THE HYPERDETERMINANT OF A SYMMETRIC TENSOR

LUKE OEDING

ABSTRACT. The hyperdeterminant of a symmetric tensor factors into several irreducible factors with multiplicities. Using geometric techniques these factors are identified along with their degrees and their multiplicities. The analogous decomposition for the μ-discriminant of a symmetric tensor is found.

1. INTRODUCTION

After degree and number of variables, perhaps the most important invariant of a polynomial is the discriminant $\Delta(f)$ - a polynomial in the coefficients of f which vanishes precisely when f has a double root. Much of the interesting behavior of f is encoded in $\Delta(f)$.

Consider a homogeneous degree d polynomial on n variables x_i

$$f = \sum_{0 \le i_j \le n} a_{i_1,\dots,i_d} \binom{d}{m_1,\dots,m_n} x_{i_1}\cdots x_{i_d},$$

where $a_{i_1,\dots,i_d}$ are constants, m_j is the number of times that the index j appears in the set $\{i_1,\dots,i_d\}$, and $\binom{d}{m_1,\dots,m_n}$ is the multinomial coefficient. In the case $d=2$, the data $(a_{i,j})_{1\le i,j\le n}$ that describe f equivalently describe a matrix $A_f = (a_{i,j})_{1\le i,j\le n}$, which is symmetric; $a_{j,i} = a_{i,j}$. It is well known that when $d=2$, the discriminant $\Delta(f)$ is equal to the determinant $\det(A_f)$. In general, the data $(a_{i_1,\dots,i_d})_{1\le i_1,\dots,i_d\le n}$ that describe f equivalently describe a d-dimensional array or tensor $A_f = (a_{i_1,\dots,i_d})_{1\le i_1,\dots,i_d\le n}$, which is symmetric for all permutations of the indices; $a_{i_{\sigma(1)},\dots,i_{\sigma(n)}} = a_{i_1,\dots,i_d}$ for all permutations $\sigma\in\mathfrak{S}_n$.

A. Cayley [4] introduced the notion of the hyperdeterminant of a multidimensional matrix (tensor) analogous to the determinant of a square matrix. The hyperdeterminant, which we define precisely below, may be thought of in analogy to the discriminant as a polynomial that tells when a tensor is singular.

After 150 years without much attention paid to hyperdeterminants, Gelfand, Kapranov and Zelevisnki brought hyperdeterminants into a modern light in their groundbreaking work [7, 8]. In particular they determined precisely when the hyperdeterminant is non-trivial and computed the degree. Inspired by their work, we study the hyperdeterminant applied to a symmetric tensor, or equivalently to a polynomial. Said another way, this is a study of the symmetrization of the hyperdeterminant. Our goal is to determine how the symmetrized hyperdeterminant factors, to determine the geometric meaning of each factor, and to determine the degrees and multiplicities of the factors.

Date: July 26, 2011.
This material is based upon work supported by the National Science Foundation under Award No. 0853000: International Research Fellowship Program (IRFP). The author also gratefully acknowledges partial support from the Mittag-Leffler institute.

The first example that is not a matrix is binary cubics. The discriminant of a binary cubic has degree 4. The hyperdeterminant of a $2 \times 2 \times 2$ tensor also has degree 4 and the formula is well known (see [8, (1.5) p.448]). One can quickly check that the symmetrization of this polynomial is the discriminant of a binary cubic. It turns out that this and the quadrics case are the only two cases that have such simple behavior.

Our curiosity was peaked by the following example that was first pointed out to us by Giorgio Ottaviani. For plane cubics, the discriminant has degree 12. The hyperdeterminant of a $3 \times 3 \times 3$ matrix has degree 36. Using Macaulay2 [9] Ottaviani used Schläfley's method to compute the hyperdeterminant, applied this to a symmetric tensor, specialized to a random line and found that the symmetrization of the hyperdeterminant is a reducible polynomial which splits into a factor of degree 12 (the discriminant) and a factor of degree 4 with multiplicity 6. The degree 4 factor turned out to be Aronhold's invariant for plane cubics and defines the variety of Fermat cubics. While Aronhold's invariant is classical, we refer the reader to [12] where one finds a matrix construction which can be applied to construct Aronhold's invariant for degree 3 symmetric forms on 3 variables, Toeplitz's invariant [14] for triples of symmetric 3×3 matrices, and Strassen's invariant [13] for $3\times 3\times 3$ tensors.

After this example, Ottaviani posed the problem to understand and describe this phenomenon in general. Indeed when d or n are larger than the preceding examples, the hyperdeterminant becomes quite complicated, with much beautiful structure, [10]. The approach of the current article is to study these algebraic objects from a geometric point of view, thus avoiding some of the computational difficulties, such as those that arise in computing an expansion of the hyperdeterminant in terms of monomials.

The outline of the article is the following. In Section 2 we recall terminology from combinatorics, namely the notion of one partition being a refinement of another and present a formula for the number of such refinements. In Section 3 we review facts from multilinear algebra necessary for our calculations. In Section 4 we recall the relevant geometric objects (including Segre-Veronese varieties, Chow varieties and projective duality). Finally in Section 5 we use geometric methods to prove our main results, which are the following:

Theorem 1.1. *The $n^{\times d}$-hyperdeterminant of a symmetric tensor of degree $d \geq 2$ on $n \geq 2$ variables splits as the product*

$$\prod_{\lambda} \Xi_{\lambda,n}^{\mathcal{M}_\lambda},$$

where $\Xi_{\lambda,n}$ is the equation of the dual variety of the Chow variety $Chow_\lambda \mathbb{P}^{n-1}$ when it is a hypersurface in $\mathbb{P}^{\binom{n-1+d}{d}-1}$, $\lambda = (\lambda_1, \dots, \lambda_s)$ is a partition of d, and the multiplicity $M_\lambda = \binom{d}{\lambda_1,\dots,\lambda_s}$ is the multinomial coefficient.

Geometrically, this theorem is essentially a statement about the symmetrization of the dual variety of the Segre variety. In fact, Theorem 1.1 is a special case of the more general result for Segre-Veronese varieties:

Theorem 1.2. *Let μ be a partition of $d \geq 2$, and V be a complex vector space of dimension $n \geq 2$. Then*

$$\mathrm{Seg}_\mu \left(\mathbb{P}V^{\times t}\right)^\vee \cap \mathbb{P}\left(S^d V^*\right) = \bigcup_{\lambda \prec \mu} Chow_\lambda \left(\mathbb{P}V\right)^\vee,$$

where $\lambda \prec \mu$ is the refinement partial order. In particular,

$$\mathcal{V}(Sym(\Delta_{\mu,n})) = \prod_{\lambda \prec \mu} \Xi_{\lambda,n}^{M_{\lambda,\mu}}$$

where $\Xi_{\lambda,n}$ is the equation of $Chow_\lambda(\mathbb{P}V)^\vee$ when it is a hypersurface in $\mathbb{P}(S^dV)$, and the multiplicity $M_{\lambda,\mu}$ is the number of refinements from μ to λ.

To properly use the previous two theorems, we need to know which dual varieties of Chow varieties are hypersurfaces.

Proposition 1.3. *Suppose $d \geq 2$, $\dim V = n \geq 2$ and $\lambda = (\lambda_1, \dots, \lambda_s) = (1^{m_1}, \dots, p^{m_p})$. Then $Chow_\lambda(\mathbb{P}V)^\vee$ a hypersurface with the only exceptions*
- *$n = 2$ and $m_1 \neq 0$*
- *$n > 2$, $s = 2$ and $m_1 = 1$ (so $\lambda = (d-1, 1)$).*

We also have a formula for the degree of the hypersurfaces in the binary case.

Theorem 1.4. *The degree of $Chow_\lambda(\mathbb{P}^1)^\vee$ with $\lambda = (1^{m_1}, 2^{m_2}, \dots, p^{m_p})$, $m_1 = 0$ and $m = \sum_i m_i$ is*

$$(m+1)\binom{m}{m_2, \dots, m_p} 1^{m_2} 2^{m_3} \cdots (p-1)^{m_p}$$

In more than 2 variables we have a recursive procedure for computing the degree which is a consequence of Theorem 1.2.

Corollary 1.5. *Suppose $\dim V \geq 2$. Let d_λ denote $\deg(Chow_\lambda(\mathbb{P}V)^\vee)$. Then the vector $(d_\lambda)_\lambda$ is the unique solution to the (triangular) system of equations*

$$\deg(\Delta_{\mu,n}) = \sum_{\lambda \prec \mu} d_\lambda M_{\lambda,\mu}.$$

The degree of $\Delta_{\mu,n}$ is given by a generating function [7, Theorem 3.1, Proposition 3.2] or also [8, page 454], the multiplicities $M_{\lambda,\mu}$ are computable via Proposition 2.2, so the corollary gives a recursive way to compute all of the degrees of the duals of the Chow varieties.

Remark 1.6. The hypersurfaces $\mathrm{Chow}_\lambda(\mathbb{P}V)^\vee$ are $\mathrm{SL}(V)$ invariant, and thus each defining polynomial is an $\mathrm{SL}(V)$ invariant for polynomials. Since invariants of polynomials have been well studied, many of the dual varieties to Chow varieties have alternative descriptions as classically studied objects, however we prefer to ignore these connections for our proofs in order to have a more uniform treatment. However we point out that Corollary 1.5 may be used as a way to determine degrees of classical invariants.

