On the Existence of a Self-Similar Coarse Graining of a Self-Similar Space

Akihiko Kitada, Tomoyuki Yamamoto, Tsuyoshi Yoshioka and Shousuke Ohmori
Laboratory of mathematical design for materials,
Faculty of Science and Engineering, Waseda University,
3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555

Abstract
A topological space homeomorphic to a self-similar space is demonstrated to be self-similar. There exists a self-similar space S whose coarse graining is homeomorphic to S. The coarse graining of S is, therefore, self-similar again. In the same way, the coarse graining of the self-similar coarse graining of S is, furthermore, self-similar. These situations succeed endlessly. Such a self-similar S is generated actually from an intense quadratic dynamics.

Keywords: self-similar set, fractals, dynamical system, Cantor set, coarse graining

1 Introduction

In the fractal sciences, the fine structure of the self-similar space is characterized by the property that every details looks similar with the whole. In the present report, we are oppositely concerned with the coarse structures of a self-similar space, that is, with the problem "what self-similar space can have a coarse graining of it with a self-similarity again?". According to A. Fernández [1], the procedure of the coarse graining or the block construction [2] of a space in the statistical physics corresponds mathematically to that of the construction of a quotient space which is defined by a classification of all points in the space through the identification of the different points based on an equivalence relation.

At first, a sufficient condition for a given topological space to be metrizable and self-similar with respect to the metric is investigated, and, second, the existence of a decomposition space [3] as a coarse graining of a self-similar space S whose self-similarity is defined by a system of weak contractions which is topologically closely related to that defining the self-similarity of S

is discussed in a quite elementary way. As a consequence, we are convinced that there exists a sequence of self-similar coarse graining of a self-similar space even for the quadratic dynamics known to be one of the simplest dynamical system. Finally, it is noted that each step of the sequence can equally generate a topological space characteristic of condensed matter such as dendrite [4].

2 A condition for a topological space to be self-similar

An answer of the problem "for what topological space, can we find a system of weak contractions which makes the space self-similar?" is simply stated as follows.

Proposition. *The existence of a self-similar space which is homeomorphic [5] to (Y, τ) is sufficient for a topological space (Y, τ) to be a metrizable space and self-similar with respect to the metric.*

Proof. Let (X, τ_d) be self-similar based on a system of weak contractions $p_j : (X, \tau_d) \to (X, \tau_d),\ d(p_j(x), p_j(x')) \leq \alpha_j(\eta) d(x, x')$ for $d(x, x') < \eta,\ 0 \leq \alpha_j(\eta) < 1,\ j = 1, \ldots, m\ (2 \leq m < \infty)$. That is, $\bigcup_{j=1}^{m} p_j(X) = X$. Using a homeomorphism $h : (X, \tau_d) \simeq (Y, \tau)$, we can define a metric ρ on Y as

$$\rho(y, y') = d(h^{-1}(y), h^{-1}(y')), \quad y,\ y' \in Y.$$

The metric topology τ_ρ is identical with the initial topology τ . From the relations 1) and 2) below, the metric space (Y, τ_ρ) is confirmed to be self-similar by a system of weak contractions $q_j : (Y, \tau_\rho) \to (Y, \tau_\rho),\ j = 1, \ldots, m$ where q_j is topologically conjugate to p_j with the above homeomorphism h, that is, $q_j = h \circ p_j \circ h^{-1}$.

1) $\rho(q_j(y), q_j(y')) = d(h^{-1}(q_j(y)), h^{-1}(q_j(y')))$
$= d(p_j(h^{-1}(y)), p_j(h^{-1}(y'))) \leq \alpha_j(\eta) d(h^{-1}(y), h^{-1}(y'))$
$= \alpha_j(\eta) \rho(y, y')$ for $\rho(y, y') < \eta$.

2) $\bigcup_{j=1}^{m} q_j(Y) = \bigcup_{j=1}^{m} q_j(h(X)) = h(\bigcup_{j=1}^{m} p_j(X)) = h(X) = Y.$ $\square$

3 Existence of a self-similar decomposition space

As an application of Proposition, we will show the existence of a self-similar decomposition space of a self-similar space.

