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EXISTENCE OF A LORENZ RENORMALIZATION FIXED POINT OF AN ARBITRARY CRITICAL ORDER

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ABSTRACT. We present a proof of the existence of a renormalization fixed point for Lorenz maps of the simplest non-unimodal combinatorial type $(\{0,1\},\{1,0,0\})$ and with a critical point of arbitrary order $\rho > 1$.

1. INTRODUCTION

E. N. Lorenz in [8] demonstrated numerically the existence of certain three-dimensional flows that have a complicated behavior. The *Lorenz flow* has a saddle fixed point with a one-dimensional unstable manifold and an infinite set of periodic orbits whose closure constitutes a global attractor of the flow.

As it is often done in dynamics, one can attempt to understand the behaviour of a three-dimensional flow by looking at the first return map to an appropriately chosen two-dimensional section. In the case of the Lorenz flow, it is convenient to chose the section as a plane transversal to the local stable manifold, and ,therefore, intersecting it along a curve γ. The first return map is discontinuous at γ.

The *geometric Lorenz flow* has been introduced in [9]: a Lorenz flow with an extra condition that the return map preserves a one-dimensional foliation in the section, and contracts distances between points in the leafs of this foliation at a geometric rate. Since the return maps is contracting in the leafs, its dynamics is asymptotically one-dimensional, and can be understood in terms of a map acting on the space of leafs (an interval). This interval map has a discontinuity at the point of the interval corresponding to γ, and is commonly called the *Lorenz maps*. More precisely,

Definition 1.1. Let $s > 0$ and $\rho > 0$. A C^s-Lorenz map $\psi : [-1, r] \mapsto [-1, r]$ is a map given by a pair (f, g), such that:

1) $f : [-1, 0) \mapsto [-1, r]$ and $g : (0, r] \mapsto [-1, r]$. f and g are continuous and strictly increasing;
2) there exists $\rho > 0$, the exponent of ψ, such that

$$f(x) = l(|x|^\rho), \quad g(x) = t(|x|^\rho),$$

l and t being C^s-diffeomorphisms.

Guckenheimer and Williams have proved in [5] that there is an open set of three-dimensional vector fields, that generate a geometric Lorenz flow with a smooth Lorenz map of $\rho < 1$. However, one can use the arguments of [5] to construct open sets of vector fields with Lorenz maps of $\rho \geq 1$. Similarly to the unimodal family, Lorenz maps with $\rho > 1$ have a richer dynamics that combines contraction with expansion.

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For any $x \in [-1, r] \setminus \{0\}$ such that $f^n(x) \neq 0$ for all $n \in \mathbb{N}$, define the itinerary $\omega(x) \in \{0,1\}^{\mathbb{N}}$ of x as the sequence $\{\omega_0(x), \omega_1(x), \dots\}$, such that

$$\omega_i = \begin{cases} 0, & f^i(x) < 0, \\ 1, & f^i(x) > 0. \end{cases}$$

If one imposes the usual order $0 < 1$, then for any two ω and $\tilde{\omega}$ in $\{0,1\}^{\mathbb{N}}$ we say that $\omega < \tilde{\omega}$ iff there exists $r \geq 0$ such that $\omega_i = \tilde{\omega}_i$ for all $i < r$ and $\omega_r < \tilde{\omega}_r$.

The limits

$$\omega(x^+) = \lim_{y \to x^+} \omega(y), \quad \omega(x^-) = \lim_{y \to x^-} \omega(y)$$

exists for all $y \in [-1, r]$.

The kneading invariant $K(f)$ of f is the pair $(K^-(f), K^+(f)) = (\omega(0^-), \omega(0^+))$. Hubbard and Sparrow have shown in [6] that (K^-, K^+) is the kneading invariant of some topologically expansive Lorenz map iff for all $n \in \mathbb{N}$

$$K_0^- = 0, \quad K_0^+ = 1, \quad \sigma(K^+) \leq \sigma^n(K^+) < \sigma(K^-), \quad \sigma(K^+) < \sigma^n(K^-) \leq \sigma(K^-),$$

here σ is the shift in $\{0,1\}^{\mathbb{N}}$.

Kneading invariants for a general Lorenz map, not necessarily expansive, satisfy a weaker condition:

$$K_0^- = 0, \quad K_0^+ = 1, \quad \sigma(K^+) \leq \sigma^n(K^\pm) \leq \sigma(K^-), \quad n \in \mathbb{N}.$$

Conversely, any sequence as above is a kneading sequence for some Lorenz map.

A Lorenz map f is called *renormalizable* if there exist p and q, $-1 < p < 0 < q < r$, such that the first return map (f^n, g^m), $n > 1, m > 1$, of $[p, q]$ is a Lorenz map.

The intervals $f^i([p, 0))$, $1 \leq i \leq n-1$, are pairwise disjoint, and disjoint from $[p, q]$. So are the intervals, $f^i((0, q])$, $1 \leq i \leq m-1$. Since these intervals do not contain zero, we can associate a finite sequence of 0 and 1 to each sequence of the intervals:

$$K^- = \{K_0^-, \dots, K_{n-1}^-\}, \quad K^+ = \{K_0^+, \dots, K_{m-1}^+\},$$

which will be called the type of renormalization. The subset of maps 1.1 which are renormalizable of type (α, β) is referred to as the domain of renormalization $\mathcal{D}_{\alpha,\beta}$ (cf. [7]).

The study of renormalizable Lorenz maps was initiated by Tresser et al. (see e.g. [1]) but a more recent paper is that of Martens and de Melo (see [7]). The latter authors consider the combinatorics of the renormalizable maps, and prove several results about the domains of renormalization and the structure of the parameter plane for two-dimensional Lorenz families.

The second author of the present paper has provided a computer assisted proof of existence of a renormalization fixed point for the renormalization operator of type $(\{0,1\}, \{1,0,0\})$ in [10]. Furthermore, issues of existence of renormalization periodic points and hyperbolicity have been addressed by the second author in [11], where it is proved that the limit set of renormalization, restricted to monotone combinatorics with the return time of one branch being large, is a Cantor set, and that each point in the limit set has a two-dimensional unstable manifold. This result holds for any real $\rho > 1$.

In this paper we give an analytic proof of the result of [10] for a general exponent of the Lorenz map $\rho > 1$.

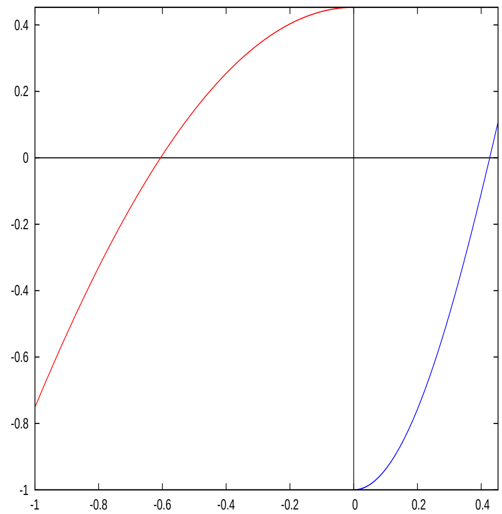

FIGURE 1. *The renormalization fixed point for* $\rho = 2$ *computed in* [10]

We consider the renormalization operator R of type $(\alpha, \beta) = (\{0,1\}, \{1,0,0\})$, specifically $R(f,g) = (\hat{f}, \hat{g})$ where

$$\hat{f}(z) = \lambda^{-1} g(f(\lambda z)), \tag{1}$$

$$\hat{g}(z) = \lambda^{-1} f(f(g(\lambda z))), \tag{2}$$

$$\lambda = -f(f(-1)). \tag{3}$$

As usual, the notation C^ω will denote the analytic class of maps.

Main Theorem. *For every* $\rho > 1$*, there exists a* C^ω*-Lorenz map* (f^*, g^*) *which is a fixed point of the renormalization of type* $(\{0,1\}, \{1,0,0\})$.

To prove the theorem we introduce an operator on an appropriate functional space of the diffeomorphic parts of the inverse branches of f and g. The crucial ingredient of our proof is a demonstration that there exists a subset in this functional space, invariant under the operator, characterized by the condition that the nonlinearities of the inverse branches are negative and bounded away from zero. It is this negativity of the nonlinearity that seems to be indispensable to complete the proof.

We would like to remark that the results of this paper could be made somewhat more general: indeed a similar method can be used to demonstrate existence of renormalization fixed points of other types for longer α and β. This is the range of α and β that was not accessible through the methods used in [11] where the condition that one of the branches has a very long return time (long α or β) was crucial.

However, we believe that at this point it would be timely to attempt to built a complete renormalization theory for Lorenz maps that would mirror that for unimodal maps. Specifically, one could attempt to extend the results of [11] to all return times, and demonstrate existence of the whole renormalization horseshoe via real or complex a priori bounds. We believe, that the negativity of the nonlinearity of the inverse branches could again play an important role in such proofs.

2. An operator on the Epstein class

Consider the action of this operator on the *little Epstein class* of functions, that is, functions f and g factorizable as $f = l \circ p_\rho \circ -id$, and $g = t \circ p_\rho$, $\rho > 1$, where p_ρ is the exponential map

$$p_\rho(z) = z^\rho,$$

and l, t are some diffeomorphisms (to be specified later) of the range of p_ρ (cf. [2], [3], [4]). Ignoring the issue of domains of maps for a moment, we get for the fixed point version of (1)-(2).

$$\begin{aligned} l \circ p_\rho \circ -id &= \lambda^{-1} \circ t \circ p_\rho \circ l \circ p_\rho \circ -\lambda \\ \lambda \circ l \circ p_\rho &= t \circ p_\rho \circ l \circ p_\rho \circ \lambda \\ t^{-1} \circ \lambda \circ l \circ p_\rho &= p_\rho \circ l \circ p_\rho \circ \lambda \\ p_{\frac{1}{\rho}} \circ t^{-1} \circ \lambda \circ l \circ p_\rho &= l \circ p_\rho \circ \lambda \\ l^{-1} \circ p_{\frac{1}{\rho}} \circ t^{-1} \circ \lambda \circ l \circ p_\rho &= p_\rho \circ \lambda \\ l^{-1} \circ p_{\frac{1}{\rho}} \circ t^{-1} \circ \lambda &= p_\rho \circ \lambda \circ p_{\frac{1}{\rho}} \circ l^{-1} \\ l^{-1} \circ p_{\frac{1}{\rho}} \circ t^{-1} \circ \lambda &= \lambda^\rho \circ l^{-1}, \end{aligned} \tag{4}$$

here, $p_{\frac{1}{\rho}}$ is the root function

$$p_{\frac{1}{\rho}}(re^{i\theta}) = r^{\frac{1}{\rho}} e^{\frac{i\theta}{\rho}}.$$

