X is a sequence and $k = \{k_n\}_{n\in\omega}$ is a strictly increasing sequence of natural numbers, then the $\sigma\mathfrak{K}$-arrow $I\colon X \circ k \to X$ defined by $I = \{i_n\}_{n\in\omega}$, where $i_n = \mathrm{id}_{x_{k_n}}$, is an isomorphism. Its inverse is $J = \{j_n\}_{n\in\omega}$, where $j_n = x_n^{k_m}$ and $m = \min\{s\colon k_s \geqslant n\}$. The composition $I \circ J$ is formally $\{j_n\}_{n\in\omega}$ regarded as an arrow from X to X. Clearly, $I \circ J$ is equivalent to the identity $\{\mathrm{id}_{x_n}\}_{n\in\omega}$. Similarly, $J \circ I$ is equivalent to the identity of $X \circ k$.

The original category $\mathfrak{K}$ may be regarded as a subcategory of $\sigma\mathfrak{K}$, identifying an object

x with a sequence

$$x \xrightarrow{\mathrm{id}_x} x \xrightarrow{\mathrm{id}_x} x \xrightarrow{\mathrm{id}_x} \cdots$$

Thus, we shall always assume that $\mathfrak{K} \subseteq \sigma\mathfrak{K}$. Given a sequence X and $n \in \omega$, we shall denote by x_n^∞ the arrow from x_n to X induced by the nth object of X. Formally, x_n^∞ is the equivalence class of $\{x_n^m\}_{m \geqslant n}$.

1.2 Fraïssé sequences

Fraïssé classes and limits can be described using categories. Let $\mathfrak{K}$ be a fixed category.

A *Fraïssé sequence* in $\mathfrak{K}$ is a sequence U satisfying the following two conditions.

(F1) For every object x in $\mathfrak{K}$ there exist $n \in \omega$ and a $\mathfrak{K}$-arrow $x \to u_n$.

(F2) For every $n \in \omega$ and for every $\mathfrak{K}$-arrow $f\colon u_n \to y$ there exist $m \geqslant n$ and a $\mathfrak{K}$-arrow $g\colon y \to u_m$ such that $g \circ f = u_n^m$.

Recall that $\mathfrak{K}$ has the *amalgamation property* if for every $\mathfrak{K}$-arrows $f\colon c \to a$, $g\colon c \to b$ there exist $\mathfrak{K}$-arrows $f'\colon a \to w$, $g'\colon b \to w$ satisfying $f' \circ f = g' \circ g$. A Fraïssé sequence exists whenever $\mathfrak{K}$ has the amalgamation property, the joint embedding property and has countably many isomorphic types of arrows. A Fraïssé sequence is unique up to isomorphism. We refer to [14] for the details.

A standard induction shows that the amalgamation property partially extends to the category of sequences. Namely:

Proposition 1.1. *Assume $\mathfrak{K}$ has the amalgamation property. Then for every $\sigma\mathfrak{K}$-arrows $f\colon c \to A$, $g\colon c \to B$ with $c \in \mathrm{Ob}(\mathfrak{K})$, there exist $\sigma\mathfrak{K}$-arrows $f'\colon A \to W$, $g'\colon B \to W$ satisfying $f' \circ f = g' \circ g$.*

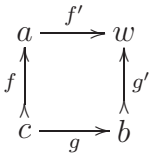
However, it is shown in [14] that in general the amalgamation property of $\mathfrak{K}$ does not imply the same property of $\sigma\mathfrak{K}$.

Now, let $\mathfrak{K} \subseteq \mathfrak{L}$ be a pair of categories such that $\mathfrak{K}$ has the same objects as $\mathfrak{L}$. For instance, $\mathfrak{K}$ is a category of finitely generated models of a fixed language with embeddings and $\mathfrak{L}$ allows all homomorphisms. Note that $\sigma\mathfrak{K}$ as a subcategory of $\sigma\mathfrak{L}$. We shall need to deal with the category $\mathfrak{R} = \sigma(\mathfrak{K}, \mathfrak{L})$ whose objects are ω-sequences in $\mathfrak{K}$ and the arrows come from $\mathfrak{L}$, i.e., $\mathrm{Ob}(\mathfrak{R}) = \mathrm{Ob}(\sigma\mathfrak{K})$ and $\mathfrak{R}(X, Y) = \sigma\mathfrak{L}(X, Y)$ for $X, Y \in \mathrm{Ob}(\mathfrak{R})$.

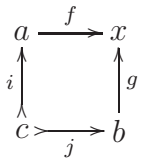
For example, if $\mathfrak{K}$, $\mathfrak{L}$ are as above, $\mathfrak{R}$ is the category of countable models with all possible homomorphisms, while $\sigma\mathfrak{K}$ is the category of countable models with embeddings.

2 Main result

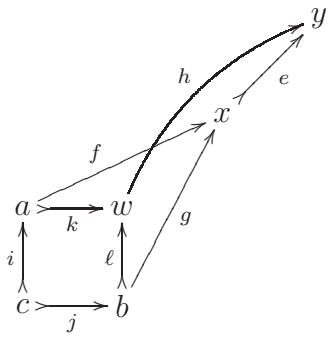
Let $\mathfrak{K} \subseteq \mathfrak{L}$ be two fixed categories with the same objects. We say that $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has the *mixed amalgamation property* if for every arrows $f\colon c \to a$ and $g\colon c \to b$ such that $f \in \mathfrak{K}$ and $g \in \mathfrak{L}$, there exist arrows $f'\colon a \to w$, $g'\colon b \to w$ satisfying $f' \circ f = g' \circ g$ and such that $g' \in \mathfrak{K}$ and $f' \in \mathfrak{L}$. The mixed amalgamation is described in the following diagram, where $\rightarrowtail$ denotes an arrow in $\mathfrak{K}$.



We say that $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has the *amalgamated extension property* if for every commutative $\mathfrak{L}$-diagram



with $i, j \in \mathfrak{K}$, there exist $\mathfrak{K}$-arrows $e\colon x \to y$, $k\colon a \to w$, $\ell\colon b \to w$ and an $\mathfrak{L}$-arrow $h\colon w \to y$ such that $e \circ f = h \circ k$, $e \circ g = h \circ \ell$ and $k \circ i = \ell \circ j$. That is, the following diagram is commutative.



We now define the following axioms for a pair of categories $\langle \mathfrak{K}, \mathfrak{L} \rangle$, needed for our main result.

($\mathfrak{H}_0$) $\mathfrak{K} \subseteq \mathfrak{L}$ and $\mathrm{Ob}(\mathfrak{K}) = \mathrm{Ob}(\mathfrak{L})$.

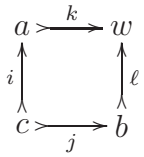
($\mathfrak{H}_1$) $\mathfrak{K}$ has both the amalgamation property and the joint embedding property.

($\mathfrak{H}_2$) $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has the mixed amalgamation property.

($\mathfrak{H}_3$) $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has the amalgamated extension property.

Definition 2.1. A pair of categories $\langle \mathfrak{K}, \mathfrak{L} \rangle$ *has property* ($\mathfrak{H}$) if it satisfies conditions ($\mathfrak{H}_0$) – ($\mathfrak{H}_3$).

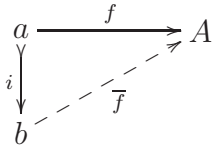
It is necessary to make some comments on the properties described above. Namely, the condition $\mathrm{Ob}\,(\mathfrak{K}) = \mathrm{Ob}\,(\mathfrak{L})$ can be removed from $(\mathfrak{H}_0)$, it appears there for the sake of convenience only. The role of $\mathfrak{L}$ is offering more arrows than $\mathfrak{K}$, some of them will be needed for constructing retractions. One can think of the $\mathfrak{K}$-arrows as "embeddings". In most cases, these will be indeed monics. Condition $(\mathfrak{H}_1)$ is needed mainly for the existence and good properties of a Fraïssé sequence in $\mathfrak{K}$. Recall that the joint embedding property follows from amalgamations, whenever $\mathfrak{K}$ has an initial object (or at least a weakly initial object). Condition $(\mathfrak{H}_2)$ will be crucial for proving that the Fraïssé sequence and its retracts are $\mathfrak{K}$-injective (see the definition below). Finally, the somewhat technical condition $(\mathfrak{H}_3)$ will be needed for the argument in the main lemma relating $\mathfrak{K}$-injective objects with the Fraïssé sequence. If $\mathfrak{L}$ has a terminal object then $(\mathfrak{H}_3)$ implies that $\mathfrak{K}$ has the amalgamation property. Summarizing, if $\mathfrak{K}$ has a weakly initial object and $\mathfrak{L}$ has a terminal object, then we may ignore condition $(\mathfrak{H}_1)$. Condition $(\mathfrak{H}_3)$ becomes trivial if $\mathfrak{K}$ has pushouts in $\mathfrak{L}$. We say that $\mathfrak{K}$ has *pushouts in* $\mathfrak{L}$ if for every pair of $\mathfrak{K}$-arrows $i\colon c \to a$, $j\colon c \to b$, there exist $\mathfrak{K}$-arrows $k\colon a \to w$, $\ell\colon b \to w$ such that



is a pushout square in $\mathfrak{L}$. It is obvious from the definition of a pushout that $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has the amalgamated extension property (with $y = x$ and $e = \mathrm{id}_x$) whenever $\mathfrak{K}$ has pushouts in $\mathfrak{L}$. Let us remark that for all examples with property $(\mathfrak{H})$ appearing in this note, the amalgamated extension property holds with $x = y$ and $e = \mathrm{id}_x$ (see the definition and diagram above).

Below is the crucial notion, whose variations appear often in the literature (see, e.g., [1], where a definition similar to ours can be found). Let $\mathfrak{K} \subseteq \mathfrak{L}$ be two categories with the same objects.

Definition 2.2. We say that $A \in \mathrm{Ob}\,(\sigma\mathfrak{K})$ is $\mathfrak{K}$-*injective in* $\sigma(\mathfrak{K}, \mathfrak{L})$ if for every $\mathfrak{K}$-arrow $i\colon a \to b$, for every $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $f\colon a \to X$, there exists a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $\overline{f}\colon b \to X$ such that $\overline{f} \circ i = f$.



This definition obviously generalizes to an arbitrary pair of categories $\mathfrak{K} \subseteq \mathfrak{R}$. We restrict attention to the special case $\mathfrak{R} = \sigma(\mathfrak{K}, \mathfrak{L})$, since more general versions will not be needed.

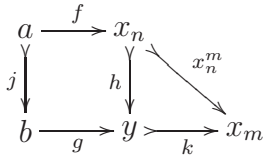
Following is a useful criterion for injectivity.