Recently there has been a considerable amount of work on hyperdeterminants, Chow varieties and related topics. Indeed we have learned from the works [1–3, 5–7, 11, 15], and we are particularly grateful for the very rich book [8] which provided us both with several useful results and techniques, as well as inspiration.

In this paper we will work over $\mathbb{C}$ (or any algebraically closed field of characteristic 0), it is likely that some these results can be extended to arbitrary characteristic, but we do not concern ourselves with this problem here. All polynomials will be assumed to be homogeneous.

2. Combinatorial Ingredients

An integer vector $\lambda = (\lambda_1, \dots, \lambda_s)$ is called a *partition* of an integer d with s parts if $d \geq \lambda_1, \dots, \lambda_s > 0$ and $\sum_i \lambda_i = d$. We often shorten this by writing $\lambda \vdash d$ and $\#\lambda = s$. We do not require the λ_i to be in order.

Often we would like to keep track of the number of repetitions that occur in λ. In this case, write $\lambda = (1^{m_1}, 2^{m_2}, \dots, p^{m_p})$. where i^{m_i} is to be interpreted as the integer i repeated m_i times.

Suppose $\lambda \vdash d$ and $\mu \vdash d$. We will say $\lambda = (\lambda_1, \dots, \lambda_s)$ is a *refinement* of $\mu = (\mu_1, \dots, \mu_t)$ and write $\lambda \prec \mu$ if there is an expression

$$\begin{aligned} \lambda_1 &= \mu_{i_{1,1}} + \cdots + \mu_{i_{1,t_1}} \\ \lambda_2 &= \mu_{i_{2,1}} + \cdots + \mu_{i_{2,t_2}} \\ &\cdots \\ \lambda_s &= \mu_{i_{s,1}} + \cdots + \mu_{i_{t,t_s}} \end{aligned} \tag{1}$$

and (after re-ordering) $\mu = (\mu_{i_{1,1}}, \dots, \mu_{i_{1,t_1}}, \dots, \mu_{i_{s,1}}, \dots, \mu_{i_{s,t_s}})$. Here we emphasize that we do not distinguish two expressions as different if only the orders of the summations in (1) change, but we do distinguish the case when different choices of indices of μ appear in different equations even if some of the μ_i take the same value.

Let $M_{\lambda,\mu}$ denote the number of distinct expressions of the form (1). We will say that $M_{\lambda,\mu}$ is the *number of refinements* from μ to λ.[1] Here are some easy properties of $M_{\lambda,\mu}$ that follow immediately from the definition.

Proposition 2.1. *Let $M_{\lambda,\mu}$ denote the number of refinements from μ to λ. Then the following properties hold.*

- $M_{(d),\mu} = 1$ *for all* $|\mu| = d$.
- $M_{\lambda,\mu} = 0$ *if* $s > t$ *or if* $s = t$ *and* $\lambda \neq \mu$*, in particular, the matrix* $(M_{\lambda,\mu})_{\lambda,\mu}$ *is lower triangular for a good choice in ordering of the indices.*
- *If* $\lambda = (1^{m_1}, 2^{m_2}, \dots, p^{m_p})$*, then* $M_{\lambda,\lambda} = m_1! \cdots m_p!$.
- $M_{\lambda,1^d} = \binom{d}{\lambda} := \binom{d}{\lambda_1,\dots,\lambda_s} = \frac{d!}{\lambda_1!\cdots\lambda_s!}$*, the multinomial coefficient.*

Computing $M_{\lambda,\mu}$ can become complicated. For example $(3,2)$ is a refinement of $(2,1,1,1)$ and we find that $(3,2) = (2+1, 1+1)$ (in 3 different ways) and $(3,2) = (1+1+1, 2)$ (in one way) so $M_{(2,1,1,1),(3,2)} = 4$.

We may compute $M_{\lambda,\mu}$ via brute force, and we state this in the following proposition, which follows directly from the definition. However we would like to know if there is a more efficient function giving the values of $M_{\lambda,\mu}$.

Proposition 2.2. *Let $B(t,s)$ denote the set of all surjective maps*

$$\phi \colon \{1, \dots, t\} \longrightarrow \{1, \dots, s\},$$

and let $\chi(a,b) = 1$ if $a = b$ and $= 0$ otherwise. Then

$$M_{\lambda,\mu} = \sum_{\phi \in B(t,s)} \prod_{i=1}^{s} \chi\left(\lambda_i, \sum_{j \in \phi^{-1}(i)} \mu_j\right).$$

[1] Unfortunately [7] uses the same symbol $M_{\lambda,\mu}$ for the Gale-Ryser number, but in [8] they use $d_{\lambda,\mu}$ for the Gale-Ryser number. We emphasize that our $M_{\lambda,\mu}$ and $d_{\lambda,\mu}$ are related, but not equal.

For example, $M_{(3,2,2),(3,1,1,1,1)} = 6$. We compute this by considering all surjections $\phi\colon \{1,2,3,4,5\}\longrightarrow\{1,2,3\}$. One finds that only the surjections that satisfy $\phi^{-1}(1) = 1$ can contribute non-zero since there is only one way to produce $\lambda_1 = 3$. The only remaining ϕ that can (and do) contribute non-zero to $\mathcal{M}_{\lambda,\mu}$ are the following 6 cases

$$\begin{array}{l}\{\phi(\{2,3\}) = 2, \phi(\{4,5\}) = 3\}, \{\phi(\{2,4\}) = 2, \phi(\{3,5\}) = 3\},\\ \{\phi(\{2,5\}) = 2, \phi(\{3,4\}) = 3\}, \{\phi(\{3,4\}) = 2, \phi(\{2,5\}) = 3\},\\ \{\phi(\{3,5\}) = 2, \phi(\{2,4\}) = 3\}, \{\phi(\{4,5\}) = 2, \phi(\{2,3\}) = 3\}.\end{array}$$

Note that this construction accounts for the ambiguity in the location of the 2's in the partition $(3,2,2)$. This is a counterexample to a naïve guess that $M_{\lambda,\mu}$ could be simply the number of unordered refinements from μ to λ times the number of permutations that fix λ.

$M_{\lambda,\mu}$ also computes the dimension of the space of (unlabeled) partial symmetrization maps

$$S^{\mu_1}V \otimes S^{\mu_2}V \otimes \cdots \otimes S^{\mu_t}V \longrightarrow S^{\lambda_1}V \otimes S^{\lambda_2}V \otimes \cdots \otimes S^{\lambda_s}V.$$

We can see this as follows: The set $B(t,s)$ over-parameterizes all choices of collections of vector spaces of the form S^{μ_i} that will be symmetrized and mapped to a vector space of the form S^{λ_j}. Then for a given $\phi \in B(t,s)$, the number of copies S^{λ_i} occurs in $\bigotimes_{j\in\phi^{-1}(i)} S^{\mu_j}V$ is, via the Pieri formula, equal to 1 precisely when $\lambda_i = \sum_{j\in\phi^{-1}(i)} \mu_i$, and zero otherwise. In Section 6 we list some examples of the matrices $(M_{\lambda,\mu})$.

3. Some multi-linear algebra

The elementary facts below will turn out to be useful later. By following the philosophy to not use coordinates unless necessary, we hope to give a more streamlined approach. As a reference and for much more regarding multilinear algebra and tensors we suggest [11], which is where we learned this perspective.

Suppose $[F]$ is a hyperplane in $\mathbb{P}V^{\otimes d}$. Then F may be considered as a linear map $F\colon V^{\otimes d}\longrightarrow\mathbb{C}$, or equivalently as a multilinear form $F\colon V^{\times d}\longrightarrow\mathbb{C}$. More explicitly, let $[v_1\otimes v_2\otimes\cdots\otimes v_d] \in \mathbb{P}\left(V^{\otimes d}\right)$. Then

$$F\left(v_1\otimes v_2\otimes\cdots\otimes v_d\right) = F\left(v_1, v_2, \ldots, v_d\right), \tag{2}$$

where on the left we are thinking of F as a linear map, and on the right as a multilinear form. Our choice of interpretation of F and how to evaluate F will be clear from the context so we will not introduce new notation for the different uses.

Now consider $\mu \vdash d$, $\mu = (\mu_1,\ldots,\mu_t)$ and

$$u_1^{\mu_1}\otimes u_2^{\mu_2}\otimes\cdots\otimes u_t^{\mu_t} \in S^{\mu_1}V\otimes\cdots\otimes S^{\mu_t}V.$$

The form F may be evaluated on points of $\mathbb{P}\left(S^{\mu_1}V\otimes\cdots\otimes S^{\mu_t}V\right)$ via the inclusion into $\mathbb{P}\left(V^{\otimes d}\right)$

$$F(u_1^{\mu_1}\otimes u_2^{\mu_2}\otimes\cdots\otimes u_t^{\mu_t}) = F(u_1,\ldots,u_1,u_2,\ldots,u_2,\ldots,u_t,\ldots,u_t),$$

where u_i is repeated μ_i times.