In a similar way, we get from (2) the following equation for inverse diffeomorphic parts l^{-1} and t^{-1} of the inverse branches of the fixed point of R:

$$t^{-1} \circ -p_{\frac{1}{\rho}} \circ l^{-1} \circ -p_{\frac{1}{\rho}} \circ l^{-1}\lambda = \lambda^\rho \circ t^{-1}. \tag{5}$$

Define diffeomorphisms U and V by setting

$$l^{-1}(z) = aU(r-z), \quad t^{-1}(z) = bV(z+1),$$

where the normalizing constants a and b will be chosen below. Then (4) and (5) become

$$\begin{aligned} \lambda^\rho U(r-z) &= U\left(r - p_{\frac{1}{\rho}}\left(bV(\lambda z + 1)\right)\right), \\ \lambda^\rho V(z+1) &= V\left(1 - p_{\frac{1}{\rho}}\left(aU\left(r + p_{\frac{1}{\rho}}\left(aU(r - \lambda z)\right)\right)\right)\right), \end{aligned}$$

or

$$\lambda^\rho U(z) = U\left(r - p_{\frac{1}{\rho}}\left(bV(\lambda(r-z)+1)\right)\right), \tag{6}$$

$$\lambda^\rho V(z) = V\left(1 - p_{\frac{1}{\rho}}\left(aU\left(r + p_{\frac{1}{\rho}}\left(aU(r - \lambda(z-1))\right)\right)\right)\right). \tag{7}$$

Set

$$b = \frac{r^\rho}{V(\lambda r + 1)}, \quad \text{and} \quad a = \frac{1}{U(r+y)}, \tag{8}$$

where $y = y(\lambda)$ solves

$$y = p_{\frac{1}{\rho}}\left(\frac{U(r+\lambda)}{U(r+y)}\right). \tag{9}$$

We will demonstrate in Section 5 that (9) has a unique solution whenever U is an appropriate functional class.

Equations (6) and (7) become

$$\lambda^\rho U = U \circ \Psi_{V,\lambda,r}, \quad \Psi_{V,\lambda,r} \equiv r - r p_{\frac{1}{\rho}}\left(\frac{V(\lambda(r-z)+1)}{V(\lambda r+1)}\right), \tag{10}$$

$$\lambda^\rho V = V \circ \Phi_{U,\lambda,r}, \quad \Phi_{U,\lambda,r} \equiv 1 - p_{\frac{1}{\rho}}\left(aU\left(r + p_{\frac{1}{\rho}}\left(aU(r-\lambda(z-1))\right)\right)\right). \tag{11}$$

Notice, that the normalization constants a and b have been chosen so that

$$\Psi_{V,\lambda,r}(0) = \Phi_{U,\lambda,r}(0) = 0.$$

We "decouple" the system $(10) - (11)$

$$\lambda^\rho U = U \circ \Psi_{V,\lambda,r}, \qquad \Psi_{V,\lambda,r} \equiv r - r p_{\frac{1}{\rho}}\left(\frac{V(\lambda(r-z)+1)}{V(\lambda r+1)}\right),$$

$$\mu^\rho V = V \circ \Phi_{U,\mu,r}, \qquad \Phi_{U,\mu,r} \equiv 1 - p_{\frac{1}{\rho}}\left(\frac{U\left(r + p_{\frac{1}{\rho}}\left(\frac{U(r-\mu(z-1))}{U(r+y)}\right)\right)}{U(r+y)}\right),$$

$$y = p_{\frac{1}{\rho}}\left(\frac{U(r+\mu)}{U(r+y)}\right), \tag{12}$$

and set

$$Z(x) = p_{\frac{1}{\rho}}\left(\frac{U(r+x)}{U(r+y)}\right), \quad W(x) = p_{\frac{1}{\rho}}\left(\frac{V(x+1)}{V(\lambda r+1)}\right). \tag{13}$$

The "decoupled" system becomes

$$\lambda^\rho U = U \circ \Psi_{V,\lambda,r}, \qquad \Psi_{V,\lambda,r}(z) \equiv r - rW(\lambda(r-z)), \tag{14}$$

$$\mu^\rho V = V \circ \Phi_{U,\mu,r}, \qquad \Phi_{U,\mu,r}(z) \equiv 1 - Z(Z(\mu(1-z))), \qquad y = Z(\mu). \tag{15}$$

We will define an operator $\mathcal{T}_r$ on pairs (U,V), that belong to an appropriate functional space, as follows. Given a pair (U,V), let $\lambda = \lambda(V,r)$ and $\mu = \mu(U,r)$ be the solutions (if they exist) of the equations

$$\lambda^\rho = \Psi'_{V,\lambda,r}(0) = \lambda \frac{r}{\rho}\frac{V'(\lambda r+1)}{V(\lambda r+1)} = \lambda r W'(\lambda r), \tag{16}$$

$$\mu^\rho = \Phi'_{U,\mu,r}(0) = \mu \frac{y}{\rho^2}\frac{U'(r+y)}{U(r+y)}\frac{U'(r+\mu)}{U(r+\mu)} = \mu Z'(y) Z'(\mu). \tag{17}$$

Define

$$(\tilde U, \tilde V) \equiv \mathcal{T}_r(U,V) = \left(\lambda^{-\rho} U \circ \Psi_{V,\lambda,r}, \mu^{-\rho} V \circ \Phi_{U,\mu,r}\right), \tag{18}$$

and

$$\tilde Z(z) = p_{\frac{1}{\rho}}\left(\frac{\tilde U(r+z)}{\tilde U(r+\tilde y)}\right), \qquad \tilde y = p_{\frac{1}{\rho}}\left(\frac{\tilde U(r+\mu)}{\tilde U(r+\tilde y)}\right),$$

$$\tilde W(z) = p_{\frac{1}{\rho}}\left(\frac{\tilde V(1+z)}{\tilde V(1+\lambda r)}\right).$$

In the following sections we will choose (U, V) in an appropriate subset $\mathcal{S}$ of a compact space of functions holomorphic in a double slit plane, and demonstrate that

a) for all $(U, V) \in \mathcal{S}$ the solutions (λ, μ) of $(16)-(17)$ exist and are unique for every r, and $\mathcal{T}_r$ is a continuous operator of $\mathcal{S}$ into itself;
b) $\lambda < \mu$ for sufficiently small r, and $\mu > \lambda$ for sufficiently large r for all $(U, V) \in \mathcal{S}$;
c) the iterates $\mathcal{T}_r^n(U_0, V_0)$ converge to a fixed point (U_r^*, V_r^*) of $\mathcal{T}_r$ uniformly in r, in particular the maps $r \to \lambda_r^* = \lambda(V_r^*, r)$ and $r \to \mu_r^* = \mu(U_r^*, r)$ are continuous for a range of positive r.

These three facts will imply that there exists a value r' of r such that $\lambda_{r'}^* = \mu_{r'}^*$, and the pair $(U_{r'}^*, V_{r'}^*)$ solves $(6)-(7)$.

To prove a), b) and c) above, we will construct a subset of a compact space of functions U and V holomorphic on a double slit plane, such that the nonlinearities of Z and W are *negative* on the real slice of the domain

$$N_Z(x) \le \Sigma < 0, \quad N_W(x) \le \Gamma < 0,$$

and we will demonstrate that $\tilde{Z}$ and $\tilde{W}$ for the image of (U, V) under $\mathcal{T}_r$ has the same bounds on the nonlinearity.

We would like to note, that assumptions on the nonlinearity seemed to be unnecessary in similar proofs of the existence of the fixed points for the unimodal maps, for example, in the proof of existence of the Feigenbaum fixed point in [2]-[4].

3. Preliminaries

3.1. **Herglotz bounds.** We will proceed with some definitions.

The upper and the lower half planes will be denoted as

$$\mathbb{C}_\pm \equiv \{z \in \mathbb{C} : \pm \Im(z) > 0\}.$$

Let $J = (-a, b) \subset \mathbb{R}$. Given such interval $J \subset \mathbb{R}$, denote

$$\mathbb{C}_J \equiv \mathbb{C}_+ \cup \mathbb{C}_- \cup J.$$

We will further define the space of Herglotz–Pick functions

$$\Omega(J) \equiv \left\{ u : u \quad \text{is} \quad \text{holomorphic} \quad \text{on} \quad \mathbb{C}_J, u(z) = \overline{u(\overline{z})}, u(0) = 0 \right\}.$$

$\Omega(J)$ is a compact metric space.

Functions in $\Omega(J)$ admit the following integral representation:

$$f(z) - f(z_0) = a(z - z_0) + \int d\nu(t) \left(\frac{1}{t - z} - \frac{1}{t - z_0} \right), \tag{19}$$

where ν is a measure supported in $\mathbb{R} \setminus (-a, b)$. This integral representation can be used to obtain the following *Herglotz bounds* on $\Omega(J)$

$$\frac{a}{x(a + x)} \le \frac{f'(x)}{f(x)} \le \frac{b}{x(b - x)}, \quad x \in (-a, b). \tag{20}$$

Notice, that the integration of the Herglotz bound (20) gives for all $y > x > 0$:

$$\frac{y(a + x)}{x(a + y)} \le \frac{f(y)}{f(x)} \le \frac{y(b - x)}{x(b - y)}. \tag{21}$$

Next, we denote by $\Omega_c(J)$ the subclass of functions $f \in \Omega(J)$ normalized at some point c, $b > c > 0$, as $f(c) = 1$. Using the integral representation (19), one can demonstrate that any $f \in \Omega_c(J)$ satisfies the following bounds

$$\frac{1}{c}\frac{a+c}{a+x} \geq \frac{f(x)}{x} \geq \frac{1}{c}\frac{b-c}{b-x}, \quad x \in (-a, c), \tag{22}$$

$$\frac{1}{c}\frac{a+c}{a+x} \leq \frac{f(x)}{x} \leq \frac{1}{c}\frac{b-c}{b-x}, \quad x \in (c, b). \tag{23}$$

Suppose $f \in \Omega(J)$, $J \neq \emptyset$. Then, for every $z \in J$ and every finite complex sequence $v_0, ..., v_N$, one has the following relation for the derivatives of f

$$\sum_{j,k=0}^{N} \frac{f^{(j+k+1)}(z)}{(j+k+1)!} v_j^* v_k \geq 0,$$

In particular, all odd derivatives of $f \in \Omega(J)$ are non-negative on J, and so is the Schwarzian

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \geq 0, \quad \text{on } J. \tag{24}$$

In particular, the nonlinearity of a Herglotz–Pick function is increasing.