Proposition 2.3. *Assume $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has the mixed amalgamation property and $X \in \mathrm{Ob}\,(\sigma\mathfrak{K})$. Then X is $\mathfrak{K}$-injective in $\sigma(\mathfrak{K}, \mathfrak{L})$ if and only if for every $n \in \omega$, for every $\mathfrak{K}$-arrow $f\colon x_n \to y$, there exist $m \geqslant n$ and an $\mathfrak{L}$-arrow $g\colon y \to x_m$ satisfying*

$$g \circ f = x_n^m.$$

Proof. Suppose X is $\mathfrak{K}$-injective and fix a $\mathfrak{K}$-arrow $f\colon x_n \to y$. Applying $\mathfrak{K}$-injectivity for $x_n^\infty\colon x_n \to X$, we find $G\colon y \to X$ such that $G \circ f = x_n^\infty$. The arrow G factors through some $\mathfrak{L}$-arrow $g\colon y \to x_m$ for some $m \geqslant n$, that is, $G = x_m^\infty \circ g$. Finally, $g \circ f = x_n^m$.

Suppose now that X satisfies the condition above and fix a $\mathfrak{K}$-arrow $j\colon a \to b$ and a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $F\colon a \to X$. Then $F = x_n^\infty \circ f$ for some $\mathfrak{L}$-arrow f, where $n \in \omega$. Applying the mixed amalgamation property, find a $\mathfrak{K}$-arrow $h\colon x_n \to y$ and an $\mathfrak{L}$-arrow $g\colon b \to y$ such that $g \circ j = h \circ f$. By assumption, there exist $m \geqslant n$ and an $\mathfrak{L}$-arrow $k\colon y \to x_m$ such that the following diagram commutes.



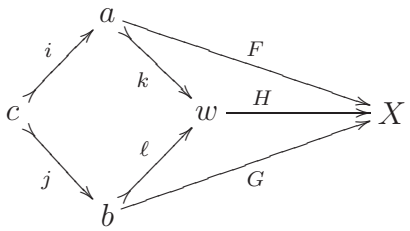
Finally, taking $G = x_m^\infty \circ k \circ g$, we get $G \circ j = F$. □

Our interest in $\mathfrak{K}$-injectivity comes from the following fact, which is an immediate consequence of the criterion above.

Proposition 2.4. *Assume $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has the mixed amalgamation property and U is a Fraïssé sequence in $\mathfrak{K}$. Then U is $\mathfrak{K}$-injective in $\sigma(\mathfrak{K}, \mathfrak{L})$.*

We shall need the following "injective" version of amalgamated extension property.

Lemma 2.5. *Assume $\langle \mathfrak{K}, \mathfrak{L} \rangle$ satisfies $(\mathfrak{H})$ and $X \in \mathrm{Ob}\,(\sigma\mathfrak{K})$ is $\mathfrak{K}$-injective in $\sigma(\mathfrak{K}, \mathfrak{L})$. Then for every $\mathfrak{K}$-arrows $i\colon c \to a$, $j\colon c \to b$ and for every $\sigma(\mathfrak{K}, \mathfrak{L})$-arrows $F\colon a \to X$, $G\colon b \to X$ such that $F \circ i = G \circ j$, there exist $\mathfrak{K}$-arrows $k\colon a \to w$, $\ell\colon b \to w$ and a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $H\colon w \to X$ such that the diagram*



commutes.

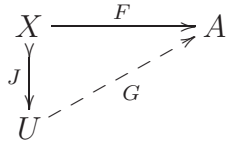
Proof. Find n such that $F = x_n^\infty \circ f$ and $G = x_n^\infty \circ g$ for some $\mathfrak{L}$-arrows f, g, where $x_n^\infty \colon x_n \to X$ is the canonical arrow induced by the nth object of the sequence X. Using property $(\mathfrak{H}_3)$, we find $\mathfrak{K}$-arrows $k \colon a \to w$, $\ell \colon b \to w$, $e \colon x_n \to y$ and an $\mathfrak{L}$-arrow $h \colon w \to y$ such that $h \circ k = e \circ f$ and $h \circ \ell = e \circ g$. Using the $\mathfrak{K}$-injectivity of X we can find a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $P \colon y \to X$ such that $P \circ e = x_n^\infty$. Let $H = P \circ h$. Then

$$H \circ k = P \circ h \circ k = P \circ e \circ f = x_n^\infty \circ f = F.$$

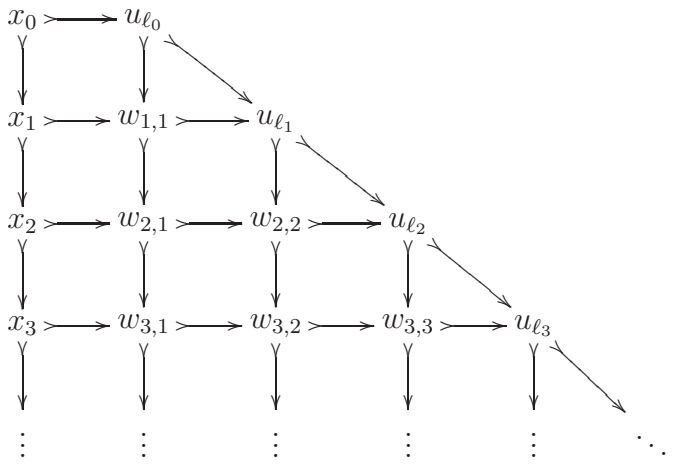
Similarly, $H \circ \ell = G$. □

The following lemma is crucial.

Lemma 2.6. *Assume $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has property $(\mathfrak{H})$ and A is a $\mathfrak{K}$-injective object in $\sigma(\mathfrak{K}, \mathfrak{L})$. Furthermore, assume U is a Fraïssé sequence in $\mathfrak{K}$ and $F \colon X \to A$ is an arbitrary $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow. Then there exist a $\sigma\mathfrak{K}$-arrow $J \colon X \to U$ and a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $G \colon U \to A$ such that $G \circ J = F$.*

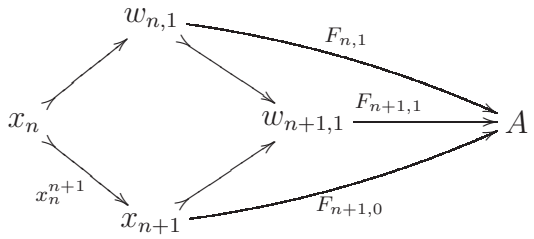


Proof. Recall that we use the usual convention for objects $x_n = X(n)$, $u_n = U(n)$, and for arrows $x_n^m = X(n, m)$, $u_n^m = U(n, m)$. We shall construct inductively the following "triangular matrix" in $\mathfrak{K}$, together with commuting $\sigma(\mathfrak{K}, \mathfrak{L})$-arrows $F_{i,j} \colon w_{i,j} \to A$ for $j \leqslant i + 1$, where we agree that $w_{i,0} = x_i$ and $w_{i,i+1} = u_{\ell_i}$.

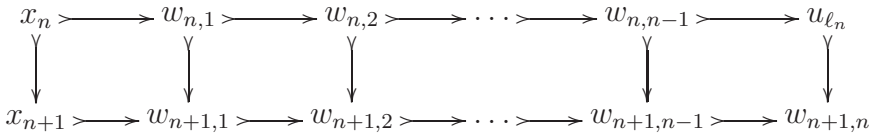


The first column in the diagram above is the sequence X, while the diagonal is a cofinal subsequence of U. Our initial assumption on $F_{i,j}$ is that $\{F_{n,0}\}_{n \in \omega} = F$. It is clear how to start the construction: Using the Fraïssé property of U, we find ℓ_0 and a $\mathfrak{K}$-arrow $e_0 \colon x_0 \to u_{\ell_0}$. Next, using the $\mathfrak{K}$-injectivity of A, we find $F_{0,1} \colon u_{\ell_0} \to A$ satisfying $F_{0,1} \circ e_0 = F_{0,0}$.

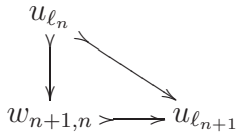
Suppose the nth row has already been constructed, together with arrows $F_{i,j}$ for $i \leqslant n$, $j \leqslant n+1$. Starting from $\mathfrak{K}$-arrows $x_n^{n+1}\colon x_n \to x_{n+1}$ and $x_n \to w_{n,1}$, using Lemma 2.5, we find $w_{n+1,1} \in \mathrm{Ob}\,(\mathfrak{K})$ and $\mathfrak{K}$-arrows $w_{n,1} \to w_{n+1,1}$, $x_{n+1} \to w_{n+1,1}$, and a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $F_{n+1,1}\colon w_{n+1,1} \to A$ such that the diagram



commutes. Continuing this way, using Lemma 2.5, we obtain the $(n+1)$st row and $\sigma(\mathfrak{K}, \mathfrak{L})$-arrows $F_{n+1,i}$ for $i \leqslant n+1$ which commute together with the following diagram.



Now, using the Fraïssé property of U we find $\ell_{n+1} > \ell_n$ and a $\mathfrak{K}$-arrow $w_{n+1,n} \to u_{\ell_{n+1}}$ making the triangle



commutative. Using Lemma 2.5 again, we get an arrow $F_{n+1,n+2}\colon u_{\ell_{n+1}} \to A$ commuting with $F_{n+1,n}$, $F_{n,n+1}$ and the triangle above.

Finally, the compositions of the horizontal arrows in the triangular "matrix" constructed above induce an arrow of sequences $J\colon X \to U$ in $\sigma\mathfrak{K}$. The inductive construction also gives a sequence of arrows $\{F_{n,n+1}\}_{n\in\omega}$ that turns into a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $G\colon U \to A$ satisfying $G \circ J = F$. This completes the proof. □

Theorem 2.7. *Let $\langle \mathfrak{K}, \mathfrak{L}\rangle$ be a pair of categories with property $(\mathfrak{H})$. Assume $\mathfrak{K}$ has a Fraïssé sequence U and let X be an arbitrary sequence in $\mathfrak{K}$. The following properties are equivalent.*

(a) X is $\mathfrak{K}$-injective in $\sigma(\mathfrak{K}, \mathfrak{L})$.

(b) There exist a $\sigma\mathfrak{K}$-arrow $J\colon X \to U$ and a $\sigma(\mathfrak{K}, \mathfrak{L})$ arrow $R\colon U \to X$ such that $R \circ J = \mathrm{id}_X$.

(c) X is a retract of U in $\sigma(\mathfrak{K}, \mathfrak{L})$.

Note that condition (c) is formally weaker than (b), since it is not required in (c) that the right inverse of a retraction $R\colon U \to X$ is a $\sigma\mathfrak{K}$-arrow.