Now suppose λ and μ are such that $M_{\lambda,\mu}$ is non-zero, and consider the inclusion

$$S^{\lambda_1}V\otimes S^{\lambda_2}V\otimes\cdots\otimes S^{\lambda_s}V \subset S^{\mu_1}V\otimes S^{\mu_2}V\otimes\cdots\otimes S^{\mu_t}V,$$

Let $v_1^{\lambda_1}\otimes\cdots\otimes v_s^{\lambda_s} \in S^{\lambda_1}V\otimes S^{\lambda_2}V\otimes\cdots\otimes S^{\lambda_s}V$. Since $v^j = v^{\otimes j}$ for any j, we may make explicit the above inclusion by writing $v_1^{\lambda_1}\otimes\cdots\otimes v_s^{\lambda_s}$ in the form

$u_1^{\mu_1}\otimes u_2^{\mu_2}\otimes\cdots\otimes u_t^{\mu_t}$, where each vector u_i is an element of $\{v_1,\dots,v_s\}$ and there is re-ordering of the factors implied by the inclusion above. In this case, we say that $u_1^{\mu_1}\otimes u_2^{\mu_2}\otimes\cdots\otimes u_t^{\mu_t}$ *symmetrizes* to $v_1^{\lambda_1}\otimes\cdots\otimes v_s^{\lambda_s}$. In addition, there is an inclusion $S^dV\subset S^{\lambda_1}V\otimes S^{\lambda_2}V\otimes\cdots\otimes S^{\lambda_s}V$, so we may further symmetrize both points to $v_1^{\lambda_1}\cdots v_s^{\lambda_s}$.

Now suppose $[F]$ is symmetric hyperplane in $\mathbb{P}V^{\otimes d}$, *i.e*, $F\in S^dV^*$. Then (2) implies that F *takes the same value at every tensor in* $\mathbb{P}V^{\otimes d}$ *that symmetrizes to* $v_1^{\lambda_1}\cdots v_s^{\lambda_s}$. We will use this fact several times in the sequel.

As a matter of notation, if $u\in\{v_1,\dots,v_n\}$ we will write $\frac{v_1\cdots v_n}{u}$ to denote the product omitting u.

4. Geometric Ingredients

Hyperdeterminants, the discriminant and their cousins, whose definitions we will recall below, are all equations of irreducible hypersurfaces in projective space, and moreover each hypersurface is the dual variety of another variety.

To say that a polynomial splits into many irreducible factors (with multiplicities) geometrically says that the associated hypersurface decomposes as the union of many hypersurfaces (with multiplicities). Geometrically, we would like to describe one dual variety as the union of other dual varieties. Our perspective is to study the relation between dual varieties and (geometric) symmetrization. In what follows we will introduce all of the geometric notions we will need to prove our main results.

4.1. **Segre-Veronese and Chow varieties.** Let V be a complex vector space of dimension n. Let $\lambda\vdash d$ with $\#\lambda=s$. Consider the Segre-Veronese embedding via $\mathcal{O}(\lambda)$, which is given by

$$\begin{array}{cccc} \mathrm{Seg}_\lambda:\mathbb{P}V^{\times s} & \xrightarrow{|\mathcal{O}(\lambda)|} & \mathbb{P}\left(S^{\lambda_1}V\otimes\cdots\otimes S^{\lambda_s}V\right) & \subseteq\mathbb{P}\left(V^{\otimes d}\right)\\ ([a_1],\dots,[a_s]) & \mapsto & [a_1^{\lambda_1}\otimes\cdots\otimes a_s^{\lambda_s}]. & \end{array}$$

We call the image of this map a *Segre-Veronese variety*, and denote it by $\mathrm{Seg}_\lambda\left(\mathbb{P}V^{\times s}\right)$. More generally, all of the vector spaces can be different, but we do not need that generality here.

Notice that when $\lambda=(1^d)=(1,\dots,1)$ this is the usual Segre embedding, whose image we will denote by $\mathrm{Seg}\left(\mathbb{P}V^{\times d}\right)$, and when $\lambda=(d)$ the map is the d^{th} Veronese embedding, whose image we will denote by $\nu_d\left(\mathbb{P}V\right)$.

It is easy to see that $\mathrm{Seg}_\lambda\left(\mathbb{P}V^{\times s}\right)$ is a smooth, non-degenerate, homogeneous variety of dimension $s(n-1)$. Moreover, it is clear that if $\sigma\in\mathfrak{S}_s$ is a permutation, then $\mathrm{Seg}_{\sigma(\lambda)}\left(\mathbb{P}V^{\times s}\right)$ is isomorphic to $\mathrm{Seg}_\lambda\left(\mathbb{P}V^{\times s}\right)$, and the isomorphism is an equality if and only if λ is fixed by σ.

Recall that a consequence of the Pieri formula is that for all $\lambda\vdash d$, there is an inclusion

$$S^dV\subset S^{\lambda_1}V\otimes\cdots\otimes S^{\lambda_s}V.$$

Since $G=\mathrm{GL}(V)$ is reductive, there is a unique G-invariant complement to S^dV in $S^{\lambda_1}V\otimes\cdots\otimes S^{\lambda_s}V$, which we will denote by W^λ.

Note that the linear span of the Segre-Veronese variety is the whole ambient space. This means, in particular, that there is always a basis of $S^{\lambda_1}V\otimes\cdots\otimes S^{\lambda_s}V$ consisting of monomials.

For each λ there is a natural projection from W^λ, namely

$$\pi_{W^\lambda}:\mathbb{P}\left(S^{\lambda_1}V\otimes\cdots\otimes S^{\lambda_s}V\right)\dashrightarrow\mathbb{P}S^dV,$$

whose definition on decomposable elements is

$$[a_1^{\lambda_1} \otimes \cdots \otimes a_s^{\lambda_s}] \quad \mapsto \quad a_1^{\lambda_1} \cdots a_s^{\lambda_s},$$

and is extended by linearity.

For each λ we define a *Chow variety*, denoted $\mathrm{Chow}_\lambda(\mathbb{P}V)$, as the image of the Segre-Veronese variety under the projection π_{W^λ}. Chow varieties are also sometimes called coincident root loci [6], or a split variety when $\lambda = (1^d)$ [**?**]. One can check that this is equivalent to the usual definition of a Chow variety, see [3]. Notice that the image of the projection is not changed by permutations acting on λ. In other words, $\mathrm{Chow}_\lambda(\mathbb{P}V^{\times s})$ is equally the projection of $\mathrm{Seg}_{\sigma(\lambda)}(\mathbb{P}V^{\times s})$ for any permutation $\sigma \in \mathfrak{S}_s$. The number of unique projections is $M_{\lambda,\lambda}$.

Notice that when $\lambda = (1^d)$, the Chow variety is the variety of polynomials that are completely reducible as a product of linear forms, and is sometimes called the split variety. For general λ, the Chow variety is the closure of the set of polynomials that are completely reducible as the product of linear forms that are respectively raised to powers $\lambda_1, \dots, \lambda_s$.

The following is well known (see [6] for example).

Proposition 4.1. *Suppose* $\lambda \vdash d$ *with* $\#\lambda = s$. *Then* $\dim(\mathit{Chow}_\lambda(\mathbb{P}V)) = s(n-1)$.

Proof. Let $\dim(V) = n$, and $d = |\lambda|$. Note that the Segre-Veronese map $\mathbb{P}V \times \cdots \times \mathbb{P}V \longrightarrow \mathbb{P}\left(S^{\lambda_1}V \otimes \cdots \otimes S^{\lambda_s}V\right)$ is an embedding, and in particular the dimension of the image is $s(n-1)$. The projection to S^dV is a finite morphism, so the image is also $s(n-1)$-dimensional. □

Remark 4.2. It is interesting to note that the refinement partial order on partitions also exactly controls the containment partial order on Chow varieties. Namely

$$\mathrm{Chow}_\lambda(\mathbb{P}V) \subset \mathrm{Chow}_\mu(\mathbb{P}V)$$

precisely when $\lambda \prec \mu$.

4.2. **Dual varieties.** Let U denote a complex, finite dimensional vector space and let U^* denote the dual vector space of linear forms $\{U \longrightarrow \mathbb{C}\}$. For a smooth projective variety $X \subset \mathbb{P}U$, the dual variety $X^\vee \subset \mathbb{P}U^*$ is the variety of tangent hyperplanes to X. Specifically, let $\widehat{T}_xX \subset U$ denote the cone over the tangent space to X at $[x] \in X$. The *dual variety* of X in $\mathbb{P}U^*$ is defined as

$$X^\vee := \left\{[H] \in \mathbb{P}U^* \mid \exists [x] \in X, \widehat{T}_xX \subset H\right\}.$$

Remark 4.3. If $X \subset \mathbb{P}U$ is not smooth the dual variety can still be defined with a bit more care. Consider the incidence variety (conormal variety)

$$\mathcal{P} = \{([x],[H]) \mid \widehat{T}_x \subset H\} \subset \mathbb{P}U \times \mathbb{P}U^* \subset \mathbb{P}(U \otimes U^*),$$

which we define only for smooth points of X and then take the Zariski closure (see [16] for a more thorough treatment). The conormal variety is equipped with projections p_1 and p_2 to the first and second factors respectively. The projection p_2 to the second factor defines $X^\vee$.

Recall that the dual variety of an irreducible variety is also irreducible. Usually, we expect the dual variety $X^\vee$ to be a hypersurface. When this does not occur, we say that X is *defective.*

The dual variety of the Veronese $\nu_d(\mathbb{P}V)^\vee$ is a hypersurface defined by the classical discriminant of a degree d polynomial on n variables, which we will denote

$\Delta_{(d),n}$, see [8, Example I.4.15, p.38]. We are told in the same passage that G. Boole in 1842 introduced this discriminant and found that $\deg(\Delta_{(d),n}) = (n)(d-1)^{n-1}$.