Let S be a self-similar, perfect [6], zero-dimensional (0-dim) [7], compact metric space, and (X, τ_d) be any compact metric space which is self-similar. Then, there exists a continuous map f from S onto X [8], and X is homeomorphic to the decomposition space $(\mathcal{D}_f, \tau(\mathcal{D}_f))$ of S with a homeomorphism $h : (X, \tau_d) \simeq (\mathcal{D}_f, \tau(\mathcal{D}_f)),\ x \mapsto f^{-1}(x)$ [9]. Here, $\mathcal{D}_f = \{f^{-1}(x) \subset S;\ x \in X\}$ and $\tau(\mathcal{D}_f) = \{\mathcal{U} \subset \mathcal{D}_f;\ \bigcup \mathcal{U} (= \bigcup_{D \in \mathcal{U}} D)$ is an open set of $S\}$. The decomposition topology $\tau(\mathcal{D}_f)$ is identical with a metric topology τ_ρ with a metric $\rho(y, y') = d(h^{-1}(y), h^{-1}(y')), y,\ y' \in \mathcal{D}_f$ [10]. Since the metric space (X, τ_d) is assumed to be self-similar, from Proposition, the decomposition space $(\mathcal{D}_f, \tau_\rho)$ must be self-similar based on a system of weak contractions each of which is topologically conjugate to each weak contraction which defines the self-similarity of X. According to the self-similarity of the selected space X, the decomposition space $\mathcal{D}_f$ of S can have various types of self-similarity.

Now, let us consider a special case where the system of contructions defining the self-similarity of the decomposition space $\mathcal{D}_f$ of S is topologically related to that defining the self-similarity of S. Let $\{S_1, \cdots, S_n\}$ be a partition of S [3] such that each S_i is a clopen (closed and open) set of S. (Concerning the existence of such partition of S, see Appendix.) Since the metric space S_1 is perfect, 0-dim, compact, it is homeomorphic to the Cantor's Middle Third Set (abbreviated to CMTS) [11] as well as the space S. Therefore, S_1 and S are homeomorphic. Let $f : S \to S_1$ be a *not one to one*, continuous, onto map. For example, the map $f : S \to S_1$ defined as $f(x) = x$ for $x \in S_1, f(x) \equiv q_2 \in S_1$ for $x \in S_2, \cdots, f(x) \equiv q_n \in S_1$ for $x \in S_n$ is a continuous, onto map. It must be noted that $\mathcal{D}_f$ is not trivial decomposition space $\{\{x\} \subset S; x \in S\}$ because the map f is not one to one [12]. Since the decomposition space $\mathcal{D}_f$ of S is homeomorphic to S_1 [9], S must be homeomorphic to $\mathcal{D}_f$. Therefore, from Proposition, $\mathcal{D}_f$ is self-similar based on a system of weak contractions each of which is topologically conjugate to each weak contraction which defines the self-similarity of S.

Since the metric space $\mathcal{D}_f$ is perfect, 0-dim and compact, the same situation as for the initial space S can take place for the decomposition space

$\mathcal{D}_f$ of S. Therefore, continuing this process endlessly, we obtain an infinite sequence of self-similar decomposition spaces or self-similar coarse graining starting from the self-similar space S, namely, a hierarchic structure of self-similar spaces as shown in Fig.1. In Fig. 1, the above mentioned decomposition space $\mathcal{D}_f$ of S is denoted by $\mathcal{D}^1$. $\mathcal{D}^1$ is self-similar due to a system of weak contractions $\{f_j^1 = h^1 \circ f_j \circ (h^1)^{-1} : \mathcal{D}^1 \to \mathcal{D}^1; j = 1, \cdots, m\}$. Here, $\{f_j : S \to S; j = 1, \cdots, m\}$ is a system of weak contractions which defines the self-similarity of S, and h^1 is a homeomorphism from S to $\mathcal{D}^1$. The decomposition space $\mathcal{D}^2$ of $\mathcal{D}^1$ is self-similar based on a system of weak contractions $\{f_j^2 = h^2 \circ f_j^1 \circ (h^2)^{-1} : \mathcal{D}^2 \to \mathcal{D}^2; j = 1, \cdots, m\}$ where h^2 is a homeomorphism from $\mathcal{D}^1$ to $\mathcal{D}^2$. We can continue the procedure in this manner.

Statement. [14, 15, 16] *Let (Z, τ_d) be a compact metric space. If the sytem $\{f_j : (Z, \tau_d) \to (Z, \tau_d),\ j = 1, \ldots, m\}$ of weak contractions*
$d(f_j(z), f_j(z')) \leq \alpha_j(\eta) d(z, z')$ for $d(z, z') < \eta,\ 0 < \alpha_j(\eta) < 1$, $\inf_{\eta>0} \alpha_j(\eta) > 0, j = 1, \cdots, m$
satisfies three conditions

i) Each f_j is one to one,

ii) The set $\bigcup_{j=1}^m \{z \in Z;\ f_j(z) = z\}$ is not a singleton,

iii) $\sum_{j=1}^m inf_{\eta>0} \alpha_j(\eta) < 1$,

then, there exists a perfect, 0-dim, compact S $(\subset Z)$ such that
$\bigcup_{j=1}^m f_j(S) = S$.

