Study on the Hilbert's Eighth Problem

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Abstract

In this paper, we first prove Riemann Hypothesis and General Riemann Hypothesis. Then, we improve the result of the prime number theorems for arithmetic progression. Finally, we provide a proof that Goldbach Conjecture and Twin Prime Conjecture are established. Thus, we have resolved the Hilbert's Eighth Problem.

Keywords: Riemann Hypothesis Goldbach Conjecture Twin Prime Conjecture

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Notation:

$\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$	Riemann zeta Function
$L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s}$	Dirichlet L-Function
$\frac{1}{\Gamma(s)}=se^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-s/n}$	$\Gamma(s)$ as Γ-function
$\gamma=0.5772\ldots$	Euler constant
χ,χ^{0},χ^{*}	Character, principal character, primitive character
$\tau(\chi)=\sum_{n=1}^{q}\chi(n)e\left(\frac{n}{q}\right)$	One of Gauss sum
$\frac{F'}{F}(s)=\frac{F'(s)}{F(s)}$	
$\phi(n)=n\prod_{p\mid n}\left(1-\frac{1}{p}\right)$	Euler function
$\Lambda(n)=\begin{cases}\log p, n=p^{m}\\ 0, otherwise\end{cases}$	Mangoldt function

$$\psi(x)=\sum_{n\le x}\Lambda(n)$$

$$\theta(x)=\sum_{p\le x}\log p$$

$$\pi(x)=\sum_{p\le x}1$$

$$\psi(x;q,l)=\sum_{\substack{n\le x\\ n\equiv l \bmod q}}\Lambda(n)$$

$$\psi(x;b,q,l)=\sum_{\substack{bn\le x\\ bn\equiv l \bmod q}}\Lambda(n)$$

$$\psi(x,\chi)=\sum_{n\le x}\Lambda(n)\chi(n)$$

$$\theta(x;q,l)=\sum_{\substack{p\equiv l \bmod q\\ 2<p\le x}}\log p$$

$$\theta(x;b,q,l)=\sum_{\substack{bp\equiv l \bmod q\\ 2<p\le x}}\log p$$

$$C(N)=\prod_{2<p|N}\frac{p-1}{p-2}\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)$$

0. Introduction

In 1900, Hilbert suggested 23 famous mathematic problems, whose Eighth Problem includes the following hypotheses: 1) Riemann Hypothesis (RH): all of non-trivial zero points of $\zeta(s)$ have real part equal to $1/2$; 2) General Riemann Hypothesis (GRH): all of non-trivial zero points of $L(s,\chi)$ have real part equal to $1/2$; 3) Goldbach Conjecture: any even integer >2 is sum of two primes; 4) Twin Prime Conjecture: the number of twin primes is infinite.

Although a number of mathematicians have attempted to prove or disproved the above four hypotheses [1-39], none of them has been completely resolved up to now. In this article, we will prove all of these hypotheses using some novel ways. To prove some relevant theorems, we need the following Lemmas.

1. Lemmas

Lemma 1: *In 1859, Riemann defined zeta-function* $\zeta(s)$ *and function* $\xi(s)$,

$$\zeta(s)=\sum_{n=0}^{\infty}n^{-s}, \tag{1.1.1}$$

and

$$\xi(s)=\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{1.1.2}$$

Functions $\zeta(s)$ *and* $\xi(s)$ *satisfied functional equations*

$$\zeta(s)=\pi^{s-½}\Gamma\left(\frac{1-s}{2}\right)\Big/\Gamma\left(\frac{s}{2}\right)\zeta(1-s), \tag{1.1.3}$$

$$\xi(s)=\xi(1-s), \tag{1.1.4}$$

and

$$\frac{\xi'}{\xi}(s)=-\frac{\xi'}{\xi}(1-s). \tag{1.1.5}$$

Lemma 2: *For Dirichlet L-function* $L(s,\chi)$ *and function* $\xi(s,\chi)$,

$$L(s,\chi)=\sum_{n=0}^{\infty}\chi(n)n^{-s}, \tag{1.2.1}$$

and

$$\xi(s,\chi)=\left(\frac{q}{\pi}\right)^{\frac{1}{2}(s+\delta)}\Gamma\left(\frac{s+\delta}{2}\right)L(s,\chi), \tag{1.2.2}$$

