CHANGE OF TOPOLOGY IN MEAN CONVEX MEAN CURVATURE FLOW

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Abstract. Consider the mean curvature flow of an $(n+1)$-dimensional compact, mean convex region in Euclidean space (or, if $n < 7$, in a Riemannian manifold). We prove that elements of the m^{th} homotopy group of the complementary region can die only if there is a shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$ singularity for some $k \le m$.

1. Introduction

Let $K(t)$ be a time-dependent closed region in a Riemannian manifold such that the boundary $\partial K(t)$ moves by mean curvature flow. Clearly the topology of the complement $K(t)^c$ can change only if there is a singularity of the flow. It is natural to ask if we can deduce properties of the singularities from the way the topology changes. In this paper, we give a rather precise answer if the regions are mean convex. In particular, consider a mean curvature flow $t \in [0, \infty) \mapsto K(t)$ of compact regions in an $(n+1)$-dimensional Riemannian manifold N such that $K(0)$ is mean convex and has smooth boundary. If $n \ge 7$, we require that the metric on N be flat[1]. We prove a theorem that implies the following:

1.1. **Theorem.** *Suppose that $0 \le a < b$. Suppose there is a map of the m-sphere into $K(a)^c$ that is homotopically trivial in $K(b)^c$ but not in $K(a)^c$.*

Then at some time t with $a \le t < b$, there is a singularity of the flow at which the Gaussian density is $\ge d_m$, the Gaussian density of a shrinking m-sphere in $\mathbf{R}^{m+1}$, and at which the tangent flow is a shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$ for some k with $1 \le k \le m$.

The following is an interesting special case:

1.2. **Corollary.** *Suppose K is a compact, mean convex subset of $\mathbf{R}^{n+1}$ with smooth boundary, and suppose that there is a map of the m-sphere into K^c that is homotopically nontrivial.*

Then the resulting mean curvature flow has a singularity with Gaussian density $\ge d_m$.

The corollary follows from the theorem because compact subsets of Euclidean space disappear in finite time under mean curvature flow. (Thus we can choose b large enough that $K(b)$ is empty.)

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[1]None of the arguments in this paper depend on dimension. However, they do require that the singularities of the flow have convex type (as defined in §2), and in high dimensions it has not been proved that all singularities have convex type except when the ambient Riemannian metric is flat.

More generally, the topological assumption in Theorem 1.1 can be replaced by the weaker assumption that there is a continuous map

$$F : \mathbf{B}^{m+1} \to K(b)^c \text{ with } F(\partial \mathbf{B}^{m+1}) \subset K(a)^c$$

that cannot be homotoped by a 1-parameter family of such maps to a map F' whose image $F'(\mathbf{B}^{m+1})$ lies in $K(a)^c$. See Theorem 3.1. (The m in Theorem 3.1 corresponds to $(m+1)$ here.)

In Theorems 1.1 and 3.1, the moving hypersurfaces $\partial K(t)$ have no boundary. Those theorems generalize to hypersurfaces with boundary, where the motion of the boundary is prescribed. See Theorem 5.4.

The reader may wonder whether Theorem 1.1 could be strengthened to say that there is a shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$ singularity with $k = m$ rather than $k \le m$. The answer is no, even under the assumption (for the given flow) that there is a unique m for which the hypothesis holds. Altschuler, Angenent, and Giga [AAG95] proved that that there is a "doubly-degenerate neckpinch" mean curvature flow in $\mathbf{R}^3$ in which the surfaces are mean convex topological spheres that are rotationally symmetric about an axis and that are smooth until they collapse to a point at which the tangent flow is not a shrinking $\mathbf{S}^2$ but rather a shrinking $\mathbf{S}^1 \times \mathbf{R}$. Compare this flow to a flow of convex spheres in $\mathbf{R}^3$. The two flows are topologically completely equivalent, yet the singularities are different. In general, for every k and n with $1 \le k <$n, one can construct a mean convex n-sphere in $\mathbf{R}^{n+1}$ that shrinks to a point but whose tangent flow at that point is a shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$.

The reader may also wonder what can be said without assuming mean convexity. Certain analogs of the theorems in this paper hold for arbitrary (i.e., not necessarily mean convex) hypersurfaces in $\mathbf{R}^2$ and $\mathbf{R}^3$; see [IW11c]. For hypersurfaces (of any dimension) that do not fatten under level set flow, there are some restrictions on the way that the topology of the complement can change, no matter what kinds of singularities occur: see [Whi95].

In [IW11c], the results of this paper are used to get lower density bounds on self-similar shrinkers for mean curvature flow, and in [IW11a] and [IW11b] they are used to get lower density bounds for densities of minimal cones.

The results of this paper rely strongly on properties of singularities of mean convex mean curvature flow that were proved in [Whi00] and [Whi03]. However, the results of this paper are vacuously true for flows with singularities with Gaussian density ≥ 2. Thus (for this paper) one only needs the results of [Whi00] and [Whi03] under the assumption that the singularities have Gaussian density < 2, and the most complicated parts of those papers are trivially true under that assumption.

2. Preliminaries

In this section, we state the facts about mean curvature flow of mean convex sets that are important for this paper. Let $t \in [0, \infty) \mapsto K(t)$ be a mean curvature flow of mean convex subsets of a smooth, $(n+1)$-dimensional Riemannian manifold. If $x \in \partial K(t)$ is a regular point, we let $\kappa_1(x) \le \kappa_2(x) \le \cdots \le \kappa_n(x)$ be the principal curvatures of $\partial K(t)$ with respect to the inward unit normal normal. (We could also write $\kappa_i(x,t)$, but since the surfaces $\partial K(t)$ for distinct values of t are disjoint, t is determined by x.) We let $h(x) = \kappa_1(x) + \cdots + \kappa_n(x) > 0$ be the scalar mean curvature. We say that a singular point $x \in \partial K(t)$ (where $t > 0$) has *convex type* provided

(1) Each tangent flow at x is a self-similarly shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$ for some $k \geq 1$.
(2) If $x_i \in \partial K(t_i)$ is a sequence of regular points converging to x, then

$$\liminf \frac{\kappa_1(x_i)}{h(x_i)} \geq 0.$$

(Actually (1) follows from (2), which in turn folllows from the seemingly weaker assumption that the lim inf in (2) must be $> -\infty$. See [Whi03].)