Proof. (a) $\implies$ (b) Applying Lemma 2.6 to the identity $\mathrm{id}_X\colon X \to X$, we get a $\sigma\mathfrak{K}$-arrow $J\colon X \to U$ and a $\sigma(\mathfrak{K},\mathfrak{L})$-arrow $R\colon U \to X$ such that $R \circ J = \mathrm{id}_X$.
(b) $\implies$ (c) This is obvious.
(c) $\implies$ (a) Let $J\colon X \to U$ and $R\colon U \to X$ be $\sigma(\mathfrak{K},\mathfrak{L})$-arrows such that $R \circ J = \mathrm{id}_X$. Fix a $\mathfrak{K}$-arrow $i\colon a \to b$ and an $\mathfrak{L}$-arrow $F\colon a \to X$. By Proposition 2.4, U is $\mathfrak{K}$-injective in $\sigma(\mathfrak{K},\mathfrak{L})$, so there exists $G\colon b \to U$ such that $G \circ i = J \circ F$. Finally, we have $R \circ G \circ i = R \circ J \circ F = F$. □

2.1 Possible extensions of the main result

Theorem 2.7 has a natural generalization to uncountable Fraïssé sequences. More precisely, let κ be an uncountable regular cardinal and assume that all sequences in $\mathfrak{K}$ of length $< \kappa$ have colimits in $\mathfrak{L}$, where the colimiting cocones are $\mathfrak{K}$-arrows. In this case we say that $\mathfrak{K}$ is *κ-continuous* in $\mathfrak{L}$. Under this assumption, a version of Lemma 2.6 for κ-sequences is true, with almost the same proof—usual induction is replaced by transfinite induction. Proposition 2.4 is valid for arbitrary Fraïssé sequences, the countable length of the sequence was never used in the proof.
Let $\mathrm{Seq}_{\leqslant\kappa}(\mathfrak{K})$ denote the category of all sequences in $\mathfrak{K}$ of length $\leqslant \kappa$, with arrows induced by natural transformations (like in the countable case). Let $\mathrm{Seq}_{\leqslant\kappa}(\mathfrak{K},\mathfrak{L})$ denote the category with the same objects as $\mathrm{Seq}_{\leqslant\kappa}(\mathfrak{K})$, and with arrows taken from $\mathrm{Seq}_{\leqslant\kappa}(\mathfrak{L})$. We can now formulate an "uncountable" version of our main result.

Theorem 2.8. *Let κ be an uncountable regular cardinal and let $\langle\mathfrak{K},\mathfrak{L}\rangle$ be a pair of categories with property $(\mathfrak{H})$, such that $\mathfrak{K}$ is κ-continuous in $\mathfrak{L}$. Assume $\mathfrak{K}$ has a Fraïssé sequence U of length κ. Given a sequence X in $\mathfrak{K}$ of length $\leqslant \kappa$, the following properties are equivalent.*

(a) X is $\mathfrak{K}$-injective in $\mathrm{Seq}_{\leqslant\kappa}(\mathfrak{K},\mathfrak{L})$.

(b) There exist a $\mathrm{Seq}_{\leqslant\kappa}(\mathfrak{K})$-arrow $J\colon X \to U$ and a $\mathrm{Seq}_{\leqslant\kappa}(\mathfrak{K},\mathfrak{L})$ arrow $R\colon U \to X$ such that $R \circ J = \mathrm{id}_X$.

(c) X is a retract of U in $\mathrm{Seq}_{\leqslant\kappa}(\mathfrak{K},\mathfrak{L})$.

Let us now come back to the countable case. Assume $\langle\mathfrak{K},\mathfrak{L}\rangle$ has property $(\mathfrak{H})$ and moreover $\mathfrak{K}$ has pushouts in $\mathfrak{L}$. Let us look at the proof of Lemma 2.6. We can assume that all squares in the infinite "triangular matrix" constructed there are pushouts in $\mathfrak{L}$. Using the notation from the proof of Lemma 2.6, let W^n denote the sequence coming from the nth column. Observe that the arrow from W^n to W^{n+1} is determined by the "horizontal" $\mathfrak{K}$-arrow $w_{n+1,n} \to w_{n+1,n+1}$. In other words, all other $\mathfrak{K}$-arrows come

as a result of the corresponding pushout square. An arrow of sequences $F\colon V \to W$ determined by pushouts from a single $\mathfrak{K}$-arrow will be called *pushout generated* from $\mathfrak{K}$. Denote by $\sigma^{\mathrm{PO}}\mathfrak{K}$ the category whose objects are ω-sequences in $\mathfrak{K}$, while arrows are pushout generated from $\mathfrak{K}$. A deeper analysis of the proof of Lemma 2.6 gives the following observation, which may be of independent interest.

Proposition 2.9. *Assume $\langle \mathfrak{K}, \mathfrak{L}\rangle$ is a pair of categories with property $(\mathfrak{H})$ and $\mathfrak{K}$ has pushouts in $\mathfrak{L}$. Let $X \in \mathrm{Ob}\,(\sigma\mathfrak{K})$ be $\mathfrak{K}$-injective in $\sigma(\mathfrak{K}, \mathfrak{L})$. Then:*

(1) X is $\sigma^{\mathrm{PO}}\mathfrak{K}$-injective in $\sigma(\mathfrak{K}, \mathfrak{L})$.

(2) Let U be a Fraïssé sequence in $\mathfrak{K}$. There exists a sequence

$$X_0 \to X_1 \to X_2 \to \dots$$

in $\sigma^{\mathrm{PO}}\mathfrak{K}$ such that $X_0 = X$ and U is the colimit of this sequence in $\sigma\mathfrak{K}$.

Clearly, (1) and (2) imply immediately that X is a retract of U.

3 Applications

We start with some more comments on property $(\mathfrak{H})$. In concrete cases (especially in model-theoretic categories), it is much easier to prove the (mixed) amalgamation property for special "primitive" arrows rather than for arbitrary arrows. In order to formalize this idea, fix a pair of categories $\langle \mathfrak{K}, \mathfrak{L}\rangle$ satisfying condition $(\mathfrak{H}_0)$ and fix a collection $\mathcal{F} \subseteq \mathfrak{K}$ (actually $\mathcal{F}$ might be a proper class). We say that $\mathfrak{K}$ is *generated* by $\mathcal{F}$ if for every $f \in \mathfrak{K}$ there exist $n \in \omega$ and $g_0, \dots, g_{n-1} \in \mathcal{F}$ such that $f = g_{n-1} \circ \dots \circ g_0$. For example, if $\mathfrak{K}$ is the category of embeddings of finite models of a fixed first-order language, $\mathcal{F}$ may be the class of embeddings $f\colon S \to T$ such that T is generated by $f[S] \cup \{b\}$ for some $b \in T$. We define the amalgamation property for $\mathcal{F}$ and the mixed amalgamation property for $\langle \mathcal{F}, \mathfrak{L}\rangle$, as before.

Proposition 3.1. *Let $\mathfrak{K} \subseteq \mathfrak{L}$ be two categories with the same objects, such that $\mathfrak{K}$ has the joint embedding property. Assume further that $\mathfrak{K}$ is generated by a family $\mathcal{F}$ such that $\mathcal{F}$ has the amalgamation property and $\langle \mathcal{F}, \mathfrak{L}\rangle$ has both the mixed amalgamation property and the amalgamated extension property. Then $\langle \mathfrak{K}, \mathfrak{L}\rangle$ has property $(\mathfrak{H})$.*

Proof. Given an arrow $f \in \mathfrak{K}$, we say that f *has length* $\leqslant n$ if $f = g_{n-1} \circ \dots \circ g_0$, where $g_0, \dots, g_{n-1} \in \mathcal{F}$. In particular, all arrows from $\mathcal{F}$ have length 1. Easy induction shows that if $i\colon c \to a$, $j\colon c \to b$ are $\mathfrak{K}$-arrows such that the length of i is $\leqslant m$ and the length of j is $\leqslant n$, then there exist $\mathfrak{K}$-arrows $k\colon a \to w$, $\ell\colon b \to w$ such that $k \circ i = \ell \circ j$ and k has length $\leqslant n$, while ℓ has length $\leqslant m$. Since every $\mathfrak{K}$-arrow has a finite length, this shows that $\mathfrak{K}$ has the amalgamation property.

A similar induction on the length of $\mathfrak{K}$-arrows shows that $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has the amalgamated extension property. Finally, using the fact that $\langle \mathcal{F}, \mathfrak{L} \rangle$ has the mixed amalgamation property, we prove by induction that for every $\mathfrak{K}$-arrow $i\colon c \to a$ of length $\leqslant n$, and for every $\mathfrak{L}$-arrow $f\colon c \to b$, there exist an $\mathfrak{L}$-arrow $g\colon a \to w$ and a $\mathfrak{K}$-arrow $\ell\colon b \to w$ of length $\leqslant n$ such that $g \circ i = \ell \circ f$. This shows that $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has the mixed amalgamation property. □

Another simplification for proving property $(\mathfrak{H})$ is the concept of mixed pushouts.

Let $\mathfrak{K} \subseteq \mathfrak{L}$ be two categories with the same objects. We say that $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has the *mixed pushout property* if for every arrows $f\colon c \to a$ and $g\colon c \to b$ such that $f \in \mathfrak{K}$ and $g \in \mathfrak{L}$, there exist arrows $f'\colon a \to w$ and $g'\colon b \to w$ such that $f' \in \mathfrak{L}$, $g' \in \mathfrak{K}$ and

$$\begin{array}{ccc} a & \xrightarrow{f'} & w \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g'} \\ c & \xrightarrow[g]{} & b \end{array}$$

is a pushout square in $\mathfrak{L}$. Note that if both f, g are $\mathfrak{K}$-arrows in the definition above, then so are f', g', by uniqueness of the pushout.

The definition above makes sense (and is applicable) in case where $\mathfrak{K}$ is an arbitrary family of arrows, not necessarily a subcategory. This is presented in the next statement.

Proposition 3.2. *Let $\mathfrak{K} \subseteq \mathfrak{L}$ be two categories with the same objects. Assume that $\mathfrak{K}$ has the joint embedding property and $\mathcal{F} \subseteq \mathfrak{K}$ is such that $\langle \mathcal{F}, \mathfrak{L} \rangle$ has the mixed pushout property and $\mathcal{F}$ generates $\mathfrak{K}$. Then $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has property $(\mathfrak{H})$.*

Proof. Suppose first that $\mathcal{F} = \mathfrak{K}$. The amalgamation property (condition $(\mathfrak{H}_1)$) follows from the remark above, namely that the pushout of two $\mathfrak{K}$-arrows consists of $\mathfrak{K}$-arrows. Mixed amalgamation property (condition $(\mathfrak{H}_2)$) is just a weaker version of the mixed pushout property. Finally, the amalgamated extension property (condition $(\mathfrak{H}_3)$) follows immediately from the definition of a pushout.