(*where* $\delta=\delta(\chi)=\frac{1}{2}(1-\chi(-1))$),

there are functional equations

$$L(s,\chi)=\frac{\tau(\chi)}{i^{\delta}\sqrt{q}}\left(\frac{q}{\pi}\right)^{\frac{1}{2}-s}\Gamma\left(\frac{1-s+\delta}{2}\right)\Big/\Gamma\left(\frac{s+\delta}{2}\right)L(1-s,\bar{\chi}), \tag{1.2.3}$$

$$\xi(s,\chi)=\frac{i^{\delta}\sqrt{q}}{\tau(\bar{\chi})}\xi(1-s,\bar{\chi}), \tag{1.2.4}$$

and

$$\frac{\xi'}{\xi}(s,\chi)=-\frac{\xi'}{\xi}(1-s,\chi). \quad (1.2.5)$$

Lemma 3: *In deleted neighborhood of anyone non-trivial zero point of function* $\zeta(s)$, *we have equations:*

$$\frac{\zeta'}{\zeta}(s)+\frac{\zeta'}{\zeta}(1-s)+2\log\pi=\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right), \quad (1.3.1)$$

and

$$\frac{\zeta'}{\zeta}(\overline{s})+\frac{\zeta'}{\zeta}(1-\overline{s})+2\log\pi=\frac{\Gamma'}{\Gamma}\left(\frac{\overline{s}}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1-\overline{s}}{2}\right). \quad (1.3.2)$$

Proof: In deleted neighborhood of anyone non-trivial zero point of function $\zeta(s)$, by (1.1.2), we have

$$\frac{\xi'}{\xi}(s)=\frac{\zeta'}{\zeta}(s)+\frac{1}{s}+\frac{1}{s-1}-\frac{1}{2}\log\pi+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right), \quad (1.3.3)$$

and

$$\frac{\xi'}{\xi}(1-s)=\frac{\zeta'}{\zeta}(1-s)-\frac{1}{s}-\frac{1}{s-1}-\frac{1}{2}\log\pi+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right). \quad (1.3.4)$$

Combining (1.1.5), we have

$$\frac{\zeta'}{\zeta}(s)+\frac{\zeta'}{\zeta}(1-s)+2\log\pi=\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right). \quad (1.3.5)$$

In a same way, we have

$$\frac{\zeta'}{\zeta}(\overline{s})+\frac{\zeta'}{\zeta}(1-\overline{s})+2\log\pi=\frac{\Gamma'}{\Gamma}\left(\frac{\overline{s}}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1-\overline{s}}{2}\right). \quad (1.3.6)$$

Lemma 3 is proved.

Lemma 4: *In deleted neighborhood of anyone non-trivial zero point of function* $L(s,\chi)$, *we have equations:*

$$\frac{L'}{L}(s,\chi)+\frac{L'}{L}(1-s,\chi)+2\log\pi=\frac{\Gamma'}{\Gamma}\left(\frac{s+\delta}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1-s+\delta}{2}\right), \quad (1.4.1)$$

and

$$\frac{L'}{L}(s,\chi)+\frac{L'}{L}(1-s,\chi)+2\log\pi=\frac{\Gamma'}{\Gamma}\left(\frac{s+\delta}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1-s+\delta}{2}\right). \quad (1.4.2)$$

Proof: In deleted neighborhood of anyone non-trivial zero point of function $L(s,\chi)$, by (1.2.2), we obtain

$$\frac{\xi'}{\xi}(s,\chi)=\frac{L'}{L}(s,\chi)+\frac{1}{s}+\frac{1}{s-1}-\frac{1}{2}\log\pi+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+\delta}{2}\right), \qquad (1.4.3)$$

and

$$\frac{\xi'}{\xi}(1-s,\chi)=\frac{L'}{L}(1-s,\chi)-\frac{1}{s}-\frac{1}{s-1}-\frac{1}{2}\log\pi+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s+\delta}{2}\right), \qquad (1.4.4)$$

combining (1.2.5), we have

$$\frac{L'}{L}(s,\chi)+\frac{L'}{L}(1-s,\chi)+2\log\pi=\frac{\Gamma'}{\Gamma}\left(\frac{s+\delta}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1-s+\delta}{2}\right). \qquad (1.4.5)$$

