THERE ARE INFINITELY MANY PRIMES OF THE FORM a^2+1

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Abstract. In this paper we founded a formal system of second order arithmetic $\langle P(N), +, \times, 0, 1, \in\rangle$ by extending the operations $+, \times$ on natural numbers to the operations on finite sets of natural numbers. We design a new algorithm on the congruence classes to obtain a recursive formula of the set sequence T_i' which approaches the set of all numbers a making a^2+1 primes. Considering that the number of elements $|T_i'|$ of the set sequence T_i' is strictly increasing and the cardinal function $|T_i'|$ is continuous with respect to the order topology of T_i', we proved that there are infinite many primes of the form a^2+1. Finally, we extend this result to attack the problem of prime infinity in general polynomials.

1. Introduction

It is an simple and longstanding problem, whether a polynomial function of natural number a assumes an infinitely many primes [1][2][3].

In 1837, G.L.Dirichlet had proved that if a and b are relatively prime positive integers, then the arithmetic progression $an+b$ contains infinitely many primes[4]. This is the only non-trivial solution known.

Like twin prime conjecture, it has never been proved that a simple quadratic polynomial a^2+1 will represent infinitely many primes. At the 1912 International Congress of Mathematicians in Cambridge, Edmund Landau listed four basic problems about primes, this is fourth problem, and said they are "unattackable at the present state of science".

In 1973, Chinese mathematician J.R.Chen had proved that for infinitely many primes p the number $p+2$ is either prime or a product of two primes[5].

In 1978, H.Iwaniec had proved similar result that there are infinitely many natural numbers a such that a^2+1 is the product of at most two primes[6].

Both results are the nearest approach to the extremely difficult prime conjecture until now. They were obtained via analytic number theory and modern sieve theory. Both results are far from proving those conjectures, the well–known "parity problem" and the problem of estimating error items in modern sieve theory prevent further progress[1].

In this paper we return to the discrete approach, and consider those conjectures from a different angle or measure.

2. A formal system

First of all we give some operations on finite sets of natural numbers.

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Let

$$A = \langle a_1, a_2, \ldots, a_i, \ldots, a_n \rangle,$$
$$B = \langle b_1, b_2, \ldots, b_j, \ldots, b_m \rangle$$

be the arbitrary finite sets of natural numbers, we define

$$A + B = \langle a_1 + b_1, a_2 + b_1, \ldots, a_i + b_j \ldots, a_{n-1} + b_m, a_n + b_m \rangle,$$
$$AB = \langle a_1 b_1, a_2 b_1, \ldots, a_i b_j \ldots, a_{n-1} b_m, a_n b_m \rangle.$$

Let $A \backslash B$ be the set subtraction.

Define the solution of the system of congruences

$$A \equiv \langle a_1, a_2, \ldots, a_i, \ldots, a_n \rangle \mod a,$$
$$B \equiv \langle b_1, b_2, \ldots, b_j, \ldots, b_m \rangle \mod b$$

be

$$X \equiv D \equiv \langle d_{11}, d_{21}, \ldots, d_{ij}, \ldots, d_{n-1m}, d_{nm} \rangle \mod ab.$$

Where $x \equiv d_{ij} \mod ab$ is a solution of the system of congruences $x \equiv a_i \mod a$ and $x \equiv b_j \mod b$. By Chinese remainder theorem the solution $X \equiv D \mod ab$ is computable and unique.

Now, we had founded a model of the second order arithmetic

$$\langle P(N), +, \times, 0, 1, \in \rangle,$$

where N is the set of natural numbers and $P(N)$ is the power set of N[7].

Mathematicians assume that $\langle N, +, \times, 0, 1 \rangle$ is the standard model of Peano theory PA, similarly, we assume that $\langle P(N), N, +, \times, 0, 1, \in \rangle$ is the standard model of the theory of second order arithmetic $PA + ZF$, which is the sum set of Peano theory PA and set theory ZF.