The positivity of the Schwarzian has the following consequences. Let $J = (-a, b)$ be non-empty. Denote $g = f''/f'$, suppose that g is non-zero in $[x, y] \subset J$, and integrate the inequality $g'(x) \geq g(x)^2/2$:

$$\frac{1}{g(x)} - \frac{1}{g(y)} \geq \frac{y-x}{2}.$$

If $g(x) > 0$, then $g(y) > 0$, and $g(x) \leq 2/(y-x)$, which is also true if $g(x) < 0$. At the same time, $g(y) \geq -2/(y-x)$. Taking the limit $y \to b$ in the first inequality, and $x \to -a$ in the second, we get

$$\frac{-2f'(x)}{a+x} \leq f''(x) \leq \frac{2f'(x)}{b-x}, \quad x \in (-a, b). \tag{25}$$

3.2. **Nonlinearity.** Next, assume that the nonlinearity

$$N_f(x) = \frac{f''(x)}{f'(x)}$$

of f is positive on J. Then, we can use the positivity of the Schwarzian derivative to obtain

$$(\ln N_f(x))' = \frac{f'''(x)}{f''(x)} - \frac{f''(x)}{f'(x)} = \frac{f'(x)}{f''(x)}\left(\frac{f'''(x)}{f'(x)} - \left(\frac{f''(x)}{f'(x)}\right)^2\right) > \frac{1}{2}N_f(x).$$

Alternatively, if the nonlinearity is negative, then

$$\begin{aligned}(\ln(-N_f(x)))' &= (\ln(-f''(x)) - \ln(f'(x)))' \\ &= \frac{f'''(x)}{f''(x)} - \frac{f''(x)}{f'(x)} = \frac{f'(x)}{f''(x)}\left(\frac{f'''(x)}{f'(x)} - \left(\frac{f''(x)}{f'(x)}\right)^2\right) < \frac{1}{2}N_f(x).\end{aligned}$$

In either case, the solution to the the initial value problems $(\ln N_f(x))' > N_f(x)/2$, $N_f(x_0)=N_0$, or $(\ln(-N_f(x)))' < N_f(x)/2$, $N_f(x_0)=N_0$, , is

$$N_f(x) > \frac{2N_0}{2-N_0(x-x_0)}, \quad x \geq x_0,$$

therefore,

$$N_f(y) \geq \frac{2N_f(x)}{2-N_f(x)(y-x)} \tag{26}$$

for all $y \geq x$ in the case of a nonlinearity of a constant sign.

Furthermore,

$$(\ln f'(x))' = N_f(x),$$

and, under the same assumption of N_f of a constant sign, the initial value problem

$$\frac{f''(x)}{f'(x)} \geq \frac{2N_0}{2-N_0(x-x_0)}, \quad x \geq x_0, \quad f'(x_0)=f_0, \quad f''(x_0)/f'(x_0)=N_0$$

is

$$f'(x) \geq \frac{4f_0}{(2-N_0(x-x_0))^2},$$

and we get

$$f'(y) \geq \frac{4f'(x)}{(2-N_f(x)(y-x))^2}, \tag{27}$$

for all $y \geq x$.

Given a real constant σ and a real $c \in J=(-a,b)$, we set

$$\Omega^c_{\gtreqless\sigma}(J) \equiv \left\{ f \in \Omega(J) : \frac{\partial_x^2 p_{\frac{1}{\rho}}(f(c+x))}{\partial_x p_{\frac{1}{\rho}}(f(c+x))} \gtreqless \sigma, \quad x \in (-c, b-c) \right\}.$$

Notice, the set $\Omega^c_{\gtreqless\sigma}(J)$ in general is not a convex subset of $\Omega(J)$.

3.3. **Schwarz Lemma.** Finally, we will mention the following easy consequence of the of Schwarz Lemma which will play an important role in our proofs below (cf [3]):

Lemma 3.1. *Suppose f is a holomorphic map of $\mathbb{C}_J$, $J=(-a,b)$, into $\mathbb{C}'_J$, $J'=(-a',b')$, which fixes 0 then*

$$|f'(0)| \leq \frac{a'b'(a+b)}{ab(a'+b')}. \tag{28}$$

4. Statement of results

We will now give a more precise statement of what will be proved in the following sections.

Set

$$\lambda_+(r) = \left(\frac{r}{r+1}\right)^{\frac{1}{\rho}}, \quad \mu_+(r) = \left(\frac{1}{(r+1)^2}\right)^{\frac{1}{\rho}}, \tag{29}$$

and

$$J_U = \left(r - \frac{r}{\lambda_+(r)\mu_+(r)}, r + \frac{1}{\lambda_+(r)}\right), \; J_V = \left(1 - \frac{1}{\lambda_+(r)\sqrt{\mu_+(r)}}, 1 + \frac{r}{\mu_+(r)}\right). \tag{30}$$

Theorem A. *For any $\rho > 1$, there exist $r_+ > r_- > 0$, and two functions $\Sigma(r) < 0$ and $\Gamma(r) < 0$, continuous on (r_-, r_+), such that:*

i) for every $r \in (r_-, r_+)$ and $(U, V) \in \Omega^r_{<\Sigma}(J_U) \times \Omega^1_{<\Gamma}(J_V)$ there is a unique solution $(\lambda, \mu) \in (0, \lambda_+(r)) \times (0, \mu_+(r))$ of the equations (16) and (17), and the functions $r \mapsto \lambda(r)$, $r \mapsto \mu(r)$ are continuous on (r_-, r_+);

ii) $\mathcal{T}_r$ is a well-defined, continuous operator of the subset $\Omega^r_{<\Sigma}(J_U) \times \Omega^1_{<\Gamma}(J_V)$ into itself, where J_U and J_V are as in (30);

iii) for any $(U_0, V_0) \in \Omega^r_{<\Sigma}(J_U) \times \Omega^1_{<\Gamma}(J_V)$ the iterates $\mathcal{T}_r^n(U_0, V_0)$ converge uniformly to a fixed point of $\mathcal{T}_r$.

Existence of a renormalization fixed point follows from the following theorem.

Theorem B. *For every $\rho > 1$ there exists $r' \in (r_-, r_+)$ such that $\lambda(r) = \mu(r)$, and, therefore, the system*

$$\lambda^\rho U = U \circ \Psi_{V,\lambda,r'}, \quad \lambda^\rho V = V \circ \Phi_{U,\lambda,r'}$$

has a solution $(\lambda^*, U^*_{r'}, V^*_{r'}) \in (0,1) \times \Omega(J_U) \times \Omega(J_V)$.

5. Existence of the scaling parameters for the decoupled system

Consider functions

$$(U, V) \in \Omega(J_U) \times \Omega(J_V),$$

and let Z and W be as in (13). Such (Z, W) are in $\Omega(J_Z) \times \Omega(J_W)$, where

$$J_Z = \left(-r, \frac{1}{\lambda_+}\right), \quad J_W = \left(-1, \frac{r}{\mu_+}\right).$$

We will start with a simple lemma that insures that the "parameter" y from (12) is well-defined.

Lemma 5.1. *For any $U \in \Omega(J_U)$, $\mu \in (0,1)$ and $\rho > 1$, the equation*

$$y = p_{\frac{1}{\rho}}\left(\frac{U(x+\mu)}{U(x+y)}\right)$$

has a unique solution $y \in (\mu, 1)$.

Furthermore, if $U \in \Omega^r_{<\sigma}(J_U)$ for some $\sigma < 0$, then $y > y_-$, where

$$y_- \equiv \frac{\sqrt{r^2 + 4(r+\mu)} - r}{2} > \sqrt{\mu}. \tag{31}$$

Proof. Consider the function

$$f(y) = y^\rho U(r+y) - U(r+\mu).$$

We have

$$f(\mu) = \mu^\rho U(r+\mu) - U(r+\mu) < 0, \quad f(1) = U(r+1) - U(r+\mu) > 1$$

we have use that $0 < \mu < 1$ and U is an increasing function. Therefore, f has a zero in $(\mu, 1)$. Furthermore, for any $y > 0$

$$f'(y) = \rho y^{\rho-1} U(r+y) + y^\rho U'(r+y) > 0,$$

f is a monotone increasing function, and its zero in $(\mu, 1)$ is unique.

To demonstrate the last claim of the Lemma, notice, that the function Z is concave whenever $U \in \Omega^r_{<\sigma}(J_U)$, therefore

$$y = Z(\mu) > Z(y)\frac{r+\mu}{r+y} = \frac{r+\mu}{r+y}$$

(notice $Z(-r) = 0$, $Z(y) = 1$). The solution of this quadratic inequality yield the lower bound (31). □

Next, observe, that

$$\begin{aligned} N_{\Phi_{U,\mu,r}}(x) = \frac{\Phi''_{U,\mu,r}(x)}{\Phi'_{U,\mu,r}(x)} &= -\mu\left(\frac{Z''(Z(\mu(1-x)))}{Z'(Z(\mu(1-x)))}Z'(\mu(1-x)) + \frac{Z''(\mu(1-x))}{Z'(\mu(1-x))}\right) \\ &= -\mu\left\{N_Z(Z(\mu(1-x)))Z'(\mu(1-x)) + N_Z(\mu(1-x))\right\}. \end{aligned}$$

This implies, that whenever $U \in \Omega^r_{<\sigma}(J_U)$ for some $\sigma < 0$, the function

$$\Phi_{U,\mu,r}(x) = 1 - Z(Z(\mu(1-x))),$$

has *positive* nonlinearity and is in $\Omega(J_\Phi)$, where

$$J_\Phi = \left(1 - \frac{y}{\mu\lambda_+}, 1 + \frac{r}{\mu}\right).$$

In particular, the analyticity of $\Phi_{U,\mu,r}$ on $\mathbb{C}_{J_\Phi}$ follows from the fact that $Z(\mu(1-x))$ maps the interval J_Φ to $(Z(y/\lambda_+), 0)$, where

$$Z\left(\frac{y}{\lambda_+}\right) \le \frac{r + \frac{y}{\lambda_+}}{r+y} \le \frac{1}{\lambda_+},$$

the first inequality following from concavity. Therefore, $(Z(y/\lambda_+), 0)$ is contained in the domain of analyticity of Z.