In many situations, singularities are known to have convex type:

2.1. **Proposition.** *Suppose that K is a compact, mean convex region in an $(n+1)$-dimensional Riemannian manifold. Let $t \mapsto K(t)$ be the mean curvature flow with $K(0) = K$. Suppose that*

(1) $n < 7$, or

(2) ∂K is smooth and the Riemannian metric on N is flat.

Then for $t > 0$, the singularities of the flow all have convex type.

See [Whi03] for the proof in the case (1) and [Whi11] for the proof in case (2).

2.2. **Proposition.** *Let $t \mapsto K(t)$ be a mean curvature flow of mean convex sets, and suppose that the singularities of the flow are of convex type. Let $t > 0$ and let x be a point in the interior of $K(t)$. Let y be a point in $\partial K(t)$ that minimizes distance to x. Then y is a regular point of the flow.*

Proof. Note that $K(t)$ contains the ball with center x and radius $\operatorname{dist}(x, y)$, from which it follows

$$\liminf_{r \to 0} \frac{\operatorname{vol}(K(t) \cap \mathbf{B}(y,r))}{\operatorname{vol}(\mathbf{B}(y,r))} \geq \frac{1}{2}.$$

On the other hand, if $z \in \partial K(t)$ is a singular point of convex type, then it is straightforward to show that

$$\lim_{r \to 0} \frac{\operatorname{vol}(K(t) \cap \mathbf{B}(z,r))}{\operatorname{vol}(\mathbf{B}(z,r))} = 0.$$

□

2.3. **Proposition** (Stone)**.** *Let x be a convex-type singularity of a mean convex mean curvature flow. Then there is a $k = k(x) \geq 1$ such that every tangent flow at x is a shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$, where k depends only on the Gaussian density Θ at the point x. (It does not depend on the sequence of dilations used to obtain the tangent flow.)*

Thus the tangent flow is unique up to rotations. For the reader's convenience, we give the idea of Stone's proof. See [Sto94, Appendix A] for details.

Proof. The Gaussian density d_k of a shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$ (where $\mathbf{S}^k$ is the unit k-sphere in $\mathbf{R}^{k+1}$) may be calculated explicitly:

$$\begin{aligned} d_k &= \left(\frac{k}{2\pi e}\right)^{k/2} \sigma_k \\ &= \left(\frac{k}{2e}\right)^{k/2} \left(\frac{2\sqrt{\pi}}{\Gamma(\frac{k+1}{2})}\right), \end{aligned}$$

where σ_k is the area of a k-dimensional sphere of radius 1. Using this formula, one can show that

$$d_1 > d_2 > \dots. \tag{1}$$

Now if a shrinking $\partial \mathbf{B}^k \times \mathbf{R}^{n-k}$ is a tangent flow to $t \mapsto K(t)$ at the point (x,t), then $d_k = \Theta$. Thus by (1), k is determined by Θ. □

2.4. **Proposition.** *Let Σ_k be the set of spacetime points (x,t) such that*

(1) $t > 0$,
(2) $x \in \partial K(t)$,
(3) x is a singular point of convex type, and
(4) the Gaussian density at (x,t) is d_k (or, equivalently, the tangent flows at x are shrinking $\mathbf{S}^k \times \mathbf{R}^{n-k}$s.)

Then Σ_k has parabolic Hausdorff dimension at most $(n-k)$.

This follows easily from standard dimension reducing. (It also a special case of the stratification theory in [Whi97, §9].) Actually, in this paper, we do not need the full strength of Proposition 2.4. All we need is the following much weaker corollary:

2.5. **Corollary.** *Suppose that at a certain time $t > 0$, the singularities all have convex type with Gaussian density $\le d_k$. Then $K(t)$ is a smooth $(n+1)$-manifold with boundary except for a closed subset of $\partial K(t)$ whose Hausdorff dimension is at most $(n-k)$.*

2.6. **Proposition.** *Let $t > 0$ and let $p \in \partial K(t)$ be a either a regular point or a convex-type singular point at which the Gaussian density $\Theta(p)$ of the flow is $\le d_m$. Let x_i be a sequence of points in the interior of $K(t)$ that converge to p. Let y_i be a point in $\partial K(t)$ that minimizes distance to x_i. Translate $K(t)$ by $-y_i$ and dilate by $1/\operatorname{dist}(x_i,y_i)$ to get a set K_i. Then a subsequence $K_{i(j)}$ converges to a convex set K' with smooth boundary, and the convergence $\partial K_{i(j)} \to \partial K'$ is smooth on bounded sets.*

Furthermore, the homotopy groups $\pi_j(\partial K')$ are trivial for $j < m$.

Proof. The assertion is trivially true if p is a regular point (in that case, the set K' is a closed halfspace), so we assume that p is a singular point. Except for the assertion about homotopy groups, this is proved in [Whi03], which also shows that, after a rotation, either

(i) ∂K is the graph of an entire function from $\mathbf{R}^n$ to $\mathbf{R}$, or
(ii) K has the form $C \times \mathbf{R}^{n-k}$ for some $k \ge 1$, where C is a compact, convex subset of $\mathbf{R}^{k+1}$.

In the first case, all the homotopy groups of ∂K are trivial. Thus we may assume that K has the form $C \times \mathbf{R}^{n-k}$ as in (ii).

Recall that the *entropy* of a hypersurface M in $\mathbf{R}^{n+1}$ is the supremum of

$$\frac{1}{(4\pi)^{n/2} r^n} \int_{y \in M} e^{-|y-x|^2/4r^2} \, d\mathcal{H}^n y$$

over all $x \in \mathbf{R}^{n+1}$ and $r > 0$. Because ∂K is a part of a limit flow at (p,t), its entropy at most the Gaussian density of the original flow at the point p:

$$\operatorname{Entropy}(\partial K) \le \Theta(p) \le d_m. \tag{2}$$

(This follows easily from Huisken's monotonicity.) On the other hand, ∂K forms an $\mathbf{S}^k \times \mathbf{R}^{n-k}$ singularity under mean curvature flow. (If this not clear, apply Huisken's Theorem [Hui84] to see that the mean curvature flow starting with C collapses to a round point, and then cross that flow with $\mathbf{R}^{n-k}$ to get a mean curvature flow starting with K and collapsing to a $(n-k)$-space with an $\mathbf{S}^k \times \mathbf{R}^{n-k}$ singularity.) By Huisken's monotonicity,

$$d_k \le \text{Entropy}(\partial K),$$

so $d_k \le d_m$ by (2). Thus $k \ge m$ (by (1)), which implies that the j^{th} homotopy group of ∂K (which is diffeomorphic to $\mathbf{S}^k \times \mathbf{R}^{n-k}$) is trivial for all $j < m$. □