When the set $\langle a \rangle$, $\langle b \rangle$ consists of the single element a,b, by the definition

$$\langle a \rangle + \langle b \rangle = \langle a + b \rangle,$$
$$\langle a \rangle \langle b \rangle = \langle ab \rangle,$$

we may identify the set $\langle a \rangle$ of single element with the number a, $\langle a \rangle = a$. So that, by lifting the type of single element we may reduce the second order model

$$\langle P(N), N, +, \times, 0, 1, \in \rangle,$$

to be a first order model

$$\langle P(N), +, \times, \langle 0 \rangle, \langle 1 \rangle, \in \rangle.$$

Thus the reasoning in the second order arithmetic $P(N), N$ is formal. We may isomorphic embed the first order model of natural numbers N in this first order model $P(N)$.

Below we use $P(N)$ to denote the second order formal system

$$\langle P(N), N, +, \times, 0, 1, \in \rangle.$$

Now, the sets of natural numbers are individuals in the second order formal system $P(N)$. An individual determined not only includes all elements in him but also includes all information of distributions of its elements.

Usually a predicate $R(a)$ in N corresponds to an individual $\{a : R(a)\}$ in $P(N)$, which is called an interpretation or model of the predicate $R(a)$.

In the first order formal system N, there are many algorithms for computing a natural number and proving its property. Example Euclid's algorithm for computing the greatest common divisor $\gcd(a, b)$ and the sieve method of Eratosthenes for

computing primes less than a given natural number x. In the second order formal system $P(N)$, we will design more algorithms to compute a set of natural numbers and to prove its property.

In the first order system of real numbers $\langle R,+,\times,0,1\rangle$ the limit $\lim x_i$ of a numerical sequence x_i may determine a number and its property, similarly, in the second order system $P(N)$ we will design a set sequence T_i', its limit $\lim T_i'$ may determine a set of some primes, example the set of all primes of the form a^2+1, and its infinity or not.

The second order language $\langle +,\times,\in\rangle$ has great expressive power. The theory of second order arithmetic $PA+ZF$ is powerful and extremely flexible, in this theory we try more deeply to discuss the sets of natural numbers.

We do not discuss further this formal system in view from logic[8].

3. A recursive formula and some elementary conclusions

Let p_i be i-th prime, $p_0=2$.

For any prime $p_i>2$, we consider the divided relation $p_i \mid a^2+1$, namely, the congruence

$$a^2+1\equiv 0 \mod p_i.$$

This is a quadric congruence, by Euler's criterion we easy prove that -1 is a quadratic residue of the prime p_i if and only if $p_i\equiv 1 \mod 4$. If -1 is not a quadratic residue of the prime p_i, then the prime p_i does not divide a^2+1, we overlook the prime p_i.

list the primes q_j of the form $4k+1$,

$$5,13,17,29\ldots,q_j,\ldots.$$

Let

$$X\equiv B_j \mod q_j$$

be the solution of the congruence

$$a^2+1\equiv 0 \mod q_j.$$

Example:

$$\begin{aligned} B_1 &\equiv \langle 2,3\rangle \mod 5,\\ B_2 &\equiv \langle 5,8\rangle \mod 13,\\ B_1 &\equiv \langle 4,13\rangle \mod 17,\\ B_1 &\equiv \langle 12,,15\rangle \mod 29, \end{aligned}$$

Let

$$m_{i+1}=\prod_0^i q_j.$$

From the set of all even numbers $x\equiv 0 \mod 2$ we delete the congruence classes $B_j \mod q_j$ successively, and obtain the congruence class $T_{i+1} \mod m_{i+1}$ such that

$$T_{i+1}\equiv\{a:\forall_{q_j\le q_i}(a^2+1\not\equiv 0 \mod q_j)\}.$$

Then, the recursive formula of T_{i+1}, which is the set of least nonnegative representatives of residue classes $\mod m_{i+1}$ is as follows:

$$\begin{aligned} T_1 &= \langle 2\rangle, \\ T_{i+1} &= (T_i + \langle m_i\rangle\langle 0,1,2,\ldots,q_i-1\rangle) \setminus D_i. \end{aligned} \tag{3.1}$$

Where $X \equiv D_i \mod m_{i+1}$ is the solution of the system of congruences

$$X \equiv T_i \mod m_i,$$

$$X \equiv B_i \mod q_i.$$

Obviously, when we delete the number a in the congruence classes $B_j \mod q_j$ successively, except $q_j = a^2+1$ itself may be the prime of the form a^2+1, other numbers of the form a^2+1 are all composites having the prime divisor q_j.

