ON THE KOSZUL PROPERTY OF TORIC FACE RINGS

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ABSTRACT. Toric face rings is a generalization of the concepts of affine semigroup rings and Stanley-Reisner rings. We characterize toric face rings having the Koszul, strongly Koszul or initially Koszul property. Firstly, we compute the graded Betti numbers of the underlying field as a module over the toric face ring. We ask whether given two conditions, that the defining ideal of the toric face ring has the monomial part generated in degree 2, and that for each cone of the supporting fan, the corresponding monoid ring is Koszul, we can conclude the ring itself is Koszul. Then we give a full characterization of strongly Koszul toric face rings. We also prove that initially Koszul homogeneous toric face rings are in fact affine semigroup rings.

1. INTRODUCTION

Let k be a fixed field, R a homogeneous affine k-algebra. We say that R is a Koszul k-algebra if as an R-module, k has linear resolution. In this note, we consider various properties of toric face rings which imply the Koszul property.

Let Σ be a *rational pointed fan* in $\mathbb{R}^d$ ($d \geq 1$ a natural number), i.e. Σ is a collection of rational pointed cones in $\mathbb{R}^d$ which is closed under taking faces of cones, and the intersection of any two cones of which is a common face of these two cones. A *monoidal complex* $\mathscr{M}$ supported on Σ is a collection of affine monoids M_C indexed by elements C of Σ, such that M_C generates C and the following compatibility condition is fullfilled: if $D \subseteq C \in \Sigma$, then $M_D = M_C \cap D$. Starting with the work of Stanley, among other authors, Bruns, Ichim, Koch and Römer in [IR] and [BKR] considered toric face rings of $\mathscr{M}$ over k, denoted by $k[\mathscr{M}]$, which are a generalization of affine semigroup rings and Stanley-Reisner rings. In some sense, toric face rings are determined by the "local" data encoded by various monoids M_C where $C \in \Sigma$ and the "global" data encoded by the fan Σ. (For instances see Proposition 2.3 below). For an algebraic treatment of affine semigroup rings and Stanley-Reisner rings, see the book of Bruns and Herzog [BH] or Bruns and Gubeladze [BG], for a more combinatorial treatment of Stanley-Reisner rings see the book of Stanley [Sta].

Among other results, the above authors get the generalization of Hochster's formula for Betti numbers of Stanley-Reisner rings and a general theorem computing the initial ideal of the defining ideal of a toric face ring using triangulation - extending a theorem of Sturmfels [Stu], formulas for local cohomology of toric face rings, and other things.

Assume that R is a Koszul k-algebra. Considering the first sygyzy, it is easy to see that R is necessarily defined by quadratic relations over some polynomial ring. For Stanley-Reisner rings, Fröberg [Fr] proved that this is also sufficient for R to be Koszul by using

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Taylor's resolution. Later, the notion of Koszul filtrations was defined by Conca, Trung and Valla [CTV].

Definition 1.1 (Conca, Trung, Valla)**.** A family $\mathscr{F}$ of ideals of R is said to be a *Koszul filtration* of R if:

(i) every ideal of $\mathscr{F}$ is generated by linear forms;
(ii) the ideal 0 and the graded maximal ideal belong to $\mathscr{F}$;
(iii) for every $I \in \mathscr{F}$ different from 0, there exists $J \in \mathscr{F}$ strictly contained in I and a linear form $x \in I$ such that $J + (x) = I$ and $J : I \in \mathscr{F}$.

Note that a Koszul filtration, if it exists, does not contain the unit ideal. Conca, Trung and Valla proved that a ring which has a Koszul filtration must be Koszul. This gives another proof to the above result of Fröberg, because in this case, the family of those ideals, each of which is generated by some variables, form a Koszul filtration. After that, several related notions of Koszulness were introduced. Herzog, Hibi and Restuccia [HHR] defined strongly Koszul algebras. A homogeneous k-algebra is *strongly Koszul* if its irrelevant ideal admits a system of generators of degree 1, namely $a_1, \dots, a_n$, such that for all increasing sequence $1 \le i_1 < \dots < i_j \le n$, the ideal $(a_{i_1}, \dots, a_{i_{j-1}}) : a_{i_j}$ is generated by a subset of $\{a_1, \dots, a_n\}$. (This is a different but equivalent version of the strongly Koszul property of Herzog, Hibi and Restuccia.)

Herzog, Hibi and Restuccia also proved that strongly Koszul algebras are Koszul. On the other hand, Blum [Bl] and Conca, Rossi, Valla [CRV] introduced initially Koszul algebras. R is *initially Koszul* (abbreviated i-Koszul) with respect to a sequence $a_1, \dots, a_n \in R_1$, if the family of ideals $\mathscr{F} = \{(a_1, \dots, a_i) : i = 0, \dots, n\}$ is a Koszul filtration of R. Algebras which are i-Koszul must also be Koszul, and in fact they even have the stronger property of having a quadratic Gröbner basis with respect to a natural order, see [CRV, Theorem 2.4], [Bl, Theorem 2.1].

Restricted to the class of toric face rings, it is natural to ask: what conditions must be satisfied by a monoidal complex so that its associated toric face ring is a homogeneous Koszul, strongly Koszul or i-Koszul algebra? To illustrate, it turns out that in the case of Stanley-Reisner rings, the answers are quite simple. Independent of the field, the Stanley-Reisner ring is Koszul if and only if the simplical complex is a flag complex, i.e. all the minimal non-faces have two vertices (this result is due to Fröberg [Fr]). Moreover, in this case, the ring is strongly Koszul, see [HHR, Corollary 2.2]. A Stanley-Reisner ring is i-Koszul in the natural sense only if the simplicial complex is a full simplex, as follows from [Bl, Proposition 2.3]. But in the situation of monoidal complexes $\mathscr{M}$ which give rise to Stanley-Reisner rings, for all cones C of the supporting fan Σ, the corresponding monoid M_C is some $\mathbb{N}^t$, which makes everything easier to handle. It would be more interesting to see what happens in the general case, where no such special hypothesis is made. For results and problems about Koszul property of algebras associated with polytopes, see for example [BGT] and [BG, Chapter 7].