Concludingly, we are convinced of the existence of a sequence as shown in Fig. 1 of self-similar coarse graining of a self-similar space based on the above quadratic dynamics $F_\mu(x)$ with a sufficiently large rate constant $\mu > 0$.

4 Generation of dendrites from each step of the sequence $S, \mathcal{D}^1, \mathcal{D}^2, \cdots$

Since all of the metric spaces $S, \mathcal{D}^1, \mathcal{D}^2, \cdots$ in Fig. 1 are perfect, 0-dim and compact, there exist continuous maps [8], k from S onto the dendrite δ as a compact metric space, k^1 from $\mathcal{D}^1$ onto δ, k^2 from $\mathcal{D}^2$ onto $\delta, \cdots$, respectively [17]. The decomposition spaces $\delta_S = \{k^{-1}(x) \subset S; x \in \delta\}$

of S due to f, $\delta_{\mathcal{D}^1} = \{(k^1)^{-1}(x) \subset \mathcal{D}^1; x \in \delta\}$ of $\mathcal{D}^1$ due to k^1, $\delta_{\mathcal{D}^2} = \{(k^2)^{-1}(x) \subset \mathcal{D}^2; x \in \delta\}$ of $\mathcal{D}^2$ due to k^2, $\cdots$ are homeomorphic to the dendrite δ, and therefore, $\delta_S, \delta_{\mathcal{D}^1}, \delta_{\mathcal{D}^2}, \cdots$ must have the dendritic structure in common (Fig. 3). For example, the self-similar space S generated from a quadratic dynamics $F_\mu(x) = \mu x(1-x)$ with a sufficiently large $\mu > 0$ is mathematically demonstrated to be able to form a dendrite through the coalescence or the rearrangement of constituents of S.

Appendix

Let S be a perfect, 0-dim T_0-space. Then, for any n, there exist n non-empty clopen (closed and open) sets $S_1, \cdots, S_n$ of S such that $S_i \cap S_{i'} = \phi$ for $i \neq i'$ and $\bigcup_{i=1}^{n} S_i = S$. For any n, there exist n non-empty clopen sets $S_{i_1}, \cdots, S_{i_n}$ of S such that $S_{i_j} \cap S_{i_{j'}} = \phi$ for $j \neq j'$ and $\bigcup_{j=1}^{n} S_{i_j} = S_i$. We can continue in this manner endlessly.

proof) To use the mathematical induction, let the statement hold for $n-1$. Since S is perfect, the open set S_{n-1} has at least two distinct points a and b. Since S is a T_0-space, there exists an open set u containing a such that $b \notin u$ without loss of generality. Since S is 0-dim, there exists a clopen set v which contains the point a and is contained in the open set $u \cap S_{n-1}$. Since $b \in S_{n-1} - v$, the clopen set $S_{n-1} - v$ is not empty. Thus, we obtain a desired n-partition $\{S_1, \cdots, S_{n-2}, v, S_{n-1} - v\}$ of S. Concerning the subspace S_i, it suffices to remember that any non-empty open set in a perfect space is perfect again. $\square$

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References

[1] A. Fernández: J. Phys. A 21 (1988) L295.

[2] S.K. Ma : *Modern theory of critical phenomena* (Benjamin, 1976) p.246.

[3] Let (A, τ) be a topological space. A partition $\mathcal{D}$ of A is a set $\{\phi \neq D \subset A\}$ of nonempty subsets of A such that $D \cap D' = \phi$ for $D \neq D'$, $D, D' \in \mathcal{D}$

and $\bigcup \mathcal{D}(=\bigcup\limits_{D\in\mathcal{D}} D) = A$. A decomposition space $(\mathcal{D}, \tau(\mathcal{D}))$ of (A, τ) is a topological space whose topology $\tau(\mathcal{D})$ on a partition $\mathcal{D}$ of A is defined by $\tau(\mathcal{D}) = \{\mathcal{U} \subset \mathcal{D}; \bigcup \mathcal{U}(=\bigcup\limits_{D\in\mathcal{U}} D) \in \tau\}$. The space $(\mathcal{D}, \tau(\mathcal{D}))$ is a kind of quotient space of (A, τ). See, for the detailed discussions, S.B. Nadler Jr., *Continuum theory* (Marcel Dekker, 1992) p.36.

[4] A dendrite is a metric space which is locally connected, connected and compact. The reference in [3], p.165.