Suppose now that $\mathcal{F} \neq \mathfrak{K}$. It suffices to prove that $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has the mixed pushout property. Like in the proof of Proposition 3.1, we use induction on the length of $\mathfrak{K}$-arrows, having in mind that the obvious "composition" of two pushout squares is a pushout square. More precisely, the inductive hypothesis says: Given a $\mathfrak{K}$-arrow $i\colon c \to a$ of length $< n$, and an $\mathfrak{L}$-arrow $f\colon c \to b$, there exist an $\mathfrak{L}$-arrow $g\colon a \to w$ and a $\mathfrak{K}$-arrow $\ell\colon b \to w$ of length $< n$ such that

$$\begin{array}{ccc} b & \xrightarrow{\ell} & w \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ c & \xrightarrow[i]{} & a \end{array}$$

is a pushout square in $\mathfrak{L}$. □

Many natural pairs of categories, in particular coming from model theory, have the mixed pushout property. Concrete well-known examples are finite graphs, partially ordered sets, semilattices; each class is considered as a pair of two categories, the first one with embeddings and the second one with all homomorphisms. These examples are mentioned in [5]. A typical example of a pair $\langle \mathfrak{K}, \mathfrak{L} \rangle$ with property $(\mathfrak{H})$, failing the mixed pushout property is the category $\mathfrak{L}$ of all finite linear orders with increasing (i.e. order preserving) functions and $\mathfrak{K}$ the category of all finite linear orders with embeddings.

In contrast to the above results, it is worth to mention a Fraïssé class that does not fit into our framework. Namely, the Fraïssé class of finite K_n-free graphs (where K_n denotes the complete graph with n vertices and $n > 2$) has the pushout property (formally the class of embeddings has pushouts in the class of all homomorphisms), yet the corresponding pair of categories fails to have mixed amalgamations. Specifically, a graph is meant to be a structure with one symmetric irreflexive binary relation, so a homomorphism of graphs cannot identify vertices connected by edges. In other words, every graph homomorphism restricted to a complete subgraph becomes an embedding. It has been proved by Mudrinski [18] that for $n > 2$, the Fraïssé limit of K_n-free graphs (called the *Henson graph* H_n) is retract rigid, i.e. identity is the only retraction of H_n. On the other hand, we have the following easy fact (also stated in [6, Example 3.3]).

Proposition 3.3. *No K_n-free graph with $n > 2$ is injective for finite K_n-free graphs.*

Proof. Suppose X is such a graph. Using injectivity for $S = \emptyset$ and $T = K_{n-1}$, we see that X contains an isomorphic copy K of K_{n-1}. Now let S be a graph with $n-1$ vertices and no edges and let $f\colon S \to X$ be a bijection onto K. Let $T = S \cup \{v\}$, where v is connected to all the vertices of S. By injectivity, there exists a homomorphism $g\colon T \to X$ extending f. But now $K \cup \{g(v)\} \subseteq X$ is a copy of K_n, a contradiction. □

Before discussing concrete examples of pairs with property $(\mathfrak{H})$, we make a simple remark on injectivity. Recall that an arrow $j\colon x \to y$ is *left-invertible* in $\mathfrak{L}$ if there exists $f \in \mathfrak{L}$ such that $f \circ j = \mathrm{id}_x$. Following is an easy consequence of our main result.

Corollary 3.4. *Let $\langle \mathfrak{K}, \mathfrak{L} \rangle$ be a pair of categories such that every $\mathfrak{K}$-arrow is left-invertible in $\mathfrak{L}$. Assume that $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has property $(\mathfrak{H})$ and U is a Fraïssé sequence in $\mathfrak{K}$. Then for every sequence $X \in \mathrm{Ob}(\sigma\mathfrak{K})$ there exist a $\sigma\mathfrak{K}$-arrow $J\colon X \to U$ and a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $R\colon U \to X$ such that $R \circ J = \mathrm{id}_X$.*

Proof. In view of Theorem 2.7, it suffices to show that every sequence is $\mathfrak{K}$-injective in $\sigma(\mathfrak{K}, \mathfrak{L})$. Fix $X \in \mathrm{Ob}(\sigma\mathfrak{K})$, a $\mathfrak{K}$-arrow $j\colon a \to b$, and a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $f\colon a \to X$. Choose an $\mathfrak{L}$-arrow $r\colon b \to a$ such that $r \circ j = \mathrm{id}_a$. Then $g = f \circ r$ has the property that $g \circ j = f$. This shows that X is $\mathfrak{K}$-injective in $\sigma(\mathfrak{K}, \mathfrak{L})$. □

This corollary applies to finite Boolean algebras (also noted in [5]) and, as we shall see later, to finite linear orderings.

3.1 Fraïssé classes and algebraically closed models

Let $\mathfrak{M}$ be a class of finitely generated models of a fixed first-order language L. It is natural to consider the category ${}^{\mathrm{hom}}\mathfrak{M}$ whose objects are all elements of $\mathfrak{M}$ and arrows are all homomorphisms (i.e. maps that preserve all relations, functions and constants). It is also natural to consider the category ${}^{\mathrm{emb}}\mathfrak{M}$ whose objects are again all elements of $\mathfrak{M}$, while arrows are embeddings only. In many cases, $\langle {}^{\mathrm{emb}}\mathfrak{M}, {}^{\mathrm{hom}}\mathfrak{M}\rangle$ has property $(\mathfrak{H})$. Simplifying the notation, we shall say that $\mathfrak{M}$ has the *pushout property* or *mixed amalgamation property* if $\langle {}^{\mathrm{emb}}\mathfrak{M}, {}^{\mathrm{hom}}\mathfrak{M}\rangle$ has such a property. Denote by $\overline{\mathfrak{M}}$ the class of all (countable) models that are unions of ω-chains of models from $\mathfrak{M}$. It is clear that $\sigma^{\mathrm{emb}}\mathfrak{M}$ is equivalent to $\overline{\mathfrak{M}}$ with embeddings and $\sigma({}^{\mathrm{emb}}\mathfrak{M}, {}^{\mathrm{hom}}\mathfrak{M})$ is equivalent to $\overline{\mathfrak{M}}$ with all homomorphisms.

Recall that a model $X \in \overline{\mathfrak{M}}$ is *algebraically closed* if for every formula

$$\varphi(x_0, \dots, x_{k-1}, y_0, \dots, y_{\ell-1})$$

that is a finite conjunction of atomic formulae, for every $a_0, \dots, a_{k-1} \in X$, if there exists an extension $X' \supseteq X$ in $\overline{\mathfrak{M}}$ satisfying

$$X' \models (\exists\, y_0, \dots, y_{\ell-1})\ \varphi(a_0, \dots, a_{k-1}, y_0, \dots, y_{\ell-1})$$

then there exist $b_0, \dots, b_{\ell-1} \in X$ such that $X \models \varphi(a_0, \dots, a_{k-1}, b_0, \dots, b_{\ell-1})$.

Proposition 3.5. *Let $\mathfrak{M}$ be a class of finitely generated models of a fixed first-order language. Every $\mathfrak{M}$-injective model in $\overline{\mathfrak{M}}$ is algebraically closed.*

Proof. Fix an $\mathfrak{M}$-injective model $X \in \overline{\mathfrak{M}}$. Fix $X' \supseteq X$ and assume $X' \models (\exists\, \vec{y})\ \varphi(\vec{a}, \vec{y})$ for some k-tuple $\vec{a}$ of elements of X, where $\varphi(\vec{x}, \vec{y})$ is a finite conjunction of atomic formulae and $\vec{x}, \vec{y}$ are shortcuts for $(x_0, \dots, x_{k-1})$ and $(y_0, \dots, y_{\ell-1})$, respectively.

Let $S \in \mathfrak{M}$ be a submodel of X that contains $\vec{a}$. Let $T \in \mathfrak{M}$ be a submodel of X' containing S and a fixed tuple $\vec{b}$ such that $X' \models \varphi(\vec{a}, \vec{b})$. Then also $T \models \varphi(\vec{a}, \vec{b})$, because this property is absolute for φ. Using the $\mathfrak{M}$-injectivity of X, find a homomorphism $f\colon T \to X$ satisfying $f \restriction S = \mathrm{id}_S$. Finally, let $\vec{c} = (f(b_0), \dots, f(b_{\ell-1}))$, where $\vec{b} = (b_0, \dots, b_{\ell-1})$. Since f is a homomorphism and φ is a conjunction of atomic formulae, we have that $X \models \varphi(\vec{a}, \vec{c})$. □

We shall say that a structure M is *n-generated* if there exists $S \subseteq M$ such that $|S| \leqslant n$ and S generates M, that is, no proper submodel of M contains S. Recall that a first-order language is *finite* if it contains finitely many predicates (constant, relation and function symbols).

Proposition 3.6. *Let $\mathfrak{M}$ be a class of finite models of a fixed first-order language L. Assume that either L is finite or for every $n \in \omega$ there exist finitely many isomorphic types of n-generated models in $\mathfrak{M}$. Assume furthermore that $\mathfrak{M}$ has the mixed amalgamation property. Then every algebraically closed L-model $X \in \overline{\mathfrak{M}}$ is $\mathfrak{M}$-injective.*

Proof. Fix $S, T \in \mathfrak{M}$ such that S is a submodel of T. Fix a homomorphism $f\colon S \to X$. Using the mixed amalgamation, we can find an extension $X' \in \overline{\mathfrak{M}}$ of X and a homomorphism $f'\colon T \to X'$ such that $f' \restriction S = f$. Let $\mathcal{G}$ be the set of all functions $g\colon T \to X$ satisfying $g \restriction S = f$. We need to show that some $g \in \mathcal{G}$ is a homomorphism.

Suppose first that there exist only finitely many $|T|$-generated structures in $\mathfrak{M}$ and let $\mathcal{N} \subseteq \mathfrak{M}$ be a finite set that contains isomorphic types of all of them.

Given $g \in \mathcal{G}$, denote by g' a fixed isomorphism from the submodel generated by $g[T]$ onto a fixed model from the collection $\mathcal{N}$. Note that g is a homomorphism if and only if $g' \circ g$ is a homomorphism. Now observe that the set $\mathcal{H} = \{g' \circ g\colon g \in \mathcal{G}\}$ is finite.

