# A Proof of Goldbach Conjecture

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## Abstract

In this paper, the Goldbach Conjecture｛1, 1｝ is proved by the complex variable integration. To prove the conjecture, a new function is introduced into Dirichlet series. And then, by using the Perron Formula of Dirichlet Series and the Residue Theorem, we conclude that any larger even integer can decompose the sum of two primes.

**Key words**: the Goldbach Conjecture; the Prime Theorem; Dirichlet series.

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## Notation:

| | |
|---|---|
| $p$ | prime number |
| $N$ | positive integer |
| $p\|N$ | $p$ exactly divides $N$ |
| $(p,N)=1$ | coprime between $p$ and $N$ |
| ｛1, 1｝ | the even integer as the sum of two primes |
| ｛a, b｝ | the even integer as the sum of product of at most a primes and product of at most b primes |
| $\phi(n)$ | the Euler function |
| $s=\sigma+it$ | complex variable |
| $\zeta(s)$ | $\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$, the Riemann zeta function |
| $\frac{\zeta'}{\zeta}(s)$ | $\frac{\zeta'}{\zeta}(s)=\frac{\zeta'(s)}{\zeta(s)}$ |

| | |
|---|---|
| $\Lambda(n)$ | $\Lambda(n)=\begin{cases}\log p, if \quad n=p^m\\ 0, \quad otherwise\end{cases}$, the Mangoldt function |
| $\pi(x)$ | $\pi(x)=\sum_{p\le x}1$ |
| $\theta(x)$ | $\theta(x)=\sum_{p\le x}\log p$ |
| $\psi(x)$ | $\psi(x)=\sum_{n\le x}\Lambda(n)$ |
| $\sigma_c$ | the convergence abscissa of Dirichlet series |
| $\sigma_a$ | the absolute convergence abscissa of Dirichlet series |
| $resf(s)$ | the residue of function $f(s)$ |
| $N(f(s))$ | the number of zero point of function $f(s)$ |
| $B=O(A), B\ll A$ | there exists a calculable position constant $c$, such that $\lvert B\rvert\le cA$, |
| $o(1)$ | the constant tending to zero |
| $\gamma$ | the Euler constant, $\gamma=0.577\cdots$ |
| $\varepsilon$ | the sufficiently small positive constant |
| $c_1$, $c_2$, $c_3$ | calculable positive constant |

## 1. Introduction

As well-known, the Goldbach Conjecture ｛1, 1｝ has not been resolved in mathematical field. The conjecture states that every even integer $N \geq 4$ can decompose the sum of two primes (e.g., 12 = 5 + 7, 20 = 3 + 17). In over past two hundreds years, several relevant proofs on this issue have been conducted. Using the sieve method, some mathematicians verified the results including｛9, 9｝(Brun, 1920) [1], ｛1, c｝(R´enyi, 1947) [2], ｛1, 5｝(Pan, 1962) [3], ｛1, 4｝(Wang, 1962) [4], ｛1, 3｝(Richert, 1969) [5], ｛1, 2｝(Chen, 1973) [6], etc. In 1975, Montgomery and Vaughan made the progress on the exceptional set in Goldbach's problem by the circle method [7]. However,｛1, 1｝has not been proven up to now. In current paper, by using the complex variable integration, we are to prove ｛1, 1｝.

First, we define a new function

$$\lambda(n)=\begin{cases}0, & if \quad n \equiv N \underset{p \leq \sqrt{N}}{\bmod} p \\ 1, & otherwise\end{cases}, \tag{1.1}$$

where $(p,N)=1$, $p=\{2,3,5,7,\ldots\}$ is the prime sequence. Hence, this function is provided with properties of sieve function.