This note is organized as follows. In Section 2, we recall the basic theory of toric face rings. We propose a natural condition on the generators of the underlying monoidal complexes under which the corresponding toric face rings are standard graded. We say that the toric face ring is homogeneous if this condition is satisfied. In Section 3, we recall the natural grading associated with a monoidal complex and a system of generators.

We apply a method of Peeva, Reiner and Sturmfels [PRS], and Herzog, Reiner, Welker [HRW] to compute the graded Betti numbers of the residue field over the toric face ring. In Section 4, we prove that if a homogeneous toric face ring is Koszul, then for each cone of the supporting fan, the corresponding affine semigroup ring is also Koszul. We give a counterexample showing that the converse is not true, even when the defining ideal is generated in degree 2. We propose the question whether the converse would be true if the "monomial part" of the defining ideal is generated in degree 2. This is one of the main problems which motivate the content of this note. In Section 5, we characterize strong Koszulness of toric face rings. To achieve this goal, we provide a formula computing the colon ideals appearing in the definition of strong Koszulness via the various "local" colon ideals. Finally, in Section 6, we prove that homogeneous i-Koszul toric face rings are indeed affine semigroup rings. In particular, homogeneous toric face rings which are universally initially Koszul in the sense of Blum [Bl] must be polynomial rings.

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2. NOTATIONS AND BACKGROUND ON TORIC FACE RINGS

Let k be a field, $d \geq 1$ a natural number, Σ be a rational pointed fan in $\mathbb{R}^d$. In other words, Σ is a collection of rational pointed cones such that:

(i) if $C \in \Sigma$ and D is a face of C then $D \in \Sigma$;
(ii) for every $C, C' \in \Sigma$, $C \cap C'$ is a face of both C and C'.

Σ is called *simplicial* if each of its cones C is generated by linearly independent vectors in $\mathbb{R}^d$. A maximal element (with respect to inclusion) of Σ is called a *facet* of Σ. A one dimensional face of a cone of Σ is called an *extremal ray*.

A monoidal complex $\mathscr{M}$ supported on Σ is a collection of affine monoids M_C, where C varies in Σ such that:

(i) $M_C \subseteq C \cap \mathbb{Z}^d$ and $\mathbb{R}_{\geq 0} M_C = C$;
(ii) for every $C, D \in \Sigma$ with $D \subseteq C$, $M_D = M_C \cap D$.

For instances, taking $M_C = C \cap \mathbb{Z}^d$ for each C we get a monoidal complex supported on the fan Σ.

The toric face ring over k associated with the monoidal complex $\mathscr{M}$ supported on Σ, denoted by $k[\mathscr{M}]$ is defined as follows. As a k-vector space we set

$$k[\mathscr{M}] = \bigoplus_{a \in \cup_{C \in \Sigma} M_C} k t^a.$$

The product on basis elements is given as follows:

$$t^a \cdot t^b = \begin{cases} t^{a+b} & \text{if for some } C \in \Sigma \text{ both } a \text{ and } b \text{ belongs to } M_C; \\ 0 & \text{otherwise.} \end{cases}$$

Sometimes we write a instead of the basis element t^a of $k[\mathscr{M}]$. In that case, instead of $t^a \cdot t^b$, we write $a \cdot b$, hence in $k[\mathscr{M}]$, without too much confuse arised:

$$a\cdot b=\begin{cases} a+b & \text{if for some } C\in\Sigma \text{ both } a \text{ and } b \text{ belongs to } M_C;\\ 0 & \text{otherwise.}\end{cases}$$

It is known that $R=k[\mathscr{M}]$ is an affine reduced commutative k-algebra with unit, which corresponds to t^0. The Krull dimension of $k[\mathscr{M}]$ equals to the maximal dimension of the facets of Σ, see for example [IR] or [BKR]. An important aspect is that $k[\mathscr{M}]$ inherits the $\mathbb{Z}^d$-grading from the embedding of the monoidal complex. Every $\mathbb{Z}^d$-graded component of $k[\mathscr{M}]$ has k-dimension less than or equal to 1.

The two basic examples of toric face rings are Stanley-Reisner rings and affine semigroup rings.

Example 2.1. Let Δ be a simplicial complex on the vertex set $[n]=\{1,\dots,n\}$. Denote by $e_1,\dots,e_n$ the standard basis vectors of R^n. For each face F of Δ, consider the cone C_F generated by the vectors $e_i, i\in F$. It is clear that the collection $\Sigma=\{C_F, F\in\Delta\}$ is a rational pointed fan in $\mathbb{R}^n$. For each $F\in\Delta$, choose $M_{C_F}=C_F\cap\mathbb{Z}^n$, then we get a monoidal complex supported on Σ, the toric face ring of which is exactly the Stanley-Reisner ring $k[\Delta]$. By definition, this is the quotient of $k[X_1,\dots,X_n]$ by the square-free monomial ideal I_Δ generated by monomials $\prod_{j\in G}X_j$ where $G\subseteq[n], G\notin\Delta$.

Example 2.2. Let M be an finitely generated submonoid of $\mathbb{N}^d$ ($d\geq 1$). Choosing Σ to have only one facet $C=\mathbb{R}_{\geq 0}M$, and $M_C=M$, the resulting toric face ring is isomorphic to the affine semigroup ring $k[M]$.