Let $S = \{s_i\}_{i<k}$ and $T \setminus S = \{t_j\}_{j<\ell}$. Fix $g \in \mathcal{G}$ and suppose it is not a homomorphism. There exists either a relation R or a function F and a finite sequence of elements of T that witness this fact. Let ψ_g be an atomic formula describing this fact. We may assume that ψ_g has $k+\ell$ free variables, the first k are supposed to denote $s_0, \dots, s_{k-1}$ and the latter ones $t_0, \dots, t_{\ell-1}$. Let φ be the conjunction of all formulae ψ_g, where $g \in \mathcal{G}$. Then $T \models \varphi(s_0, \dots, s_{k-1}, t_0, \dots, t_{\ell-1})$ and, since f' is a homomorphism,

$$X' \models \varphi\Big(f(s_0), \dots, f(s_{k-1}), f'(t_0), \dots, f'(t_{\ell-1})\Big). \tag{1}$$

Using the fact that X is algebraically closed, find $\vec{u} = (u_0, \dots, u_{\ell-1})$ in X such that

$$X \models \varphi\Big(f(s_0), \dots, f(s_{k-1}), u_0, \dots, u_{\ell-1}\Big). \tag{2}$$

Let $g \in \mathcal{G}$ be such that $g(t_j) = u_j$ for $j < \ell$. Then g is a homomorphism. Indeed, otherwise there would be a witness (a relation or a function, plus some elements of T) saying that $g' \circ g$ is not a homomorphism; however φ "knows" all these witnesses, which gives rise to a contradiction.

Suppose now that L is finite and consider again the set $\mathcal{G}$. For each $g \in \mathcal{G}$, if g is not a homomorphism, this is witnessed by an atomic formula ψ_g and some elements of T. Now, even though the set $\mathcal{G}$ may be infinite, the number of atomic formulae with parameters in T is finite. As before, let $\varphi(\vec{x}, \vec{y})$ collect all of them. Again, $X \models \varphi(\vec{s}, \vec{t})$ and consequently (1) holds. Since X is algebraically closed, we can find $\vec{u}$ such that (2) holds. Finally, $g \in \mathcal{G}$ satisfying $g(t_j) = u_j$ $(j < \ell)$ is the desired homomorphism. □

Following Dolinka [5], we say that a class of models $\mathfrak{M}$ has the *1-point homomorphism extension property* (briefly: 1PHEP) if for every embedding $i\colon A \to B$ and for every surjective homomorphism $f\colon A \to C$, where $A, B, C \in \mathfrak{M}$ and B is generated by $A \cup \{b\}$ for some $b \in B$, there exist an embedding $j\colon C \to D$ and a homomorphism $g\colon B \to D$ for which the diagram

$$\begin{array}{ccc} C & \xrightarrow{\ j\ } & D \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ A & \xrightarrow[\ i\ ]{} & B \end{array}$$

commutes. Let us say that an embedding $i\colon A \to B$ is *primitive* if B is generated by one element from $i[A]$. Clearly, every embedding is a composition of primitive embeddings. Furthermore, every homomorphism is the composition of a surjective homomorphism and an embedding. These facts, together with easy induction (see Proposition 3.1) show that 1PHEP is equivalent to the mixed amalgamation property of $\langle {}^{\mathrm{emb}}\mathfrak{M}, {}^{\mathrm{hom}}\mathfrak{M}\rangle$.

Combining Theorem 2.7, Propositions 3.5, 3.6 and the remarks above, we obtain a strengthening of Dolinka's result [5]:

Corollary 3.7. *Let $\mathfrak{M}$ be a Fraïssé class of finite models of a fixed first-order language L. Assume that L is finite or for every $n \in \omega$ the number of isomorphism types of n-generated structures in $\mathfrak{M}$ is finite. Assume further that $\mathfrak{M}$ has the pushout property and the 1PHEP. Let $U \in \overline{\mathfrak{M}}$ be the Fraïssé limit of $\mathfrak{M}$. For a model $X \in \overline{\mathfrak{M}}$ the following conditions are equivalent.*

(a) X is a retract of U.

(b) X is algebraically closed.

The "pushout property" in the statement above means that ${}^{\mathrm{emb}}\mathfrak{M}$ has pushouts in ${}^{\mathrm{hom}}\mathfrak{M}$. This assumption may of course be replaced by a weaker one, namely, that $\langle {}^{\mathrm{emb}}\mathfrak{M}, {}^{\mathrm{hom}}\mathfrak{M}\rangle$ has the amalgamated extension property.

Note that a Fraïssé class of finite models of a finite language mail fail the condition concerning the number of n-generated models. For example, let L consist of a unique unary function symbol P and let $\mathfrak{M}$ be the class of all finite L-models. That is, every model $S \in \mathfrak{M}$ is endowed with a function $P^S\colon S \to S$ and $f\colon S \to T$ is a homomoprhism iff $f(P^S(x)) = P^T(f(x))$ for every $x \in S$. It is an easy exercise to check that $\mathfrak{M}$ is a Fraïssé class with the mixed pushout property, therefore the corollary above applies. On the other hand, for each $n \in \omega$ there exists a 1-generated structure $S_n \in \mathfrak{M}$ of cardinality n. Namely, $S_n = \{0, \dots, n-1\}$ with the function P defined by $P(n-1) = 0$ and $P(i) = i+1$ for $i < n-1$. Thus, there are infinitely many 1-generated structures in $\mathfrak{M}$. Note that a countable L-structure $\langle X, P\rangle$ belongs to $\overline{\mathfrak{M}}$ if and only if for every finite set $A \subseteq X$ there exists a finite set $S \subseteq X$ such that $A \subseteq S$ and $P[S] \subseteq S$.

There are some natural Fraïssé classes of finite models of infinite languages and with infinitely many n-generated structures, for which Proposition 3.6 (and consequently the corollary above) still hold. In section 3.3 below, we shall investigate Fraïssé classes of metric spaces, showing that the possibility of characterizing injectivity by "being algebraic closed" depends on the language specifying the objects.

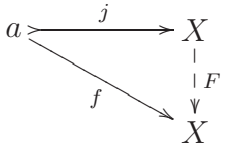
3.2 A note on homomorphism-homogeneous structures

In connection with (classical model-theoretic) Fraïssé limits, there is an interesting notion of homomorphism-homogeneous structures, introduced recently by Cameron and Nešetřil [4] and studied already by several authors (see [16], [10], [3], [20], [17]). Namely,

a (usually countable) structure M is *homomorphism-homogeneous* if every homomorphism between its finitely-generated substructures extends to an endomorphism of M. It is clear that this notion can be defined in category-theoretic language, using a pair of categories $\langle \mathfrak{K}, \mathfrak{L} \rangle$ as before, where $\mathfrak{L}$-arrows mean "homomorphisms" and $\mathfrak{K}$-arrows mean "embeddings". It turns out that homomorphism-homogeneity is strictly related to injectivity, as we show below.

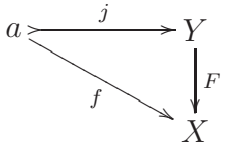
As usual, fix two categories $\mathfrak{K} \subseteq \mathfrak{L}$ with the same objects.

Definition 3.8. We say that an object $X \in \mathrm{Ob}(\sigma\mathfrak{K})$ is $\mathfrak{L}$-*homogeneous in* $\sigma(\mathfrak{K}, \mathfrak{L})$ if for every $\sigma\mathfrak{K}$-arrow $j\colon a \to X$ such that $a \in \mathrm{Ob}(\mathfrak{K})$, for every $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $f\colon a \to X$, there exists a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $F\colon X \to X$ satisfying $F \circ j = f$. This is described in the diagram below.



Note that the arrow f is of the form $x_n^\infty \circ f'$ for some $f' \in \mathfrak{L}$. That is why the definition above really speaks about $\mathfrak{L}$-homogeneity, not $\sigma(\mathfrak{K}, \mathfrak{L})$-homogeneity. It can actually be viewed as a variation on the mixed amalgamation property, which is witnessed by the results below.

Lemma 3.9. *Let $\langle \mathfrak{K}, \mathfrak{L} \rangle$ be a pair of categories such that $\mathfrak{K} \subseteq \mathfrak{L}$ and let $X, Y \in \mathrm{Ob}(\sigma\mathfrak{K})$ be such that X is $\mathfrak{K}$-injective in $\sigma(\mathfrak{K}, \mathfrak{L})$. Then for every $\sigma\mathfrak{K}$-arrow $j\colon a \to Y$ with $a \in \mathrm{Ob}(\mathfrak{K})$, for every $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $f\colon a \to X$, there exists a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $F\colon Y \to X$ for which the diagram*



commutes.

Proof. The arrow j factorizes through some y_k, that is, $j = y_k^\infty \circ i$ for some $\mathfrak{K}$-arrow $i\colon a \to y_k$. Using $\mathfrak{K}$-injectivity, we construct inductively $\mathfrak{L}$-arrows $f_n\colon y_n \to X$ for $n \geqslant k$ so that $f_k \circ i = f$ and $f_{n+1} \circ y_n^{n+1} = f_n$ for $n > k$. This gives rise to an arrow of sequences $F = \{f_n\}_{n \geqslant k}$ satisfying $F \circ j = f$. □

Letting $X = Y$ in the lemma above, we obtain:

Corollary 3.10. *Let $\mathfrak{K} \subseteq \mathfrak{L}$ be a pair of categories. Every $\mathfrak{K}$-injective object is $\mathfrak{L}$-homogeneous in $\sigma(\mathfrak{K}, \mathfrak{L})$.*

The equivalence (b)$\Longleftrightarrow$(c) in the next statement, in the context of model theory, has been noticed by Dolinka [6, Prop. 3.8].

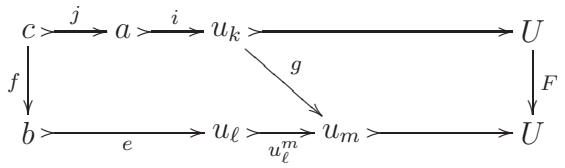
Proposition 3.11. *Let $\mathfrak{K} \subseteq \mathfrak{L}$ be a pair of categories and let $\mathfrak{K}$ have a Fraïssé sequence $U \in \mathrm{Ob}\,(\sigma\mathfrak{K})$. The following properties are equivalent:*

(a) U is $\mathfrak{K}$-injective in $\sigma(\mathfrak{K}, \mathfrak{L})$.

(b) U is $\mathfrak{L}$-homogeneous in $\sigma(\mathfrak{K}, \mathfrak{L})$.

(c) $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has the mixed amalgamation property.

Proof. Implication (c) $\Longrightarrow$ (a) has been proved in Proposition 2.4. Implication (a) $\Longrightarrow$ (b) is a consequence of Corollary 3.10. It remains to show that (b) $\Longrightarrow$ (c).
Suppose U is $\mathfrak{L}$-homogeneous in $\sigma(\mathfrak{K}, \mathfrak{L})$ and fix a $\mathfrak{K}$-arrow $j\colon c \to a$ and an $\mathfrak{L}$-arrow $f\colon c \to b$. Using the property of being a Fraïssé sequence, find $\mathfrak{K}$-arrows $i\colon a \to u_k$ and $e\colon b \to u_\ell$ with some $k, \ell < \omega$. Since U is $\mathfrak{L}$-homogeneous, there exists a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $F\colon U \to U$ satisfying $F \circ u^\infty_k \circ i \circ j = u^\infty_\ell \circ e \circ f$. Finally, find an $\mathfrak{L}$-arrow $g\colon u_k \to u_m$ with $m > \ell$, such that $u^\infty_m \circ g = F \circ u^\infty_k$. The situation is described in the following diagram.