Based on the series: $\zeta(s)$, $\pi(x)$, $\theta(x)$ and $\psi(x)$, we obtain the new following series:

$$\zeta(s,\lambda)=\sum_{n=1}^{\infty}\lambda(n)n^{-s}, \tag{1.2}$$

$$\pi(x,\lambda)=\sum_{p\leq x}\lambda(p), \tag{1.3}$$

$$\theta(x,\lambda)=\sum_{p\leq x}\lambda(p)\log p, \tag{1.4}$$

$$\psi(x,\lambda)=\sum_{n\leq x}\lambda(n)\Lambda(n), \tag{1.5}$$

and

$$\Psi(s,\lambda)=\sum_{n=1}^{\infty}\lambda(n)\Lambda(n)n^{-s}. \tag{1.6}$$

In this case, if $x=N$, such that $\pi(N,\lambda)>0$, then ｛1, 1｝ is true.

By the Prime Number Theorem [8], we have

$$\theta(x,\lambda)=\pi(x,\lambda)\log x-\int_2^x \pi(u,\lambda)u^{-1}du, \tag{1.7}$$

$$\pi(x,\lambda)=\theta(x,\lambda)(\log x)^{-1}+\int_2^x \theta(x,\lambda)\left(u\log^2 u\right)^{-1}du, \tag{1.8}$$

and

$$\psi(x,\lambda)=\theta(x,\lambda)+O\left(x^{1/2}\right). \tag{1.9}$$

So we can estimate the value of function $\pi(x,\lambda)$ by function $\psi(x,\lambda)$.

In present paper, applying the Perron Formula of Dirichlet Series and the Residue Theorem, we will verify following theorems.

**Theorem 1.** *If* $N$ *is any larger even integer, then*

$$\psi(N,\lambda)=N\prod_{\substack{p\le\sqrt{N}\\(p,N)=1}}\left(1-\frac{1}{p-1}\right)+O\left(N\,\mathrm{e}^{-c\sqrt{\log N}}\right). \tag{1.10}$$

**Theorem 2.** *If* $N$ *is any larger even integer, then*

$$\pi(N,\lambda)=\frac{N}{\log N}\prod_{\substack{p\le\sqrt{N}\\(p,N)=1}}\left(1-\frac{1}{p-1}\right)+O\left(Ne^{-c\sqrt{\log N}}\right). \tag{1.11}$$

**Remark:** The propositions (1.10) and (1.11) are equivalent.

By Theorem 2 we have $\pi(N,\lambda)>0$, the Goldbach Conjecture (i.e., {1, 1}) is established.

To prove Theorem 1 and Theorem 2, we need the following lemmas.

## 2. Lemmas

**Lemma 1.** The Perron Formula of Dirichlet Series.

*For the Dirichlet series*

$$A(s)=\sum_{n=1}^{\infty}a(n)n^{-s},\quad \sigma_a<+\infty, \tag{2.1}$$

*if there exist increasing functions* $H(u)$ *and* $B(u)$, *such that*

$$|a(n)|\le H(n),\quad n=1,2,3\ldots, \tag{2.2}$$

and

$$\sum_{n=1}^{\infty}|a(n)|n^{-\sigma}\le B(\sigma),\quad \sigma>\sigma_a, \tag{2.3}$$

*for any* $s_0=\sigma_0+it_0$, *let* $b_0\ge\sigma_0+b>\sigma_a$, $T\ge1$, $x\ge1$ (*when x is a positive integer*), *then*

$$\sum_{n\le x}a(n)n^{-s_0}+\frac{1}{2}a(x)x^{-s_0}=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}A(s_0+s)\frac{x^s}{s}ds$$

$$+O\left(\frac{x^b B(b+\sigma_0)}{T}\right)+O\left(x^{1-\sigma_0} H(2x)\min\left(1,\frac{\log x}{T}\right)\right), \tag{2.4}$$

*the constant implied in* $O$ *depends on* $\sigma_a$ *and* $b_0$.

*Let* $s_0=0$, *then*

$$\sum_{n\le x} a(n)+\frac{1}{2}a(x)=\frac{1}{2\pi i}\int_{b-iT}^{b+iT} A(s)\frac{x^s}{s}ds+O\left(\frac{x^b B(b)}{T}\right)$$

$$+O\left(xH(2x)\min\left(1,\frac{\log x}{T}\right)\right) \tag{2.5}$$

(for the proof of Lemma 1, see [8]).