The number of elements of the set T_{i+1} is

$$|T_{i+1}| = \prod_1^i (q_j - 2). \tag{3.2}$$

For example, the first a few terms of the sets T_i are

$$\begin{aligned} T_1 &= \langle 2\rangle, \\ T_2 &= (\langle 2\rangle + \langle 0,2,4,6,8\rangle) \setminus \langle 2,8\rangle = \langle 4,6,10\rangle, \\ T_3 &= (\langle 4,6,10\rangle + \langle 0,10,20,\ldots,110,120\rangle) \setminus \langle 34,44,60,70,86,96\rangle \\ &= \langle 4,6,10,14,16,20,24,26,30,36,40,46,50,54,56,64,66,\ldots,126,130\rangle, \\ T_4 &= (\langle 4,6,10,\ldots,126,130\rangle + \langle 0,130,260,\ldots,2080\rangle) \setminus \langle 4,30,64,\ldots,2206\rangle \\ &= \langle 6,10,14,16,20,24,26,36,40,\ldots,2200,2204,2210\rangle. \end{aligned}$$

It is easy to prove the formula (3.1),(3.2) by using the mathematical induction.

Now we list some elementary conclusion from the recursive formula T_i, their proof is easy.

(1) Let $s_i = \min T_i$ be the smallest number of the set T_i, then a^2+1 is a prime if and only if

$$a = s_i \bigwedge a^2 + 1 = q_i.$$

This criterion recursively enumerates all numbers a making a^2+1 primes.

(2) Using the recursive formula T_i, we easy compute the primes of the form a^2+1, in fact, we had computed out the first few primes of the form a^2+1 for $a = 2,4,6,10,14,16,20,24,26,36,40$.

(3) If $a^2+1 \geq q_i$ is a prime, then the natural number a belongs the congruence classes $T_i \mod m_i$.

(4) "Let $P(N)$ be the largest prime factor of the natural number N. We have known for more than fifty years that $P(n^2+1)$ tends infinity with n."[1]

The recursive formula T_i provides a fine proof for this fact again.

4. The recursive formula T_i' and its main conclusion

The recursive formula T_i expresses a sieving progress, which is complete new variation on the historical Eratosthenes sieve. Entire set of natural numbers is sieved, without estimating error items, this is different from the traditional sieve method.

If delete all natural numbers a making a^2+1 composites in congruence classes $B_1, B_2, \ldots, B_i, \ldots$ from entire set of even natural numbers successively, what result will obtain? We consider limit of the sieving process to obtain the set of all numbers a making a^2+1 primes.

Now we quote the definition of the limit of set sequences of natural numbers[9].

$\limsup_{n=\infty} F_n$ (limit superior of the set sequences $F_0, F_1, \ldots$),

$\liminf_{n=\infty} F_n$ (limit inferior of the set sequences $F_0, F_1, \ldots$) defined as follows:

$$\limsup_{n=\infty} F_n = \bigcap_{n=0}^{\infty} \bigcup_{i=0}^{\infty} F_{n+i},$$

$$\liminf_{n=\infty} F_n = \bigcup_{n=0}^{\infty} \bigcap_{i=0}^{\infty} F_{n+i}.$$

It is easy to check that $\limsup F_n$ is the set of those elements x which belong to F_n for infinitely many n. Analogously, x belongs to $\liminf F_n$ if and only if it belongs to F_n for almost all n, that is, if it belongs to all but a finite number of F_n.

It is easily seen that

$$\liminf_{n=\infty} F_n \subset \limsup_{n=\infty} F_n$$

If the inclusion sign can be replaced by the equality sign, that is, if the superior and inferior limits are equal, then their common value is denoted by

$$\lim_{n=\infty} F_n,$$

and is called the limit of the sequence $F_0, F_1, \ldots$. In this case we also say that the sequence is convergent.