Thus, j and f are amalgamated by a $\mathfrak{K}$-arrow $u^m_\ell \circ e$ and an $\mathfrak{L}$-arrow $g \circ i$. □

Under certain natural assumptions, we are able to characterize homomorphism-homogeneous objects. In the next statement we speak about countable categories, but what we really have in mind is the existence of countably many isomorphic types of arrows. For example, the category of finite sets is a proper class, yet it can obviously be regarded as a countable category.

Theorem 3.12. *Let $\mathfrak{K} \subseteq \mathfrak{L}$ be a pair of categories such that $\langle \mathfrak{K}, \mathfrak{L} \rangle$ has the mixed pushout property, $\mathfrak{L}$ is countable, and $\mathfrak{K}$ has the initial object 0. For a sequence $X \in \mathrm{Ob}\,(\sigma\mathfrak{K})$, the following properties are equivalent.*

(a) X is $\mathfrak{L}$-homogeneous in $\sigma(\mathfrak{K}, \mathfrak{L})$.

(b) There exists a subcategory $\mathfrak{K}_0$ of $\mathfrak{K}$ such that 0 is initial in $\mathfrak{K}_0$, $X \in \mathrm{Ob}\,(\sigma\mathfrak{K}_0)$, $\langle \mathfrak{K}_0, \mathfrak{L} \rangle$ has the mixed pushout property, and X is $\mathfrak{K}_0$-injective in $\sigma(\mathfrak{K}_0, \mathfrak{L})$.

(c) There exists a subcategory $\mathfrak{K}_0$ of $\mathfrak{K}$ such that 0 is initial in $\mathfrak{K}_0$, $X \in \mathrm{Ob}\,(\sigma\mathfrak{K}_0)$, $\langle \mathfrak{K}_0, \mathfrak{L} \rangle$ has the mixed pushout property, and X is a retract of a Fraïssé sequence in $\mathfrak{K}_0$.

The existence of the initial object in $\mathfrak{K}$ is not essential, but removing it we would have to make more technical assumptions involving the joint embedding property.

Proof. The equivalence (b)$\Longleftrightarrow$(c) is contained in Theorem 2.7. The fact that $\mathfrak{K}_0$ is countable has been used here for the existence of a Fraïssé sequence. Implication (b) $\Longrightarrow$ (a) is contained in Corollary 3.10. It remains to show that (a) $\Longrightarrow$ (b).

We may assume that $x_0 = 0$ in the sequence X. Let $\mathcal{S} = \{x_n^m \colon n \leqslant m,\ n, m \in \omega\}$. Then $\mathcal{S}$ is a subcategory of $\mathfrak{K}$ that contains the intial object 0. We first check that X is $\mathcal{S}$-injective. Fix a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $f \colon x_n \to X$ and fix $m > n$. Since X is $\mathfrak{L}$-homogeneous, there exists a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $F \colon X \to X$ satisfying $F \circ x_n^\infty = f$. Note that $x_n^\infty = x_m^\infty \circ x_n^m$, therefore $(F \circ x_m^\infty) \circ x_n^m = f$, which shows the $\mathcal{S}$-injectivity of X.

Now let $\mathfrak{K}_0$ consist of all $\mathfrak{K}$-arrows $j \colon c \to a$ such that X is j-injective in $\sigma(\mathfrak{K}, \mathfrak{L})$ and there exists at least one $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow from c to X. That is, for every $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $f \colon c \to X$, there exists a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $g \colon a \to X$ satisfying $g \circ j = f$. The second assumption is needed for keeping 0 initial in $\mathfrak{K}_0$, namely, X should also be injective for the (unique) arrow $0 \to a$. It is clear that $\mathfrak{K}_0$ is a subcategory of $\mathfrak{K}$ containing $\mathcal{S}$. In particular, $X \in \mathrm{Ob}\,(\sigma\mathfrak{K}_0)$. It remains to show that $\langle \mathfrak{K}_0, \mathfrak{L}\rangle$ has the mixed pushout property. For this aim, fix an $\mathfrak{K}_0$-arrow $j \colon c \to b$, an $\mathfrak{L}$-arrow $p \colon c \to a$, and let $k \colon a \to w$, $q \colon b \to w$ be such that $k \in \mathfrak{K}$, $q \in \mathfrak{L}$ and

$$\begin{array}{ccc} b & \xrightarrow{q} & w \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle k} \\ c & \xrightarrow[p]{} & a \end{array}$$

is a pushout square in $\mathfrak{L}$. Fix a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $f \colon a \to X$. Since X is j-injective, there exists a $\sigma(\mathfrak{K}, \mathfrak{L})$-arrow $g \colon b \to X$ satisfying $g \circ j = f \circ p$. Both arrows f and g are factorized through some x_n, namely, $f = x_n^\infty \circ f'$ and $g = x_n^\infty \circ g'$ for some $\mathfrak{L}$-arrows f', g'. Using the property of a pushout, we find a unique $\mathfrak{L}$-arrow $h \colon w \to x_n$ satisfying $h \circ k = f'$ and $h \circ q = g'$. In particular, $\overline{f} = x_n^\infty \circ h$ has the property that $\overline{f} \circ k = f$. This shows that X is k-injective in $\sigma(\mathfrak{K}, \mathfrak{L})$ and completes the proof. □

Unfortunately, the result above is not fully applicable to Fraïssé classes. Namely, in case where $\mathfrak{K}$ is a countable Fraïssé class, $\mathfrak{K}_0$ may not be a full subcategory of $\mathfrak{K}$. This is demonstrated below, for the class of finite graphs.

Example 3.13. Let X be the two-element complete graph. It is clear that X is homomorphism-homogeneous (and also ultrahomogeneous). We consider graphs without loops, therefore every endomorphism of X is an automorphism. More precisely, we consider the pair $\langle \mathfrak{K}, \mathfrak{L}\rangle$, where $\mathrm{Ob}\,(\mathfrak{K}) = \mathrm{Ob}\,(\mathfrak{L})$ are all finite simple graphs, the $\mathfrak{K}$-arrows are embeddings and the $\mathfrak{L}$-arrows are graph homomorphisms.

Let $\mathfrak{K}_0$ be any subcategory of $\mathfrak{K}$ that has pushouts in $\mathfrak{L}$ and contains all the embeddings of subgraphs of X. So, $\mathrm{Ob}\,(\mathfrak{K}_0)$ contains the empty graph and complete subgraphs of size $\leqslant 2$. The pushout with embeddings of the empty graph is just the coproduct (disjoint sum), there $\mathrm{Ob}\,(\mathfrak{K}_0)$ contains the 2-element graph D with no edges. Furthermore, $\mathrm{Ob}\,(\mathfrak{K}_0)$ contains the graph G whose set of vertices is $\{-1, 0, 1\}$ and the edges are

$\{-1, 0\}$ and $\{0, 1\}$. Such a graph comes from the pushout of two embeddings of the one-element graph into X. Now consider an embedding $j\colon D \to G$ such that $j[D] = \{-1, 1\}$. Let $f\colon D \to X$ be one-to-one. Clearly, f is a homomorphism and no homomorphism $g\colon G \to X$ satisfies $g \circ j = f$. This shows that X is not j-injective. In particular, $\mathfrak{K}_0$ is not a full subcategory of $\mathfrak{K}$.

3.3 Metric spaces

We shall now discuss a concrete model-theoretic application of our result: Retracts of the universal metric space of Urysohn. Let $\mathfrak{M}$ be the category of finite metric spaces with isometric embeddings. The objects of $\mathfrak{M}$ are models of a first-order language: For each rational $r > 0$ we can define the binary relation $D_r(x, y) \iff d(x, y) < r$, where d denotes the metric on a fixed set X. The axioms of a metric can be rephrased in terms of the relations D_r. For example, the triangle inequality follows from the following (infinitely many) formulae:

$$D_r(x, z) \wedge D_s(z, y) \implies D_{r+s}(x, y).$$

It is clear that, in this language, a homomorphism of metric spaces is a non-expansive map. Recall that $f\colon X \to Y$ is *non-expansive* if $d_Y(f(p), f(q)) \leqslant d_X(p, q)$ for every $p, q \in X$, where d_X, d_Y denote the metrics on X and Y respectively.

It is also possible to describe a metric space by similar relations D_r, now meaning that the distance is $\leqslant r$. We shall see later that, even though both languages describe the same objects, the notion of being algebraically closed is completely different.

Clearly, the language of metric spaces is infinite and there exist infinitely many types of 2-element metric spaces (even when restricting to rational distances), therefore one cannot apply Dolinka's result here. Moreover, $\mathfrak{M}$ is formally not a Fraïssé class, because it contains continuum many pairwise non-isomorphic objects. It becomes a Fraïssé class when restricting to spaces with rational distances. However, in that case we cannot speak about complete metric spaces. In any case, our main result is applicable to the complete metric space of Urysohn, as we show below.

The following lemma, in a slightly different form, can be found in [7, Lemma 3.5].

Lemma 3.14. *Let $f\colon X \to Y$ be a non-expansive map of nonempty finite metric spaces. Assume $X \cup \{a\}$ is a metric extension of X. Then there exists a metric extension $Y \cup \{b\}$ of Y such that*

$$\begin{array}{ccc} Y & \rightarrowtail & Y \cup \{b\} \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ X & \rightarrowtail & X \cup \{a\} \end{array}$$

where $g \restriction X = f$ and $g(a) = b$, is a pushout square in the category of metric spaces with non-expansive maps. Furthermore

$$d(y,b) = \min_{x\in X}\Big(d(y,f(x)) + d(x,a)\Big) \tag{M}$$

for every $y \in Y$.

The statement obviously fails when $X = \emptyset$ and $Y \neq \emptyset$.