In a same way, we have

$$\frac{L'}{L}(s,\chi)+\frac{L'}{L}(1-s,\chi)+2\log\pi=\frac{\Gamma'}{\Gamma}\left(\frac{s+\delta}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1-s+\delta}{2}\right). \qquad (1.4.6)$$

Lemma 4 is proved.

Lemma 5: *There exists the inequality, for* $\chi^{*}\neq\chi^{0}$, *and* $\rho=\beta+i\gamma$ *is non-trivial zero point of* $L(s,\chi)$,

$$\psi(x,\chi^{*})=\sum_{|\gamma|\le T}\frac{x^{\beta}}{1+|\gamma|}+O\left(\frac{x\log^{2}(xqT)}{T}\right). \qquad (1.5.1)$$

Lemma 6: *According to equivalent relation about* $\theta(x)$ *and* $\psi(x)$, *we have*

$$\left|\theta(x;q,l)-x/\phi(q)\right|=\left|\psi(x;q,l)-x/\phi(q)\right|, \qquad (1.6.1)$$

$$\sum_{2\le q\le D}\left|\theta(x;q,l)-x/\phi(q)\right|=\sum_{2\le q\le D}\left|\psi(x;q,l)-x/\phi(q)\right|, \qquad (1.6.2)$$

and

$$\sum_{2\le q\le D}\left|\theta(x;b,q,l)-\frac{x/b}{\phi(q)}\right|=\sum_{2\le q\le D}\left|\psi(x;b,q,l)-\frac{x/b}{\phi(q)}\right|. \qquad (1.6.3)$$

Lemma 7: *We define a new weighted sieve function*

$$S(A,z)=\sum_{\substack{p\equiv l \bmod a\\ (a,P(z))=1}}\log p, \qquad (1.7.1)$$

where $P(z)=\prod_{2\le p\le z}p$, *and* $a\in A=\{a:2\le a\le x\}$.

Let $\omega(d)=\dfrac{d}{\phi(d)}$, then

$$0\le\frac{\omega(p)}{p}=\frac{1}{\phi(p)}<1,$$

and

$$\left|\sum_{w\le p\le z}\omega(p)\log p/p-\log(z/w)\right|\le L, \text{ (}L \text{ is a positive constant).}$$

Therefore $S(A,z)$ is a line sieve function.

According to Rosser sieve method, let $2\le z\le y^{1/2}, A=\{a:a=p-l, 2\le p\le N\}$, and $X=\theta(N)=N$, we have

$$S(A,z)\ge 2e^{-\gamma}C(N)f(u)\left(1+O\left(\frac{1}{\log z}\right)\right)\frac{N}{\log z}-\sum_{d\le y}|r_d|, \tag{1.7.2}$$

$$S(A,z)\le 2e^{-\gamma}C(N)\left(1+O\left(\frac{1}{\log z}\right)\right)F(u)\frac{N}{\log z}+\sum_{d\le y}|r_d|, \tag{1.7.3}$$

where

$$r_d=\theta(N;d,l)-\frac{N}{\phi(d)}, \tag{1.7.4}$$

$$u=\log y/\log z, \tag{1.7.5}$$

$$f(u)=2e^{\gamma}u^{-1}\log(u-1), \text{ if } 2\le u\le 4, \tag{1.7.6}$$

and

$$F(u)=2e^{\gamma}u^{-1}, \text{ if } 2\le u\le 3. \tag{1.7.7}$$

2. Theorems

Theorem 1: *All of non-trivial zero points of* $\zeta(s)$ *have real part equal to* $1/2$.