**Lemma 2.** *There exists a positive constant* $c_1$, *such that* $\zeta(\sigma+it,\lambda)$ *has not zero point in the range*

$$\sigma\ge 1-c_1\log^{-1}(|t|+2) \tag{2.6}$$

**Proof.** Let $\zeta(s,\bar{\lambda})$ be defined by

$$\zeta(s,\bar{\lambda})=\sum_{n=1}^{\infty}(1-\lambda(n))n^{-s}. \tag{2.7}$$

When $\sigma\ge 1-c_1\log^{-1}(|t|+2)$, we obtain

$$|\zeta(s,\lambda)|\le|\zeta(s,\bar{\lambda})|, \tag{2.8}$$

and

$$|\zeta(s,\lambda)+\zeta(s,\bar{\lambda})|=|\zeta(s)|. \tag{2.9}$$

By the Rouché Theorem, we deduce

$$N(\zeta(s,\lambda))=N(\zeta(s)). \tag{2.10}$$

Since $\zeta(\sigma+it)$ has not zero point in the range $\sigma\ge 1-c_1\log^{-1}(|t|+2)$, then $\zeta(\sigma+it,\lambda)$ has not zero point in the range $\sigma\ge 1-c_1\log^{-1}(|t|+2)$ too.

This proves Lemma 2.

**Lemma 3.** *For Dirichlet series* $\Psi(s,\lambda)=\sum_{n=1}^{\infty}\lambda(n)\Lambda(n)n^{-s}$, *we have*

$$\sigma_c=\sigma_a=1. \tag{2.11}$$

**Proof.** By the Prime Number Theorem for Arithmetic Progressions, when $x \to \infty$ and $(q, N) = 1$, we have

$$\lim_{x\to\infty} \sum_{\substack{n\equiv N \bmod q \\ n\le x}} \Lambda(n) = \frac{x}{\phi(q)} \tag{2.12}$$

$$\lim_{x\to\infty} \sum_{n\le x} \lambda(n)\Lambda(n) = x \prod_{\substack{p\le\sqrt{N} \\ (p,N)=1}} \left(1 - \frac{1}{\phi(p)}\right)$$

$$= x \prod_{\substack{p\le\sqrt{N} \\ (p,N)=1}} \left(1 - \frac{1}{p-1}\right)$$

hence

$$\sigma_c = \lim_{x\to\infty} \log^{-1} x \log \left| \sum_{n=1}^{x} \lambda(n)\Lambda(n) \right| = 1. \tag{2.13}$$

Since $\lambda(n) \ge 0$, So $\sigma_c = \sigma_a = 1$.

This proves Lemma 3.

**Lemma 4.** *At* $s = 1$, $\Psi(s, \lambda) = \sum_{n=1}^{\infty} \lambda(n)\Lambda(n) n^{-s}$ *exists a pole, and the residue of function* $\Psi(s, \lambda)$ *is that*

$$\mathop{res}_{s=1} \Psi(s, \lambda) = \prod_{\substack{p\le\sqrt{N} \\ (p,N)=1}} \left(1 - \frac{1}{p-1}\right). \tag{2.14}$$

**Proof.** Because

$$\lim_{s\to 1} (1-s) \sum_{n=1}^{\infty} \Lambda(n) n^{-1} = 1, \tag{2.15}$$

and

$$\lim_{x\to\infty} \sum_{\substack{n\le x \\ n\equiv N \bmod q}} \Lambda(n) n^{-1} = \frac{1}{\phi(q)} \sum_{n=1}^{\infty} \Lambda(n) n^{-1}, \tag{2.16}$$

consequently

$$\lim_{x\to\infty} \sum_{n\le x} \lambda(n)\Lambda(n) n^{-1} = \prod_{\substack{p\le\sqrt{N} \\ (p,N)=1}} \left(1 - \frac{1}{\phi(p)}\right) \sum_{n=1}^{\infty} \Lambda(n) n^{-1}$$

$$= \prod_{\substack{p \le \sqrt{N} \\ (p,N)=1}} \left(1 - \frac{1}{p-1}\right) \sum_{n=1}^{\infty} \Lambda(n) n^{-1} \tag{2.17}$$