At the same time, the function

$$\Psi_{V,\lambda,r}(x) = r - rW(\lambda(r-x))$$

has positive nonlinearity and is in $\Omega(J_\Psi)$, where

$$J_\Psi = \left(r - \frac{r}{\lambda\mu_+}, r + \frac{1}{\lambda}\right).$$

We are now ready to prove the following Lemma:

Lemma 5.2. *Let $(U,V) \in \Omega^r_{<\sigma}(J_U) \times \Omega^1_{<\gamma}(J_V)$ for some $\sigma > 0$ and $\gamma > 0$.*

Then the equations (16) and (17) have a unique solution (λ, μ) in the set

$$(\lambda_-(r), \lambda_+(r)) \times (\mu_-(r), \mu_+(r)),$$

where $\lambda_+(r)$ and $\mu_+(r)$ are as in (29), and

$$\lambda_-(r) = \left(\frac{r}{\rho}\frac{1 - \sqrt{\mu_+}\lambda_+}{(\lambda_+ r + 1)(1 + \sqrt{\mu_+}\lambda_+^2 r)}\right)^{\frac{1}{\rho-1}}$$

$$\mu_-(r) = \left(\frac{y_- r^2}{\rho^2}\frac{(1-\lambda_+\mu_+)^2}{(r+1)(r+\mu_+)(r+\lambda_+\mu_+)(r+\lambda_+\mu_+^2)}\right)^{\frac{1}{\rho-1}}.$$

Furthermore, the map $(U,V) \mapsto (\lambda,\mu)$ is continuous from $\Omega^r_{<\sigma}(J_U) \times \Omega^1_{<\gamma}(J_V)$ to $\lambda_-(r), \lambda_+(r)) \times (\mu_-(r), \mu_+(r))$.

Proof. One can obtain the lower bounds on $\Psi'_{V,\lambda,r}(0)$ and $\Phi'_{U,\mu,r}(0)$ straightforwardly from (16), (17) and (20).

$$\Psi'_{V,\lambda,r}(0) \geq \frac{\lambda r}{\rho} \frac{\frac{1}{\lambda_+\sqrt{\mu_+}} - 1}{(\lambda r+1)(\frac{1}{\lambda\sqrt{\mu_+}} + \lambda r)}, \tag{32}$$

$$\begin{aligned} \Phi'_{U,\mu,r}(0) &\geq \mu \frac{y}{\rho^2} \frac{\left(\frac{r}{\lambda_+\mu_+} - r\right)^2}{(r+\mu)(r+y)(\frac{r}{\lambda_+\mu_+} + \mu)(\frac{r}{\lambda_+\mu_+} + y)} \\ &\geq \mu \frac{y_-}{\rho^2} \frac{(r - r\lambda_+\mu_-+)^2}{(r+\mu_+)(r+1)(r+\mu_+^2\lambda_+)(r+\lambda_+\mu_+)}. \end{aligned} \tag{33}$$

On the other hand, we can use the Schwarz Lemma 3.1 to bound $\Psi'_{V,\lambda,r}(0)$ and $\Phi'_{U,\mu,r}(0)$ from above. First, notice, that since the nonlinearities, and hence the second derivatives, of $\Psi_{V,\lambda,r}$ and $\Phi_{U,\mu,r}$ are positive $\Psi'_{V,\lambda,r}(t) < \Psi'_{V,\lambda,r}(0)$ and $\Phi'_{U,\mu,r}(t) < \Phi'_{U,\mu,r}(0)$ for all negative t in the domain of these functions, and

$$\Psi_{V,\lambda,r}(-t) > -t\Psi'_{V,\lambda,r}(0), \qquad r - \frac{r}{\lambda\mu_+} < -t < 0,$$
$$\Phi_{U,\mu,r}(-t) > -t\Phi'_{U,\mu,r}(0), \qquad 1 - \frac{y}{\mu\lambda_+} < -t < 0.$$

Therefore,

$$\Psi_{V,\lambda,r} : \left(-t, r + \frac{1}{\lambda}\right) \mapsto \left(-t\Psi'_{V,\lambda,r}(0), r\right),$$
$$\Phi_{U,\mu,r} : \left(-t, 1 + \frac{r}{\mu}\right) \mapsto \left(-t\Phi'_{U,\mu,r}(0), 1 - \frac{yr}{r+\mu}\right) \subset \left(-t\Phi'_{U,\mu,r}(0), \frac{y}{r+y}\right),$$

where we have used the concavity of Z to get

$$Z(0) > Z(\mu)\frac{r}{r+\mu} = y\frac{r}{r+\mu} > Z(y)\frac{r}{r+y} = \frac{r}{r+y} \implies$$
$$\Phi_{U,\mu,r}\left(1 + \frac{r}{\mu}\right) = 1 - Z(0) \leq 1 - \frac{r}{r+y} = \frac{y}{r+y}.$$

We can now use (28) to find upper bounds on $\Psi'_{U,\mu,r}(0)$ and $\Phi'_{V,\lambda,r}(0)$:

$$\Psi'_{V,\lambda,r}(0) \leq \frac{t\Psi'_{V,\lambda,r}(0) r \left(r + \frac{1}{\lambda} + t\right)}{t\left(r + \frac{1}{\lambda}\right)\left(t\Psi'_{V,\lambda,r}(0) + r\right)} \implies \Psi'_{V,\lambda,r}(0) \leq \frac{\lambda r}{\lambda r + 1}$$

$$\Phi'_{U,\mu,r}(0) \leq \frac{t\Phi'_{U,\mu,r}(0)\frac{y}{r+y}\left(1 + \frac{r}{\mu} + t\right)}{t\left(1 + \frac{r}{\mu}\right)\left(t\Phi'_{U,\mu,r}(0) + \frac{y}{r+y}\right)} \implies \Phi'_{U,\mu,r}(0) \leq \frac{y}{r+y}\frac{\mu}{r+\mu}.$$

Consider solutions of the equations $\lambda^\rho = \Psi'_{V,\lambda,r}(0)$ with the upper and lower bounds on $\Psi'_{V,\lambda,r}(0)$ substituted for the right hand side:

$$\lambda^\rho = \frac{\lambda r}{\rho} \frac{\frac{1}{\lambda_+\sqrt{\mu_+}} - 1}{(\lambda r+1)(\frac{1}{\lambda\sqrt{\mu_+}} + \lambda r)} \implies \lambda \geq \lambda_-(r) = \left(\frac{r}{\rho} \frac{1 - \sqrt{\mu_+}\lambda_+}{(\lambda_+ r + 1)(1 + \sqrt{\mu_+}\lambda_+^2 r)}\right)^{\frac{1}{\rho-1}}$$

$$\lambda^\rho = \frac{\lambda r}{\lambda r + 1} \implies \lambda \leq \lambda_+(r) = \left(\frac{r}{r+1}\right)^{\frac{1}{\rho}}. \tag{34}$$

The function

$$f(\lambda, r) = \lambda^{\rho-1} - \frac{1}{\lambda}\Psi'_{V,\lambda,r}(0) = \lambda^{\rho-1} - rW'(\lambda r)$$

satisfies $f(\lambda_+, r) \geq 0$ and $f(\lambda_-, r) \leq 0$, and

$$\partial_\lambda f(\lambda, r) = (\rho - 1)\lambda^{\rho-2} - r\partial_\lambda W'(\lambda r).$$

We have

$$\begin{aligned} W'(x) &= \frac{1}{\rho}\frac{1}{V(\lambda r+1)^{\frac{1}{\rho}}} V'(x+1)V(x+1)^{\frac{1}{\rho}-1}, \\ W'(\lambda r) &= \frac{1}{\rho}\frac{V'(\lambda r+1)}{V(\lambda r+1)}, \\ \partial_\lambda W'(\lambda r) &= \frac{r}{\rho}\left(\frac{V''(\lambda r+1)}{V(\lambda r+1)} - \frac{V'(\lambda r+1)^2}{V(\lambda r+1)^2}\right), \end{aligned}$$

while

$$\begin{aligned} W''(x) &= \frac{1}{\rho}\frac{1}{V(\lambda r+1)^{\frac{1}{\rho}}}\left(V''(x+1)V(x+1)^{\frac{1}{\rho}-1} + \left(\frac{1}{\rho}-1\right)V'(x+1)^2V(x+1)^{\frac{1}{\rho}-2}\right), \\ W''(\lambda r) &= \frac{1}{\rho}\left(\frac{V''(x+1)}{V(x+1)} + \left(\frac{1}{\rho}-1\right)\frac{V'(x+1)^2}{V(x+1)^2}\right). \end{aligned}$$

Therefore $\partial_\lambda W'(\lambda r) < rW''(\lambda r) < 0$, since the nonlinearity, and hence the second derivative, of W is negative. It follows that f is monotone and has a unique zero in the interval (λ_-, λ_+). Continuity of λ in V follows from the fact that f is continuous in V.

Similarly, the function

$$g(\mu, r) = \mu^{\rho-1} - \frac{1}{\mu}\Phi'_{U,\mu,r}(0) = \mu^{\rho-1} - Z'(y)Z'(\mu) \tag{35}$$

has a zero in the interval (μ_-, μ_+), where

$$\mu_- = \left(\frac{y_- r^2}{\rho^2}\frac{(1-\lambda_+\mu_+)^2}{(r+1)(r+\mu_+)(r+\lambda_+\mu_+)(r+\lambda_+\mu_+^2)}\right)^{\frac{1}{\rho-1}}, \tag{36}$$

$$\mu_+ = \left(\frac{1}{(r+1)^2}\right)^{\frac{1}{\rho}}. \tag{37}$$

We will now show that this zero is unique. First,

$$\partial_\mu g(\mu, r) = (\rho-1)\mu^{\rho-2} - (\partial_\mu Z'(y))\, Z'(\mu) - Z'(y)\partial_\mu Z'(\mu) \tag{38}$$

Next,

$$\begin{aligned} y^\rho = \frac{U(r+\mu)}{U(r+y)} \implies \\ \partial_\mu y = \frac{U'(r+\mu)}{\rho y^{\rho-1}U(r+y) + y^\rho U'(r+y)} \leq \frac{U'(r+\mu)}{\rho y^{\rho-1}U(r+y)} = \frac{1}{\rho}y\frac{U'(r+\mu)}{U(r+\mu)} \end{aligned}$$