Proof. We first need to show that (M) defines a metric on $Y \cup \{b\}$. Of course, only the triangle inequality requires an argument. Fix $y, y_1 \in Y$. Find $x_1 \in X$ such that $d(y_1,b) = d(y_1,f(x_1)) + d(x_1,a)$. Using the triangle inequality in Y, we get

$$d(y,b) \leqslant d(y,f(x_1)) + d(x_1,a) \leqslant d(y,y_1) + d(y_1,f(x_1)) + d(x_1,a) = d(y,y_1) + d(y_1,b).$$

Now find $x \in X$ such that $d(y,b) = d(y,f(x)) + d(x,a)$. Using the triangle inequalities in X and Y, and the fact that $d(f(x),f(x_1)) \leqslant d(x,x_1)$, we obtain

$$\begin{aligned} d(y,b) + d(y_1,b) &= d(y,f(x)) + d(x,a) + d(y_1,f(x_1)) + d(x_1,a) \\ &\geqslant d(y,f(x)) + d(x,x_1) + d(y_1,f(x_1)) \\ &\geqslant d(y,f(x)) + d(f(x),f(x_1)) + d(y_1,f(x_1)) \\ &\geqslant d(y,y_1). \end{aligned}$$

Thus, d defined by (M) fulfills the triangle inequality.
Given $x \in X$, we have $d(g(x),g(a)) = d(f(x),b) \leqslant d(f(x),f(x)) + d(x,a) = d(x,a)$. This shows that g is non-expansive.
Finally, assume $p\colon X \cup \{a\} \to W$ and $q\colon Y \to W$ are non-expansive maps such that $p \restriction X = q \circ f$. We need to show that there exists a unique non-expansive map $h\colon Y \cup \{b\} \to W$ satisfying $h \circ g = p$ and $h \restriction Y = q$. The uniqueness of h is clear, namely $h(b) = h(g(a)) = p(a)$. It remains to verify that h is non-expansive.
Suppose otherwise and fix $y \in Y$ such that $d(h(y),h(b)) > d(y,b)$. Find $x \in X$ such that $d(y,b) = d(y,f(x)) + d(x,a)$. So we have

$$d(h(y),p(a)) > d(y,f(x)) + d(x,a). \tag{*}$$

Knowing that p and q are non-expansive, we get

$$d(p(x),p(a)) \leqslant d(x,a) \qquad \text{and} \qquad d(q(y),q(f(x)) \leqslant d(y,f(x)). \tag{**}$$

Note that $q(f(x)) = p(x)$ and $q(y) = h(y)$. Finally, (*) and (**) give

$$d(h(y),p(a)) > d(p(x),p(a)) + d(h(y),p(x))$$

which contradicts the triangle inequality in W. This completes the proof. □

We say that a metric space $\langle X, d\rangle$ is *finitely hyperconvex* if for every finite family of closed balls

$$\mathcal{A} = \big\{\overline{\mathrm{B}}(x_0, r_0), \overline{\mathrm{B}}(x_1, r_1), \dots, \overline{\mathrm{B}}(x_{n-1}, r_{n-1})\big\}$$

such that $\bigcap \mathcal{A} = \emptyset$, there exist $i, j < n$ such that

$$d(x_i, x_j) > r_i + r_j.$$

This is a weakening of the notion of a *hyperconvex metric space*, due to Aronszajn & Panitchpakdi [2], where the family above may be of arbitrary cardinality. Actually, the authors of [2] had already considered κ-hyperconvex metric spaces; finite hyperconvexity corresponds to $\aleph_0$-hyperconvexity. A variant of finite hyperconvexity (with closed balls replaced by open balls) has been recently studied by Niemiec [19] in the context of topological absolute retracts.

The following facts relate this definition to our main topic. The first one should be well known for readers familiar with the notion of a hyperconvex metric space, namely, every metric space embeds isometrically into a hyperconvex one.

Lemma 3.15. *Let X be a finite metric space and let $\mathcal{A} = \{\overline{\mathrm{B}}(x_i, r_i)\}_{i<N}$ be a family of closed balls such that $N \in \omega$ and $d(x_i, x_j) \leqslant r_i + r_j$ for every $i, j < N$. Then there exists a metric extension $X \cup \{a\}$ of X such that $d(a, x_i) \leqslant r_i$ for every $i < N$.*

Proof. Fix $a \notin X$ and define

$$(*) \qquad d(a, x) = \min_{i<N}\Big(d(x, x_i) + r_i\Big).$$

Obviously, $d(a, x_i) \leqslant r_i$. It remains to check that (*) indeed defines a metric on $X \cup \{a\}$. It is the triangle inequality that requires a proof. Fix $x, y \in X$ and fix $k < N$ such that $d(a, y) = d(y, x_k) + r_k$. Then

$$d(a, x) \leqslant d(x, x_k) + r_k \leqslant d(x, y) + d(y, x_k) + r_k = d(x, y) + d(y, a).$$

This shows that $d(x, a) \leqslant d(x, y) + d(y, a)$. Now fix $i < N$ such that $d(a, x) = d(x, x_i) + r_i$. We have that $d(x_i, x_k) \leqslant r_i + r_k$, therefore

$$\begin{aligned} d(a, x) + d(a, y) &= d(x, x_i) + r_i + d(y, x_k) + r_k \\ &\geqslant d(x, x_i) + d(y, x_k) + d(x_i, x_k) \geqslant d(x, y). \end{aligned}$$

This shows that d defined by (*) satisfies the triangle inequality. □

Lemma 3.16. *A metric space is finitely hyperconvex if and only if it is injective with respect to isometric embeddings of finite metric spaces.*

Proof. Let X be a finitely hyperconvex metric space and fix a non-expansive map $f\colon S \to X$, where S is a finite metric space. It suffices to show that f can be extended to a non-expansive map $f'\colon T \to X$ whenever T is a metric extension of S and $T \setminus S = \{a\}$. Fix $T = S \cup \{a\}$ and let

$$\mathcal{A} = \{\overline{\mathrm{B}}(f(s), r_s)\colon s \in S\},$$

where $r_s = d(s, a)$. Given $s, s_1 \in S$, we have that $d(f(s), f(s_1)) \leqslant d(s, s_1) \leqslant r_s + r_{s_1}$. Since X is finitely hyperconvex, there exists $b \in \bigcap \mathcal{A}$. This means that $d(b, f(s)) \leqslant d(s, a)$ for every $s \in S$. Thus, setting $f'(a) = b$ and $f' \restriction S = f$, we obtain a non-expansive extension of f. This shows the "only if" part.

For the "if" part, fix a family $\mathcal{A} = \{\overline{\mathrm{B}}(x_i, r_i)\}_{i<N}$ in X, so that $d(x_i, x_j) \leqslant r_i + r_j$ for $i, j < N$. Let $S = \{x_0, x_1, \dots, x_{N-1}\}$ and endow S with the metric inherited from X. Let $T = S \cup \{a\}$ be a metric extension of S such that $d(a, x_i) \leqslant r_i$ for $i < N$. It exists by Lemma 3.15. Applying the injectivity of X, we can find a non-expansive extension $g\colon T \to X$ of the inclusion $S \subseteq X$. Let $b = g(a)$. Then $d(b, x_i) \leqslant d(a, x_i) \leqslant r_i$ for $i < N$. This shows that $\bigcap \mathcal{A} \neq \emptyset$. □

Theorem 3.17. *Given a Polish space $\langle X, d\rangle$, the following properties are equivalent:*

(a) $\langle X, d\rangle$ is a non-expansive retract of the universal Urysohn space $\mathbb{U}$.

(b) $\langle X, d\rangle$ is finitely hyperconvex.

(b') $\langle X, d\rangle$ is injective with respect to isometric embeddings of finite metric spaces.

Proof. The equivalence (b)$\Longleftrightarrow$(b') is contained in Lemma 3.16.

(a) $\Longrightarrow$ (b') Assume $X \subseteq \mathbb{U}$ and $r\colon \mathbb{U} \to X$ is a non-expansive retraction. Fix finite metric spaces $S \subseteq T$ and a non-expansive map $f\colon S \to X$. Using the mixed pushout property (a consequence of Lemma 3.14 and Proposition 3.2), we can find an isometric embedding $j\colon f[S] \to W$ and a non-expansive map $g\colon T \to W$ such that W is a finite metric space and $g \restriction S = j \circ f$. Using the ultrahomogeneity of $\mathbb{U}$, we can find an isometric embedding $h\colon W \to \mathbb{U}$ such that $h \circ j$ is the inclusion $f[S] \subseteq \mathbb{U}$. Finally, let $p = r \circ h \circ g$. Then $p\colon T \to X$ is a non-expansive map and $p \restriction S = f$.

(b') $\Longrightarrow$ (a) Fix a Polish space X satisfying (b'). Fix a countable dense set $D \subseteq X$. Let $K_0 = \mathbb{Q} \cup \{d(x, y)\colon x, y \in D\}$ and let K be the subsemigroup of $\langle \mathbb{R}, +\rangle$ generated by K_0. Consider the category of nonempty finite metric spaces with distances in K (we call them *K-metric spaces*). This category is countable, therefore it has a Fraïssé sequence. This Fraïssé sequence defines a countable metric space E whose completion is, by uniqueness, the Urysohn space. Enlarging D to a countable set, we may assume that it is injective with respect to isometric embeddings of finite K-metric spaces. By Theorem 2.7, D is a non-expansive retract of E and consequently X is a non-expansive retract of $\mathbb{U}$. □

It is easy to check that the results above remain valid when restricting to S-metric spaces, where S is a fixed subsemigroup of $\langle [0,+\infty),+\rangle$. In case S is proper and closed in $[0,+\infty)$, the universal ultrahomogeneous S-metric space is different from the Urysohn space. Further generalization is obtained by bounding the distances by some constanct $C \in S \cup \{+\infty\}$. Namely, one can consider the class $\mathfrak{M}_S^C$ of finite metric spaces $\langle X,d\rangle$, where $d[X\times X] \subseteq S \cap [0,C]$. It is clear that analogues of Lemmata 3.14, 3.15 and 3.16 hold for the class $\mathfrak{M}_S^C$, when $C = +\infty$. For the case $C < +\infty$, notice that there is a natural functor mapping $\langle X,d\rangle \in \mathfrak{M}_S^{+\infty}$ to $\langle X,d_C\rangle \in \mathfrak{M}_S^C$, where

$$d_C(x,y) = \min\{d(x,y), C\}.$$

Applying this functor, we can easily conclude that Lemmata 3.14, 3.15 and 3.16 hold for arbitrary classes of the form $\mathfrak{M}_S^C$. However, we need to specify the more general version of hyperconvexity. Namely, we say that $\langle X,d\rangle$ is *finitely (S,C)-hyperconvex* if for every finite family of closed balls $\mathcal{B} = \{\overline{\mathrm{B}}(x_i,r_i)\}_{i<n}$ with $r_i \in S \cap (0,C]$ for $i<n$, it holds that $\bigcap \mathcal{B} \neq \emptyset$ whenever $d(x_i,x_j) \leqslant r_i + r_j$ for every $i,j<n$. Finally, we obtain the following generalization of Theorem 3.17.