3. The Main Theorem

We begin by recalling some topological terminology. Suppose that Y is a topological space and that X is a subset of Y. We write

$$F : (\mathbf{B}^k, \partial \mathbf{B}^k) \to (Y, X)$$

to indicate that F is a continuous map of the pair $(\mathbf{B}^k, \partial \mathbf{B}^k)$ into (Y, X), i.e, a continuous map of $\mathbf{B}^k$ into Y such that $F(\partial \mathbf{B}^k) \subset X$. Two such maps

$$F, G : (\mathbf{B}^k, \partial \mathbf{B}^k) \to (Y, X)$$

are called homotopic in (Y, X) provided there is a homotopy $H : \mathbf{B}^k \times [0,1] \to Y$ from F to G such that

$$H(\cdot, s) : (\mathbf{B}^k, \partial \mathbf{B}^k) \to (Y, X)$$

for all $s \in [0,1]$.

We say that the pair (Y, X) is **m-connected** if for every $k \le m$, every continuous map

$$F : (\mathbf{B}^k, \partial \mathbf{B}^k) \to (Y, X)$$

is homotopic in (Y, X) to a map G whose image $G(\mathbf{B}^k)$ lies in X.

We can now state the main theorem:

3.1. **Theorem.** *Let $t \in [0, \infty) \mapsto K(t)$ be a mean curvature flow of compact, mean convex subsets of a Riemannian manifold N. Suppose that $0 < a < b$ and that each singularity during the the time interval $a \le t < b$ has convex type and has Gaussian density $\le d_m$, the Gaussian density of a shrinking m-sphere in $\mathbf{R}^{m+1}$.*

Then the pair $(K(b)^c, K(a)^c)$ is m-connected.

In particular, the conclusion implies that if a map of $\mathbf{S}^k$ (for $k < m$) into $K(a)^c$ is contractible in $K(b)^c$, then it is also contractible in $K(a)^c$. Thus Theorem 3.1 implies 1.1.

Recall (Proposition 2.1) that if the metric on N is flat and if $\partial K(0)$ is smooth, or if $\dim(N) \le 7$, then the singularities all have convex type.

Note also that if $\partial K(0)$ is smooth, then we can also allow $a = 0$ in Theorem 3.1, because the topology cannot not change before the first singular time.

Proof of Theorem 3.1. The theorem follows immediately from Theorem 4.4 and Proposition 4.3 below. □

4. An Abstract Form of the Main Theorem

In this section, we state and prove an abstract version (Theorem 4.4) of Theorem 3.1. The abstract version is no harder to prove than the special case, but it has the advantage of also applying to some variants of mean curvature flow.

4.1. **Definition.** Suppose K is a closed set in the interior[2] of smooth $(n+1)$-dimensional Riemannian manifold. A point in K is a **regular** point of K if it is an interior point of K or if it is a boundary point with a neighborhood U such that $K \cap U$ is smoothly diffeomorphic to a closed halfspace in $\mathbf{R}^{n+1}$. The **singular set** $\operatorname{sing}(K)$ of K is the set of points in K that are not regular points of K.

Note that the singular set of K is a closed subset of ∂K.

The following definition captures some of the key features of "having convex type singularities whose Gaussian densities are all $\le d_m$".

4.2. **Definition.** Suppose K is a closed set in the interior of a smooth $(n+1)$-dimensional Riemannian. We define $Q(K)$ to be the largest integer m with the following properties:

(1) The singular set $\operatorname{sing}(K)$ has Hausdorff dimension $\le n - m$.
(2) If x is an interior point of K and y is a singular point of K, then $\operatorname{dist}(x, y) > \operatorname{dist}(x, \partial K)$.
(3) Let x_i be a sequence of points in the interior of K converging to a point in $\operatorname{sing}(K)$. Translate K by $-x_i$ and dilate by $1/\operatorname{dist}(x_i, \partial K)$ to get K_i. Then a subsequence of the K_i converges to a convex subset K' of $\mathbf{R}^{n+1}$ with smooth boundary, and the convergence is smooth on bounded sets.
(4) If K' is as in (3), then $\partial K'$ has trivial k^{th} homotopy for every $k < m$.

If no such integer exists, we let $Q(K) = -\infty$.

Note that if K has no interior, then (2), (3), and (4) are vacuously true, and $\operatorname{sing}(K) = K$, so in that case $Q(K)$ is the largest integer less than or equal to $n - \dim(\operatorname{sing}(K))$.

The following proposition describes for mean curvature flow how $Q(K(t))$ is related to the Gaussian densities of the singularities at time t:

4.3. **Proposition.** *Let $t \in [0, T] \mapsto K(t)$ be a mean curvature flow of mean-convex regions in the interior of a smooth Riemannian $(n+1)$-manifold. If $t \in (0, T]$ and if the singularities at time t all have convex type with Gaussian density $\le d_m$, then $Q(K(t)) \ge m$.*

Proof. The result follows immediately from Proposition 2.2, Corollary 2.5, and Proposition 2.6. □

For mean convex mean curvature flow, $Q(K(t))$ will typically equal the smallest m such that there is a singularity at time t with Gaussian density d_m. However, there are degenerate situations in which $Q(K(t))$ is strictly less than that m. For example, at the singular time for the doubly-degenerate neckpinch in $\mathbf{R}^3$ mentioned in the introduction, $K(t)$ is a single point and thus $Q(K(t)) = 2 - 0 = 2$, but the Gaussian density at that singularity is d_1.

[2]For now, the reader may as well assume that the ambient manifold has no boundary. In §5, we will consider sets K that contain portion of the boundary of the ambient manifold.

4.4. **Theorem.** *Let* $t \in [a,b] \mapsto K(t)$ *be a one-parameter family of compact subsets of a smooth, Riemannian* $(n+1)$*-manifold. Assume that*
(1) $K(t) \subset \text{interior}(K(T))$ *for* $a \le T < t \le b$.
(2) $K(T) = \cap_{t<T} K(t)$ *for* $T \in (a,b]$.
(3) $\text{interior}(K(T)) = \cup_{t>T} \text{interior}(K(t))$ *for* $T \in [a,b)$.
(4) $Q(K(T)) \ge m$ *for each* $T \in [a,b)$.
Then the pair $(K(b)^c, K(a)^c)$ *is* m*-connected.*

Remark. The assumption that the $K(t)$'s are compact can be replaced by the weaker assumption that they are closed and that $K(a) \setminus K(b)$ has compact closure. No changes are required in the proof.