Proof: It is well known that all non-trivial zero points of $\zeta(s)$ are in complex region $0\le \mathrm{Re}(s)<1$. Let $\rho=\beta+i\gamma$ is anyone non-trivial zero point of $\zeta(s)$, then $\overline{\rho}$, $1-\rho$ and $1-\overline{\rho}$ are also non-trivial zero points of $\zeta(s)$. Because $\zeta(\overline{s})=\overline{\zeta(s)}$, in deleted neighborhood of non-trivial zero point of $\zeta(s)$, we have

$$\lim_{s\to\rho}\frac{\zeta'}{\zeta}(s)=\frac{\zeta'}{\zeta}(\overline{s}), \tag{2.1.1}$$

and

$$\lim_{s\to\rho}\frac{\zeta'}{\zeta}(1-s)=\frac{\zeta'}{\zeta}(1-\overline{s}). \tag{2.1.2}$$

Combining (2.1.1), (2.1.2), (1.3.1) and (1.3.2), in deleted neighborhood $0<|s-\rho|<\varepsilon$, we obtain

$$\lim_{s\to\rho}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{\overline{s}}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{1-\overline{s}}{2}\right)=0. \tag{2.1.3}$$

Since

$$\frac{\Gamma'}{\Gamma}(z)=\frac{1}{z}+\gamma+\sum_{n=1}^{\infty}\left(\frac{1}{n+z}-\frac{1}{n}\right), \tag{2.1.4}$$

$$\frac{\Gamma'}{\Gamma}(z_1)-\frac{\Gamma'}{\Gamma}(z_2)=\sum_{n=0}^{\infty}\left(\frac{1}{n+z_2}-\frac{1}{n+z_1}\right), \tag{2.1.5}$$

then

$$\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{\overline{s}}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{1-\overline{s}}{2}\right)$$
$$=\sum_{n=0}^{\infty}\left(\frac{1}{n+\frac{1-\overline{s}}{2}}-\frac{1}{n+\frac{s}{2}}\right)+\left(\frac{1}{n+\frac{\overline{s}}{2}}-\frac{1}{n+\frac{1-s}{2}}\right). \tag{2.1.6}$$

Defining $s=\frac{1}{2}+\alpha+i\gamma$, $\overline{s}=\frac{1}{2}+\alpha-i\gamma$, $1-s=\frac{1}{2}-\alpha-i\gamma$, and $1-\overline{s}=\frac{1}{2}-\alpha+i\gamma$, where $0\le\alpha\le\frac{1}{2}$, by (2.1.3) and (2.1.6), we have

$$\lim_{s\to\rho}\sum_{n=0}^{\infty}\left(\frac{1}{n+\frac{1}{4}-\frac{\alpha}{2}+\frac{i\gamma}{2}}-\frac{1}{n+\frac{1}{4}+\frac{\alpha}{2}+\frac{i\gamma}{2}}\right)+\sum_{n=0}^{\infty}\left(\frac{1}{n+\frac{1}{4}+\frac{\alpha}{2}-\frac{i\gamma}{2}}-\frac{1}{n+\frac{1}{2}-\frac{\alpha}{2}-\frac{i\gamma}{2}}\right)=0$$

$$\Rightarrow\lim_{s\to\rho}\left(\sum_{n=0}^{\infty}\frac{\alpha\gamma\left(n+\frac{1}{4}\right)}{\left(\left(n+\frac{1}{4}\right)^2-\frac{\gamma^2}{4}-\frac{\alpha^2}{4}\right)^2+\gamma^2\left(n+\frac{1}{4}\right)^2}\right)=0. \tag{2.1.7}$$

Since all non-trivial zero points of $\zeta(s)$ are complex zero points, $\gamma \neq 0$, and

$$\left(\sum_{n=0}^{\infty} \frac{\gamma\left(n+\frac{1}{4}\right)}{\left(\left(n+\frac{1}{4}\right)^2-\frac{\gamma^2}{4}-\frac{\alpha^2}{4}\right)^2+\gamma^2\left(n+\frac{1}{4}\right)^2} \right) \neq 0, \qquad (2.1.8)$$

then

$$\lim_{s\to\rho} \frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{\overline{s}}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{1-\overline{s}}{2}\right)=0 \Rightarrow \alpha=0. \qquad (2.1.9)$$

Furthermore we have

$$\lim_{s\to\rho} \zeta(s)=0 \Rightarrow \operatorname{Re}(s)=\tfrac{1}{2}. \qquad (2.1.10)$$

This deduction is suitable for all of non-trivial zero points of $\zeta(s)$. Consequently, Theorem 1 is established and RH is true

Theorem 2: *All of non-trivial zero points of* $L(s,\chi)$ *have real part equal to* $\frac{1}{2}$.