Theorem 3.18. *Let S be a closed subsemigroup of $\langle [0,+\infty),+\rangle$ and let $C \in S$ be a positive constant. Given a separable C-bounded complete S-metric space X, the following conditions are equivalent.*

(a) X is a non-expansive retract of the C-bounded S-metric Urysohn space $\mathbb{U}_S^C$.

(b) X is finitely (S,C)-hyperconvex.

(c) X is injective with respect to isometric embeddings of finite C-bounded S-metric spaces.

Note that a particular case $S = \omega$ and $C = 2$ can be regarded as the class of finite simple graphs, where $d(x,y) = 1$ iff there is an edge between x and y. Being finitely hyperconvex is rephrased to "algebraically closed" in the language of graphs[1].

The theorem above speaks about complete metric spaces, so in most cases it is not about Fraïssé classes. However, we can also formulate a version dealing with a countable (not necessarily closed) subsemigroup S of $[0,+\infty)$. In that case, $\mathfrak{M}_S$ is a Fraïssé class (since it is countable) and we can consider its Fraïssé limit $U_S \in \overline{\mathfrak{M}}_S$, a (usually non-complete) countable ultrahomogeneous S-metric space. By the remarks above, we conclude that $X \in \overline{\mathfrak{M}}_S$ is a non-expansive retract of U_S if and only if it is finitely S-hyperconvex. This gives rise to the announced example showing that "being algebraically closed" for metric spaces may or may not be equivalent to injectivity.

[1] Actually, when viewing a graph as a metric space, a homomorphism allows identifying vertices connected by edges.

Example 3.19. Consider the class $\mathfrak{M}_{\mathbb{Q}}$ of finite rational metric spaces. Assume that the language consists of relations D_r ($r \in \mathbb{Q}$), where $D_r(x,y)$ means "$d(x,y) < r$". Using finite conjunctions of atomic formulae, there is no way to say that X is finitely $\mathbb{Q}$-hyperconvex. Indeed, consider $\mathbb{Q}$ as a metric space with the usual distance and take $X = \mathbb{Q} \setminus \{1\}$. Clearly, $\mathbb{Q}$ is finitely $\mathbb{Q}$-hyperconvex, hence algebraically closed (see Proposition 3.5). Thus, X is algebraically closed too, because of the strict inequalities in the relations D_r. On the other hand, X is obviously not $\mathfrak{M}_{\mathbb{Q}}$-injective: The inclusion $\{0,2\} \subseteq X$ has no non-expansive extension onto $\{0,1,2\}$.

Finally, consider the same language for $\mathfrak{M}_{\mathbb{Q}}$, but with a different interpretation. Namely, let $D_r(x,y)$ mean "$d(x,y) \leqslant r$" ($r \in \mathbb{Q}$). Now it is clear that "being algebraically closed" implies "being finitely $\mathbb{Q}$-hyperconvex", because of a version of Lemma 3.15 for rational metric spaces. Thus, both properties are equivalent and now it is true that a countable rational metric space is $\mathfrak{M}_{\mathbb{Q}}$-injective if and only if it is algebraically closed.

3.4 Linear orders

Let κ be an infinite cardinal and let $\mathfrak{LO}_{<\kappa}$ denote the class of all linearly ordered sets of cardinality $< \kappa$. A homomorphism of linearly ordered sets will be called an *increasing map*. As mentioned before, $\mathfrak{LO}_{<\omega}$ gives a natural example of a pair $\langle {}^{\mathrm{emb}}\mathfrak{LO}_{<\omega}, {}^{\mathrm{hom}}\mathfrak{LO}_{<\omega}\rangle$ failing the pushout property. However, we have the following

Proposition 3.20. *For every infinite cardinal κ, the pair $\langle {}^{\mathrm{emb}}\mathfrak{LO}_{<\kappa}, {}^{\mathrm{hom}}\mathfrak{LO}_{<\kappa}\rangle$ has property $(\mathfrak{H})$.*

Proof. Condition $(\mathfrak{H}_1)$ follows from $(\mathfrak{H}_3)$, because ${}^{\mathrm{emb}}\mathfrak{LO}_{<\kappa}$ has an initial object (the empty set) and ${}^{\mathrm{hom}}\mathfrak{LO}_{<\kappa}$ has a terminal object, the 1-element linearly ordered set. It remains to show $(\mathfrak{H}_2)$ and $(\mathfrak{H}_3)$.

Call an embedding $j\colon A \to B$ *primitive* if $|B \setminus j[A]| \leqslant 1$. It is clear that every increasing embedding is the colimit of a transfinite sequence of primitive embeddings. We shall use an uncountable version of Proposition 3.1, which can be easily proved by transfinite induction, using the fact that the category ${}^{\mathrm{emb}}\mathfrak{LO}_{<\kappa}$ is κ-continuous in ${}^{\mathrm{hom}}\mathfrak{LO}_{<\kappa}$.

Denote by $\mathcal{P}$ the class of all primitive embeddings in ${}^{\mathrm{emb}}\mathfrak{LO}_{<\kappa}$. Let us prove first that $\langle \mathcal{P}, {}^{\mathrm{hom}}\mathfrak{LO}_{<\kappa}\rangle$ has the amalgamated extension property (condition $(\mathfrak{H}_3)$). Fix linearly ordered sets C, A, B such that $A = C \cup \{a\}$ and $B = C \cup \{b\}$. Fix increasing maps $f\colon A \to L$ and $g\colon B \to L$ such that $f \restriction C = g \restriction C$. Formally, we have to assume that $a \neq b$. Let $W = A \cup B$. We let $a < b$ if $f(a) < g(b)$; we let $a > b$ otherwise. It is clear, using the compatibility of f and g, that this defines a linear order on W, extending the orders of A and B. The unique map $h\colon W \to L$ satisfying $h \restriction A = f$ and $h \restriction B = g$ is increasing. This shows $(\mathfrak{H}_3)$.

Now fix linearly ordered sets C, A, B such that $A = C \cup \{a\}$ with $a \notin C$, and fix an

increasing map $f\colon C \to B$. Let

$$L = \bigcup_{c<a} (\leftarrow, f(c)] \qquad \text{and} \qquad R = \bigcup_{c>a} [f(c), \rightarrow),$$

where $(\leftarrow, x] = \{p\colon p \leqslant x\}$ and $[x, \rightarrow) = \{p\colon p \geqslant x\}$. Note that $B = L \cup R$ and $L \cap R$ is either empty or a singleton. Let $W = B \cup \{w\}$, where either $w \in L \cap R$ or $w \notin B$ in case where $L \cap R = \emptyset$. In the latter case, define $x < w$ and $w < y$ for $x \in L$, $y \in R$. Define $g\colon A \to W$ by setting $g(a) = w$ and $g \restriction C = f$. Clearly, g is increasing and the inclusion $B \subseteq W$ is primitive. This shows $(\mathfrak{H}_2)$ and completes the proof. □

Note that every increasing embedding of finite linear orders is left-invertible. Thus, as a corollary, we obtain the fact that every countable linear order is an increasing retract of the rational numbers. Of course, this result can be proved directly, realizing that $X \cdot \mathbb{Q}$ with the lexicographic ordering is isomorphic to $\mathbb{Q}$, whenever X is a countable linear order.

Passing to the uncountable case, let us note that $\mathfrak{LO}_{<\omega_1}$ has the Fraïssé limit if and only if the Continuum Hypothesis holds. Denote this Fraïssé limit by $\mathbb{Q}_{\omega_1}$. It is easy to check that a linearly ordered set X of cardinality ω_1 is injective for countable linear orders (isomorphic to $\mathbb{Q}_{\omega_1}$) if and only if for every countable sets $A, B \subseteq X$ such that $a < b$ for $a \in A$, $b \in B$, there exists $x \in X$ such that $a \leqslant x \leqslant b$ ($a < x < b$) whenever $a \in A$, $b \in B$ (one of the sets A, B may be empty here). For example, the closed unit interval $[0, 1]$ satisfies this condition, therefore it can be embedded as an increasing retract of $\mathbb{Q}_{\omega_1}$.

We finally make a short remark on reversed Fraïssé sequences, in the context of linear orders. General theory of reversed Fraïssé limits of finite models (of a first-order language) was developed in [11]. The idea comes just by considering the opposite category. More specifically, fix a class $\mathfrak{M}$ of finite models and consider the pair $\langle {}^{\mathrm{quo}}\mathfrak{M}, {}^{\mathrm{hom}}\mathfrak{M}\rangle$, where ${}^{\mathrm{quo}}\mathfrak{M}$ is the category whose objects are the elements of $\mathfrak{M}$ and the arrows are quotient maps. Now property $(\mathfrak{H})$ is defined by reversing the arrows in all the diagrams. For example, amalgamation is replaced by "reversed amalgamation" and pushouts are replaced by pulbacks. Sequences are now contravariant functors and it is natural to consider their limits endowed with the topology (inherited from the product of finite sets). It is not hard to see that precisely the continuous homomorphisms are induced by arrows between sequences. It is worth noting that if $\mathfrak{M}$ is closed under finite products and substructures then ${}^{\mathrm{quo}}\mathfrak{M}$ has pullbacks in ${}^{\mathrm{hom}}\mathfrak{M}$. The pullback of two quotient maps $f\colon z \to x$, $g\colon z \to y$ is provided by the structure

$$w = \{\langle s, t\rangle \in x \times y\colon f(s) = g(t)\}.$$

Coming back to finite linear orders, consider the pair $\langle {}^{\mathrm{quo}}\mathfrak{LO}_{<\omega}, {}^{\mathrm{hom}}\mathfrak{LO}_{<\omega}\rangle$. It is straight to see that ${}^{\mathrm{quo}}\mathfrak{LO}_{<\omega}$ has no pullbacks in ${}^{\mathrm{hom}}\mathfrak{LO}_{<\omega}$. On the other hand, it is easy and standard to check that this pair has (the reversed variant of) property $(\mathfrak{H})$. Note that

every increasing quotient of finite linearly ordered sets is right-invertible. Thus, all sequences in ${}^{\mathrm{quo}}\mathfrak{OO}_{<\omega}$ are "finitely projective". It is clear that the inverse Fraïssé limit of $\mathfrak{OO}_{<\omega}$ is the Cantor set endowed with the standard linear order. Thus, using Theorem 2.7 (or, more precisely, Corollary 3.4), we obtain the following well known fact that belongs to the folklore.

Corollary 3.21. *Every compact metric totally disconnected linearly ordered space is a continuous increasing retract of the standard Cantor set.*

Of course, again it is not hard to prove this directly, by showing that a metric compact totally disconnected linearly ordered space K can be isomorphically embedded into the Cantor set and constructing the retraction "manually". Note that the reversed Fraïssé theory would only say that K is a continuous increasing quotient of the Cantor set, however not all continuous increasing quotient maps of the Cantor are right-invertible.

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