The *hyperdeterminant* of format $n^{\times d}$, denoted $HD_{n,d}$, is the equation of the (irreducible) hypersurface $\mathrm{Seg}\left(\mathbb{P}V^{\times d}\right)^\vee \subset \mathbb{P}\left(V^{\otimes d}\right)^*$. Note that $HD_{n,d}$ is a polynomial of degree $N(n,d)$ on $\left(V^{\otimes d}\right)^*$ where $N(n,d)$ can be computed via the generating functions found in [7, Theorem 3.1, Proposition 3.2] or also [8, Theorem XIV.2.4, p 454].

Segre-Veronese varieties and their duals are well-studied objects. In particular, it is known precisely when they are hypersurfaces [8, Proposition XIII.2.3 p.441], and their degree is given in [8, Theorem XIII.2.4 p.441]. For $\mu \vdash d$ we denote by $\Delta_{\mu,n}$ the *μ-discriminant*, which is the equation of $\mathrm{Seg}_\mu\left(\mathbb{P}V^{\times t}\right)^\vee$ when non-trivial. (Note this notion is often called an A-discriminant in [8].)

When the dual variety $\mathrm{Chow}_\lambda\left(\mathbb{P}V\right)^\vee$ is a hypersurface (see Proposition 1.3), we will let $\Xi_{\lambda,n}$ denote its equation, which is unique up to multiplication by a non-zero scalar.

4.3. **Projections of dual varieties.** The focus of this article is the symmetrization of the hyperdeterminant. In general *the symmetrization of a polynomial* is the map induced by the map that symmetrizes the variables. This may be described invariantly as follows. If $f \in S^e(V^{\otimes d})^*$ is a degree e homogeneous polynomial on $V^{\otimes d}$, then $Sym(f)$ is the image of f under the projection $S^e(V^{\otimes d})^* \longrightarrow S^e(S^dV)^*$. While this map can be described in bases in complete detail, we do not need this for the current work.

To study the dual varieties of the varieties we have introduced, we need a statement about the relation between taking dual variety and taking projection. This is the content of the following proposition, which can be found in Landsberg's book, Proposition 9.2.6.1 v.2.25.11:

Proposition 4.4 ([11])**.** *Let $X \subset \mathbb{P}V$ be a variety and let $W \subset V$ be a linear subspace. Consider the rational map $\pi\colon \mathbb{P}V \dashrightarrow \mathbb{P}(V/W)$. Assume $X \not\subset \mathbb{P}W$. Then*

$$\pi_W(X)^\vee \subseteq \mathbb{P}W^\perp \cap X^\vee$$

and if $\pi_W(X) \cong X$, then equality holds.

When $W = W^\lambda = S^dV^\perp \subset S^{\lambda_1}V \otimes \cdots \otimes S^{\lambda_s}V$, the map π_W is symmetrization and we will denote it by Sym.

In the sequel we will perform an intermediate projection. For this we record the following useful lemma, which follows immediately from the proposition.

Lemma 4.5. *Suppose $\mathbb{P}W \subset \mathbb{P}U \subset \mathbb{P}V$ for vector spaces W, U, V, and $X \subset \mathbb{P}V$, but $X \not\subset \mathbb{P}W$ and $X \not\subset \mathbb{P}V$. Consider projections*

$$\pi_U^V\colon \mathbb{P}V \dashrightarrow \mathbb{P}V/U$$
$$\pi_W^U\colon \mathbb{P}V/U \dashrightarrow \mathbb{P}V/W$$
$$\pi_W^U\colon \mathbb{P}V \dashrightarrow \mathbb{P}V/W.$$

Then

$$\pi_W^U(X \cap \mathbb{P}V/W)^\vee \subseteq (X \cap \mathbb{P}V/U)^\vee \cap \mathbb{P}W^\perp = X^\vee \cap \mathbb{P}W^\perp.$$

The following statement, which follows directly from the definition, relates the symmetrization of the μ-discriminant to the geometric setting.

Proposition 4.6 (Proposition/Definition)**.** *The symmetrization of the μ-discriminant is the μ-discriminant of a symmetric tensor*

$$\mathcal{V}(Sym(\Delta_{\mu,n})) = \mathrm{Seg}_\mu\left(\mathbb{P}V^{\times t}\right)^\vee \cap \mathbb{P}\left(S^dV^*\right).$$

In particular, the symmetrization of the hyperdeterminant is the hyperdeterminant of a symmetric multi-linear form;

$$\mathcal{V}(Sym(HD_{d,n})) = \mathrm{Seg}\left(\mathbb{P}V^{\times d}\right)^\vee \cap \mathbb{P}\left(S^dV^*\right).$$

4.4. **Plane cubics again.** As a prototypical example, we return to plane cubics.

Let V be a complex vector space with $\dim(V) = 3$ and let $d = 3$. Consider the variety of decomposable plane cubics, denoted $\mathrm{Chow}_{1,1,1}\mathbb{P}V$. More concretely, $\mathrm{Chow}_{1,1,1}\mathbb{P}V = \{[f] \in \mathbb{P}S^3V \mid f = l_1l_2l_3,\ 0 \neq l_i \in V\} \subset \mathbb{P}S^3V$.

Proposition 4.7. *$Chow_{1,1,1}(\mathbb{P}V)^\vee$ is the orbit of the Fermat cubic,* i.e. *the 3^{rd} secant variety to the cubic Veronese:*

$$Chow_{1,1,1}(\mathbb{P}V)^\vee = \sigma_3(\nu_3\mathbb{P}V).$$
$$= \overline{\{h \in \mathbb{P}S^3V^* \mid h = e_1^3 + e_2^3 + e_3^3, e_i \in V^*\}} \subset \mathbb{P}S^3V^*.$$

In particular, $Chow_{1,1,1}(\mathbb{P}V)^\vee$ is a hypersurface.

Lemma 5.1 below implies that since $\mathrm{Chow}_{1,1,1}(\mathbb{P}V)^\vee$ is a hypersurface its equation must divide the symmetrization of the hyperdeterminant of format $3 \times 3 \times 3$.

Proof. The dual variety is the variety of all tangent hyperplanes. So we need to calculate $T_f\mathrm{Chow}_{1,1,1}(\mathbb{P}V)$ for all $f \in \mathrm{Chow}_{1,1,1}(\mathbb{P}V)$ for f a general point, and consider all hyperplanes H such that $T_fH \supset T_f\mathrm{Chow}_{1,1,1}(\mathbb{P}V)$. Let $f(t) = l_1(t)l_2(t)l_3(t)$, $t \in \mathbb{C}$ be a curve in $\mathrm{Chow}_{1,1,1}(\mathbb{P}V)$ such that $f(0) = f = l_1l_2l_3$. Then $\frac{df(t)}{dt}_{|t=0} = l_1'l_2l_3 + l_1l_2'l_3 + l_1l_2l_3'$, where l_i' are arbitrary tangent vectors. By allowing the direction of the curve through f to vary, we obtain,

$$T_f\mathrm{Chow}_{1,1,1}(\mathbb{P}V) = V \cdot l_2l_3 + V \cdot l_1l_3 + V \cdot l_1l_2 = V \cdot \langle l_1l_2, l_1l_3, l_2l_3\rangle.$$

Carlini provides a nice geometric interpretation of this tangent space in his study of Chow varieties, see [3, Prop. 3.4].

A hyperplane $H \subset \mathbb{P}S^3V$ is the zero-set $\mathcal{V}(h)$ where $h \in S^3V^*$ is a linear form on the vector space S^3V. Since f was assumed to be general, we may assume that l_1, l_2, l_3 are linearly independent and hence form a basis of V with dual basis e_1, e_2, e_3 of V^*. Then we write h in the e_i basis as

$$\begin{aligned} h = \lambda_{3,0,0}e_1^3 + \lambda_{2,1,0}e_1^2e_2 + \lambda_{2,0,1}e_1^2e_3 \\ +\lambda_{0,3,0}e_2^3 + \lambda_{1,2,0}e_1e_2^2 + \lambda_{0,2,1}e_2^2e_3 \\ +\lambda_{0,0,3}e_3^3 + \lambda_{1,0,2}e_1e_3^2 + \lambda_{0,1,2}e_2e_3^2 \\ +\lambda_{1,1,1}e_1e_2e_3, \end{aligned}$$

where the $\lambda_{r,s,t} \in \mathbb{C}$ are arbitrary parameters. Now $h(f) = 0$ implies that $\lambda_{1,1,1} = 0$. We apply the condition $h(vl_1l_2) = 0$ for $v \in V$. This implies when $v = l_1$ that $h(l_1^2l_2) = 0$ and hence $\lambda_{2,1,0} = 0$. Continuing in this manner, we find that $h = \lambda_{3,0,0}e_1^3 + \lambda_{0,3,0}e_2^3 + \lambda_{0,0,3}e_3^3$, by taking closure, this completes the proof of the first part.

To prove statement about dimension, we count free parameters. The choice of 3 points e_i from $\mathbb{P}^2$ amounts to the choice of 6 parameters, and the choice of $[\lambda_{3,0,0}, \lambda_{0,3,0}, \lambda_{0,0,3}]$ is a choice from another $\mathbb{P}^2$. These choices are generically independent so we have 8 free parameters in total. Thus we have a hypersurface in $\mathbb{P}^9 = \mathbb{P}S^3\mathbb{C}^3$. $\square$

Because of our generality assumption, there are six different tensors – $l_1 \otimes l_2 \otimes l_3$ and its permutations – that symmetrize to $l_1 l_2 l_3$. This fact implies that there are 6 copies of the equation of $\mathrm{Chow}_{1,1,1}(\mathbb{P}V)^\vee$ in the symmetrized hyperdeterminant.