We use this bound in an estimate on $\partial_\mu Z'(\mu)$ in the third line below:

$$
\begin{aligned}
Z'(x) &= \frac{1}{\rho}\frac{1}{U(r+y)^{\frac{1}{\rho}}} U'(r+x)U(r+x)^{\frac{1}{\rho}-1},\\
Z'(\mu) &= \frac{1}{\rho} y \frac{U'(r+\mu)}{U(r+\mu)},\\
Z''(x) &= \frac{1}{\rho}\frac{1}{U(r+y)^{\frac{1}{\rho}}}\left(U''(r+x)U(r+x)^{\frac{1}{\rho}-1} + \left(\frac{1}{\rho}-1\right)U'(r+x)^2 U(r+x)^{\frac{1}{\rho}-2}\right),\\
\partial_\mu Z'(\mu) &= \frac{1}{\rho}\left(\partial_\mu y\right)\frac{U'(r+\mu)}{U(r+\mu)} + \frac{1}{\rho} y \left(\frac{U''(r+\mu)}{U(r+\mu)} - \frac{U'(r+\mu)^2}{U(r+\mu)^2}\right)\\
&\le \frac{1}{\rho^2} y \frac{U'(r+\mu)^2}{U(r+\mu)^2} + \frac{1}{\rho} y \left(\frac{U''(r+\mu)}{U(r+\mu)} - \frac{U'(r+\mu)^2}{U(r+\mu)^2}\right)\\
&\le y Z''(\mu) < 0.
\end{aligned}
$$

At the same time

$$
\partial_\mu Z'(y) = \left(\partial_\mu y\right)\frac{1}{\rho}\left(\frac{U''(r+y)}{U(r+y)} - \frac{U'(r+y)^2}{U(r+y)^2}\right) \le \left(\partial_\mu y\right) Z''(y) < 0.
$$

Therefore, the right hand side of (38) is positive, and g is a monotone increasing function. The zero of g in (μ_-, μ_+) is unique. The fact that the map $U \mapsto \mu$ is continuous follows from the continuity of the function g (see (35)) in U. □

We will now demonstrate that the unique solutions of (16) and (17) have to satisfy $\mu > \lambda$ for sufficiently small r, and $\mu < \lambda$ for sufficiently large r.

Lemma 5.3. *For every $\rho > 1$ there exist $r_+ = r_+(\rho) > r_- = r_-(\rho) > 0$, such that the unique solution (λ, μ) of (16) and (17) satisfy $\mu > \lambda$ for all $r < r_-$, and $\lambda > \mu$ for all $r > r_+$.*

Proof. First, we look at small r's.

According to the formula (34),

$$
\lambda^\rho r + \lambda^{\rho-1} < r, \tag{39}
$$

and $\lambda = O(r^{\frac{1}{\rho}})$. These two facts, in turn, imply that the first term in (39) is $O(r^2)$, the second — $O(r^{\frac{\rho-1}{\rho}})$, and consequently, for small r the inequality (39) becomes

$$
\lambda^{\rho-1} < Cr \implies \lambda = O(r^{\frac{1}{\rho-1}}),
$$

(C here and below will denote an irrelevant constant, not necessarily one and the same). At the same time, according to (33)

$$
\begin{aligned}
\mu^{\rho-1} &\ge C\frac{y}{r+y}\frac{r^2\left(1-\lambda_+\mu_+\right)^2}{(r+\mu)(r+\lambda_+\mu_+\mu)(r+\lambda_+\mu_+ y)}\\
&\ge C\frac{y}{r+y}\frac{r^2}{(r+\mu)^2(r+O(r^{\frac{1}{\rho-1}}))}.
\end{aligned}
$$

Notice, that $y/(r+y)$ is an increasing function of y, therefore its minimum is achieved at y_-. For small r, $y_- = O(\sqrt{r+\mu})$, i.e., for small r, $y_-/(r+y_-) = O(1)$.

$$
\mu^{\rho-1} \ge C\frac{r^2}{(r+\mu)^2(r+O(r^{\frac{1}{\rho-1}}))}.
$$

We consider two cases $\rho < 2$ and $\rho \geq 2$. In the first case

$$\mu^{\rho-1} \geq C \frac{r^2}{(r+\mu)^2(r+O(r^{\frac{1}{\rho-1}}))} \geq C \frac{r}{(r+\mu)^2} \implies \mu^{\frac{\rho-1}{2}}(r+\mu) \geq C r^{\frac{1}{2}} \implies \mu \geq O(r^{\frac{1}{\rho+1}}).$$

in the second case

$$\mu^{\rho-1} \geq C \frac{r^2}{(r+\mu)^2(r+O(r^{\frac{1}{\rho-1}}))} \geq C \frac{r^{2-\frac{1}{\rho-1}}}{(r+\mu)^2} \implies \mu^{\frac{\rho-1}{2}}(r+\mu) \geq C r^{1-\frac{1}{2(\rho-1)}} \implies \mu \geq O(r^{\frac{2\rho-3}{(\rho-1)(\rho+1)}}).$$

In both cases, for sufficiently small r, $\mu > \lambda$.

We will now look at large r. First, consider (32) for large r:

$$\begin{aligned}\lambda^{\rho-1} &\geq \frac{r}{\rho} \frac{1-\lambda_+\sqrt{\mu_+}}{(\lambda_+ r+1)(1+\lambda_+\sqrt{\mu_+} r)} \geq \frac{1}{\rho} \frac{r\left(1+O\left(\frac{1}{r}\right)\right)}{(\lambda_+ r+1)\left(2+O\left(\frac{1}{r}\right)\right)} \\ &\geq \frac{1}{\rho} \frac{r\left(1+O\left(\frac{1}{r}\right)\right)}{\left(\left(1+O\left(\frac{1}{r}\right)\right)r+1\right)\left(2+O\left(\frac{1}{r}\right)\right)} \geq \frac{1}{2\rho}\left(1+O\left(\frac{1}{r}\right)\right).\end{aligned}$$

On the other hand, $\mu_+ = O\left(\frac{1}{r^2}\right)$ (cf. (37)). Therefore, for sufficiently large r, $\lambda > \mu$. □

6. Bounded nonlinearity

We will now look at the images of the nonlinearities N_Z and N_W under the operator $\mathcal{T}_r$. Let $(U,V) \in \Omega^r_{<\sigma}(J_U) \times \Omega^1_{<\gamma}(J_V)$ for some negative σ and γ. Then, the equations (16) and (17) have a unique solution (λ, μ). Denote

$$(\tilde{U}, \tilde{V}) \equiv \mathcal{T}_r(U,V) = \left(\lambda^{-\rho} U \circ \Psi_{V,\lambda,r}, \mu^{-\rho} V \circ \Phi_{U,\mu,r}\right).$$

Also, for brevity, denote $p_{\frac{1}{\rho}} U = U_\rho$ and $p_{\frac{1}{\rho}} V = V_\rho$, then

$$\begin{aligned}N_{\tilde{Z}}(x) &= \frac{\left(\tilde{U}_\rho(r+x)\right)''}{\left(\tilde{U}_\rho(r+x)\right)'} = \frac{\left(U_\rho\left(\Psi_{V,\lambda,r}(r+x)\right)\right)''}{\left(U_\rho\left(\Psi_{V,\lambda,r}(r+x)\right)\right)'} = \frac{\left(U_\rho\left(r+\hat{\Psi}_{V,\lambda,r}(x)\right)\right)''}{\left(U_\rho\left(r+\hat{\Psi}_{V,\lambda,r}(x)\right)\right)'} \\ &= \frac{U''_\rho\left(r+\hat{\Psi}_{V,\lambda,r}(x)\right)\hat{\Psi}'_{V,\lambda,r}(x)^2 + U'_\rho\left(r+\hat{\Psi}_{V,\lambda,r}(x)\right)\hat{\Psi}''_{V,\lambda,r}(x)}{U'_\rho\left(r+\hat{\Psi}_{V,\lambda,r}(x)\right)\hat{\Psi}'_{V,\lambda,r}(x)} \\ &= N_Z\left(\hat{\Psi}_{V,\lambda,r}(x)\right)\hat{\Psi}'_{V,\lambda,r}(x) + \frac{\hat{\Psi}''_{V,\lambda,r}(x)}{\hat{\Psi}'_{V,\lambda,r}(x)} \\ &= N_Z\left(\hat{\Psi}_{V,\lambda,r}(x)\right)\hat{\Psi}'_{V,\lambda,r}(x) + N_{\Psi_{V,\lambda,r}}(r+x).\end{aligned}$$

where $\hat{\Psi}_{V,\lambda,r}(x) = \Psi_{V,\lambda,r}(r+x) - r$.

$$
\begin{aligned}
N_{\tilde W}(x) &= \frac{\left(\tilde V_\rho(1+x)\right)''}{\left(\tilde V_\rho(1+x)\right)'} = \frac{\left(V_\rho\left(\Phi_{U,\mu,r}(1+x)\right)\right)''}{\left(V_\rho\left(\Phi_{U,\mu,r}(r+x)\right)\right)'} = \frac{\left(V_\rho\left(r+\hat\Phi_{U,\mu,r}(x)\right)\right)''}{\left(V_\rho\left(r+\hat\Phi_{U,\mu,r}(x)\right)\right)'} \\
&= \frac{V_\rho''\left(r+\hat\Phi_{U,\mu,r}(x)\right)\hat\Phi'_{U,\mu,r}(x)^2 + V_\rho'\left(r+\hat\Phi_{U,\mu,r}(x)\right)\hat\Phi''_{U,\mu,r}(x)}{V_\rho'\left(r+\hat\Phi_{U,\mu,r}(x)\right)\hat\Phi'_{U,\mu,r}(x)} \\
&= N_W\left(\hat\Phi_{U,\mu,r}(x)\right)\hat\Phi'_{U,\mu,r}(x) + \frac{\hat\Phi''_{U,\mu,r}(x)}{\hat\Phi'_{U,\mu,r}(x)} \\
&= N_W\left(\hat\Phi_{U,\mu,r}(x)\right)\hat\Phi'_{U,\mu,r}(x) + N_{\Phi_{U,\mu,r}}(1+x).
\end{aligned}
$$

where $\hat\Phi_{U,\mu,r}(x) = \Phi_{U,\mu,r}(1+x) - 1$.

We will also require the following relation between the nonlinearity of $\Phi_{U,\mu,r}$ and Z:

$$
N_{\Phi_{U,\mu,r}}(x) = -\mu\left\{N_Z(Z(\mu(1-x)))Z'(\mu(1-x)) + N_Z(\mu(1-x))\right\}. \tag{40}
$$

Recall, that for Herglotz–Pick functions

$$
N_f'(x) = \frac{f'''(x)}{f'(x)} - \left(\frac{f''(x)}{f'(x)}\right)^2 \geq \frac{1}{2}N_f(x)^2,
$$

which follows from the positivity of the Schwarzian derivative. Therefore, the nonlinearity of these functions is monotone increasing.