Proof: Let $\rho=\beta+i\gamma$ is any of non-trivial zero points of $L(s,\chi)$, then

$$\lim_{s\to\rho} \frac{L'}{L}(s,\chi)=\frac{L'}{L}(\overline{s},\chi), \qquad (2.2.1)$$

and

$$\lim_{s\to\rho} \frac{L'}{L}(1-s,\chi)=\frac{L'}{L}(1-\overline{s},\chi). \qquad (2.2.2)$$

Combining (2.21), (2.22), (1.4.1) and (1.4.2), in deleted neighborhood $0<|s-\rho|<\varepsilon$, we have

$$\lim_{s\to\rho} \frac{\Gamma'}{\Gamma}\left(\frac{s+\delta}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1-s+\delta}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{\overline{s}+\delta}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{1-\overline{s}+\delta}{2}\right)=0. \qquad (2.2.3)$$

Similarly, defining $s=\frac{1}{2}+\alpha+i\gamma$, $\overline{s}=\frac{1}{2}+\alpha-i\gamma$, $1-s=\frac{1}{2}-\alpha-i\gamma$ and $1-\overline{s}=\frac{1}{2}-\alpha+i\gamma$, where $0\le\alpha\le\frac{1}{2}$, in deleted neighborhood $0<|s-\rho|<\varepsilon$, we have

$$\lim_{s\to\rho} \left(\sum_{n=0}^{\infty} \frac{\alpha\gamma\left(n+\frac{\delta}{2}+\frac{1}{4}\right)}{\left(\left(n+\frac{\delta}{2}+\frac{1}{4}\right)^2-\frac{\gamma^2}{4}-\frac{\alpha^2}{4}\right)^2+\gamma^2\left(n+\frac{\delta}{2}+\frac{1}{4}\right)^2} \right)=0. \qquad (2.2.4)$$

Since

$$\lim_{s\to\rho}\left(\sum_{n=0}^{\infty}\frac{\gamma\left(n+\frac{\delta}{2}+\frac{1}{4}\right)}{\left(\left(n+\frac{\delta}{2}+\frac{1}{4}\right)^2-\frac{\gamma^2}{4}-\frac{\alpha^2}{4}\right)^2+\gamma^2\left(n+\frac{\delta}{2}+\frac{1}{4}\right)^2}\right)\neq 0, \quad (2.2.5)$$

then

$$\lim_{s\to\rho}\frac{\Gamma'}{\Gamma}\left(\frac{s+\delta}{2}\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1-s+\delta}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{\overline{s}+\delta}{2}\right)-\frac{\Gamma'}{\Gamma}\left(\frac{1-\overline{s}+\delta}{2}\right)=0\Rightarrow\alpha=0, \quad (2.2.6)$$

namely

$$\lim_{s\to\rho}L(s,\chi)=0\Rightarrow \mathrm{Re}(s)=\tfrac{1}{2}. \quad (2.2.7)$$

This deduction is suitable for all of non-trivial zero points of $L(s,\chi)$. Consequently, Theorem 2 is proved, and GRH is true

Theorem 3: *There exists a calculable constant for* O *and* $=$ *, such that*

$$\psi(x;q,l)=x/\phi(q)+O\left(x^{1/2}\log^2 x\right), \quad (2.3.1)$$

or

$$E(x;q,l)=\psi(x;q,l)-x/\phi(q)=\ x^{1/2}\log^2 x. \quad (2.3.2)$$

Proof: Let $\chi\neq\chi^0,(q,l)=1$, then

$$\psi(x;q,l)=x\phi^{-1}(q)+\phi^{-1}(q)\sum_{\chi \bmod q}\overline{\chi}(l)\psi(x,\chi), \quad (2.3.3)$$

and

$$E(x;q,l)=\phi^{-1}(q)\sum_{\chi \bmod q}\overline{\chi}(l)\psi(x,\chi)$$
$$=\ \phi^{-1}(q)\sum_{\chi \bmod q}\left|\overline{\chi}(l)\right|\psi(x,\chi). \quad (2.3.4)$$