This example is characteristic of the theme of the rest of the article. The splitting of the hyperdeterminant of a polynomial will depend on the dimensions and multiplicities of the dual varieties of Chow varieties. We also will show that it this is sufficient.

Remark 4.8. One may carry out a similar calculation for $\mathrm{Chow}_\lambda(\mathbb{P}V)$ for any $\lambda \vdash d$, and $\#\lambda = s \leq n$. Let H be a generic symmetric hyperplane tangent to $\mathrm{Chow}_\lambda(\mathbb{P}V)$ at $l_1^{\lambda_1} \cdots l_s^{\lambda_s}$. Suppose the l_i are linearly independent and by choosing more linearly independent forms if necessary to form a basis of V and let e_i be a dual basis. The form $h \in S^d V$ associated to H may be written as

$$h = \sum_{i_1,\dots,i_d \leq n} \lambda_{i_1,\dots,i_n} e_1^{i_1} \cdots e_n^{i_n},$$

where $\lambda_{i_1,\dots,i_n}$ are constants. The condition that H be tangent to $\mathrm{Chow}_\lambda(\mathbb{P}V)$ at $l_1^{\lambda_1} \cdots l_s^{\lambda_s}$ does not impose any conditions on the $\lambda_{0,\dots 0,d,0,\dots 0}$ if $s \geq 2$. This fact implies that (as long as $s \leq n$),

$$\mathrm{Chow}_\lambda(\mathbb{P}V)^\vee \supset \sigma_d(\nu_d(\mathbb{P}V)),,$$

where $\sigma_d(\nu_d(\mathbb{P}V))$ is the variety of points on secant $d-1$-planes to the Veronese variety $\nu_d\mathbb{P}V$. Equality does not hold in general. One may use the dual of Chow varieties as a source for equations for secant varieties of Veronese varieties. The utility of this fact is limited by the degree of the equations obtained.

4.5. **Dimension.** Now we would like to prove Proposition 1.3 about the dimension of the duals of Chow varieties. For convenience, we repeat that we need to show that for $\lambda = (1^{m_1}, \dots, p^{m_p})$, and $n = \dim(V)$, $\mathrm{Chow}_\lambda(\mathbb{P}V)^\vee$ a hypersurface with the only exceptions

- $n = 2$ and $m_1 \neq 0$
- $n > 2$, $s = 2$ and $m_1 = 1$ (so $\lambda = (d-1, 1)$).

The case $d = 2$ is already well understood, so we will assume $d > 2$.

Proof of Proposition 1.3. The dimension of a dual variety can be calculated via the Kac dimension formula, essentially calculating the Hessian at a general point, but we prefer to work geometrically.

A dual variety $X^\vee$ is a hypersurface unless a general tangent hyperplane is tangent to X in a positive dimensional space. This is the condition that we will apply in both cases. Our proof follows a standard proof about the non-degeneracy for the dual of Segre-Veronese varieties.

For any n, the Chow variety $\mathrm{Chow}_\lambda(\mathbb{P}V)$ does not contain any linear spaces if $m_1 = 0$, so in this case the dual is a hypersurface.

Now suppose $m_1 > 0$. We must then show that a generic hyperplane is tangent to $\mathrm{Chow}_\lambda(\mathbb{P}V)$ along (at least) a line precisely when $n = 2$ or when $n > 2$ and $\lambda = (d-1, 1)$.

Consider a general point $[x] \in \mathrm{Chow}_\lambda(\mathbb{P}V)$ with $\lambda = (1^{m_1}, \dots, p^{m_p})$ and $m_1 > 0$. Then we may write $x = lf$, where f is completely decomposable. Then the tangent space is

$$\begin{array}{rcl} T_x\mathrm{Chow}_\lambda(\mathbb{P}V) & = & \{lf, wf, lf'z \mid w, z \in V\} \\ & = & V \cdot f + \sum_i V \cdot l \cdot \frac{f}{y_i}. \end{array},$$

where y_i are the factors of f.

Notice that the linear space $\mathbb{P}L = \mathbb{P}(V \cdot f)$ is contained in $\mathrm{Chow}_\lambda(\mathbb{P}V)$, and up to reordering of the factors of x, every linear space on $\mathrm{Chow}_\lambda(\mathbb{P}V)$ is of this form.

Suppose H is a general hyperplane that contains a general tangent space $T_x := T_x\mathrm{Chow}_\lambda(\mathbb{P}V)$ as above, we want to count the number of conditions on a element $l'f \in V \cdot f$ so that H be tangent to $\mathrm{Chow}_\lambda(\mathbb{P}V)$ at $l'f$. The dimension of choices for $l'f$ avoiding the line through x is $n-1$, in other words we must choose l' from $l^\perp \subset V$.

Now consider two cases, first $\#\lambda = s = 2$ and later $s > 2$.

If $s = 2$, consider $x = l_1 \cdot f$, with $l_2^{d-1} = f$, and generically we may assume l_1 and l_2 are linearly independent.

Now let $y = l_3 l_2^{d-1}$ be a general point in $V l_2^{d-1}$, where l_3 is assumed to be independent of l_1 so that x and y are independent.

Since H annihilates T_x, we should calculate T_y modulo T_x. The vectors that remain are all of the form $l_3 l_2' l_2^{d-2}$. Now if l_2' is in the line $[l_2]$ then $l_3 l_2' l_2^{d-2}$ is contained in T_x. Additionally, if l_2' is in the line $[l_1]$, then $l_2 l_3' l_3^{d-2}$ is not on $\mathrm{Chow}(1, d-1)\mathbb{P}V$. So we generically have a non-trivial condition $H(l_2 l_3' l_3^{d-2}) = 0$ for each $l_2' \in \{l_1, l_2\}^\perp$, which is at most $n-2$ conditions. Therefore the dimension of the space of possible points $l'f$ is at least $n - 1 - (n-2) = 1$, therefore a generic hyperplane is tangent along a line.

Now suppose $s > 2$. We will consider first the case $s = 3$ and later argue that considering this case suffices.

Let $x = l_1 l_2^i l_3^j$, where $i + j + 1 = d$ and $i, j > 0$, else we revert to the previous case. Consider $y = l_4 l_2^i l_3^j \in V l_2^i l_3^j$. Now compute T_y modulo T_x. Points on T_y have the form

$$l_4' l_2^i l_3^j + i \cdot l_4 l_2' l_2^{i-1} l_3^j + j \cdot l_4 l_2^i l_3 l_3^{j-1},$$

which reduces to

$$i \cdot l_4 l_2' l_2^{i-1} l_3^j + j \cdot l_4 l_2^i l_3 l_3^{j-1},$$

modulo T_x. As before, the maximum number of independent conditions we can impose on the choices of $y \in V l_2^i l_3^j$ will come from the cases when

$$l_2' \in \{l_1, l_2\}^\perp \text{ and } l_3' \in \{l_1, l_3\}^\perp,$$

which are $n - 2 + n - 2 = 2n - 4$ conditions, and for generic H, this bound will be achieved. Note when $n = 2$ no additional conditions are imposed and $\mathrm{Chow}_{1,i,j}\mathbb{P}^1$ is not a hypersurface. On the other hand, $2n - 4$ independent conditions imposed on a space of dimension $n - 1$ will not have positive dimension as soon as $n \geq 3$, and thus $\mathrm{Chow}_{1,i,j}\mathbb{P}V$ is a hypersurface whenever $\dim V \geq 3$.

Finally, when $s > 3$ the analogous calculation provides at least as many conditions to impose on the $n - 1$ choices of possible additional points in Vf where a generic hyperplane may be tangent to $\mathrm{Chow}_\lambda(\mathbb{P}V)$, so the dimension of the resulting space will not be positive for $\dim V \geq 3$. $\square$

5. Proof of main results

We aim to prove Theorem 1.2 in two steps. The first step is the following.

Lemma 5.1. *Suppose $\lambda \vdash d$ with $\#\lambda = s$. Then for every $\mu \vdash d$ with $\#\mu = t$ such that $\lambda \prec \mu$ (by refinement)*

$$Chow_\lambda\left(\mathbb{P}V\right)^\vee \subset \mathrm{Seg}_\mu\left(\mathbb{P}V^{\times t}\right)^\vee \cap \mathbb{P}\left(S^dV^*\right). \tag{3}$$

Moreover when $Chow_\lambda\left(\mathbb{P}V\right)^\vee$ is a hypersurface it occurs with multiplicity $M_{\lambda,\mu}$ in $\mathrm{Seg}_\mu\left(\mathbb{P}V^{\times t}\right)^\vee \cap \mathbb{P}\left(S^dV^*\right)$, where $M_{\lambda,\mu}$ is the number of refinements from μ to λ.

A formula for $M_{\lambda,\mu}$ is given in Proposition 2.2.

Proof. Suppose F is a symmetric hyperplane tangent to $\mathrm{Chow}_\lambda\left(\mathbb{P}V^{\times s}\right)$ at a general point $[v_1^{\lambda_1}\cdots v_s^{\lambda_s}]$. Then we have

$$F\left(w\frac{v_1^{\lambda_1}\cdots v_s^{\lambda_s}}{v_i}\right) = 0 \tag{4}$$

for all $1 \leq i \leq s$ and for all $w \in V$.