For each $C\in\Sigma$, let $R_C=k[M_C]$, which is naturally a subring of R. We have natural surjections $R\to R_C$ defined by:

$$t^a\longmapsto\begin{cases} t^a & \text{if } a \text{ belongs to } M_C;\\ 0 & \text{otherwise.}\end{cases}$$

The homomorphism $R_C\to R$ follows by $R\to R_C$ is the identity on R_C, in other words R_C is an *algebra retract* of R for every $C\in\Sigma$.

Following [BKR], we say that the finite set $\{a_1,\dots,a_n\}$ is a system of generators of $\mathscr{M}$ if $a_i\in\cup_{C\in\Sigma}M_C$ for every $i\in[n]$, and the subset $\{a_1,\dots,a_n\}\cap M_C$ is a system of generator of M_C for every $C\in\Sigma$. This system of generators gives a surjection $\varphi: S=k[X_1,\dots,X_n]\to R$. Let $I=\operatorname{Ker}\varphi$. For each cone C of Σ, denote $S_C=k[X_i: a_i\in M_C]$, we also have a map $\varphi_C: S_C\to k[M_C]$, whose kernel is denoted by I_C.

Denote by $\Delta_{\mathscr{M}}$ the following simplicial complex on the set $[n]$: a subset $F\subseteq[n]$ is a face of $\Delta_{\mathscr{M}}$ if and only if there exists some cone $C\in\Sigma$ such that $\{a_j, j\in F\}\subseteq M_C$. The following proposition (which is [BKR, Proposition 2.3]) shows the dependence of I on $\Delta_{\mathscr{M}}$ and the "local" data about defining ideals.

Proposition 2.3. *Denote by $C_1,\dots,C_r$ all the facets of Σ. Then*

$$I=A_{\mathscr{M}}+\sum_{i=1}^{r}S\cdot I_{C_i},$$

where $A_{\mathscr{M}}$ is generated by square-free monomials $\prod_{j\in G}X_j$, for which $G\notin\Delta_{\mathscr{M}}$.

Loosely speaking, I consists of the "monomial part" $A_{\mathscr{M}}$ and the "binomial part" consisting of various "local" binomial ideals.

Our additional assumption in dealing with various Koszulness notions is that $k[\mathscr{M}]$ possesses a standard $\mathbb{Z}$-grading.

Definition 2.4. The finite set $\{a_1,\dots,a_n\}$ is called a *standard system of generators* of $\mathscr{M}$ if for every facet C of Σ, the ring $k[M_C]$ is a homogeneous k-algebra which is generated in degree 1 by $\{a_1,\dots,a_n\}\cap M_C$. We call $k[\mathscr{M}]$ a *homogeneous toric face ring* if $\mathscr{M}$ has a standard system of generators.

Given a standard system of generators of $\mathscr{M}$, the $\mathbb{Z}$-gradings on the rings $k[M_C]$ where $C\in\Sigma$ induce a $\mathbb{Z}$-grading on $k[\mathscr{M}]$. Note that we do not require $\{a_1,\dots,a_n\}\cap M_C$ to minimally generates the graded maximal ideal of $k[M_C]$. The reason is in dealing with strongly Koszul and initially Koszul properties, the minimality will follow automatically. We also do not require the $\mathbb{Z}$-grading to be compatible with the existing $\mathbb{Z}^d$-grading, as we can use the two gradings separately.

3. Betti numbers of toric face rings

A homogeneous k-algebra R is Koszul if and only if $\beta^R_{i,j}(k)=0$ for all $i\neq j$, where $\beta^R_{i,j}(k)=\dim_k \operatorname{Tor}^R_i(k,k)_j$ are the Betti numbers of k as a graded R-module. Laudal and Sletsjøe [LS] computed Betti numbers of affine semigroup rings, and the result was later reproved and generalized in [PRS] and [HRW]. In this section, using the method employed in [PRS] and [HRW], we derive a formula for the graded Betti numbers of the ground field k as a module over the toric face ring. We will use the natural grading associated with a monoidal complex as defined in [BKR].

In details, let Σ be a rational pointed fan in $\mathbb{R}^d$ and $\mathscr{M}$ be a monoidal complex supported on Σ with $\{a_1,\dots,a_n\}$ being a system of generators. Use again the notation of Proposition 2.3. Denote by $B_{\mathscr{M}}$ the ideal of S generated by all the binomials in I. Consider the following monoid H associated to $\mathscr{M}$ and the system of generators. Let $\sim$ denote the relation in $\mathbb{N}^n$ such that $a\sim b$ if and only if $X^a-X^b\in B_{\mathscr{M}}$. This relation is compatible with vector sum in the sense that $a\sim b$ implies $a+c\sim b+c$ for all $a,b,c\in\mathbb{N}^n$. Let H be the set of equivalent classes of $\mathbb{N}^n/\sim$ with the addition inherited from that of $\mathbb{N}^n$.

We say that a monoid is *positive* if for elements λ,μ of this monoid with $\lambda+\mu=0$, we must have $\lambda=\mu=0$. We say that a monoid *cancellative with respect to* 0 if an equation $\lambda+\mu=\lambda$ in the monoid implies that $\mu=0$. It is not hard to see that H is a commutative positive monoid. Moreover, we have the following result which is Lemma 4.4 in [BKR].