From above definition it is easy to prove some simple properties of the limit of sequences of sets:

(1) If $F_0 \supset F_1 \supset F_2 \supset \ldots$, then $\bigcap F_n = \lim F_n$.
(2) If $F_0 \subset F_1 \subset F_2 \subset \ldots$, then $\bigcup F_n = \lim F_n$.
(3) $\limsup(A_n \bigcap B_n) = \limsup A_n \bigcap \limsup B_n$.
(4) $\limsup(A_n \bigcup B_n) = \limsup A_n \bigcup \limsup B_n$.
(5) $\liminf A_n \bigcup \liminf B_n \subset \liminf(A_n \bigcup B_n)$.
(6) $\liminf(A_n \bigcap B_n) \subset \liminf A_n \bigcap \liminf B_n$.

In above textbook K.Kuratowski and A.Mostowski looked those properties as exercises. We directly use them.

Call this limit be set theoretic limit.

Let us use the set theoretic limit of set sequence of natural numbers to determine the set of all number a making a^2+1 primes.

Now we use a predicate $R(2,a)$ to denote a^2+1 is a prime.

As the congruence classes

$$X_i \equiv T_i \mod m_i,$$

there is an inclusion relation

$$X_1 \supset X_2 \supset \cdots \supset X_i.$$

Thus the set sequences of natural numbers $X_1, X_2, \ldots, X_i, \ldots$ have the limit $\lim X_i$ by the property (2).

Since when we delete the congruence classes $B_j, j < i$ to take out the number $a = s_i \bigwedge a^2 + 1 = q_i$, next we will remove number a by $q_i \mid a^2 + 1$. Thus

$$\lim X_i = \emptyset.$$

Obviously, $T_i \subset X_i$, by the property (4) obtain

$$\lim T_i = \emptyset. \tag{4.1}$$

With the $\lim T_i$ we would prove nothing.

We modify the set B_j to be

$$B'_j = \{a : a^2 + 1 \equiv 0 \mod q_j \text{ except } R(2, a)\}.$$

Namely, we delete the congruence class $B_j \mod q_j$ but save the number a if $a^2 + 1$ is a prime, $q_j = a^2 + 1$.

Let A_i be the set of number a such that $a^2 + 1$ less than q_i and $a^2 + 1$ is a prime

$$A_i = \{a : a^2 + 1 < q_i \bigwedge R(2, a)\}.$$

We modify the set T_i to be

$$T'_i = A_i \bigcup T_i. \tag{4.2}$$

Except saving all numbers a making $a^2 + 1$ primes, both set sequences T'_i and T_i are same.

Now we use the limit of the set sequences T'_i to prove the main theorem in this paper.

Theorem 4.1. *There are infinitely many primes of the form* $a^2 + 1$.

Proof. First of all let us consider the number of elements of the set sequence T'_i.

Let $|A_i|$ be the number of all numbers a making $a^2 + 1$ primes and $a^2 + 1 < q_i$, then the number of elements of the set sequence T'_i is

$$|T'_i| = |A_i| + |T_i|,$$

$$|T'_i| \geq |T_i|.$$

As i goes to infinity, from formula (3.2), may obtain that the number of elements of set sequence T'_i is strictly increasing, thus the limit of the number of elements of set sequence T'_i is infinite

$$\lim |T'_i| = \infty. \tag{4.3}$$

Where ∞ denotes the smallest infinite cardinal $\aleph_0$ or the first infinite ordinal ω.

Call this limit be a numerical limit.

Next let us consider the set theoretic limit of the set sequence T'_i.