Proof. Let $k \le m$ and let

$$F_0 : (\mathbf{B}^k, \partial \mathbf{B}^k) \to (K(b)^c, K(a)^c).$$

be a continuous map. Let $\mathcal{F}$ be the set of all continuous maps

$$F : (\mathbf{B}^k, \partial \mathbf{B}^k) \to (K(b)^c, K(a)^c)$$

such that F is homotopic in $(K(b)^c, K(a)^c)$ to F_0. We must show that $\mathcal{F}$ contains a map whose image lies in $K(a)^c$, i.e., a map whose image is disjoint from $K(a)$.

Equivalently, if J is the set of $t \in [a,b]$ such that $\mathcal{F}$ contains a map F whose image is disjoint from $K(t)$, then we must show that $a \in J$.

We will prove that $J = [a,b]$ (and therefore that $a \in J$) by proving the following four statements:

(i) $b \in J$.
(ii) J is a relatively open subinterval of $[a,b]$.
(iii) If T is in the closure of J, then $\mathcal{F}$ contains a map F whose image is contained in the union of $K(T)^c$ and the regular part of $\partial K(T)$.
(iv) If T is in the closure of J, then $T \in J$.

Statements (i), (ii), and (iv) imply that J is a nonempty subinterval of $[a,b]$ that is both open and closed in $[a,b]$, and therefore that J is all of $[a,b]$.

Statement (i) is trivially true (since $F_0 \in \mathcal{F}$ and $F_0(\mathbf{B}^k)$ is disjoint from $K(b)$.)

Next we prove statement (ii). For $F \in \mathcal{F}$, let

$$J_F := \{t \in [a,b] : F(\mathbf{B}^k) \cap K(t) = \emptyset\}.$$

Note that

$$J = \cup_{F \in \mathcal{F}} J_F. \tag{3}$$

By definition of $\mathcal{F}$, the set J_F contains b. Since the $K(t)$'s are nested, if $a \le t \le t' \le b$ and if t is in J_F, then t' is also in J_F. Thus J_F is an interval containing b. We claim that J_F is relatively open in $[a,b]$. If $a \in J_F$, then $J_F = [a,b]$, which is certainly relatively open in $[a,b]$. Thus suppose $a \notin J$, i.e., that $K(a)$ intersects $F(\mathbf{B}^k)$. By hypotheses (1) and (2) of the theorem, there is a last time t such that $K(t)$ intersects $F(\mathbf{B}^k)$. Hence $J_F = (t,b]$, which is relatively open in $[a,b]$. We have shown that each J_F is a relatively open subinterval interval of $[a,b]$ containing b. Hence their union J is also such a subinterval of $[a,b]$. This proves statement (ii).

Next we observe that statement (iii) implies statement (iv). For suppose T is in the closure of J. Then, assuming that statement (iii) holds, $\mathcal{F}$ contains a map F that lies in the union of $K(T)^c$ with the regular part of $\partial K(T)$. Now we simply push $F(\mathbf{B}^k)$ into $K(T)^c$ by pushing it (where it touches the regular part of $\partial K(T)$)

in the direction of the outward unit normal to $K(T)$. Thus $T \in J$, which completes the proof that statement (iii) implies statement (iv).

(The sentence "now we simply push…" may be made more precise as follows. Let $S = F(\mathbf{B}^k) \cap \partial K(T)$. Let $\mathbf{v}$ be a compactly supported vectorfield defined on the regular part of $\partial K(T)$ such that $\mathbf{v}$ is nonzero at every point of S and such that at each point, $\mathbf{v}$ is a nonnegative multiple of the outward unit normal to $\partial K(T)$. Now extend $\mathbf{v}$ to be a smooth vectorfield that vanishes outside of $K(a)$. The flow generated by $\mathbf{v}$ homotopes F to a map in $\mathcal{F}$ whose image is disjoint from $K(T)$.)

It remains only to show statement (iii). Suppose $T \in [a, b)$ is in the closure of J. Let $\epsilon > 0$ (to be specified later). By statements (i) and (ii), there exist $T^* \in J \cap (T, b]$ arbitrarily close to T. Choose such a T^* sufficiently close to T that every point in $K(T) \setminus K(T^*)$ is within distance $< \epsilon$ of $\partial K(T)$. (This is possible by hypotheses (1) and (3) of the theorem.)

Since $T^* \in J$, there is a map $F \in \mathcal{F}$ such that $F(\mathbf{B}^k)$ is disjoint from $K(T^*)$. We may assume that F is smooth since the C^∞ maps are dense in the set of continuous maps. Now

$$\dim(\mathrm{sing}(K(T)) \le n - Q(K(T)) \le n - m,$$

and therefore since $k \le m$,

$$\dim(\mathrm{sing}(\partial K(T)) + k \le n < n + 1.$$

Consequently, we may assume, by putting F in general position, that $F(\mathbf{B}^k)$ contains no singular points of $\partial K(T)$. (See the appendix if this is not clear.)

We will construct a map G from $\mathbf{B}^k$ such that the image of G is contained in $K(T)^c$ together with the regular part of $\partial K(T)$. We will also construct a homotopy from F to G in $(K(b)^c, K(a)^c)$. The homotopy shows that $G \in \mathcal{F}$, thus establishing statement (iii).

Let $\Omega \subset \mathbf{B}^k$ be the inverse image under F of the interior of $K(T)$. We now describe the construction of the map G on the open set Ω.