Lemma 3.1. *Denote the class of $a\in\mathbb{N}^n$ in H by $\overline{a}$. We have:*

(i) *If $\overline{a}+\overline{c}=\overline{b}+\overline{c}$ for $a,b,c\in\mathbb{N}^n$ then $X^a-X^b\in I$.*
(ii) *H is cancellative with respect to* 0.
(iii) *If $X^a-X^b\in I$ and $X^a,X^b\notin I$ then $\overline{a}=\overline{b}$ in H.*

It is easy to see that $S=k[X_1,\dots,X_n]$ and $R=k[\mathscr{M}]$ are H-graded. Note also that $S/B_{\mathscr{M}}$ is exactly the monoid algebra $k[H]$ of H. In the case of Example 2.2, the monoidal complex $\mathscr{M}$ is a positive affine semigroup M, we can choose $\{a_1,\dots,a_n\}$ to be the minimal set of generators of M. Here $H=M$ and $B_{\mathscr{M}}$ is the toric ideal defining $k[M]$. In other words, the H-grading is the semigroup grading induced by M. In the case of Example 2.1, we have $B_{\mathscr{M}} = 0$ and $H = \mathbb{N}^n$, the H-grading is simply the fine grading.

Denote by J the ideal $I/B_{\mathscr{M}}$ of $k[H]$. Denote by $e_g, g = 1,\dots,n$ the standard basis vectors in $\mathbb{R}^n$. Then J is a *semigroup ideal* of $k[H]$ in the sense that it is generated by elements $\overline{\sum_{g\in G} e_g}$ in H, where $G \subseteq [n]$ such that there is no face of Σ containing all the $a_g, g \in G$. We know from Proposition 2.3 that $k[\mathscr{M}] = k[H]/J$.

For elements $\lambda, \mu \in H$, we say that $\lambda < \mu$ if $\lambda \neq \mu$ and $\mu - \lambda \in H$. Then $<$ makes H into a partially ordered set. For each $\lambda \in H$, denote by Δ_λ the set of all chains of elements $\alpha_1 < \ldots < \alpha_i \in H$ such that $0 = \alpha_0 < \alpha_1$ and $\alpha_i < \lambda = \alpha_{i+1}$.

We denote by $\Delta_{\lambda,J}$ the subset of Δ_λ consisting of chains $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_i < \lambda = \alpha_{i+1}$ in Δ_λ such that for some $0 \le j \le i$, the element $X^{\alpha_{j+1}-\alpha_j}$, as element of the group ring $k[H]$, belongs to J. Note that for each $\lambda \in H$, the sets $\Delta_\lambda, \Delta_{\lambda,J}$ are simplicial complexes. We denote by $\tilde{H}_j(\Delta_\lambda, \Delta_{\lambda,J}; k)$ the j-th reduced, relative simplicial homology with coefficient in k of the pair $(\Delta_\lambda, \Delta_{\lambda,J})$.

For each $i \ge 0, \lambda \in H$, let $\beta^R_{i,\lambda}(k)$ denotes the bi-graded Betti number $\dim_k \operatorname{Tor}^R_i(k,k)_\lambda$ of k as an H-graded R-module.

Note that if in addition, $\{a_1,\dots,a_n\}$ is a standard system of generators of $\mathscr{M}$ then there's a function $|.| : H \to \mathbb{Z}$ mapping $\lambda = \overline{a} \in H$ to $|\lambda|$ which is the sum of coordinates of a.

Proposition 3.2. *With the above notations,*

$$\beta^R_{i,\lambda}(k) = \dim_k \tilde{H}_{i-2}(\Delta_\lambda, \Delta_{\lambda,J}; k),$$

for every $\lambda \in H$ and $i > 0$.

In particular, if $\{a_1,\dots,a_n\}$ is a standard system of generators for $\mathscr{M}$ then the followings are equivalent:

(i) *$k[\mathscr{M}]$ is a Koszul algebra;*
(ii) *$\tilde{H}_{i-2}(\Delta_\lambda, \Delta_{\lambda,J}; k) = 0$ for all $i > 0$ and $\lambda \in H$ such that $|\lambda| > i$.*

Proof. Apply the same argument using the bar resolution as in the proof of Theorem 2.1 in [HRW]. □

4. Koszul toric face rings

Let Σ be a rational pointed fan in $\mathbb{R}^d$ and $\mathscr{M}$ be a monoidal complex supported on Σ. Let $\{a_1,\dots,a_n\}$ be a standard system of generators of $\mathscr{M}$. We use again the notations of Proposition 2.3. Of course $k[\mathscr{M}]$ is a standard graded k-algebra. We want to have a characterization of Σ and $\mathscr{M}$ when $R = k[\mathscr{M}]$ a Koszul algebra. Naturally, we would try to relate the Koszul property of R with the Koszul property of the rings R_C with C in Σ.

Recall that given an inclusion of graded rings $R \subset S$, R is called an algebra retract of S if there's a homogeneous morphism $\varepsilon : S \to R$ (the retraction map) such that ε restricts to the identity on R.

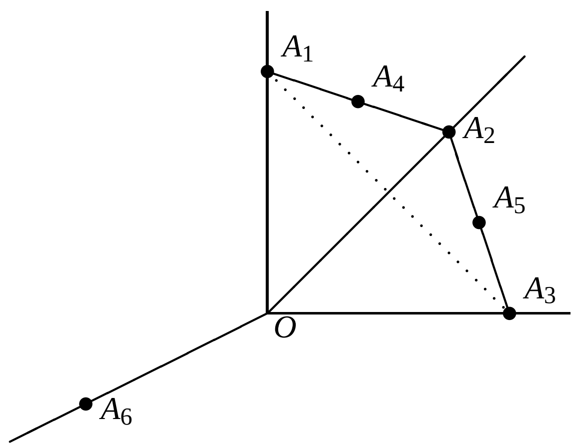
Proposition 4.1. *If $k[\mathscr{M}]$ is Koszul then for any $C \in \Sigma$, the ring $k[M_C]$ is also Koszul.*

Proof. This follows from a general fact: If $R \subset S$ is an algebra retract of homogeneous k-algebras with the retraction map $\varepsilon : S \to R$ then S is Koszul if and only if R is Koszul and R has linear resolution as an S-module via ε. See [OHH, Proposition 1.4]. □

Remark 4.2. It is easy to see that if R is Koszul then $I = \operatorname{Ker}\varphi$ is generated by quadrics. However in general, even when the ring $k[M_C]$ is a Koszul algebra for each facet $C \in \Sigma$ and the defining ideal I is generated by quadratic polynomials, $k[\mathscr{M}]$ is not necessarily Koszul, as the next example shows.