Obviously,

$$A_1 \subset A_2 \subset \cdots \subset A_i \subset \cdots,$$

by the property (2) the set sequence $A_1, A_2, \ldots, A_i, \ldots$ have the limit $\lim A_i$, and this $\lim A_i$ is the set of all numbers a making $a^2 + 1$ primes

$$\lim A_i = \{a : R(2,a)\}. \tag{4.4}$$

By the property (4), we have

$$\begin{aligned} T_i' &= A_i \bigcup T_i. \\ \liminf T_i' &\supset \liminf A_i = \lim A_i. \\ \limsup T_i' &= \limsup(A_i \bigcup T_i) \\ &= \limsup A_i \bigcup \limsup T_i \\ &= \limsup A_i \bigcup \emptyset \\ &= \limsup A_i \\ &= \lim A_i. \end{aligned}$$

By the property (5) we have

$$\liminf T_i' \supset \liminf A_i = \lim A_i.$$

Thus it is proved that the set sequence T_i' has a limit and this limit is the set of all numbers a making a^2+1 primes.

$$\lim T_i' = \lim A_i = \{a : R(2,a)\}. \tag{4.5}$$

The set theoretic limit is defined by infinite operations of sets not involving topology, so that we can not obtain that the set of all natural numbers a making a^2+1 primes is an infinite set directly from the numerical limit (4.3).

Further explore in the sieving progress, we find that the set sequence T_i' arbitrarily approaches the infinite set of all numbers a making a^2+1 primes, can endow it with an order topology, and the cardinal function $|T_i'|$ is continuous with respect to this order topology.

Obtained set T_i' from formula (3.1), let $a \in T_i'$, if $a < q_i$, then number a making a^2+1 prime, which belongs to all T_r' for $r > i$ and will never be deleted.

Obtained set T_i' from formula (3.1), let $a \in T_i'$, and $a \geq q_i$, then a is a good candidates making a^2+1 primes, assuming both numbers of primes of form $4k+1$ and $4k+3$ are roughly equal in the interval $[3, q_i]$, then the a^2+1 do not contain first $2i$ primes as factor. In this case, if a making a^2+1 prime, then a belongs to all T_r' for $r > i$, otherwise the a is an error item, there is a prime q_s such that $q_s | a^2+1$, a does not belong to any T_r' for $r > s$. Our sieve method itself will delete all error items and need not estimate the number of error items.

As i goes to infinity, we delete more and more numbers a making a^2+1 composites B_i', exhibit more and more a making a^2+1 primes or candidates making a^2+1 primes in the set T_i', the set sequence T_i' gets as close as we want to the $\lim T_i'$.

If i was extremely large, example $i = c = 10^{10^{100}}$, theoretically we can construct a set T_c' by formula (3.1), which has approximated to the $\lim T_i'$.

The set T_c' has

$$|T_c'| > \prod_{1}^{10^{10^{100}}} (q_i - 2)$$

elements a such that a^2+1 do not contain first $2\times 10^{10^{100}}-1$ primes as factor except itself, namely if a^2+1 has factors except itself, by the prime theorem, they are large than

$$2.3\times 10^{100}\times 2\times 10^{10^{100}}.$$

In approximate sense, the set T_c' nearly may be regarded as a set of all numbers a making a^2+1 primes and the number of elements of this set nearly may be regarded as infinity. Against our daily standard, the set T_c' is the infinite set of all numbers a making a^2+1 primes.

Ultimately, as the limit of the set sequence T_i', we have deleted all congruence classes B_i', and have obtained infinitely many natural numbers a such that a^2+1 not contain any prime as factor except itself, these infinitely many natural numbers a constitute exactly the set of all numbers a such that a^2+1 is a prime,.

We give a formal proof with an order topology.

We know that the T_i' is the set of numbers $a<m_i$ such that a^2+1 do not contain any prime $p<q_i$ as factor except itself by the formula (4.2). According to this order relation we use recursive definition to list a well ordered set with the order type ω

$$T_1', T_2', \ldots, T_i', \ldots.$$

Let T be the set theoretic limit of set sequence T_i'

$$T=\lim T_i'=\lim A_i=\{a: R(2,a)\}.$$

By transfinite recursive definition, according to above order relation, we list a well ordered set with the order type $\omega+1$

$$T_1', T_2', \ldots, T_i', \ldots; T. \tag{4.6}$$

Take the order topology for this well ordered set, its open sets are the sets that are the unions of open intervals (c,d) and rays $(c,d]$ [10], may obtain again

$$\lim T_i'=T=\{a, R(2,a)\}. \tag{4.7}$$

Call this limit be an order topological limit.