Proposition 6.1. *There exist functions $\Sigma(r) < 0$ and $\Gamma(r) < 0$, continuous in r, such that $\mathcal{T}_r$ is a continuous operator of the set $\Omega^r_{<\Sigma(r)}(J_U) \times \Omega^1_{<\Gamma(r)}(J_V)$ into itself.*

Proof. For a fixed r, suppose the nonlinearity of Z is bounded by some negative σ on all of $J_Z = (-r, 1/\lambda_+)$, while that of W is bounded by some γ on $J_W = (-1, r/\mu_+)$:

$$
N_Z\left(\frac{1}{\lambda_+}\right) \leq \sigma < 0, \quad N_W\left(\frac{r}{\mu_+}\right) \leq \gamma < 0, \tag{41}
$$

Notice, due to the bounds (25),

$$
\sigma \geq -\frac{2}{r+\frac{1}{\lambda_+}} \equiv \sigma_-, \quad \gamma \geq -\frac{2}{1+\frac{r}{\mu_+}} \equiv \gamma_-. \tag{42}
$$

Let (λ, μ) be the unique solution of (16) and (17). Below, we will assume a certain form of the bounds σ and γ, and we will show that $N_{\tilde Z}(1/\lambda_+)$ and $N_{\tilde W}(r/\mu_+)$ satisfy bounds of the same form. In fact, we will estimate the maximum of $N_{\tilde Z}(1/l)$ for any $\lambda < l < \lambda_+$ and the maximum of $N_{\tilde W}(r/m)$ for any $\mu < m < \mu._+$, and use the fact that $N_{\tilde Z}(1/l) > N_{\tilde Z}(1/\lambda_+)$ and $N_{\tilde W}(r/m) > N_{\tilde W}(r/\mu_+)$. The exact reason for why the nonlinearities are estimated at points $1/l > 1/\lambda_+$ and $r/m > r/\mu_+$ will be given at the end of Step 1).

Step 1). We start with $N_{\tilde Z}(1/l)$:

$$
N_{\tilde Z}\left(\frac{1}{l}\right) = N_Z\left(\hat\Psi_{V,\lambda,r}\left(\frac{1}{l}\right)\right)\hat\Psi'_{V,\lambda,r}\left(\frac{1}{l}\right) + N_{\Psi_{V,\lambda,r}}\left(r+\frac{1}{l}\right).
$$

Since $\hat{\Psi}_{V,\lambda,r}$ is an increasing function,

$$\hat{\Psi}_{V,\lambda,r}\left(\frac{1}{l}\right) \le \hat{\Psi}_{V,\lambda,r}\left(\frac{1}{\lambda}\right) = \Psi_{V,\lambda,r}\left(r+\frac{1}{\lambda}\right) - r = 0,$$

and

$$N_{\tilde{Z}}\left(\frac{1}{l}\right) \le N_Z(0)\hat{\Psi}'_{V,\lambda,r}\left(\frac{1}{l}\right) + N_{\Psi_{V,\lambda,r}}\left(r+\frac{1}{l}\right).$$

Notice, that

$$N_{\Psi_{V,\lambda,r}}(x) = -\lambda N_W(\lambda(r-x)),$$

therefore,

$$N_{\Psi_{V,\lambda,r}}\left(r+\frac{1}{l}\right) = -\lambda N_W\left(-\frac{\lambda}{l}\right), \quad N_{\Psi_{V,\lambda,r}}(0) = -\lambda N_W(\lambda r). \tag{43}$$

The estimate (26) can be used to bound $N_Z(0)$ and $N_W(\lambda r)$ from above:

$$N_Z(0) \le \frac{2N_Z\left(\frac{1}{\lambda_+}\right)}{2+N_Z\left(\frac{1}{\lambda_+}\right)\frac{1}{\lambda_+}}, \quad N_W(\lambda r) \le \frac{2N_W\left(\frac{r}{\mu_+}\right)}{2+N_W\left(\frac{r}{\mu_+}\right)\left(\frac{r}{\mu_+}-\lambda r\right)}. \tag{44}$$

We also use the bound (27) to estimate $\hat{\Psi}'_{V,\lambda,r}(1/l)$ from below.

$$\hat{\Psi}'_{V,\lambda,r}\left(\frac{1}{l}\right) = \Psi'_{V,\lambda,r}\left(r+\frac{1}{l}\right) \ge \frac{4\lambda^\rho}{\left(2-N_{\Psi_{V,\lambda,r}}(0)\left(r+\frac{1}{l}\right)\right)^2}. \tag{45}$$

We collect the estimates (44), (45) and (43), and use (25) on $N_W(-\lambda/l)$ to get

$$\begin{aligned} N_{\tilde{Z}}\left(\frac{1}{l}\right) &\le N_Z\left(\frac{1}{\lambda_+}\right)\frac{8\lambda^\rho}{\left(2+N_Z\left(\frac{1}{\lambda_+}\right)\frac{1}{\lambda_+}\right)\left(2-N_{\Psi_{V,\lambda,r}}(0)\left(r+\frac{1}{l}\right)\right)^2} \\ &\quad -\lambda N_W\left(-\frac{\lambda}{l}\right) \\ &\le N_Z\left(\frac{1}{\lambda_+}\right)\frac{8\lambda^\rho}{\left(2+N_Z\left(\frac{1}{\lambda_+}\right)\frac{1}{\lambda_+}\right)\left(2+\lambda N_W(\lambda r)\left(r+\frac{1}{l}\right)\right)^2} \\ &\quad +\lambda\frac{2}{1-\frac{\lambda}{l}}. \end{aligned} \tag{46}$$

To demonstrate that there are σ and γ such that the nonlinearities $N_{\tilde{Z}}$ and $N_{\tilde{W}}$ satisfy the bounds in (41), it is sufficient to come up with a choice of these constants so that the upper bound $\tilde{\sigma}$ on $N_{\tilde{Z}}(1/l)$ is less than σ:

$$\sigma > \tilde{\sigma}$$

where

$$\tilde{\sigma} = \sigma\frac{8\lambda^\rho}{f} + \lambda\frac{2}{1-\frac{\lambda}{l}}, \quad f(\sigma,\gamma,l) = \left(2+\frac{\sigma}{\lambda_+}\right)\left(2+\lambda a(\gamma)\left(r+\frac{1}{l}\right)\right)^2, \quad \text{(cf. (46))},$$

$$a(\gamma) = \frac{2\gamma}{2+\gamma\left(\frac{r}{\mu_+}-\lambda r\right)} \quad \text{(cf. the second equation of 44)}.$$

Recall the definition (42) of σ_- and γ_-, and set

$$\Sigma = -\frac{2-(m-\mu)}{r+\frac{1}{\lambda_+}}, \quad \Gamma = -\frac{2-(l-\lambda)}{1+\frac{r}{\mu_+}},$$

Notice, that

$$f(\Sigma,\Gamma,l)=\frac{4(rl+\mu_+(2+\lambda))(2r\lambda_++(m-\mu))}{\left(-2\mu_+(1+\lambda r)+r(\lambda-l)(1-\lambda\mu_+)\right)^2(\lambda_+r+1)l^2}(l-\lambda)^2$$
$$=O(1)(l-\lambda)^2,$$

where $O(1)$ is a positive function of r, λ, l, λ_+ and μ_+ of order 0 in $(l-\lambda)$. Consider the function

$$g(\sigma,\gamma,l)=(\tilde{\sigma}-\sigma)f(\sigma,\gamma,l)=\sigma 8\lambda^\rho+\frac{2\lambda l}{l-\lambda}f(\sigma,\gamma,l)-\sigma f(\sigma,\gamma,l).$$

We have

$$g(\Sigma,\Gamma,l)=-8\lambda^\rho\cdot\frac{2-(m-\mu)}{r+\frac{1}{\lambda_+}}+O(1)(l-\lambda)+O(1)(l-\lambda)^2. \tag{47}$$

For any $\lambda_-<\lambda<\lambda_+$, we can choose l, sufficiently close to λ, so that $g(\Sigma,\Gamma,l)\le 0$. Therefore, for such l,

$$N_{\tilde{Z}}\left(\frac{1}{\lambda_+}\right)-\sigma\le\frac{g(\Sigma,\Gamma,l)}{f(\Sigma,\Gamma,l)}\le 0,$$

as required.

It is clear at this point why we chose to estimate $N_{\tilde{Z}}$ at $1/l$ and not at $1/\lambda_+$: had l been chosen equal to λ_+ from the beginning, one would have to deal with the positive second and third terms in (47). Specifically, one would have to show that g, a complicated function of r and λ, is negative for a wide range of r's. We have circumvented this problem by estimating the nonlinearity at a point $1/l>1/\lambda_+$ (recall, $N_{\tilde{Z}}(1/l)>N_{\tilde{Z}}(1/\lambda_+)$), and using the fact that l can be freely chosen to be close to λ, the solution of (16), so that the second and the third terms in (47) are small in the absolute value compared to the first one.

The solution (λ,μ) clearly depends on (U,V), and, seemingly, so do Σ and Γ. However, we can set

$$l=\lambda+\delta(r),\quad m=\mu+\epsilon(r),$$

and choose $\delta(r)$ and $\epsilon(r)$ to be continuous positive functions of r *only*, sufficiently (but not necessarily infinitesimally) small, so that $g<0$. Then

$$\Sigma=-\frac{2-\delta(r)}{r+\frac{1}{\lambda_+(r)}},\quad \Gamma=-\frac{2-\epsilon(r)}{1+\frac{r}{\mu_+(r)}}$$

are continuous functions of r *only*.