Now we apply the ideas outlined in Section 3. Since $\lambda \prec \mu$, we can consider the inclusion

$$S^{\lambda_1}V \otimes S^{\lambda_2}V \otimes \cdots \otimes S^{\lambda_s}V \subset S^{\mu_1}V \otimes S^{\mu_2}V \otimes \cdots \otimes S^{\mu_t}V. \tag{5}$$

This implies that $v_1^{\lambda_1}\cdots v_s^{\lambda_s} \in S^{\mu_1}V \otimes \cdots \otimes S^{\mu_t}V$, and there exists a tensor $u_1^{\mu_1} \otimes u_2^{\mu_2} \otimes \cdots \otimes u_t^{\mu_t}$, where each vector u_i is an element of $\{v_1, \ldots, v_s\}$, and in particular $u_1^{\mu_1}u_2^{\mu_2}\cdots u_t^{\mu_t} = v_1^{\lambda_1}\cdots v_s^{\lambda_s}$. In other words $u_1^{\mu_1} \otimes u_2^{\mu_2} \otimes \cdots \otimes u_t^{\mu_t}$ symmetrizes to $v_1^{\lambda_1}\cdots v_s^{\lambda_s}$, and thus $F(u_1^{\mu_1} \otimes u_2^{\mu_2} \otimes \cdots \otimes u_t^{\mu_t}) = 0$ (see Section 3).

We claim that F is tangent to $\mathrm{Seg}_\mu\left(\mathbb{P}V^{\times t}\right)$ at each $[u_1^{\mu_1} \otimes u_2^{\mu_2} \otimes \cdots \otimes u_t^{\mu_t}]$.

Any tangent vector through $u_1^{\mu_1} \otimes u_2^{\mu_2} \otimes \cdots \otimes u_t^{\mu_t}$ can be written as

$$u_1^{\mu_1} \otimes \cdots \otimes u_{i-1}^{\mu_{i-1}} \otimes w \cdot u_i^{\mu_i - 1} \otimes u_{i+1}^{\mu_{i+1}} \otimes \cdots \otimes u_t^{\mu_t}$$

for $1 \leq i \leq t$ and $w \in V$. This tensor symmetrizes to

$$w \cdot \frac{u_1^{\mu_1}\cdots u_t^{\mu_t}}{u_i} = w \cdot \frac{v_1^{\lambda_1}\cdots v_s^{\lambda_s}}{v_j}$$

where the equality holds because $u_1^{\mu_1} \otimes u_2^{\mu_2} \otimes \cdots \otimes u_t^{\mu_t}$ symmetrizes to $v_1^{\lambda_1}\cdots v_s^{\lambda_s}$ and moreover $u_i = v_j$ for some j. Since F is symmetric and takes the same value at every tensor that symmetrize to the same form, (4) implies that F is tangent to $\mathrm{Seg}_\mu\left(\mathbb{P}V^{\times t}\right)$ at each $[u_1^{\mu_1} \otimes u_2^{\mu_2} \otimes \cdots \otimes u_t^{\mu_t}]$.

The number of points $u_1^{\mu_1} \otimes u_2^{\mu_2} \otimes \cdots \otimes u_t^{\mu_t} \in S^{\mu_1}V \otimes \cdots \otimes S^{\mu_t}V$ that symmetrize to $v_1^{\lambda_1}\cdots v_s^{\lambda_s}$ is computed by $M_{\lambda,\mu}$ and is a lower bound for the multiplicity of Chow_λ in $\mathrm{Seg}_\mu\left(\mathbb{P}V^{\times t}\right) \cap \mathbb{P}S^dV^*$.

On the other hand, suppose $\mathrm{Chow}_\lambda(\mathbb{P}V)^\vee$ is a hypersurface and is contained in $\mathrm{Seg}_\mu\left(\mathbb{P}V^{\times t}\right) \cap \mathbb{P}S^dV^*$.

Suppose $u_1^{\mu_1} \otimes u_2^{\mu_2} \otimes \cdots \otimes u_t^{\mu_t} \in S^{\mu_1}V \otimes \cdots \otimes S^{\mu_t}V$ is a tensor which does not symmetrize to $v_1^{\lambda_1}\cdots v_s^{\lambda_s}$ but still $[u_1^{\mu_1}\cdots u_t^{\mu_t}] \in \mathrm{Chow}_\lambda\mathbb{P}V$. In particular $[u_1^{\mu_1}\cdots u_t^{\mu_t}] \neq [v_1^{\lambda_1}\cdots v_s^{\lambda_s}]$.

If $[u_1^{\mu_1}\cdots u_t^{\mu_t}] \in T_{[v_1^{\lambda_1}\cdots v_s^{\lambda_s}]}\mathrm{Chow}_\lambda(\mathbb{P}V) \subset [F]$, then F is not tangent to $\mathrm{Chow}_\lambda(\mathbb{P}V)$ at $[u_1^{\mu_1}\cdots u_t^{\mu_t}]$ else this would violate the hypersurface condition.

If $[u_1^{\mu_1}\cdots u_t^{\mu_t}]$ is not in the tangent space and $\mathrm{Chow}(\mathbb{P}V)$ is not the whole ambient space, a generic F satisfying $s(n-1)$ independent conditions will miss a point, thus we can choose an F that does not vanish at $[u_1^{\mu_1}\cdots u_t^{\mu_t}]$.

So $M_{\lambda,\mu}$ is also the maximum multiplicity of a hypersurface $\mathrm{Chow}_\lambda(\mathbb{P}V)^\vee$ in $\mathrm{Seg}_\mu(\mathbb{P}V^{\times t})^\vee \cap \mathbb{P}S^dV^*$. $\square$

Remark 5.2. Notice that Lemma 4.5 almost provides an effortless proof of Lemma 5.1, but it only implies a lower bound for the multiplicity. So, one may quote Lemma 4.5 and argue directly as we did for a shorter proof.

The second step of the proof of the main theorem is the following.

Lemma 5.3. *Suppose $F \subset V^{\otimes d}$ is a symmetric hyperplane which is tangent to the Segre-Veronese variety $\mathrm{Seg}_\mu(\mathbb{P}V^{\times t})$ at $[u_1^{\otimes\mu_1}\otimes\cdots\otimes u_t^{\otimes\mu_t}]$. Suppose $\lambda \prec \mu$. Then $[u_1^{\mu_1}\cdots u_t^{\mu_t}] \in Chow_\lambda(\mathbb{P}V)$ and F is also tangent to $Chow_\lambda(\mathbb{P}V)$ at $[u_1^{\mu_1}\cdots u_t^{\mu_t}]$.*

Proof. By hypothesis since $\lambda \prec \mu$, there is a symmetrization of $u_1^{\otimes\mu_1}\otimes\cdots\otimes u_t^{\otimes\mu_t}$ so that $u_1^{\mu_1}\cdots u_t^{\mu_t} = v_1^{\lambda_1}\cdots v_s^{\lambda_s}$ and for all $1\le i\le s$, $v_i \in \{u_1,\dots,u_t\}$.

The conditions that tangent to $\mathrm{Seg}_\mu(\mathbb{P}V^{\times t})$ at $[u_1^{\otimes\mu_1}\otimes\cdots\otimes u_t^{\otimes\mu_t}]$ are

$$F(u_1^{\mu_1}\otimes\cdots\otimes u_{i-1}^{\mu_{i-1}}\otimes w\cdot u_i^{\mu_i-1}\otimes u_{i+1}^{\mu_{i+1}}\otimes\cdots\otimes u_t^{\mu_t}) = 0$$

for $1\le i\le t$ and $w\in V$. Now apply the ideas in Section 3. Indeed $u_1^{\mu_1}\otimes\cdots\otimes u_{i-1}^{\mu_{i-1}}\otimes w\cdot u_i^{\mu_i-1}\otimes u_{i+1}^{\mu_{i+1}}\otimes\cdots\otimes u_t^{\mu_t}$ symmetrizes to $\frac{u_1^{\mu_1}\cdots u_t^{\mu_t}}{u_i}w$, so

$$F(\tfrac{u_1^{\mu_1}\cdots u_t^{\mu_t}}{u_i}w) = 0, \quad \text{for all } 1\le i\le p \quad \text{for all } w\in V.$$

But since $u_1^{\mu_1}\cdots u_t^{\mu_t} = v_1^{\lambda_1}\cdots v_s^{\lambda_s}$ and $u_i = v_j$ for some i,j,

$$F(\frac{u_1^{\mu_1}\cdots u_t^{\mu_t}}{u_i}w) = F(\frac{v_1^{\lambda_1}\cdots v_s^{\lambda_s}}{v_j}w) = 0.$$

This holds for all $w\in V$, and these are the conditions that F be tangent to $\mathrm{Chow}_\lambda(\mathbb{P}V)$ at $[u_1^{\mu_1}\cdots u_t^{\mu_t}] = [v_1^{\lambda_1}\cdots v_s^{\lambda_s}]$ so we are done. $\square$

Proof of Theorem 1.2. Lemma 5.1 showed that

$$\bigcup_{\lambda\prec\mu} \mathrm{Chow}_\lambda(\mathbb{P}V)^\vee \subset \mathrm{Seg}_\mu\left(\mathbb{P}V^{\times t}\right)^\vee \cap \mathbb{P}\left(S^dV^*\right),$$

and moreover that each $\mathrm{Chow}_\lambda(\mathbb{P}V)^\vee$ that is a hypersurface occurs with multiplicity $M_{\lambda,\mu}$.