Since $\phi^{-1}(q)\sum_{\chi \bmod q}1<1,\ \left|\chi(n)\right|=1$, we have

$$E(x;q,l)=\ \max\psi(x,\chi)$$
$$=\ \max\psi\left(x,\chi^*\right)+O(\log x\log q). \quad (2.3.5)$$

Let $\rho=1/2+i\gamma$ is non-trivial zero point of $L(s,\chi)$, we obtain

$$E(x;q,l)=\sum_{|\gamma|\le T}\frac{x^{1/2}}{1+|\gamma|}+O\left(\frac{x\log^2(xqT)}{T}\right)+O(\log x\log q). \quad (2.3.6)$$

Let $2\le q\le x, T=x^{1/2}$, then

$$E(x;q,l)= x^{1/2}\log^2 x. \quad (2.3.7)$$

Theorem 3 is true.

Theorem 4: *There exist positive numbers* $B\ge A+2$, *such that*

$$\sum_{2\le q\le x\log^{-B}x}|E(x;q,l)|= x\log^{-A}x. \quad (2.4.1)$$

Proof: Let $\rho(q)=1/2+i\gamma(q)$ is non-trivial zero point of $L(s,\chi \bmod q)$, then

$$\sum_{2\le q\le D}|E(x;q,l)|=\sum_{2\le q\le D}\sum_{|\gamma(q)|\le T}\frac{x^{1/2}}{1+|\gamma(q)|}+O\left(\frac{xD\log^2(xDT)}{T}\right)+O(D\log x\log D) \quad (2.4.2)$$

Since $\chi^*(q_1)\ne\chi^*(q_2)\Rightarrow L(s,\chi^* \bmod q_1)\ne L(s,\chi^* \bmod q_2)$, all of non-trivial zero point of $L(s,\chi \bmod q_n)$ are different, then

$$\sum_{2\le q\le D}\sum_{|\gamma(q)|\le T}\frac{x^{1/2}}{1+|\gamma(q)|}=\sum_{|\gamma(q_n)|\le T}\frac{x^{1/2}}{1+|\gamma(q_n)|}=\sum_{|\gamma|\le T}\frac{x^{1/2}}{1+|\gamma|}, \quad (2.4.3)$$

where $|\gamma|\le T$ includes all of the different non-trivial zero points of $L(s,\chi \bmod q_n)$. Furthermore we obtain

$$\sum_{|\gamma|\le T}\frac{x^{1/2}}{1+|\gamma|}= x^{1/2}\int_{-T}^{T}\frac{1}{1+t}dt= x^{1/2}\log T. \quad (2.4.4)$$

Therefore we have

$$\sum_{2\le q\le D}|E(x;q,l)|= x^{1/2}\log T+O\left(\frac{xD\log^2(xDT)}{T}\right)+O(D\log x\log D). \quad (2.4.5)$$

Let $T=x, D=x\log^{-B}x$, $B\ge A+2$, then

$$\sum_{2\le q\le D}|E(x;q,l)|= x\log^{-A}x. \quad (2.4.6)$$

Theorem 4 is established.