First some terminology. Recall that a d-simplex is the convex hull of $(d+1)$ points in a Euclidean space provided those $(d+1)$ points do not lie in any affine subspace of dimension $< d$. The points are called vertices of the simplex. If the distance between each pair of vertices is 1, we say that the simplex is **standard**. Note that any two d-simplices are affinely isomorphic. In particular, given any d-simplex Δ, there is an affine bijection $\sigma : \Delta \to \Delta_s$ from Δ to a standard simplex Δ_s. We define the **standardized distance** $d_s(\cdot, \cdot)$ on Δ by

$$d_s(x, y) = |\sigma(x) - \sigma(y)|.$$

Given a map F from Δ into a metric space Z, we define the **standardized Lipschitz constant** $\mathrm{Lip}_s(F)$ of F to be the Lipschitz constant of F with respect to the standardized distance on Δ:

$$\mathrm{Lip}_s(F) = \sup_{x \ne y} \frac{\mathrm{dist}(F(x), F(y))}{d_s(x, y)}.$$

We now describe the map G on the portion Ω of $\mathbf{B}^k$. (Later we will extend G to all of $\mathbf{B}^k$ by letting $G = F$ on $\mathbf{B}^k \setminus \Omega$.) First, triangulate Ω. By refining the triangulation, we may assume that for each simplex Δ of the triangulation,

$$\mathrm{diam}(F(\Delta)) < \epsilon \, \mathrm{dist}(F(\Delta), \partial K(T)).$$

Here $\mathrm{dist}(X, Y)$ denotes the infimum of $\mathrm{dist}(x, y)$ among all $x \in X$ and $y \in Y$.

We define G on Ω inductively by defining it first on the 0-skeleton of the triangulation of Ω, then on the 1-skeleton, and so on. For each vertex v in the 0-skeleton, we choose a point $q \in \partial K(T)$ that minimizes $\operatorname{dist}(q, F(v))$, and we then let $G(v)$ be that chosen q. Note that q is a regular point of $\partial K(T)$ (since $Q(K(T)) > -\infty$.) Having defined G on the $(j-1)$-skeleton of Ω, we extend it to the j-skeleton as follows. For each j-simplex Δ in the triangulation, we choose a map

$$g : \Delta \to \partial K(T)$$

that minimizes $\operatorname{Lip}_s(g)$ among all maps $g : \Delta \to \partial K(T)$ such that $g = G$ on $\partial\Delta$. Having chosen such a g, we let $G(x) = g(x)$ for $x \in \Delta$. (In Lemma 4.5 below, any map G constructed by this inductive procedure will be called "F-optimal".)

Of course we must check that the procedure does not break down in going from the $(j-1)$-skeleton to the j-skeleton. That it does not break down is proved below in Lemma 4.5 (provided $\epsilon > 0$ is sufficiently small). The lemma shows (for all sufficiently small $\epsilon > 0$) that:

(4) $G(\Omega)$ lies in the regular part of $\partial K(T)$, and
(5) $\operatorname{dist}(F(x), G(x)) \le C \operatorname{dist}(F(x), \partial K(T))$ for all $x \in \Omega$. (See (6) in the lemma.)

By (5), the map G extends continuously to $\mathbf{B}^k$ by setting $G(x) = F(x)$ for $x \in \mathbf{B}^k \setminus \Omega$.

Now define a homotopy $H : \mathbf{B}^k \times [0, 1] \to K$ from F to G by setting

$$H(x, s) = (1 - s)F(x) + sG(x)$$

if the ambient space is Euclidean. More generally, we define H by letting $H(x, \cdot) : [0, 1] \to K(T)$ be the unique shortest geodesic (parametrized with constant speed) joining $F(x)$ to $G(x)$. (By (5), the shortest geodesic will be unique if $\epsilon > 0$ is sufficiently small, since $\operatorname{dist}(F(x), \partial K(T)) < \epsilon$.)

It remains only to show that (if $\epsilon > 0$ is sufficiently small) the image of H is disjoint from $K(b)$, i.e., that for $x \in \Omega$, the geodesic from $F(x)$ to $G(x)$ is disjoint from $K(b)$. Choose ϵ with

$$0 < \epsilon < \frac{\operatorname{dist}(\partial K(T), K(b))}{C}.$$

(This is possible since $\partial K(T)$ and $K(b)$ are disjoint.) Thus by (5),

$$\operatorname{dist}(F(x), G(x)) < \operatorname{dist}(\partial K(T), K(b)).$$

This means that the geodesic from $G(x)$ (which is in $\partial K(T)$) to $F(x)$ is too short to reach $K(b)$. Thus that geodesic is disjoint from $K(b)$.

We have proved that the image of the homotopy H is disjoint from $K(b)$. The homotopy proves that $G \in \mathcal{F}$. This completes the proof of Theorem 4.4. □

We now turn to the lemma that was used in the proof of Theorem 4.4. First we need some terminology. Fix a $T > 0$ and let $K = K(T)$. Let X be a simplicial complex and let F be a map from X to K. We say that a map $G : X \to \partial K$ is ***F*-optimal** provided:

(1) For each vertex v of X, $G(v)$ realizes the minimum distance from a point in ∂K to $F(v)$:

$$\operatorname{dist}(F(v), \partial K) = \operatorname{dist}(F(v), G(v)).$$

(2) For each simplex Δ of X, the restriction $G|\Delta$ is a Lip_s-minimizing map from Δ to ∂K. That is, if $g : \Delta \to \partial K$ is any map such that $g|\partial\Delta = G|\partial\Delta$, then

$$\text{Lip}_s(G|\Delta) \le \text{Lip}_s(g).$$

4.5. **Lemma.** *Let K be a compact subset of the interior of a smooth, $(n+1)$-dimensional manifold. Let Δ be a simplex of dimension $k \le Q(K)$. Then there is an $\epsilon < 0$ and a $C < \infty$ with the following property. If $F : \Delta \to K$ is a map such that*

$$\text{diam}(F(\Delta)) < \epsilon \, \text{dist}(F(\Delta), \partial K)$$

and such that

$$\text{dist}(F(\Delta), \partial K) < \epsilon,$$

then each F-optimal map from $\partial\Delta$ to ∂K extends to an F-optimal map G from Δ to ∂K, and for any such extension G,

$$\text{Lip}_s(G) \le C \; \text{dist}(F(\Delta), \partial K),$$

and

$$\text{diam}(F(\Delta) \cup G(\Delta)) \le C \; \text{dist}(F(\Delta), \partial K). \tag{6}$$

Note we may assume that the simplex Δ is standard (since the statement of the theorem is not affected by affine reparametrizations of the domain.) For purposes of proof, it is convenient to restate the lemma as follows:

4.6. **Lemma.** *Let K be as in Lemma 4.5, and let Δ be a standard simplex of dimension $k \le Q(K)$. Let $\epsilon_i \to 0$, and suppose that $F_i : \Delta \to K$ is a sequence of maps such that*

$$\text{diam}(F_i(\Delta)) \le \epsilon_i \, \text{dist}(F_i(\Delta), \partial K) \tag{7}$$

and such that

$$\text{dist}(F_i(\Delta), \partial K) < \epsilon_i. \tag{8}$$

Suppose also that $\Gamma_i : \partial\Delta \to \partial K$ is a sequence of F_i-optimal maps. Then for all sufficiently large i, there exists an F_i-optimal map $G_i : \Delta \to \partial K$ that extends Γ_i, and such a G_i must (for all sufficiently large i) have the following properties:

(i) *$G_i(\Delta)$ is contained in the regular part of ∂K.*

(ii) *The quantities*

$$\frac{\text{Lip}\, G_i}{\text{dist}(F_i(\Delta), \partial K)}$$

(if $k > 0$) and

$$\frac{\text{diam}(F_i(\Delta) \cup G_i(\Delta))}{\text{dist}(F_i(\Delta), \partial K)}$$

are bounded above as $i \to \infty$.