Therefore

$$\underset{s=1}{res}\, \Psi(s,\lambda) = \lim_{s \to 1} (1-s) \sum_{n=1}^{\infty} \lambda(n) \Lambda(n) n^{-1}$$

$$= \prod_{\substack{p \le \sqrt{N} \\ (p,N)=1}} \left(1 - \frac{1}{p-1}\right) \lim_{s \to 1} (1-s) \sum_{n=1}^{\infty} \Lambda(n) n^{-1}$$

$$= \prod_{\substack{p \le \sqrt{N} \\ (p,N)=1}} \left(1 - \frac{1}{p-1}\right). \tag{2.18}$$

Lemma 4 is proved.

## 3. Proof of Theorem 1

For Dirichlet series (1.6)

$$\Psi(s,\lambda) = \sum_{n=1}^{\infty} \lambda(n) \Lambda(n) n^{-s},$$

let $a = 1 - c_1 \log^{-1}(T+2)$, $b = 1 + \log^{-1} x$, $\log T = (\log x)^{1/(1+\alpha)}$ $(0 < \alpha < 1)$, we have

$$H(u) \le \log u, \tag{3.1}$$

and

$$B(u) \le c_2 \log x, \tag{3.2}$$

where $c_2$ is a positive constant.

By (2.5) in Lemma 1, we have

$$\psi(x,\lambda) = \sum_{n \le x} \lambda(n) \Lambda(n)$$

$$= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \Psi(s,\lambda) \frac{x^s}{s} ds + R_1, \tag{3.3}$$

where

$$R_1 \ll \frac{x \log^2 x}{T} \le x^{\tau} \log^2 x, 0 < \tau < 1. \tag{3.4}$$

By letting $a \pm iT$, $b \pm iT$ to be the fixed points for closed contour $\Gamma$, we then have

$$\psi(x,\lambda)=\frac{1}{2\pi i}\int_{\Gamma}\Psi(s,\lambda)\frac{x^s}{s}ds+R_1+R_2, \tag{3.5}$$

where

$$R_2 \ll \frac{1}{2\pi i}\left(\int_{b-iT}^{a-iT}+\int_{a-iT}^{a+iT}+\int_{a+iT}^{b+iT}\right)\Psi(s,\lambda)\frac{x^s}{s}ds$$

$$\le \frac{1}{2\pi i}\left(\int_{b-iT}^{a-iT}+\int_{a-iT}^{a+iT}+\int_{a+iT}^{b+iT}\right)\sum_{n=1}^{\infty}\Lambda(n)n^{-s}\frac{x^s}{s}ds$$

$$=\frac{1}{2\pi i}\left(\int_{b-iT}^{a-iT}+\int_{a-iT}^{a+iT}+\int_{a+iT}^{b+iT}\right)-\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds$$

$$\ll \frac{1}{2\pi i}\left(\int_{b-iT}^{a-iT}+\int_{a-iT}^{a+iT}+\int_{a+iT}^{b+iT}\right)\log^2|t|\frac{x^s}{s}ds$$

$$\ll xe^{-c\sqrt{\log x}}. \tag{3.6}$$

**Remark:** When $\sigma>1-c_3\log^2|t|$, $\sum_{n=1}^{\infty}\Lambda(n)n^{-s}=-\frac{\zeta'}{\zeta}(\sigma+it)\ll\log^2|t|$ .See [8].

By Lemma 2 and Lemma 4, $\zeta(s,\lambda)$ has not zero point in closed contour $\Gamma$. So function $\Psi(s,\lambda)\frac{x^s}{s}$ only exists a pole at $s=1$, we obtain

$$\operatorname*{res}_{s=1}\Psi(s,\lambda)\frac{x^s}{s}=\frac{1}{2\pi i}\int_{\Gamma}\Psi(s,\lambda)\frac{x^s}{s}ds$$

$$=x\prod_{\substack{p\le\sqrt{N}\\(p,N)=1}}\left(1-\frac{1}{p-1}\right). \tag{3.7}$$