Theorem 5: *there exist positive numbers* $B\ge A+2$, *such that*

$$\sum_{2\le q\le x\log^{-B}x}\sum_{1\le b\le x^{1/2}}|E(x;b,q,l)|= x\log^{-A}x, \quad (2.5.1)$$

where

$$E(x;b,q,l)=\psi(x;b,q,l)-(x/b)/\phi(q). \tag{2.5.2}$$

Proof: Let $\chi \neq \chi^0$, then

$$\begin{aligned} E(x;b,q,l) &= \phi^{-1}(q)\sum_{\chi \bmod q}\bar{\chi}(l)\chi(b)\sum_{bn\le x}\Lambda(n)\chi(n) \\ &= \max\sum_{bn\le x}\Lambda(n)\chi(n) \\ &= \max\psi(x/b,\chi) \\ &= \max\psi(x/b,\chi^*)+O(\log(x/b)\log q). \end{aligned} \tag{2.5.3}$$

Furthermore, we have

$$E(x;b,q,l)=\sum_{|\gamma(q)|\le T}\frac{(x/b)^{1/2}}{1+|\gamma(q)|}+O\left(\frac{x/b\log^2(xqT)}{T}\right)+O(\log(x/b)\log q), \tag{2.5.4}$$

and

$$\begin{aligned} &\sum_{2\le q\le D}\sum_{1\le b\le x^{1/2}}|E(x;b,q,l)| \\ &= x^{1/2}\log x\log T+O\left(\frac{xD\log x\log^2(xDT)}{T}\right)+O(D\log^2 x\log D). \end{aligned} \tag{2.5.5}$$

Let $T=x\log x$, $D=x\log^{-B}x$, $B\ge A+2$, then

$$\sum_{2\le q\le D}\sum_{1\le b\le x^{1/2}}|E(x;b,q,l)|= x\log^{-A}x. \tag{2.5.6}$$

Theorem 5 is proved.

Theorem 6: *Any large even integer is sum of two primes, and we have*

$$\sum_{p=N-p_1}\log p>4(2\log 2-\log 3)C(N)\frac{N}{\log N}+O\left(\frac{N}{\log^A N}\right), \tag{2.6.1}$$

where $2<p<N$.

Proof: Let N is any large even integer, $A=\{a:a=N-p,2<p\le N\}$, then

$$\sum_{p=N-p_1}\log p\ge S(A,x^{1/2})\ge S(A,x^{1/3})-\sum_{\substack{N^{1/3}\le p\le N^{1/2}\\ p_2p=N-p_1}}S(A(p),x^{1/3}). \tag{2.6.2}$$

Let $X=N$, $X(p)=N/p$, and $\omega(d)=\dfrac{d}{\phi(d)}$, then

$$\frac{\omega(d)}{d}=\frac{1}{\phi(d)}, \tag{2.6.3}$$

$$r_d=\theta(N;d,N)-\frac{N}{\phi(d)}=\left|E(N;d,N)\right|, \tag{2.6.4}$$

and

$$r_d(p)=\theta(N;p,d,N)-\frac{N/p}{\phi(d)}=\left|E(N;p,d,N)\right|. \tag{2.6.5}$$

By Theorems 5 and 6, we have

$$\sum_{d\le N\log^{-B}N}\left|r_d\right|=\sum_{d\le N\log^{-B}N}\left|E(N;d,N)\right|= N\log^{-A}N, \tag{2.6.6}$$

and

$$\sum_{d\le N\log^{-B}N}\sum_{N^{1/3}\le p\le N^{1/2}}\left|r_d(p)\right|=\sum_{d\le N\log^{-B}N}\sum_{N^{1/3}\le p\le N^{1/2}}\left|E(N;p,d,N)\right|= N\log^{-A}N. \tag{2.6.7}$$

By Lemma 7,

$$S\left(A,N^{1/3}\right)\ge 6e^{-\gamma}\left(1+o(1)\right)f(3)C(N)\frac{N}{\log N}+O\left(\frac{N}{\log^A N}\right), \tag{2.6.8}$$

$$S\left(A(p),N^{1/3}\right)\le 6e^{-\gamma}\left(1+o(1)\right)F(3)C(N)\frac{N/p}{\log N}+O\left(\sum_{d\le N\log^{-B}N}\left|r_d(p)\right|\right), \tag{2.6.9}$$

and

$$\sum_{N^{1/3}\le p\le N^{1/2}}S\left(A(p),N^{1/3}\right)\le 6e^{-\gamma}\left(1+o(1)\right)F(3)C(N)\frac{N}{\log N}\sum_{N^{1/3}\le p\le N^{1/2}}\frac{1}{p}+O\left(\frac{N}{\log^A N}\right). \tag{2.6.10}$$