Now for the other direction, apply Lemma 5.3. Suppose $F \in \mathrm{Seg}_\mu\left(\mathbb{P}V^{\times t}\right)^\vee \cap \mathbb{P}\left(S^dV^*\right)$. Then F is a symmetric hyperplane, and moreover, F must be tangent to $\mathrm{Seg}_\mu(\mathbb{P}V^{\times t})$ in some point $[v_1^{\otimes\mu_1}\otimes\cdots\otimes v_t^{\otimes\mu_t}]$, and tangent to $\mathrm{Chow}_\lambda(\mathbb{P}V)$ for every λ such that $v_1^{\mu_1}\cdots v_t^{\mu_t}\in\mathrm{Chow}_\lambda(\mathbb{P}V)$ and more specifically for every $\lambda\prec\mu$. This means that $F\in\mathrm{Chow}_\lambda(\mathbb{P}V)^\vee$ for such λ, and therefore

$$\mathrm{Seg}_\mu\left(\mathbb{P}V^{\times t}\right)^\vee \cap \mathbb{P}\left(S^dV^*\right) \subset \bigcup_{\lambda\prec\mu}\mathrm{Chow}_\lambda(\mathbb{P}V)^\vee.$$

$\square$

Theorem 1.1 is a specific case of Theorem 1.2, we only need to note that $M_{\lambda,\lambda} = \binom{d}{\lambda}$ is the binomial coefficient (see Section 2).

Corollary 1.5 also follows from Theorem 1.2. This is because in Section 2 we also showed that the multiplicities $M_{\lambda,\mu}$ can be both computed and organized in a lower triangular matrix. Using the generating function for $D_\mu = \deg(\mathrm{Seg}_\mu(\mathbb{P}V^{\times t}))$, found [7, Theorem 3.1, Proposition 3.2] or also [8, page 454], we can compute the vector of degrees $(D_\mu)_\mu$, so we can then solve the linear system $(D_\mu)_\mu = (M_{\lambda,\mu})_{\lambda,\mu}(d_\lambda)_\lambda$, where d_λ denotes the degree of $\mathrm{Chow}_\lambda(\mathbb{P}V^{\times s})$. See the appendix for a few examples.

5.1. **Degree formula in the binary case.**

Theorem 5.4. *Suppose $V = \mathbb{C}^2$. Let $\lambda = (1^{m_1}, 2^{m_2}, \dots, p^{m_p})$, with $m = \sum_i m_i$ and suppose $m_1 = 0$. The degree of $Chow_\lambda(\mathbb{P}^1)^\vee$*

$$(m+1)\binom{m}{m_2, \dots, m_p} 1^{m_2} 2^{m_3} \cdots (p-1)^{m_p} \tag{6}$$

Proof. To prove this, we want to find a way to relate this dual variety to a resultant whose degree is equal to the degree we have written above.

If a polynomial is homogeneous of degree d_i in the each set of k_i+1 variables, we say that a polynomial is a multi-homogeneous form of type $(k_1, \dots, k_p; d_1, d_2, \dots, d_p)$.

We will use [8, Proposition XIII.2.1], which states that the degree of a the resultant $R(f_0, \dots, f_k)$ of type $(k_1, \dots, k_r; d_1, \dots, d_r)$ with $k = \sum_i k_i$ is

$$(k+1)\binom{k}{k_1, \dots, k_r} d_1^{k_1} \cdots d_r^{k_r}. \tag{7}$$

Consider the simplest case when $\lambda = (d)$ and $\dim V = n$. If $[F]$ is a hyperplane tangent to $\mathrm{Chow}_{(d)}$ at v^d we must have

$$F(v^d) = 0 \text{ and } F(v^{d-1}w) = 0 \text{ for all } w \in V.$$

Notice that the second condition includes the first when $v = w$, so We write these conditions as

$$(\frac{\partial}{\partial w}F)(v) = 0,$$

for all $w \in V$. Then the condition is that n different polynomials $(\frac{\partial}{\partial w}F)$ of degree $d-1$ have a common root at v. This is a resultant of n polynomials of type $(n-1; d-1)$, which has degree $n\binom{n-1}{n-1}(d-1)^{n-1} = n(d-1)^{n-1}$. This agrees with the usual formula for the degree of the discriminant of a degree d polynomial on n variables.

Now we proceed analogously. Let $[x]$ be a general point on $\mathrm{Chow}_\lambda\mathbb{P}^1$ with λ as in the hypotheses. By arranging the factors of x we may write

$$\begin{aligned} x &= (w_{2,1}\cdots w_{2,m_2})^2 \cdots (w_{p,1}\cdots w_{p,m_p})^p \\ &= (w_{2,1}\cdots w_{2,m_2}\cdots w_{p,1}\cdots w_{p,m_p}) \\ &\cdot (w_{2,1}\cdots w_{2,m_2})(w_{3,1}\cdots w_{3,m_3})^2 \cdots (w_{p,1}\cdots w_{p,m_p})^{p-1} \end{aligned},$$

with the $w_{i,j}$ distinct (set $w_{i,j}$ equal to 1 if $m_i = 0$). For each i, let y_i denote the quantity $y_i = w_{i,1}\cdots w_{i,m_i} \in S^{m_i}\mathbb{C}^2$, so that $x = (y_2 y_3 \cdots y_p) y_2 y_3^2 \cdots y_p^{p-1}$, and we are now thinking of each y_i as a vector in $\mathbb{C}^{m_i+1} \cong S^{m_i}\mathbb{C}^2$.

Now suppose $[F] \subset \mathbb{P}S^dV$ is a hyperplane which is tangent to $\mathrm{Chow}_\lambda\mathbb{P}^1$ at $[x]$. Then we must have

$$F(x) = 0, \text{ and } F\left(\frac{x}{w_{i,j}}v\right) = 0, \tag{8}$$

for all $2 \leq i \leq p$,$1 \leq j \leq m_i$ and all $v \in V$.

We will now translate the conditions (8) as the condition that $m+1$ multi-homogeneous forms of type $(m_2, m_3, \dots, m_p; 1, 2, \dots, p-1)$ have a common root.

Consider

$$G = \left(\frac{\partial^p}{\partial y_2 \partial y_3 \cdots \partial y_p} F \right)(\cdot) \colon \mathbb{C}^{m_2+1} \otimes \cdots \otimes \mathbb{C}^{m_p+1} \longrightarrow \mathbb{C}.$$

This is a multi-homogenous function of type $(m_2, m_3, \dots, m_p; 1, 2, \dots, p-1)$. Note that (8) implies that $G(y_2, y_3, \dots, y_p) = 0$.

Next, construct for every $2 \leq i \leq p$ and $1 \leq j \leq m_i$,

$$G_{i,j} = \left(\frac{\partial^p}{\partial y_2 \partial y_3 \cdots \partial y_{i-1} \partial g_{i,j} \partial y_{i+1} \cdots \partial y_p} F \right) : \mathbb{C}^{m_2+1} \otimes \cdots \otimes \mathbb{C}^{m_p+1} \longrightarrow \mathbb{C},$$

where $g_{i,j} = \frac{v_{i,j}}{w_{i,j}} y_i$ and $v_{i,j}$ is a (fixed) vector so that $\{w_{i,j}, v_{i,j}\}$ form a basis of $\mathbb{C}^2$. Note that each $G_{i,j}$ is also multi-homogeneous of type $(m_2, m_3, \dots, m_p; 1, 2, \dots, p-1)$. There are $\sum_i m_i$ such G.

The conditions on F now imply that the $m = \sum_{i=1}^p m_i$ functions $G_{i,j}$ have a common root, with G namely $(y_2, y_3, \dots, y_p)$. Therefore [8, Proposition XIII.2.1] applies, and substituting $\sum_i m_i = m = k$, $m_{i+1} = k_i$, $p = r$, $d_i = i$ into the above formula (7) gives the degree of $\mathrm{Chow}_\lambda \mathbb{P}^1$ as claimed. $\square$

We would like to know if the degree of $\mathrm{Chow}_\lambda \mathbb{P}V$ may also be calculated as the degree of some resultant, but we failed in our initial attempts to formulate this correctly.

6. Computing degrees of duals of Chow varieties

As we mentioned above, Theorem 1.2 and elementary facts about the quantities $M_{\lambda,\mu}$ provide a recursive way to compute the degrees of the dual varieties to the Chow varieties. We illustrate our methods with a few examples that are non-trivial but don't take up too much space.

Throughout the following we let d_λ denote the degree of Ξ_λ, and let D_μ denote the degree of $\Delta_{\mu,n}$. First we compute $M_{\lambda,\mu}$. Next we use the generating function from [8] and a easy implementation in Maple to compute D_μ. Then Corollary 1.5 implies that to find the d_λ we can find the unique solution to $M_{\lambda,\mu} d_\lambda = D_\mu$. Of course it suffices to compute an lower triangular sub-matrix of $M_{\lambda,\mu}$. We only record a subset of the columns that corresponds to the set of λ for which $\mathrm{Chow}_\lambda(\mathbb{P}V)^\vee$ is a hypersurface. So the matrix we display contains all of the multiplicities of the factors of the symmetrized μ-discriminants, including the hyperdeterminant, which occupies the last row.