Proof. We prove it by induction on the dimension of Δ.

If Δ is 0-dimensional, it is a single point p. Let $G_i(p)$ be a point in the interior of ∂K such that

$$\text{dist}(F_i(p), G_i(p)) = \text{dist}(F_i(p), \partial K).$$

This implies that $G_i(p)$ is a regular point (since $Q(K) > -\infty$), so (i) holds. The second ratio in (ii) are both trivially equal to 1, so (ii) also holds. This completes the proof of the lemma when Δ is 0-dimensional.

Now suppose that $1 \le k = \dim(\Delta) \le Q(K)$. By induction, we may assume that the lemma is true for each face of Δ. Let p_i be a point in $F_i(\Delta)$ that minimizes the distance from $F_i(p_i)$ to ∂K.

Translate K by $-F(p_i)$ and dilate by

$$\lambda_i = \frac{1}{\mathrm{dist}(F_i(p), \partial K)}$$

to get a set K_i'. Let $F_i' : S \to K_i'$ and $\Gamma_i' : \partial\Delta \to \partial K_i'$ be the maps corresponding to F_i and Γ_i. Note that

$$0 \in F_i'(\Delta) \subset K_i' \tag{9}$$

and that

$$1 = \mathrm{dist}(0, \partial K') = \mathrm{dist}(F_i'(\Delta), \partial K_i'). \tag{10}$$

By passing to a subsequence, we may assume that the K_i' converge smoothly to a convex set K' with

$$0 \in K' \text{ and } \mathrm{dist}(0, \partial K') = 1. \tag{11}$$

By (7) and (10),

$$\mathrm{diam}(F_i'(\Delta)) \le \epsilon_i \ \mathrm{dist}(F_i'(\Delta), \partial K_i') = \epsilon_i \to 0,$$

so by (9),

$$F_i'(\cdot) \to 0 \text{ uniformly.} \tag{12}$$

Thus by (11),

$$\mathrm{dist}(F_i'(\cdot), \partial K_i') \to 1 \quad \text{uniformly.} \tag{13}$$

By induction we can assume that (ii) holds for the restrictions of F_i and Γ_i to each face Δ^* of Δ. Thus

$$\mathrm{Lip}(\Gamma_i | \Delta^*) \le c \, \mathrm{dist}(F_i(\Delta^*), \partial K)$$

for some constant c, which implies by rescaling that

$$\mathrm{Lip}(\Gamma_i' | \Delta^*) \le c \, \mathrm{dist}(F_i'(\Delta^*), \partial K_i').$$

By (12) and (13), the right hand side tends to c, so

$$\limsup_i \left(\mathrm{Lip}(\Gamma_i' | \Delta^*) \right) \le c. \tag{14}$$

If v is a vertex of Δ, then $\Gamma_i'(v)$ is a point in $\partial K_i'$ closest to $F_i'(v)$. Since since $F_i'(\cdot) \to 0$ and since $K_i' \to K'$ smoothly, this implies that

$$\limsup_{i \to \infty} \mathrm{dist}(\Gamma_i'(v), 0) = \mathrm{dist}(\partial K', 0) = 1. \tag{15}$$

By (14) and (15), the Γ_i' form an equicontinuous family, so after passing to a subsequence, we can assume that the Γ_i' converge uniformly to a Lipschitz map

$$\Gamma' : \partial\Delta \to \partial K'.$$

Now $\partial K'$ is smooth. Also, $k = \dim(\Delta) \le Q(K)$, so by definition of $Q(K)$, the $(k-1)$-dimensional homotopy of $\partial K'$ is trivial. Thus the map Γ' extends to a Lipschitz map $G' : \Delta \to \partial K'$.

By the smooth convergence $K_i' \to K'$ and the by the bounded Lipschitz norm convergence $\Gamma_i' \to \Gamma'$, it follows that (for all sufficiently large i) there is a Lipschitz map

$$G_i' : \Delta \to \partial K_i'$$

such that G_i' extends Γ_i' and such that

$$\mathrm{Lip}(G_i') \le \mathrm{Lip}(G') + \delta_i \tag{16}$$

where $\delta_i \to 0$. We may assume that $G_i' : \partial\Delta \to K_i'$ is the extension of smallest Lipschitz norm. (This minimizing extension exists because $\partial K_i'$ is compact.) By passing to a subsequence, the G_i' converge uniformly to a limit map, which we may assume to be G'. (Otherwise redefine G' to be that limit map.)

In particular, the smooth convergence $\partial K_i' \to \partial K'$ implies that G_i' maps Δ to the regular part of $\partial K_i'$ (if i is sufficiently large).

Note that

$$\frac{\mathrm{Lip}(G_i)}{\mathrm{dist}(F_i(\Delta), \partial K)} = \frac{\mathrm{Lip}(G_i')}{\mathrm{dist}(F_i'(\Delta), \partial K_i')} = \frac{\mathrm{Lip}(G_i')}{1}$$

which is bounded as $i \to \infty$ by (16).

Similarly we have

$$\frac{\mathrm{diam}(F_i(\Delta) \cup G_i(\Delta))}{\mathrm{dist}(F_i(\Delta), \partial K)} = \frac{\mathrm{diam}(F_i'(\Delta) \cup G_i'(\Delta))}{\mathrm{dist}(F_i'(\Delta), \partial K_i')} = \frac{\mathrm{diam}(F_i'(\Delta) \cup G_i'(\Delta)}{1} \tag{17}$$

which converges to $\mathrm{diam}(\{0\} \cup G'(\Delta))$ as $i \to \infty$ (since $F_i' \to 0$ and $G_i' \to G'$ uniformly.) In particular, (17) is bounded as $i \to \infty$. □

5. Manifolds with Boundary

So far in this paper, the moving hypersurfaces $\partial K(t)$ under consideration have been hypersurfaces without boundary. Now we consider the case of hypersurfaces with boundary, the motion of the boundary being prescribed and the motion away from the boundary being by mean curvature flow (or possibly by other analogous flows.)