Let $f: X\to Y$ be the function from topological space X to topological space Y

$$X:\ T_1',\ T_2',\ \ldots,\ T_i', \ldots;\ T,$$
$$Y: |T_1'|, |T_2'|, \ldots, |T_i'|, \ldots; \infty,$$

Obviously, for every open set $(|c|,|d|)$, $(|c|,\infty]$ in Y, its preimage (c,d), $(c,T]$ is an open set also in X, thus the cardinal function $f: X\to Y$ is sequentially continuous, it preserves limits by the topological theorem. We obtain

$$|\{a: R(2,a)\}|=|\lim T_i'|=\lim |T_i'|=\infty. \tag{4.8}$$

We had computed out some patterns of the first few natural number a such that a^2+1 is a prime,

$$a=2,4,6,10,14,16,20,24,26,36,40.$$

Hance there is a set $\{a: R(2,a)\}$ and it is nonempty, the predicate $R(2,a)$ has a model, its theory is consistent by Gödel completeness theorem—A theory has a model if and only if it is consistent [11]. It is impossible to prove that the number of primes of the form a^2+1 is finite. Otherwise, if there is no patter of primes for

polynomials of some forms, our reasoning will invoke a Ross-Littwood paradox, in last section we consider this paradox in the detail.

We have proved validly that the number of primes of the form a^2+1 is infinite or this set is an infinite set.

□

5. A extension of the main theorem

Let $f(a)$ be a polynomial, let $B_i \mod p_i$ be the solution of the congruence

$$f(a) \equiv 0 \mod p_i.$$

Repeat above reasoning, we extend the main theorem to any quadratic polynomial an^2+bn+c or general polynomials $f(a)$, then determine whether there are infinitely many primes of the form $f(a)$ or not.

We can extend the main theorem to attack Diksin's prime k-tuple conjecture and its generalization—Schinzel's hypothesis. Here consider Schinzel's hypothesis.

In 1958 Schinzel and Sierpinski proposed a hypothesis[12]:

Let k be a positive integer and let

$$f_1(x), f_2(x), \ldots, f_k(x)$$

be irreducible polynomials with integral coefficients and positive leading coefficients. Assume also that there is not a prime p which divides the product $f_1(m)f_2(m)\ldots f_k(m)$ for every integer m. Then there exists a positive integer n making

$$f_1(n), f_2(n), \ldots, f_k(n)$$

all primes.

If there one positive integer n making these polynomials simultaneously primes, then there are infinitely many such n.

I can not extend above result to prove original Schinzel's hypothesis, however we can sharpen this hypothesis to a special form like the primes of form a^2+1 and prove it.

Given any positive integer n and let $f_1(a), f_2(a), \ldots, f_n(a)$ be polynomials with integral coefficients and positive leading coefficients.

Let k be the degree of the polynomial

$$f_1(a)f_2(a)\ldots f_n(a).$$

If the solutions of the congruence

$$f_1(a)f_2(a)\ldots f_n(a) \equiv 0 \mod p$$

do not run through the complete system of residues $\mod p$ for every prime $p \le k$. we say the polynomials are admissible.

If there one natural number a making these polynomials simultaneously primes, we say the polynomials have prime patterns.

We use the predicate $R(k,a)$ denote the number a such that $f_1(a), f_2(a), \ldots, f_n(a)$ are primes simultaneously.

Extend above main theorem, we obtain a theorem.

Theorem 5.1. *If the polynomials*

$$f_1(a), f_2(a), \ldots, f_k(a)$$

are admissible and have prime pattern, then there are infinitely many number a making they simultaneously primes.

Proof. Let $X \equiv B_i \mod p_i$ be the solutions of the congruence

$$f_1(x)f_2(x)\dots f_k(x) \equiv 0 \mod p_i.$$

Let w_i be the number of this solutions. If this congruence has no solution for mod p_i, we overlook the prime p_i.