Example 4.3. Take $k = \mathbb{Q}$. Consider the points in $\mathbb{R}^3$ with the following coordinates $A_1 = (2,0,0), A_2 = (0,2,0), A_3 = (0,0,2), A_4 = (1,1,0), A_5 = (0,1,1)$. The semigroup ring generated by those 5 points is $k[X_1,\dots,X_5]/I_1$ where $I_1 = (X_1X_2 - X_4^2, X_2X_3 - X_5^2)$. Let $O = (0,0,0)$ be the origin of $\mathbb{R}^3$.

Take the point $A_6 = (-1,-1,-1)$. Consider the rational pointed fan in $\mathbb{R}^3$ with the following facets, which are all simplicial cones: $OA_1A_2A_3$, $OA_1A_3A_6, OA_2A_6$.



The toric face ring of this monoidal complex is $R = k[X_1,\dots,X_6]/I$ with the defining ideal $I = I_1 + (X_4X_6, X_5X_6)$. For example $X_1X_2X_6 \in I$ because $X_1X_2X_6 = X_6 \cdot (X_1X_2 - X_4^2) + X_4 \cdot X_4X_6$. So the defining ideal is generated by quadratic polynomials.

The affine semigroup rings supported on the maximal cones, which are $k[X_1,\dots,X_5]/I_1$, $k[X_1,X_3,X_6]$ and $k[X_2,X_6]$, are Koszul. In fact, in the reverse lexicographic order with $X_1 < X_2 < \ldots < X_5$, the polynomials $\{X_1X_2 - X_4^2, X_2X_3 - X_5^2\}$ form a quadratic Gröbner basis for I_1. So $k[X_1,\dots,X_5]/I_1$ is Koszul.

However, we can check by Macaulay 2 that the upper-left part of the Betti table of the maximal graded ideal $\mathfrak{m}$ of R considered as R-module is as follow:

	0	1	2	3	4	5	6	7
total:	1	6	19	46	101	217	468	1016
0:	1	6	19	45	92	173	309	534
1:	.	.	.	1	9	44	158	470
2:	.	.	.	.	.	.	1	12

Hence R is not a Koszul algebra, because $\mathfrak{m}$ has a non-linear second syzygy.

We are interested in the following question.

Question 4.4. *Let Σ be a rational pointed fan in $\mathbb{R}^d$, $\mathscr{M}$ is a monoidal complex supported on Σ. Let $\{a_1,\dots,a_n\}$ be a standard system of generators of $\mathscr{M}$. Assume that the ideal $A_{\mathscr{M}}$ in Proposition 2.3 is generated by quadrics, and for every facet C of Σ, the ring $k[M_C]$ is a Koszul algebra. Is it true that $k[\mathscr{M}]$ is a Koszul algebra?*

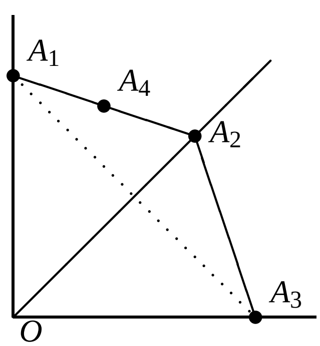
We will provide in the following some evidences to expect an affirmative answer to this question, see the subsequent Corollary 4.8, Theorem 5.4, Remark 5.5 and Theorem 6.4. Note also that in the situation of Example 2.1, this question is answered in positive by Fröberg's theorem, which can be obtained from the implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) of Corollary 5.7. In the case of Example 2.2 where Σ is a cone, the question is *a priori* answered in positive.

Remark 4.5. On the other hand, it is not true that the ideal $A_{\mathscr{M}}$ is generated by quadratic monomials given R being Koszul, as in the following example.

Example 4.6. Take $k=\mathbb{Q}$ and consider the points in $\mathbb{R}^3$ with the following coordinates $O=(0,0,0), A_1=(2,0,0), A_2=(0,2,0), A_3=(0,0,2), A_4=(1,1,0)$.

Consider the rational pointed fan in $\mathbb{R}^3$ with the following facets $OA_1A_2, OA_1A_3, OA_2A_3$. Denote by $\mathscr{M}$ the monoidal complex supported on this fan with the generators of the monoids corresponding to the above facets are $\{A_1,A_2,A_4\},\{A_1,A_3\},\{A_2,A_3\}$. The toric face ring of $\mathscr{M}$, which is $k[\mathscr{M}]=k[X_1,X_2,X_3,X_4]/(X_1X_2-X_4^2, X_3X_4)$, is Koszul. Indeed, in the lexicographic order induced by $X_1>X_2>X_3>X_4$, the set $\{X_1X_2-X_4^2, X_3X_4\}$ is a quadratic Gröbner basis for the defining ideal of this ring.

However, the ideal $A_{\mathscr{M}}=(X_3X_4, X_1X_2X_3)$ is not generated by quadrics.