The first case we consider is $n = 2$ and $d = 7$ and solve

$$\begin{pmatrix} 1 & & & \\ 1 & & & \\ 1 & 1 & & \\ 1 & 1 & & \\ 1 & & 1 & \\ 1 & 1 & 1 & \\ 1 & 3 & 1 & \\ 1 & & 2 & \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 3 & 2 \\ 1 & 6 & 5 & 6 \\ 1 & 3 & 3 & 6 \\ 1 & 5 & 7 & 7 \\ 1 & 11 & 15 & 50 \\ 1 & 21 & 35 & 210 \end{pmatrix} \begin{pmatrix} d_{(7)} \\ d_{(5,2)} \\ d_{(4,3)} \\ d_{(3,2,2)} \end{pmatrix} = \begin{pmatrix} D_{(7)} \\ D_{(6,1)} \\ D_{(5,2)} \\ D_{(5,1,1)} \\ D_{(4,3)} \\ D_{(4,2,1)} \\ D_{(4,1^3)} \\ D_{(3,3,1)} \\ D_{(3,2,2)}, \\ D_{(3,2,1,1)} \\ D_{(3,1^4)} \\ D_{(2,2,2,1)} \\ D_{(2,2,1^3)} \\ D_{(2,1^5)} \\ D_{(1^7)}) \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 36 \\ 36 \\ 48 \\ 72 \\ 120 \\ 84 \\ 144 \\ 216 \\ 480 \\ 336 \\ 720 \\ 2016 \\ 6816 \end{pmatrix}.$$

The unique solution to $M_{\lambda,\mu} d_\lambda = D_\mu$ is

$$(d_{(7)}, d_{(5,2)}, d_{(4,3)}, d_{(3,2,2)}) = (12, 24, 36, 24).$$

The previous calculation shows that the hyperdeterminant of format $2^{\times 7}$, which has degree 6816, when symmetrized splits into 4 factors of degrees 12, 24, 36 and 24 with multiplicities 1, 21, 35 and 210 respectively. Geometrically, the 4 factors are the discriminant hypersurface, and three other dual varieties to Chow varieties. The other μ-discriminants have the same factors, with different multiplicities encoded by $M_{\lambda,\mu}$. The following examples also have analogous interpretations.

Next we use only a relevant lower-triangular sub-matrix of $M_{\lambda,\mu}$, when $d = 8$ and $n = 2$ where here we have omitted several rows that are unnecessary for computing the degrees of Ξ_λ.

$$\begin{pmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & & 1 & & & & \\ 1 & & & 2 & & & \\ 1 & 2 & & 4 & 2 & & \\ 1 & 1 & 2 & & & 2 & \\ 1 & 4 & & 6 & 12 & & 120 \\ 1 & 28 & 56 & 70 & 420 & 560 & 2520 \end{pmatrix} \begin{pmatrix} (d_{(8)} \\ d_{(6,2)} \\ d_{(5,3)} \\ d_{(4,4)} \\ d_{(4,2,2)} \\ d_{(3,3,2)} \\ d_{(2,2,2,2)}) \end{pmatrix} = \begin{pmatrix} (D_{(8)} \\ D_{(6,2)} \\ D_{(5,3)} \\ D_{(4,2,2)} \\ D_{(3,3,2)} \\ D_{(2,2,2,2)} \\ D_{(1^8)}) \end{pmatrix} = \begin{pmatrix} 14 \\ 44 \\ 62 \\ 116 \\ 656 \\ 848 \\ 60032 \end{pmatrix}.$$

The unique solution to $M_{\lambda,\mu} d_\lambda = D_\mu$ is

$$(d_{(8)}, d_{(6,2)}, d_{(5,3)}, d_{(4,4)}, d_{(4,2,2)}, d_{(3,3,2)}, d_{(2,2,2,2)}) = (14, 30, 48, 27, 36, 48, 5).$$

Proceeding in the same way, here are two examples when $n = 3$. Here is the case $d = 4$.

$$\begin{pmatrix} 1 & & & \\ 1 & & & \\ 1 & 2 & & \\ 1 & 2 & 2 & \\ 1 & 6 & 12 & 24 \end{pmatrix} \begin{pmatrix} d_{(4)} \\ d_{(2,2)} \\ d_{(2,1,1)} \\ d_{(1^4)} \end{pmatrix} = \begin{pmatrix} D_{(4)} \\ D_{(3,1)} \\ D_{(2,2)} \\ D_{(2,1,1)} \\ D_{(1^4)} \end{pmatrix} = \begin{pmatrix} 27 \\ 27 \\ 129 \\ 225 \\ 1269 \end{pmatrix}.$$

The unique solution to $M_{\lambda,\mu} d_\lambda = D_\mu$ is

$$(d_{(4)}, d_{(2,2)}, d_{(2,1,1)}, d_{(1^4)}) = (27, 51, 48, 15).$$

For $d = 5$, we have

$$\begin{pmatrix} 1 & & & & & \\ 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 1 & 2 & & & \\ 1 & 2 & & 2 & & \\ 1 & 4 & 6 & 6 & 6 & \\ 1 & 10 & 20 & 30 & 60 & 120 \end{pmatrix} \begin{pmatrix} d_{(5)} \\ d_{(3,2)} \\ d_{(3,1,1)} \\ d_{(2,2,1)} \\ d_{(2,1^3)} \\ d_{(1^5)} \end{pmatrix} = \begin{pmatrix} D_{(5)} \\ D_{(4,1)} \\ D_{(3,2)} \\ D_{(3,1,1)} \\ D_{(2,1,1)} \\ D_{(2,1^3)} \\ D_{(1^5)} \end{pmatrix} = \begin{pmatrix} 48 \\ 48 \\ 360 \\ 576 \\ 1440 \\ 7128 \\ 68688 \end{pmatrix}.$$

The unique solution to $M_{\lambda,\mu} d_\lambda = D_\mu$ is

$$(d_{(5)}, d_{(3,2)}, d_{(3,1,1)}, d_{(2,2,1)}, d_{(2,1^3)}, d_{(1^5)}) = (48, 312, 108, 384, 480, 192).$$

Acknowledgements

The author would like to thank G. Ottaviani for suggesting this work and for the many useful discussions and encouragement, as well as for his hospitality while the author was a post-doc under his supervision at the University of Florence. Part of the research for this work was done while visiting the Mittag-Leffler Institute. The author would like to thank the participants for useful discussions and for the stimulating environment made by their passion for mathematics. J.M. Landsberg, C. Peterson, K. Ranestad, J. Buczynski, and F. Block also provided particularly useful discussions.

References

1. G. Boffi and J. Weyman, *Koszul complexes and hyperdeterminants*, J. Algebra **230** (2000), no. 1, 68–88. MR 1774758 (2001h:13023)
2. E. Briand, *Covariants vanishing on totally decomposable forms*, Liaison, Schottky problem and invariant theory, Progr. Math., vol. 280, Birkhäuser Verlag, Basel, 2010, pp. 237–256. MR 2664658
3. E. Carlini, *Codimension one decompositions and Chow varieties*, Projective varieties with unexpected properties, Walter de Gruyter GmbH & Co. KG, Berlin, 2005, pp. 67–79. MR 2202247 (2007f:14056)
4. A. Cayley, *On the theory of linear transformations*, Cambridge Math. J. **4** (1845), 1–16, Reprinted in his *Collected Mathematical Papers, Vol. 1*, p. 80–94, Cambridge Univ. Press, 1889 [microform].
5. J. Chipalkatti, *On equations defining coincident root loci*, J. Algebra **267** (2003), no. 1, 246–271. MR 1993476 (2004h:13009)
6. ———, *Invariant equations defining coincident root loci*, Arch. Math. (Basel) **83** (2004), no. 5, 422–428. MR 2102640 (2005g:12001)
7. I. M. Gel′fand, M. M. Kapranov, and A. V. Zelevinsky, *Hyperdeterminants*, Adv. Math. **96** (1992), no. 2, 226–263. MR 1196989 (94g:14023)
8. I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications, Boston: Birkhäuser, Boston, MA, 1994. MR 1264417 (95e:14045)
9. Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2/, 2010.
10. P. Huggins, B. Sturmfels, J. Yu, and D. Yuster, *The hyperdeterminant and triangulations of the 4-cube*, Math. Comp. **77** (2008), no. 263, 1653–1679. MR 2398786 (2009c:52021)
11. J. M. Landsberg, *The geometry of tensors: Applications to complexity, statistics, and other areas*, in preparation, 2011.
12. G. Ottaviani, *An invariant regarding Waring's problem for cubic polynomials*, Nagoya Math. J. **193** (2009), 95–110. MR MR2502909 (2010c:14057)
13. V. Strassen, *Rank and optimal computation of generic tensors*, Linear Algebra Appl. **52/53** (1983), 645–685. MR 709378 (85b:15039)

14. Emil Toeplitz, *Über ein Flächennetz zweiter Ordnung*, Math. Ann. **11** (1877), 434–463.
15. J. Weyman and A. Zelevinsky, *Singularities of hyperdeterminants*, Ann. Inst. Fourier (Grenoble) **46** (1996), no. 3, 591–644. MR 1411723 (97m:14050)
16. F. L. Zak, *Tangents and secants of algebraic varieties*, Translations of Mathematical Monographs, vol. 127, Providence: American Mathematical Society, 1993, Translated from the Russian manuscript by the author. MR 1234494 (94i:14053)

E-mail address: `oeding@math.unifi.it`

Dipartimento di Matematica "U. Dini", Universitá degli Studi di Firenze, Firenze, Italy

Department of Mathematics, University of California Berkeley, Berkeley, CA, USA
E-mail address: `oeding@math.unifi.it`