5.1. **Definition.** Let N be a smooth $(n+1)$-dimensional manifold-with-boundary. Let K be a closed subset of N. A point $x \in K$ is called a **regular point** of K provided

(1) x is an interior point of K, or
(2) $x \in N \setminus \partial N$ and N has a neighborhood U of x such that $K \cap U$ is diffeomorphic to a closed half-space in $\mathbf{R}^{n+1}$, or
(3) $x \in \partial N$ and N has a neighborhood U of x for which there is a diffeomorphism that maps U onto $\{x \in \mathbf{R}^{n+1} : x_1 \ge 0\}$ and that maps $K \cap U$ onto $\{x \in \mathbf{R}^{n+1} : x_1 \ge 0, x_2 \ge 0\}$.

Points in K that are not regular points are called singular points of K.

The following theorem should be thought of as a theorem about a moving hypersurface-with-boundary. At time t, the hypersurface is

$$M(t) := \partial K(t) = K(t) \cap \overline{N \setminus K(t)},$$

and its boundary is $\Gamma(t) := \partial K(t) \cap \partial N$. In practice, the initial surface would be prescribed by prescribing $K(0)$, and the motion of the boundary would be prescribed by prescribing $\Gamma(t)$ or, equivalently, by prescribing $K(t) \cap N$. The geometric flow would then determine the moving region $K(t)$ or, equivalently, the moving hypersurface $M(t)$.

5.2. **Theorem.** *Let $t \in [a,b] \mapsto K(t)$ be a one-parameter family of compact subsets of a smooth, $(n+1)$-dimensional Riemannian manifold-with-boundary. Assume that*

(1) $K(t) \subset K(T)$ *for* $T \le t$.
(2) $K(t) \setminus \partial N \subset \operatorname{interior}(K(T))$ *for* $T < t$.
(3) $K(T) = \cap_{t<T} K(t)$ *for* $T \in (a,b]$.
(4) $\operatorname{interior}(K(T)) = \cup_{t>T} \operatorname{interior}(K(t))$ *for* $T \in [a,b)$.
(5) *For each* $t \in [a,b]$*, the singular points of* $K(t)$ *all lie in the interior of* N.
(6) $Q(K(t)) \ge m$ *for all* $t \in [a,b)$.

Then the pair $(K(b)^c, K(a)^c)$ *is* m*-connected.*

The fact that we are assuming (2) rather than $K(t) \subset \operatorname{interior}(K(T))$ for $T < t$ (as was assumed in Theorem 4.4) allows for the boundaries $\Gamma(t)$ of the surfaces $M(t)$ to touch either other for distinct values of t. In particular, it allows for the boundary $\Gamma(t)$ to be fixed.

Theorem 5.2 can be proved exactly as Theorem 4.4 proved, except for two slight complications, which we now discuss.

The first complication arises because at the end of the proof of Theorem 4.4, we used the following:

Claim. *Let $a \le T < b$ and let $0 < C < \infty$. Then there is an $\epsilon > 0$ such that if $x \in K(T) \setminus K(b)$, if $\operatorname{dist}(x, \partial K(T)) < \epsilon$, and if y is a point in $\partial K(T)$ such that*

$$\operatorname{dist}(x,y) \le C \operatorname{dist}(x, \partial K(T)),$$

then the shortest path joining x to y in N is disjoint from $K(b)$.

Unfortunately, the hypotheses of Theorem 5.2 are not quite enough to imply the claim. (The argument given at the end of the proof of Theorem 4.4 to establish the claim does not work here because hypothesis (2) of Theorem 5.2 allows $\partial K(T)$ and $K(b)$ to intersect along ∂N.) However, we can get around this difficulty as follows. Neither the hypotheses nor the conclusion of Theorem 5.2 depend on the Riemannian metric on N. Thus we may choose a metric that is convenient. In particular, by choosing a suitable metric, we may assume that:

(18) The surface $\partial K(b)$ is totally geodesic in some small tubular neighborhood of ∂N, and $\partial K(b)$ and ∂N are orthorgonal at all points in their intersection.

Under this assumption, the claim is true:

Proof of claim (assuming (18)). Suppose $x_i \in K(T) \setminus K(b)$, $y_i \in \partial K(T)$, and

$$\frac{\operatorname{dist}(x_i, y_i)}{C} \le \operatorname{dist}(x_i, \partial K(T)) \to 0. \tag{19}$$

It suffices to show that (for all sufficiently large i), the geodesic joining x_i to y_i does not intersect $K(b)$. By passing to a subsequence, we may assume that x_i converges to a limit point p. By (19), y_i converges to the same point p.

Case 1: $p \notin K(b)$. Since x_i and y_i both converge to p, this means that (for large i) the geodesic joining them does not intersect $K(b)$.

Case 2: $p \in K(b)$. But $p = \lim_i y_i$ is also in $\partial K(T)$. Thus $p \in \partial K(b) \cap \partial N$ by hypothesis (2) of Theorem 5.2. But now assumption (18) implies that the geodesic joining x_i to y_i is, for large i, disjoint from $K(b)$.

This completes the proof of the claim. □

The second complication is that in a few steps in the proof of Theorem 4.4, it is convenient for a certain map $F : (\mathbf{B}^k, \partial\mathbf{B}^k) \to (K(T)^c, K(a)^c)$ to have its image the interior of N. (This was trivially the case in Theorem 4.4, since the $K(t)$'s were assumed to lie in the interior of N.) The following lemma shows that we can arrange for $F(\mathbf{B}^k)$ to be in the interior of N:

5.3. **Lemma.** *Under the hypotheses of Theorem 5.2, if $a < T < b$ and if*

$$F : (\mathbf{B}^k, \partial\mathbf{B}^k) \to (K(T)^c, K(a)^c)$$

then F is homotopic in $(K(T)^c, K(a)^c)$ to a map whose image lies in the interior of N.