Proposition 4.7. *Let $<$ be a monomial on $S=k[X_1,\dots,X_n]$ and $<_i$ be the induced monomial order on $S_{C_i}=k[X_j|a_j\in M_{C_i}]$, where $C_1,\dots,C_r$ are the facets of Σ. Then:*

(i) *If I has a quadratic Gröbner basis with respect to a monomial order $<$ on S then I_{C_i} has a quadratic Gröbner basis with respect to the monomial order $<_i$ on S_{C_i} for every $i=1,\dots,r$.*

(ii) *If $A_{\mathscr{M}}$ is generated by quadrics and for some monomial order $<$ on S, the ideal I_{C_i} has a quadratic Gröbner basis with respect to the monomial order $<_i$ on S_{C_i} for every $i=1,\dots,r$, then I has a quadratic Gröbner basis with respect to $<$.*

Proof. The two statements follow from the formula:

$$\mathrm{in}_<(I) = A_{\mathscr{M}} + \sum_{i=1}^{r} S \cdot \mathrm{in}_{<_i}(I_{C_i}).$$

The proof of this is similar to the proof of [BKR, Proposition 3.2]. □

Corollary 4.8. *Under the hypothesis of Proposition 4.7, assume in addition that $\mathscr{M}$ admits $\{a_1,\dots,a_n\}$ as a standard system of generators. Assume also that $A_{\mathscr{M}}$ is generated by quadrics and for every $i = 1,\dots,r$, the ideal I_{C_i} has a quadratic Gröbner basis with respect to the order $<_i$ on S_{C_i}. Then $k[\mathscr{M}]$ is a Koszul algebra.*

Note that Example 4.6 shows that the converse of Proposition 4.7 is not true, because in this case the ring $k[\mathscr{M}]$ has a quadratic Gröbner basis with respect to the lex order but $A_{\mathscr{M}}$ is not generated by quadrics.

5. Strongly Koszul toric face rings

Let us recall the notion of strongly Koszul algebras.

Definition 5.1 (Herzog, Hibi, Restuccia)**.** Let R be a homogeneous algebra over a field k, $\mathfrak{m}$ is its graded maximal ideal. Suppose that the elements $a_1,\dots,a_n$ belong to $\mathfrak{m}$, generate $\mathfrak{m}$, and are homogeneous of degree 1. Then R is called *strongly Koszul with respect to the sequence* $a_1,\dots,a_n$ if for every $1 \le i_1 < \ldots < i_j \le n$, the ideal $(a_{i_1},\dots,a_{i_{j-1}}) : a_{i_j}$ is generated by a subset of $\{a_1,\dots,a_n\}$.

Note that if R is strongly Koszul with respect to the sequence $a_1,\dots,a_n$ then $\{a_1,\dots,a_n\}$ is a minimal set of generators for $\mathfrak{m}$. Moreover, for every sequence $1 \le i_1 < \ldots < i_j \le n$ and all $l = 1,\dots,j$, the ideal $(a_{i_1},\dots,a_{i_{l-1}}) : a_{i_l}$ is generated by a subset of $\{a_1,\dots,a_n\}$. Hence this is an equivalent rephrasing of the strongly Koszul notion in [HHR].

The definition of the strongly Koszul property appears to be dependent on the order of the sequence of generators. At least for affine semigroup rings, this is not the case. We say that an affine semigroup M is *homogeneous* if M is a disjoint union

$$M = \cup_{j\ge 0} M_j$$

with $M_0 = \{0\}, M_j + M_l \subseteq M_{j+l}$ for all j,l, and M is generated by M_1. The elements of M_j are called elements of degree j. The following theorem is due to Herzog, Hibi, Restuccia [HHR, Proposition 1.4].

Theorem 5.2. *Let M be a homogeneous semigroup, and let $a_1,\dots,a_n$ be generators of degree 1 of M. Then the following are equivalent:*

(i) *$k[M]$ is strongly Koszul with respect to $a_1,\dots,a_n$;*
(ii) *the divisor poset of M is locally upper semimodular (also called wonderful);*
(iii) *the ideals $(a_i)\cap(a_j)$ are generated in degree 2 for all $i \ne j$.*

Next we consider Stanley-Reisner rings as defined in Example 2.1. It is not hard to see that if I_Δ is generated by quadrics then $k[\Delta]$ is strongly Koszul with respect to the sequence $X_1, X_2,\dots,X_n$. We will see from Corollary 5.7 that the converse is also true.

In the case of toric face rings, the following lemma describes the relation between the colon ideals appearing in the definition of strong Koszulness with the various "local" colon ideals.

Lemma 5.3. *Assume that $\{a_1,\dots,a_n\}$ is a standard system of generators of $\mathscr{M}$ and $1 \le i \le n$. Let $\mathfrak{I} = (a_1,\dots,a_{i-1}) :_R a_i$. Let $C_1,\dots,C_r$ be the facets of Σ. For each $C \in \Sigma$, denote by $\mathfrak{I}_C$ the following ideal of R_C :*

$$\mathfrak{I}_C = \begin{cases} (a_j | j < i, a_j \in M_C) :_{R_C} a_i & \text{if } a_i \in M_C; \\ 0 & \text{if } a_i \notin M_C. \end{cases}$$

Then:

(i) *For each $C \in \Sigma$, we have $(a_1,\dots,a_i) \cap R_C = (a_j | j \le i, a_j \in M_C)$.*
(ii) *For each $C \in \Sigma$ such that $a_i \in M_C$, we have $\mathfrak{I} \cap R_C = \mathfrak{I}_C$.*
(iii) *$\mathfrak{I} = (0 :_R a_i) + \sum_{l=1}^{r} R \cdot \mathfrak{I}_{C_l}$.*

Proof. This is a standard utilization of the $\mathbb{Z}^d$-grading. For (i), take $y \in (a_1,\dots,a_i) \cap R_C$ to be homogeneous with respect to the $\mathbb{Z}^d$-grading. If a_j divides y then because $y \in M_C$, we have $a_j \in M_C$. Thus (i) is proved.

For (ii), we see directly that $\mathfrak{I}_C \subseteq \mathfrak{I} \cap R_C$.