Since

$$\begin{aligned}\sum_{N^{1/3}\le p\le N^{1/2}}\frac{1}{p}&=\log\frac{\log\left(N^{1/2}\right)}{\log\left(N^{1/3}\right)}+O\left(\frac{1}{\log N}\right)\\&=\log 3-\log 2+O\left(\frac{1}{\log N}\right),\end{aligned} \tag{2.6.11}$$

therefore

$$\sum_{N^{1/3}\le p\le N^{1/2}}S\left(A(p),N^{1/3}\right)\le 6e^{-\gamma}\left(\log 3-\log 2\right)F(3)C(N)\frac{N}{\log N}+O\left(\frac{N}{\log^A N}\right). \tag{2.6.12}$$

Since $f(3)=\frac{2}{3}e^{\gamma}\log 2$, and $F(3)=\frac{2}{3}e^{\gamma}$, we obtain

$$\sum_{p=N-p_1} \log p \geq S\left(A, x^{1/3}\right) - \sum_{N^{1/3} \leq p \leq N^{1/2}} S\left(A(p), x^{1/3}\right)$$

$$\geq 4\left(2\log 2 - \log 3\right)C(N)\frac{N}{\log N} + O\left(\frac{N}{\log^{A} N}\right). \quad (2.6.13)$$

Since

$$\sum_{p=N-p_1} \log p > 0 \Rightarrow \sum_{N=p+p_1} 1 > 0, \quad (2.6.14)$$

Theorem 6 is proved, and Goldbach Conjecture is true.

Theorem 7: *Let* $2 < p < N$, *we have*

$$\sum_{p=p_1+2} \log p > 4\left(2\log 2 - \log 3\right)C(N)\frac{N}{\log N} + O\left(\frac{N}{\log^{A} N}\right). \quad (2.7.1)$$

Proof: Let $A = \{a : a = p+2, 2 < p \leq N\}$, then

$$\sum_{p=p_1+2} \log p \geq S\left(A, N^{1/2}\right) \geq S\left(A, N^{1/3}\right) - \sum_{\substack{N^{1/3} \leq p \leq N^{1/2} \\ p_2 p = p_1+2}} S\left(A(p), N^{1/3}\right). \quad (2.7.2)$$

Let $X = N$, $X(p) = N/p$, and $\omega(d) = \dfrac{d}{\phi(d)}$, then

$$r_d = \theta(N; d, 2) - \frac{N}{\phi(d)} = \left|E(N; d, 2)\right|, \quad (2.7.3)$$

and

$$r_d(p) = \theta(N; p, d, 2) - \frac{N/p}{\phi(d)} = \left|E(N; p, d, 2)\right|. \quad (2.7.4)$$

By Theorems 5 and 6, we have

$$\sum_{d \leq N\log^{-B} N} \left|r_d\right| = \sum_{d \leq N\log^{-B} N} \left|E(N; d, 2)\right| = N\log^{-A} N, \quad (2.7.5)$$

and

$$\sum_{d \leq N\log^{-B} N} \sum_{N^{1/3} \leq p \leq N^{1/2}} \left|r_d(p)\right| = \sum_{d \leq N\log^{-B} N} \sum_{N^{1/3} \leq p \leq N^{1/2}} \left|E(N; p, d, 2)\right| = N\log^{-A} N. \quad (2.7.6)$$

By Lemmas 7, we obtain

$$\sum_{p=p_1+2} \log p \geq 4\left(2\log 2 - \log 3\right)C(N)\frac{N}{\log N} + O\left(\frac{N}{\log^{A} N}\right). \quad (2.7.7)$$

Theorem 7 is proved. In addition, we have

$$N \to \infty \Rightarrow \sum_{p=p_1+2} \log p \to \infty \Rightarrow \sum_{p=p_1+2} 1 \to \infty . \quad (2.7.8)$$

That is to say, the number of twin primes is infinite, and Twin Prime Conjecture is true.

In conclusion, all the hypotheses of the Hilbert's Eighth Problem are proved to be true.

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