Proof. Note that there is a smooth vectorfield on N supported in a small tubular neighborhood of ∂N such that

$$\mathbf{v}(x) \cdot \nu(x) > 0$$

for all $x \in \partial N$, where ν is the unit normal vectorfield to ∂N that points into N, and such that $\mathbf{v}$ is tangent to $\partial K(T)$ along $\partial K(T)$. (This is possible because $K(T)$ is assumed to have no singular points in ∂N.) Let $\Phi : N \times [0, \infty) \to N$ be the flow generated by $\mathbf{v}$, and consider the homotopy

$$H : \mathbf{B}^k \times [0, \delta] \to N$$
$$H(x, s) = \Phi(F(x), s)$$

Note that the image of H is disjoint from $K(T)$. Also, if we choose $\delta > 0$ sufficiently small, then $H(\partial\mathbf{B}^k \times [0, \delta])$ will be disjoint from $K(T)$. (This is true because $F(\partial\mathbf{B}^k)$ is disjoint from $K(T)$.) Let $G(\cdot) = H(\cdot, \delta)$. Then $G(\mathbf{B}^k)$ lies in the interior of N, and F and G are homotopic in $(K(T)^c, K(a)^c)$ by the homotopy H. □

In the case of mean curvature flow, we have:

5.4. **Theorem.** *Let N be a smooth, compact, connected $(n+1)$-dimensional Riemannian manifold with boundary with $n < 7$. Let $t \in [0, \infty) \mapsto V(t)$ be a smooth, one-parameter family of compact, smooth, n-dimensional manifolds with boundary in ∂N such that $V(t') \subset V(t)$ for $t \geq t'$. Let K be a closed subset of N such that ∂K is a smooth, compact, connected manifold-with-boundary such that $K \cap \partial N = V(0)$, and such that ∂K is smooth with mean curvature at each point a nonnegative multiple of the unit normal that points into K, and such that ∂K is nowhere tangent to ∂N.*

If ∂K is a minimal surface (i.e., has mean curvature 0 at all points), assume[3] *also that $V(t) \neq V(0)$ for $t \neq 0$.*

Let $t \in [0, \infty) \mapsto M(t)$ be the solution obtained by elliptic regularization of mean curvature flow such that $M(0) = \partial K$ and such that $\partial M(t) = \partial V(t)$ for all t.

Then each $M(t)$ is the boundary in N of a region $K(t) \subset K$. The singularities of the flow form a compact subset of the interior of N and all have convex type.

In particular, if the Gaussian densities of the singularities in the time interval $a \leq t < b$ are all $< d_m$, then $t \mapsto K(t)$ satisfies all the hypotheses of Theorem 5.2, and therefore the pair $(K(b)^c, K(a)^c)$ is m-connected.

[3]This assumption guarantees that the surface starts moving immediately.

If $n \geq 7$, the Theorem remains true provided the metric on N is flat and provided $M(t)$ is smooth for some $t \geq b$.

The theorem should be true for all n without the somewhat peculiar assumptions in the last sentence of the theorem. Those assumptions are needed only because without them we do not know how to prove that the singularities of the flow have convex type.

Proof. Except for the assertion that the pair $(K(b)^c, K(a)^c)$ is m-connected, this is proved in [Whi11]. The m-connectivity of $(K(b)^c, K(a)^c)$ then follows by Theorem 5.2. □

6. Appendix

Here we give a proof of the general position principle used in the proof of Theorem 4.4.

6.1. **Proposition.** *Let N be a smooth d-dimensional manifold without boundary and let S be a subset of N with Hausdorff $(d-k)$-dimensional measure 0. Then the collection $\mathcal{C}$ of smooth maps $F : \mathbf{B}^k \to N$ such that $F(\mathbf{B}^k) \cap S = \emptyset$ is dense in the set of all smooth maps from $\mathbf{B}^k$ to N.*

Proof. First consider the case $N = \mathbf{R}^d$. Let $F : \mathbf{B}^k \to N$ be a smooth map. We will prove the proposition in this case by showing

(20) If $F \in C^\infty(\mathbf{B}^k, \mathbf{R}^d)$, then $F(\cdot) + v \in \mathcal{C}$ for almost every $v \in \mathbf{R}^d$.

The set $\Pi^{-1}(S) = \mathbf{B}^k \times S$ has $k+(d-k)$-dimensional (i.e., d-dimensional) measure 0. (Here $\Pi : \mathbf{B}^k \times \mathbf{R}^d \to \mathbf{R}^d$ is the projection map.) Therefore its diffeomorphic image $\phi(\Pi^{-1}(S))$ under the diffeomorphism

$$\phi : (x, y) \in \mathbf{B}^k \times \mathbf{R}^n \mapsto (x, y - F(x))$$

has d-dimensional measure 0. Hence the projected image $\Pi(\phi(\Pi^{-1}(S)))$ of $\phi(\Pi^{-1}(S))$ in $\mathbf{R}^d$ has Lebesgue measure 0:

$$\mathcal{L}^d(\Pi(\phi(\Pi^{-1}(S))) = 0. \tag{21}$$

Now

$$v \in \Pi(\phi(\Pi^{-1}(S))) \iff (x, v) \in \phi(\Pi^{-1}(S)) \text{ for some } x \in \mathbf{B}^k, \tag{22}$$

and

$$\begin{aligned} (x, v) \in \phi(\Pi^{-1}(S)) &\iff \phi^{-1}(x, v) \in \Pi^{-1}(S) \\ &\iff (x, F(x) + v) \in \Pi^{-1}S \\ &\iff F(x) + v \in S \end{aligned} \tag{23}$$

The desired conclusion (20) follows immediately from (21), (22), and (23). This completes the proof in the case $N = \mathbf{R}^d$.

For a general manifold N, we may assume that N is a smooth submanifold of some Euclidean space $\mathbf{R}^{d+j}$. Let U be an open subset of $\mathbf{R}^{d+j}$ that contains N and for which the nearest point retraction $\pi : U \to N$ exists and is smooth.

Let $F : \mathbf{B}^k \to N$ be a smooth map, and let $\delta = d(F(\mathbf{B}^k), U^c)$. Now the set $\pi^{-1}(S)$ has $(d-k)+j$ dimensional measure 0, or, equivalently $(d+j)-k$-dimensional measure 0. Thus by (20), the map $F(\cdot) + v$ has image disjoint from $\pi^{-1}(S)$ for almost every $v \in \mathbf{R}^{d+j}$ with $|v| < \delta$. Therefore the map $\pi(F(\cdot) + v)$ has image disjoint from S for almost every $v \in \mathbf{R}^{d+j}$ with $|v| < \delta$. □

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