By (3.5) and (3.7), we have

$$\psi(x,\lambda)=x\prod_{\substack{p\le\sqrt{N}\\(p,N)=1}}\left(1-\frac{1}{p-1}\right)+R_1+R_2. \tag{3.8}$$

Thus, combining (3.8) with (3.4) and (3.6) we have

$$\psi(x,\lambda)=x\prod_{\substack{p\le\sqrt{N}\\(p,N)=1}}\left(1-\frac{1}{p-1}\right)+O\left(xe^{-c\sqrt{\log x}}\right). \tag{3.9}$$

Let $x = N$, and $N$ is any larger number, we then obtain (1.10), namely,

$$\psi(N,\lambda) = N \prod_{\substack{p\le\sqrt{N} \\ (p,N)=1}} \left(1-\frac{1}{p-1}\right) + O\left(N\,\mathrm{e}^{-c\sqrt{\log N}}\right).$$

This completely proves Theorem 1.

## 4. Proof of Theorem 2

Combining (3.9) with (1.7), (1.8) and (1.9) we have

$$\pi(x,\lambda) = \frac{x}{\log x} \prod_{\substack{p\le\sqrt{N} \\ (p,N)=1}} \left(1-\frac{1}{p-1}\right) + R_3 \tag{4.1}$$

where

$$R_3 \ll \int_2^x \frac{\psi(u,\lambda)}{u\log^2 u}du + \frac{xe^{-c\sqrt{\log x}}}{\log x} \le \int_2^x \frac{\psi(u)}{u\log^2 u}du + xe^{-c\sqrt{\log x}}. \tag{4.2}$$

By the Prime Number Theorem

$$\psi(x) = x + O\left(xe^{-c\sqrt{\log x}}\right), \tag{4.3}$$

we have

$$\int_2^x \frac{\psi(u)}{u\log^2 u}du \ll \int_2^x \frac{u}{u\log^2 u}du + \int_2^x \frac{ue^{-c\sqrt{\log u}}}{u\log^2 u}du \ll xe^{-c\sqrt{\log x}}. \tag{4.4}$$

By (4.1), (4.2) and (4.4), we have

$$\pi(x,\lambda) = \frac{x}{\log x} \prod_{\substack{p\le\sqrt{N} \\ (p,N)=1}} \left(1-\frac{1}{p-1}\right) + O\left(xe^{-c\sqrt{\log x}}\right). \tag{4.5}$$

Let $x = N$, and $N$ is any larger number, we then obtain (1.11), i.e.,

$$\pi(N,\lambda) = \frac{N}{\log N} \prod_{\substack{p\le\sqrt{N} \\ (p,N)=1}} \left(1-\frac{1}{p-1}\right) + O\left(Ne^{-c\sqrt{\log N}}\right).$$

Theorem 2 is proved. Therefore, when $N$ is any larger number we have $\pi(N,\lambda) > 0$. Thus, the Goldbach Conjecture｛1, 1｝ is established.

**References**


[1] V. Brun, De cribe d'Eratosth6ne et le th6orhme de Goldbach, Videnskabs-selskabt: Kristiania Skrifter I, Mate-Naturvidenskapelig Klasse. 3 (1920) 1–36.

[2] A. R´enyi. On the representation of an even number as the sum of a single prime and a single almost-prime number. Dokl. Akad. Nauk SSSR. 56 (1947), 455–458.

[3] C.T. Pan, On the representation of an even number as the sum of a prime and of an almost prime, Acta Mathematica Sinica. 12 (1962) 95–106.

[4] Y. Wang, On the representation of a large integer as the sum of a prime and of an almost prime, Sci. Sin. 11 (1962) 1033–1054.

[5] H.E. Richert, Selberg's sieve with weights, Mathemetika. 16 (1969) 1–22.

[6] J.R. Chen, On the representation of a large even integer as the sum of a prime and the product of at most two primes, Sci. Sin. 17 (1973) 157–176.

[7] H.L. Montgomery, R.C. Vaughan, The exceptional set in Goldbach's problem, Acta. Arith. 27 (1975) 353–370.

[8] C.D. Pan, C.B. Pan, Basic Analytic Number Theory, Science Press, Beijing, 1999.