If the solutions of the congruence run through the complete system of residues mod p_i, then the set T_{i+1} is empty, obviously, the number of a making these polynomials simultaneously primes is finite. So that we Assume that the solutions of the congruence do not run through the complete system of residues mod p, namely, the polynomials are admissible.

Delete the residue class $X \equiv B_j \mod p_j$ we obtain the recursive formula T_i, the number of elements of the set T_i is

$$|T_i| = \prod_1^{i-1}(p_j - w_j). \tag{5.1}$$

Where w_j is the number of the solutions $B_i \mod p_j$ of the congruence, k is the degree of above polynomial, it is easy to prove $w_i < k$. Thus if $i > k$ the cardinal $|T_i|$ is strictly increasing. We obtain

$$\lim |T_i| = \infty. \tag{5.2}$$

Delete the residue class $B_j \mod p_j$ but save the number a making these polynomials simultaneously primes. Like above proof, we obtain

$$\lim |T_i'| = \infty. \tag{5.3}$$

By the set theoretic limit we obtain:

$$\lim T_n' = \lim A_a = \{a : R(k, a)\}. \tag{5.4}$$

Since the set T_i' approximates the set of all a making these polynomials simultaneously primes as i tends infinity, list a well ordered set with the order type $\omega + 1$

$$T_1', T_2', \dots, T_i', \dots; T. \tag{5.5}$$

Take the order topology for this well ordered set, we prove that the cardinal function $|T_i'|$ is continuous with respect to this order topology, thus

$$|\{a : R(k, a)\}| = |\lim T_i'| = \lim |T_i'| = \infty. \tag{5.6}$$

Similarly the primes of the form $a^2 + 1$ we have assume that there is a prime pattern of the form $f_1(a), f_2(a), \dots, f_n(a)$, thus there is a set $\{a : R(k, a)\}$ and it is nonempty, the predicate $R(k, a)$ has a model, its theory is consistent by Gödel completeness theorem.

Under above conditions we proved validly that there are infinitely many natural numbers a such that $f_1(a), f_2(a), \dots, f_n(a)$ are simultaneously primes. □

Next section we discuss that the prime pattern is necessary condition.

6. The Ross-Littwood paradox

In above reasoning a restrictive condition— have prime patterns —is necessary. If we have not found any prime pattern of admissible polynomials

$$f_1(a), f_2(a), \ldots, f_n(a)$$

or can not prove that there exist some patterns, we do not know whether there exists a set $\{a : R(k,a)\}$ as the model of the predicate $R(k,a)$ in the formal system $P(N)$. We do not know whether the theory about the predicate $R(k,a)$ is consistent or not.

Example, let $k \geq 41$, until now we do not know whether there exists a number a such that

$$x^2 - x + a$$

represents primes for $x = 0, 1, 2, \ldots, k$.

For an admissible polynomials $f_1(a), f_2(a), \ldots, f_n(a)$ which have not any prime pattern, suppose that there exist no set of numbers a making they primes in the formal system $P(N)$, we prove nothing.

For an admissible polynomials $f_1(a), f_2(a), \ldots, f_n(a)$ which have not any prime pattern, suppose that there exists a set of numbers a making they primes in the formal system $P(N)$, we would prove that the set is empty by the set theoretic limit and would prove that the set is infinite by the order topological limit, this is a contradiction. One call this contradiction be a Ross-Littwood paradox[13, 14]. This paradox is not an error of reasoning.

Example to consider the primes of form a^2-1 by above algorithm or to consider the limit of set sequence $T_i = \langle i, i+1, i+2, \ldots, 2i \rangle$, we obtain the Ross-Littwood paradox.

In informal argument, Ross-Littwood paradox opened out the contradiction between numerical limit formula (5.3) (not equal 0) and set theoretic limit formula(5.4)(empty). It seems, one adds an extra premise — a physical law that the balls or the natural numbers as the objects have continuous space-time paths. To take either the numerical limit or set theoretic limit all leave one in the embarrassing situation.