Take $z \in \mathfrak{I} \cap R_C$ which is $\mathbb{Z}^d$-graded. Since $za_i \in (a_1,\dots,a_{i-1})$, in view of the $\mathbb{Z}^d$-grading, we have $za_i = wa_j$ for some $j < i$. But $wa_j \in M_C$, so both w and a_j belong to M_C, hence using the projection from R to R_C, we get $z \in \mathfrak{I}_C$. Thus (ii) is proved.

For (iii), firstly the ideal on the right side of (iii) is contained in $\mathfrak{I}$. We prove the other inclusion. Take an element $z \in \mathfrak{I}$ which is $\mathbb{Z}^d$-graded. If $z \in 0 : a_i$, there's nothing to do. Otherwise, there is a facet C_l such that both $z, a_i \in M_{C_l}$. Hence $z \in \mathfrak{I} \cap R_{C_l} = \mathfrak{I}_{C_l}$ by (ii). ☐

The next theorem characterizes strongly Koszul toric face rings.

Theorem 5.4. *Let Σ be a rational pointed fan in $\mathbb{R}^d$, $\mathscr{M}$ is a monoidal complex supported on Σ. Let $\{a_1,\dots,a_n\}$ be a standard system of generators of $\mathscr{M}$. Then the followings are equivalent:*

(i) *$k[\mathscr{M}]$ is strongly Koszul with respect to the sequence $\{a_1,\dots,a_n\}$,*
(ii) (a) *for each $i = 1,\dots,n$, we have $0 :_{k[\mathscr{M}]} a_i = (a_{i_1},\dots,a_{i_j})$, for some elements $a_{i_1},\dots,a_{i_j}$ in $\{a_1,\dots,a_n\}$;*
  (b) *for each facet C of Σ, the ring $k[M_C]$ is strongly Koszul with respect to the sequence $\{a_1,\dots,a_n\} \cap M_C$.*

Proof. With Lemma 5.3.(iii), we see that part (ii) of the theorem implies part (i).

Assuming that we have (i). Then part (a) of (ii) is clear.

Consider a facet C of Σ, and a subsequence of $\{a_1,\dots,a_n\} \cap M_C$. Without loss of generality we can assume this subsequence to be $a_1,\dots,a_i$. We have $(a_1,\dots,a_{i-1}) :_{k[\mathscr{M}]} a_i = (a_{i_1},\dots,a_{i_k})$ by (i).

Using Lemma 5.3.(i), the ideal $(a_1,\dots,a_{i-1}) :_{k[M_C]} a_i = (a_{i_l} | a_{i_l} \in M_C)$. This concludes the proof of part (b). ☐

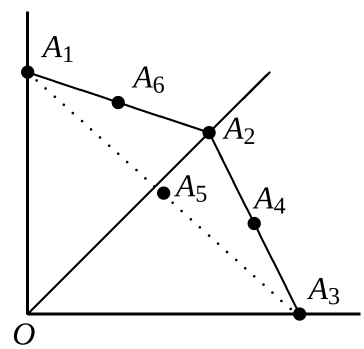
Remark 5.5. Part (a) of (ii) in the above theorem is true if we require the ideal $A_{\mathscr{M}}$ in Proposition 2.3, the "monomial part" of I, to be generated by quadrics. However, the converse is not true as the next example demonstrates.

Example 5.6. In $\mathbb{R}^3$ take six points with the following coordinates:

$A_1=(2,0,0), A_2=(0,2,0), A_3=(0,0,2), A_4=(0,1,1), A_5=(1,0,1), A_6=(1,1,0)$.

Consider the rational pointed fan in $\mathbb{R}^3$ with three maximal cones C_1, C_2, C_3, where C_1 is generated by the points A_1, A_2, A_6, the cone C_2 is generated by A_3, A_1, A_5 and the cone C_3 is generated by A_2, A_3, A_4 with the semigroup relations $A_1A_2 - A_6^2 = A_2A_3 - A_4^2 = A_3A_1 - A_5^2 = 0$.

Take M_1 to be generated by A_1, A_2, A_6 and similarly we have two other maximal semigroups M_2 and M_3 of a monoidal complex.



Denote the residue class of X_i in the quotient rings of $k[X_1,\dots,X_6]$ simply by x_i. Then $k[\mathscr{M}]$ is homogeneous strongly Koszul with respect to the sequence x_1,x_2,x_3,x_4,x_5,x_6 because of Theorem 5.4. Indeed, part (b) of (ii) is true, because for example, $k[M_1]$ is strongly Koszul with respect to the sequence x_1,x_2,x_6.

Part (a) if (ii) is true because of the following two identities:

(i) $0 :_{k[\mathscr{M}]} x_1 = (x_4)$

(ii) $0 :_{k[\mathscr{M}]} x_4 = (x_1,x_5,x_6)$

and four other similar identities.

Note however that the monomial $X_1X_2X_3$ belongs to the defining ideal of $k[\mathscr{M}]$ but X_1X_2, X_2X_3 and X_3X_1 do not. This example shows that the "monomial part" of the defining ideal of a strongly Koszul toric face ring need not to be generated by quadrics.

Corollary 5.7. *Let Δ be a simplicial complex on $[n]$. The following are equivalent:*

(i) $k[\Delta]$ *is Koszul;*

(ii) I_Δ *is generated by quadrics, or equivalently every minimal non-face of* Δ *has two elements;*

(iii) $k[\Delta]$ *is strongly Koszul.*

Proof. The implication (i) $\Rightarrow$ (ii) is due to Fröberg [Fr] and (iii) $\Rightarrow$ (i) follows from [HHR, Theorem 1.2]. We only need to prove that (ii) implies (iii).

In the case of Stanley-Reisner rings, all the rings $k[M_C]$ are polynomial rings, so they satisfy condition (ii)(b) of Theorem 5.4. We check condition (ii)(a), we can assume that $i = 1$.