Step 2). We will now consider the maximum of $N_{\tilde{W}}(r/m)$, in a similar way. For any $\mu<m<\mu_+$

$$N_{\tilde{W}}\left(\frac{r}{m}\right)=N_W\left(\hat{\Phi}_{U,\mu,r}\left(\frac{r}{m}\right)\right)\hat{\Phi}'_{U,\mu,r}\left(\frac{r}{m}\right)+N_{\Phi_{U,\mu,r}}\left(1+\frac{r}{m}\right). \tag{48}$$

First, by concavity of Z,

$$\frac{Z(0)}{r}\ge\frac{Z(y)}{r+y}=\frac{1}{r+y},$$

therefore,

$$\hat{\Phi}_{U,\mu,r}\left(\frac{r}{m}\right)=-Z\left(Z\left(-\frac{\mu}{m}r\right)\right)\le -Z(0)\le-\frac{r}{r+y}\le-\frac{r}{r+1},$$

and

$$N_{\tilde{W}}\left(\frac{r}{m}\right) \le N_W\left(-\frac{r}{r+1}\right)\hat{\Phi}'_{U,\mu,r}\left(\frac{r}{m}\right) + N_{\Phi_{U,\mu,r}}\left(1+\frac{r}{m}\right).$$

We use the bound (27) to estimate $\hat{\Phi}'_{U,\mu,r}\,(r/m)$ from below.

$$\hat{\Phi}'_{U,\mu,r}\left(\frac{r}{m}\right) = \Phi'_{U,\mu,r}\left(1+\frac{r}{m}\right) \ge \frac{4\mu^\rho}{\left(2 - N_{\Phi_{U,\mu,r}}(0)\left(1+\frac{r}{m}\right)\right)^2}. \tag{49}$$

The bound (26) can be used to bound $N_W(-r/(r+1))$ from above:

$$N_W\left(-\frac{r}{r+1}\right) \le \frac{2N_W\left(\frac{r}{\mu_+}\right)}{2 + N_W\left(\frac{r}{\mu_+}\right)\left(\frac{r}{r+1}+\frac{r}{\mu_+}\right)}. \tag{50}$$

We substitute the estimates (50) and (49) in (48), together with the estimate (25) for $N_{\Phi_{U,\mu,r}}(1+r/m)$, to get

$$\begin{aligned} N_{\tilde{W}}\left(\frac{r}{m}\right) &\le N_W\left(\frac{r}{\mu_+}\right)\frac{8\mu^\rho}{\left(2+N_W\left(\frac{r}{\mu_+}\right)\left(\frac{r}{r+1}+\frac{r}{\mu_+}\right)\right)\left(2-N_{\Phi_{U,\mu,r}}(0)\left(1+\frac{r}{m}\right)\right)^2} \\ &\quad + N_{\Phi_{U,\mu,r}}\left(1+\frac{r}{m}\right) \\ &\le N_W\left(\frac{r}{\mu_+}\right)\frac{8\mu^\rho}{\left(2+N_W\left(\frac{r}{\mu_+}\right)\left(\frac{r}{r+1}+\frac{r}{\mu_+}\right)\right)\left(2-N_{\Phi_{U,\mu,r}}(0)\left(1+\frac{r}{m}\right)\right)^2} \\ &\quad + \frac{2}{\frac{r}{\mu}-\frac{r}{m}}. \end{aligned} \tag{51}$$

Next, by the relation (40)

$$N_{\Phi_{U,\mu,r}}(0) = -\mu\left\{N_Z(y)Z'(\mu) + N_Z(\mu)\right\} \ge -\mu N_Z(\mu). \tag{52}$$

Again, we use the bound (26) to estimate $N_Z(\mu)$ from above.

$$N_Z(\mu) \le \frac{2N_Z\left(\frac{1}{\lambda_+}\right)}{2 + N_Z\left(\frac{1}{\lambda_+}\right)\left(\frac{1}{\lambda_+}-\mu\right)}. \tag{53}$$

It is sufficient to come up with a choice of σ and γ so that the upper bound on $N_{\tilde{W}}\,(r/m)$ is less than γ:

$$\gamma \ge \tilde{\gamma},$$

where

$$\tilde{\gamma} = \gamma\frac{8\mu^\rho}{h} + \frac{2}{\frac{r}{\mu}-\frac{r}{m}}, \quad h(\sigma,\gamma,m) = \left(2+\gamma\left(\frac{r}{r+1}+\frac{r}{\mu_+}\right)\right)\left(2+\mu b(\sigma)\left(1+\frac{r}{m}\right)\right)^2,$$

$$b = \frac{2\sigma}{2+\sigma\left(\frac{1}{\lambda_+}-\mu\right)} \quad \text{(cf. (52), (53))}.$$

Let be Σ and Γ be as in (47). Notice, that

$$\begin{aligned} h(\Sigma,\Gamma,m) &= \frac{4(2\lambda_+ r + (m+\mu\lambda_+ r))^2(2\mu_+ + r(l-\lambda)(\mu_+ + r + 1))}{(2\lambda_+(r+\mu) + (m-\mu)(1-\mu\lambda_+))^2(r+1)(r+\mu_+)m^2}(m-\mu)^2 \\ &= O(1)(m-\mu)^2. \end{aligned}$$

!

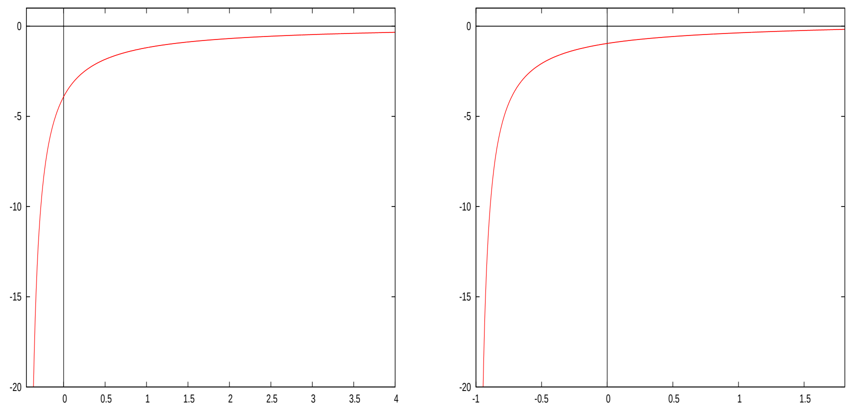

FIGURE 2. *Nonlinearities N_Z (a)) and N_W (b)) for the fixed point ($\rho = 2$) computed in* [10].

Consider the function

$$z(\sigma, \gamma, m) = (\tilde{\gamma} - \gamma)h(\sigma, \gamma, m) = \gamma 8\mu^\rho + \frac{2\mu m}{r} \frac{h(\sigma, \gamma, m)}{(m - \mu)} - \gamma h(\sigma, \gamma, m).$$

We have

$$z(\Sigma, \Gamma, m) = -8\mu^\rho \frac{2 - (l - \lambda)}{r + \frac{1}{\lambda_+}} + O(1)(m - \mu) + O(1)(m - \mu)^2.$$

As we have already discussed, for any $\mu_- < \mu < \mu_+$ we can choose

$$m = \mu + \epsilon(r)$$

where $\epsilon(r)$ is positive, sufficiently small and continuous, so that $z(\Sigma, \Gamma, m) \leq 0$. Therefore,

$$N_{\tilde{W}}\left(\frac{r}{m}\right) - \gamma \leq \frac{z(\Sigma, \Gamma, m)}{h(\Sigma, \Gamma, m)} \leq 0$$

as needed.

Step 3). We shall now prove the claim about the continuity of the operator $\mathcal{T}_r$. Recall, that according to Lemma (5.2), for every fixed r, the map $(U, V) \mapsto (\lambda(V), \mu(U))$, is continuous from $\Omega^r_{<\Sigma(r)}(J_U) \times \Omega^1_{<\Gamma(r)}(J_V)$ to $(\lambda_-(r), \lambda_+(r)) \times (\mu_-(r), \mu_+(r))$. This, together with the continuity of $\Psi_{V,\lambda,r}$ in U and λ, and $\Phi_{U,\mu,r}$ in U and μ, implies that the map

$$(U, V) \mapsto \left(\lambda^{-\rho}(V) U \circ \Psi_{V,\lambda(V),r}, \mu^{-\rho}(U) V \circ \Phi_{U,\mu(U),r}\right)$$

is continuous from $\Omega^r_{<\Sigma(r)}(J_U) \times \Omega^1_{<\Gamma(r)}(J_V)$ to itself.

□

7. Existence of a renormalization fixed point

In this section we prove Theorem B.

Proof. In Lemmas 5.2 and 5.3, and in Prop. 6.1 we have shown that there exists an interval (r_-, r_+), $0 < r_- < r_+$, of parameter values r, and, for every $r \in (r_-, r_+)$, intervals J_U and J_V and bounds $\Sigma(r)$ and $\Gamma(r)$, continuous in r, such that the operator $\mathcal{T}_r$ maps the relatively compact set $\Omega^r_{<\Sigma(r)}(J_U) \times \Omega^1_{<\Gamma(r)}(J_V)$ into itself, and, furthermore, $\lambda^\rho(V, r)$ and $\mu^\rho(U, r)$ are contained in some subinterval $[\Delta, (1-\Delta)^2] \subset (0, 1)$, $\mu_-(r_-) > \lambda_+(r_+)$ and $\lambda_-(V, r_+) > \mu_+(r_+)$.

Now, consider a sequence $(U_n, V_n) \equiv \mathcal{T}^n_r(U_0, V_0)$ (cf. (18)) with $(U_0, V_0) \in \Omega^r_{<\Sigma}(J_U) \times \Omega^1_{<\Gamma}(J_V)$. Notice, since $\Sigma(r)$ and $\Gamma(r)$ are continuous functions of r, and since the dependence of J_U and J_V on r is also continuous, such (U_0, V_0) can be chosen in such a way that the the map $r \mapsto (r, U_0, V_0)$ is continuous from (r_-, r_+) to $\sqcup_{r \in (r_-, r_+)} \Omega^r_{<\Sigma(r)}(J_U) \times \Omega^1_{<\Gamma(r)}(J_V)$ (here $\sqcup$ stands for a disjoint union).