Now we have formalzed Ross-Littwood paradox. It is easy to resolve this paradox in the formal system $P(N)$. By the reduction to absurdity, from above contradiction we obtain that there exists not any set of numbers a making the admissible polynomials $f_1(a), f_2(a), \ldots, f_n(a)$ simultaneously primes in the formal system $P(N)$ as a model of the predicate $R(k,a)$. Namely, the Ross-Littwood paradox is an Impossible Super-Task by J.P.Van Bendegem [15].

This phenomenon is like that in rational number formal system $\langle Q, +, \times, 0, 1 \rangle$ there is no irrational number$\sqrt{2}$. Suppose that there is the irrational number $\sqrt{2}$ in the rational number system Q, we obtain a contradiction.

Only one sieve method which have no survivor or pattern may lead to the Ross-Littwood paradox, thus the usual solution of the Ross-Littwood paradox is an empty, Of course this is an argument from different logical level.

Thus for the admissible polynomials $f_1(a), f_2(a), \ldots, f_n(a)$, if we have not found a prime pattern or can not prove that there is a prime pattern, we can not prove there are infinite many numbers a making those polynomials primes simultaneously, our proof itself needs a restrictive condition.

References

[1] R. Guy, *Unsolved Problems in Number Theory*, 3rd edition, Springer, (2004).

[2] K.H.Rosen, *Elementary Number Theoty and its Applications* 4rd edition, Peason Edication Asia Lid.,(2004)

[3] P. Ribenboim, *The new book of prime number records*, 3rd edition, Springer-Verlag, New York, (1996).

[4] L.E.Dikson, *A new extension of Dirichlet's theorem on prime numbers* , Massenger of Mathematics,**Vol** 33, 155–161(1903–04).

[5] Chen.J.R. *On the Representetion of a Large Even Interger as the Sum of a prime and the Product of at Most Two Primes*. Kexue Tongbao 17,385–386,(1966).
Chen.J.R. *On the Representetion of a Large Even Interger as the Sum of a prime and the Product of at Most Two Primes*. I.Sci.Sinica 16,157–176,(1973).
Chen.J.R. *On the Representetion of a Large Even Interger as the Sum of a prime and the Product of at Most Two Primes*. II.Sci.Sinica 16,412–430,(1978).

[6] H.Iwanies, *Almost-primes representated by quadratic polynomials*, Ivent.Math. 47(1978) 171–188.

[7] Fengsui Liu, *On the prime k-tuple conjecture I*, WSEAS Transactions on mathematics,**Vol** 3, 744–747(2004).
Fengsui Liu, *On the prime k-tuple conjecture*, WSEAS international conference on applied mathematics, Corfu,Greece, August 17–19, 488-264 (2004).

[8] Jouko vaananrn, *Second-order logic and foundations of mathematics*, The bulletin of symbolic logic, **Vol** 7, 504–520(2001).

[9] K.Kuratowsky and A. Mostowsky, *set theory, With an introduction to descriptive set theory*, North-Holland Publishingcom.(1976) 118–120.

[10] J.R.Munkres,*Topology*, 2rd edition,Prentice Hall, Upper Saddle River, (2000).

[11] J.R.Shoenfield, *Mathematical logic*, Addison-Wesley Publishing com. (1967).

[12] A.Schinzel and W.Sierpinski, Sur certaines hypotheses concernant les numbers premiers, Acta Arithmetica, vol4 (1958), pp.185-208. A.Schinzel, Remarks on the paper.“ Sur certaines hypotheses concernant les numbers premiers', Acta Arithmetica, vol7 (1961), pp.1-8.

[13] J.E.Littlewood,[1953], *Littlewood's Miscellany* (ed. Bela Bollobas) Cambridge University Press, Cambridge, (1986), p.26.

[14] S.Ross, *A First Course in Probability*, third edition, New York and London, Macmillan, (1988)

[15] J.P.Van Bendegem, *'Ross' paradox is an Impossible Super-Task'*, British Journal for the Philosophy of Science, vol 45(1994), pp.743-8.

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