Note that $0 :_{k[\Delta]} X_1$ is generated by classes of monomials of $S = k[X_1,\dots,X_n]$. Assume that q is a monomial of S such that $X_1 q \in I_\Delta$ and $q \notin I_\Delta$. Because I_Δ is quadrics, there is a variable X_j dividing q such that $X_1 X_j \in I_\Delta$. This concludes the proof of the corollary. □

6. Initially Koszul toric face rings

In the followings, we consider initially and universally initially Koszul algebras in the sense of Blum [Bl, Definition 1.3] and Conca, Rossi, Valla [CRV, Definition 2.2].

Definition 6.1. Let R be a homogeneous k-algebra, and $a_1,\dots,a_n \in R_1$. The ring R is called *initially Koszul* (or *i-Koszul*) *with respect to the sequence* $a_1,\dots,a_n$ if the set

$$\mathscr{F} = \{(a_1,\dots,a_i) : i = 0,\dots,n\}$$

is a Koszul filtration for R in the sense of Definition 1.1.

We say that R is *universally initially Koszul* (or *u-i-Koszul*) if R is i-Koszul with respect to any k-basis of R_1.

In this section, whenever we have a homogeneous quotient ring $R = k[X_1,\dots,X_n]/I$, it is convenient to use the convention that i-Koszulness means i-Koszulness with respect to the sequence $X_1,\dots,X_n$. Blum [Bl, Theorem 2.1] proved the following.

Theorem 6.2. *Let $>$ be the reverse lexicographic order induced by $X_n > \ldots > X_2 > X_1$. The following statements are equivalent:*

(i) *$R = k[X_1,\dots,X_n]/I$ is i-Koszul;*
(ii) *$R' = k[X_1,\dots,X_n]/\operatorname{in}_>(I)$ is i-Koszul;*
(iii) *I has a quadratic Gröbner basis with respect to $>$ and if $X_iX_j \in \operatorname{in}_>(I)$ for some $i < j$ then $X_iX_l \in \operatorname{in}_>(I)$ for all $i \le l < j$.*

In particular, i-Koszulness implies Koszulness, because the possession of a quadratic Gröbner already does. The last property in the last theorem helps to characterize i-Koszul quotients of polynomial rings by monomial ideals.

Corollary 6.3. *Let $R = k[X_1,\dots,X_n]/I$ where I is a monomial ideal. The following statements are equivalent:*

(i) *R is i-Koszul;*
(ii) *I is generated by quadrics and if $X_iX_j \in I$ for some $i < j$ then $X_iX_l \in I$ for all $i \le l < j$.*

This result of Blum [Bl, Proposition 2.3] follows easily from the above theorem. From the last corollary, we see that a Stanley-Reisner ring $k[\Delta]$ of Example 2.1 is i-Koszul if and only if $I_\Delta = 0$, in other words Δ is the full simplex.

We now prove that homogeneous i-Koszul toric face rings must be isomorphic to i-Koszul affine semigroup rings.

Theorem 6.4. *Let Σ be a rational pointed fan in $\mathbb{R}^d$ and $\mathscr{M}$ be a monoidal complex supported on Σ. Assuming that $\{a_1,\dots,a_n\}$ is a standard system of generators of $\mathscr{M}$. If $k[\mathscr{M}]$ is i-Koszul with respect to the sequence $a_1,\dots,a_n$, then Σ is a cone, and hence $k[\mathscr{M}]$ is an affine semigroup ring.*

In the proof of the theorem, we need the following lemma.

Lemma 6.5. *Under the assumptions of Theorem 6.4, if a_i generates an extremal ray of Σ then $(a_1,\dots,a_{i-1}):a_i=(a_1,\dots,a_{i-1})$.*

Proof. Note that from the i-Koszulness assumption $(a_1,\dots,a_{i-1}):a_i=(a_1,\dots,a_j)$. If $j\geq i$, then $a_i\in(a_1,\dots,a_{i-1}):a_i$. Thus $a_i^2=ba_l$ for some $l<i, b\in\cup_{C\in\Sigma}M_C$. As $a_l\neq a_i$, consider the projection to the extremal ray spanned by a_i, we have $a_i^2=0$. This is a contradiction. □

Proof. (of Theorem 6.4)

We only need to prove that all the extremal rays of Σ belong to some face of Σ. Assume that this is not the case. If $a_{i_1},\dots,a_{i_t}$ generate the extremal rays of Σ with $1\leq i_1<\dots<i_t\leq n$ then $a_{i_1}a_{i_2}\cdots a_{i_t}=0$ because there's no face of Σ containing all of them.

Of course $a_{i_2}\cdots a_{i_t}\in(a_1,\dots,a_{i_1-1}):a_{i_1}$. From Lemma 6.5, we have that $a_{i_2}\cdots a_{i_t}\in(a_1,\dots,a_{i_1-1})$. This in turn implies $a_{i_3}\cdots a_{i_t}\in(a_1,\dots,a_{i_2-1}):a_{i_2}$. So again from Lemma 6.5, we get $a_{i_3}\cdots a_{i_t}\in(a_1,\dots,a_{i_2-1})$. Iterate this argument, in the end we have $a_{i_t}\in(a_1,\dots,a_{i_{t-1}-1})$. This is a contradiction and hence we are done. □

The following result is a consequence of Theorem 6.4.

Corollary 6.6. *If $k[\mathscr{M}]$ is a homogeneous u-i-Koszul toric face ring then $k[\mathscr{M}]$ is a polynomial ring.*

Proof. From Theorem 6.4, $k[\mathscr{M}]$ is an affine semigroup ring. The Corollary follows from [Bl, Proposition 5.5], which says that affine semigroup rings which are u-i-Koszul must be polynomial rings. □

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