Since the set $\Omega^r_{<\Sigma}(J_U) \times \Omega^1_{<\Gamma}(J_V)$ is invariant under $\mathcal{T}_r$, the scalings $\lambda^\rho_n = \Psi'_{V_n,\lambda_n,r}(0)$ and $\mu^\rho_n = \Phi'_{U_n,\mu_n,r}(0)$ satisfy the bounds from Lemma 5.2, and are in $[\Delta, (1-\Delta)^2] \subset (0, 1)$. Consider

$$U_n = \frac{1}{\lambda^\rho_{n-1} \ldots \lambda^\rho_1 \lambda^\rho_0} \Psi_{V_{n-1},\lambda_{n-1},r} \circ \ldots \circ \Psi_{V_1,\lambda_1,r} \circ \Psi_{V_0,\lambda_0,r}, \quad \lambda^\rho_k = \Psi'_{V_k,\lambda_k,r}(0),$$

$$V_n = \frac{1}{\mu^\rho_{n-1} \ldots \mu^\rho_1 \mu^\rho_0} \Phi_{U_{n-1},\mu_{n-1},r} \circ \ldots \circ \Phi_{U_1,\mu_1,r} \circ \Phi_{U_0,\mu_0,r}, \quad \mu^\rho_k = \Phi'_{U_k,\mu_k,r}(0).$$

For each k, there exist neighborhoods D^k_U and D^k_V of zero, such that

$$|\Psi_{V_k,\lambda_k,r}(z)| < A_k |z|, \quad z \in D^k_U, \quad \text{and} \quad |\Phi_{U_k,\mu_k,r}(z)| < B_k |z|, \quad z \in D^k_V,$$

for some $A_k < 1$ and $B_k < 1$, such that $A_k^2 < \lambda^\rho_k < A_k$ and $B_k^2 < \mu^\rho_k < B_k$. Specifically, one can choose $A_k = \lambda^\rho_k(1+\Delta)$ and $B_k = \mu^\rho_k(1+\Delta)$. Notice, that since $A_k \geq \lambda^\rho_k + \Delta^2 > \Delta + \Delta^2$ (and similarly for B_k), the domains D^k_U and D^k_V do not shrink to zero, that is, there exists $s > 0$, such that $\mathbb{D}_s(0) \subset D^k_U$ and $\mathbb{D}_s(0) \subset D^k_V$ for all $k \geq 0$ (here, $\mathbb{D}_s(0)$ denotes a disk around 0 of radius s in $\mathbb{C}$). Thus, for any $z \in \mathbb{D}_s(0)$

$$\left|\Psi_{V_{n-1},\lambda_{n-1},r} \circ \ldots \circ \Psi_{V_1,\lambda_1,r} \circ \Psi_{V_0,\lambda_0,r}(z)\right| \leq \prod_{k=0}^{n-1} A_k s,$$

$$\left|\Phi_{U_{n-1},\mu_{n-1},r} \circ \ldots \circ \Phi_{U_1,\mu_1,r} \circ \Phi_{U_0,\mu_0,r}(z)\right| \leq \prod_{k=0}^{n-1} B_k s.$$

Furthermore, for all $z \in \mathbb{D}_s(0)$, $|\Psi_{V_k,\lambda_k,r}(z) - \lambda^\rho_k z| \leq K|z^2|$ for some constant K, therefore

$$\left|\Psi_{V_n,\lambda_n,r} \circ \ldots \circ \Psi_{V_0,\lambda_0,r}(z) - \lambda^\rho_n \Psi_{V_{n-1},\lambda_{n-1},r} \circ \ldots \circ \Psi_{V_0,\lambda_0,r}(z)\right| \leq K \left(\prod_{k=0}^{n-1} A_k s\right)^2,$$

$$\left|\Phi_{U_n,\mu_n,r} \circ \ldots \circ \Phi_{U_0,\mu_0,r}(z) - \mu^\rho_n \Phi_{U_{n-1},\mu_{n-1},r} \circ \ldots \circ \Phi_{U_0,\mu_0,r}(z)\right| \leq K \left(\prod_{k=0}^{n-1} B_k s\right)^2,$$

while, with our choice of A_k and B_k,

$$|U_{n+1}(z)-U_n(z)| \le \frac{Ks^2}{\lambda_n^\rho}\prod_{k=0}^{n-1}\frac{A_k^2}{\lambda_k^\rho} \le \frac{Ks^2}{\Delta}(1-\Delta^2)^{2n}, \tag{54}$$

$$|V_{n+1}(z)-V_n(z)| \le \frac{Ks^2}{\mu_n^\rho}\prod_{k=0}^{n-1}\frac{B_k^2}{\mu_k^\rho} \le \frac{Ks^2}{\Delta}(1-\Delta^2)^{2n}. \tag{55}$$

We therefore obtain that for every $r \in (r_-, r_+)$ the sequences (U_n, V_n) converges uniformly on $\mathbb{D}_s(0)$, and, in fact, as (54) and (55) show, the rate of convergence is independent of r. Since every $(U_n, V_n) \in \Omega(J_U)\times\Omega(J_V)$ this convergence is uniform on every compact subset of $\mathbb{C}_{J_U}\times\mathbb{C}_{J_V}$. The limit of this sequence, (U_r^*, V_r^*), satisfies

$$\lambda^\rho(V_r^*, r)\ U_r^* = U_r^* \circ \Psi_{V_r^*,\lambda(V_r^*,r),r},$$
$$\mu^\rho(U_r^*, r)\ V_r^* = V_r^* \circ \Phi_{U_r^*,\mu(U_r^*,r),r},$$

on any compact subset of $\mathbb{C}_{J_U}\times\mathbb{C}_{J_V}$, and, by extension, on all of $\mathbb{C}_{J_U}\times\mathbb{C}_{J_V}$.

We will proceed to demonstrate by induction that the map $r \mapsto (r, U_{n+1}, V_{n+1}, \lambda_n, \mu_n)$ is continuous from (r_-, r_+) to $\sqcup_{r\in(r_-,r_+)}\Omega^r_{<\Sigma(r)}(J_U)\times\Omega^1_{<\Gamma(r)}(J_V)\times[\Delta,(1-\Delta)^2]^2$.

Consider the functions

$$f(\lambda;(r,V)) = \lambda^{\rho-1} - \frac{r}{\rho}\frac{V'(\lambda r+1)}{V(\lambda r+1)}, \quad f_0(\lambda, r) \equiv f(\lambda;(r,V_0)).$$

Since both λ and μ are contained in $[\Delta,(1-\Delta)^2]$, the points $\lambda r+1$, $r+\mu$ and $r+y$ (recall, $y<1$) are always contained compactly in J_V and J_U respectively, and the map

$$(\lambda;(r,V)) \mapsto f(\lambda;(r,V))$$

is clearly continuous from $[\Delta,(1-\Delta)^2]\times\sqcup_{r\in(r_-,r_+)}\Omega^1_{<\Gamma(r)}(J_V)$ to $C^0([\Delta,(1-\Delta)^2]\times\sqcup_{r\in(r_-,r_+)}\Omega^1_{<\Gamma(r)}(J_V),\mathbb{R})$.

Recall, that according to the Lemma (5.2), for each $r\in(r_-,r_+)$ the function f_0 has a single zero in $(\lambda_-(r),\lambda_+(r))$. Since (U_0,V_0) have been chosen to be continuous functions of r, and since the map $z\mapsto V_0(z)$ is continuously differentiable in $J_V \ni \lambda r+1$, the map $(\lambda,r)\mapsto f_0(\lambda,r)$ is continuous, and so is the unique zero of $f_0(\cdot,r)$: the map $r\mapsto\lambda_0(r)$ is continuous from (r_-,r_+) to $[\Delta,(1-\Delta)^2]$. One can argue in a similar way, that the map $r\mapsto\mu_0(r)$ is continuous from (r_-,r_+) to $[\Delta,(1-\Delta)^2]$ as well.

This, together with the continuity of Ψ in V, λ and r, and Φ in U, μ and r, implies that the map

$$r\mapsto(r,U_1,V_1) = \left(r, \lambda_0^{-\rho}U_0\circ\Psi_{V_0,\lambda_0,r}, \mu_0^{-\rho}V_0\circ\Phi_{U_0,\mu_0,r}\right)$$

is continuous from (r_-,r_+) to $\sqcup_{r\in(r_-,r_+)}\Omega^r_{<\Sigma(r)}(J_U)\times\Omega^1_{<\Gamma(r)}(J_V)$.

Next, assume, that $r\mapsto(r,U_n,V_n,\lambda_{n-1},\mu_{n-1})$ is continuous. Then, one can argue identically to the case $n=1$ above (substituting (U_n,V_n) for (U_0,V_0)) that

$$r\mapsto(r,U_{n+1},V_{n+1},\lambda_n,\mu_n)$$

is continuous from (r_-,r_+) to $\sqcup_{r\in(r_-,r_+)}\Omega^r_{<\Sigma(r)}(J_U)\times\Omega^1_{<\Gamma(r)}(J_V)\times[\Delta,(1-\Delta)^2]^2$.

According to (54) and (55), (U_n,V_n) converge uniformly in r. Therefore, the functions $f_n(\lambda,r)\equiv f(\lambda;(r,V_n))$ converge uniformly on $[\Delta,(1-\Delta)^2]\times(r_-,r_+)$, and so do their unique zeros $\lambda_n(r)$. Arguing in a similar way, one can obtain that $\mu_n(r)$ converge uniformly on (r_-,r_+).

To summarize, we have argued that the maps $r \mapsto (U_{n+1}, V_{n+1}, \lambda_n, \mu_n)$ are continuous, while the iterates $(U_{n+1}, V_{n+1}, \lambda_n, \mu_n)$ converge uniformly in r. This implies that the map

$$r \mapsto (U^*_r, V^*_r, \lambda(V^*_r, r), \mu(U^*_r, r))$$

is continuous on (r_-, r_+). Since $\mu(U^*_{r_-}, r_-) > \lambda(V^*_{r_-}, r_-)$ and $\mu(U^*_{r_+}, r_+) < \lambda(V^*_{r_+}, r_+)$, the continuous functions $\lambda^*(V^*_r, r)$ and $\mu^*(U^*_r, r)$ must assume the same value at some point r'. We have for $r = r'$:

$$\lambda^\rho(V^*_{r'}, r')\ U^*_{r'} = U^*_{r'} \circ \Psi_{V^*_{r'}, \lambda(V^*_{r'}, r'), r'},$$
$$\lambda^\rho(U^*_{r'}, r')\ V^*_{r'} = V^*_{r'} \circ \Phi_{U^*_{r'}, \lambda(U^*_{r'}, r'), r'},$$

on $\mathbb{C}_{J_U} \times \mathbb{C}_{J_V}$. This implies that the following Lorenz map is a renormalization fixed point of type $(\{0,1\}, \{1,0,0\})$:

$$(l \circ p_\rho, t \circ p_\rho),$$

where

$$l(z) = r' + (U^*_{r'})^{-1}(z/a), \quad t(z) = (V^*_{r'})^{-1}(z/b) - 1,$$

are analytic diffeomorphisms on

$$a \circ U^*_{r'} \circ \zeta_{r'}(\mathbb{C}_{J_U}), \quad a(z) \equiv az, \quad \zeta_r(z) = r - z,$$

and

$$b \circ V^*_{r'} \circ \xi(\mathbb{C}_{J_V}), \quad b(z) \equiv bz, \quad \xi(z) = z + 1,$$

respectively, and a and b are as in (8). □

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