[5] Topological spaces (E, τ) and (F, τ') are said to be homeomorphic provided that there exists a continuous, one to one, open (or closed) map from E onto F. If E and F are homeomorphic, all of the topological properties in $E(F)$ are preserved in $F(E)$. See, for example, A. Kitada: *Isoukuukan to sono ouyou* (Akakura Shoten, 2007) p.24 [in Japanese].

[6] A topological space (A, τ) is said to be perfect provided that any set $\{x\}$ composed of single point $x \in A$ is not an open set, that is, $\{x\} \notin \tau$.

[7] A topological space (A, τ) is said to be 0-dim provided that at any point $x \in A$, and for any open set U containing x, there exists a closed and open set (so-called a clopen set) u containing x such that $u \subset U$. See, for the detailed discussions, W. Hurewicz and H. Wallman, *Dimension Theory* (Princeton University Press, Princeton, 1941) p.10.

[8] The reference in [3], p.106 and p.109.

[9] The reference in [3], p.44. The decomposition space $\mathcal{D}_f$ of S can have various types of topological structure. For example, if X is a dendrite (the reference in [3], p.165), also $\mathcal{D}_f$ must be dendrite.

[10] To topologize the set $\mathcal{D}_f$, we use the metric ρ defined by means of the homeomorphism h rather than the Vietoris topology (see, for example, A. Illanes and S.B. Nadler Jr., *Hyperspaces* (Marcel Dekker, 1999) p.9) which has been customarily employed for the topological discussions of the phenomena in the Chaos-Fractal sciences (see, for example, J. Banks: Chaos, Solitons & Fractals **25** (2005) 681). Since X and $\mathcal{D}_f$ are homeomorphic, the employment of the metric topology τ_ρ which is identical with the decomposition topology $\tau(\mathcal{D}_f)$ seems to be quite natural.

[11] The reference in [3], p.109.

[12] One of the simplest example of such decomposition $\mathcal{D}_f$ is as follows. Let the self-similar, perfect, 0-dim, compact metric space S be CMTS itself and let the partition $\{S_1, S_2\}$ of S be the set {CMTS $\cap[0, 1/3]$, CMTS $\cap[2/3, 1]$}. Then, the set $\{f^{-1}(x) \subset$ CMTS $, x \in S_1\}$ where $f^{-1}(x) = \{x\} \subset X$ for $x \in S_1 - \{q\}$ and $f^{-1}(q) = \{q\} \cup S_2$, is a decomposition $\mathcal{D}_f$ of CMTS.

[13] R.L. Devaney: *An introduction to chaotic dynamical systems* (Westview Press, 2003) 2nd ed., p.35.

[14] A. Kitada and Y. Ogasawara: Chaos, Solitons & Fractals **24** (2005) 785; A. Kitada and Y. Ogasawara: Chaos, Solitons & Fractals **25** (2005) 1273.

[15] A. Kitada, T. Konishi and T. Watanabe: Chaos, Solitons & Fractals 13 (2002) 363.

[16] S. Nakamura, T. Konishi and A. Kitada, J. Phys. Soc. Jpn. 64 (1995) 731.

[17] It is noted that continuous maps $k, k^1, k^2, \cdots$ must not be one to one. In fact, if they are one to one, they must be homeomorphisms between 0-dim (i.e., totally disconnected) spaces $S, \mathcal{D}^1, \mathcal{D}^2, \cdots$ and a connetcted space δ. It is impossible.

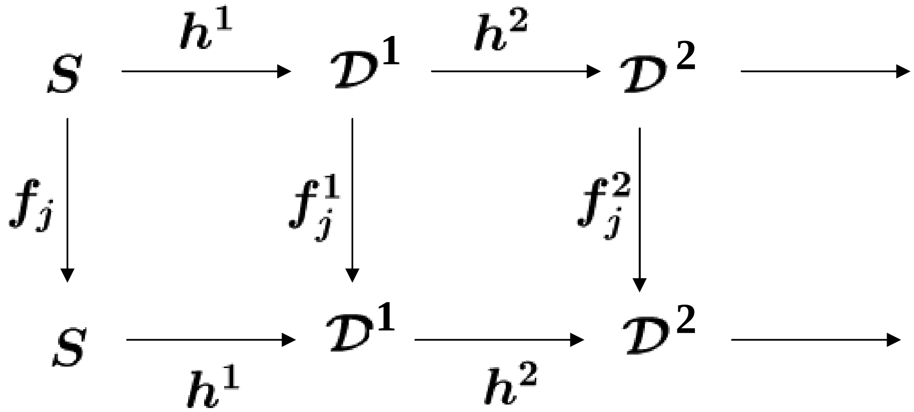

Figure 1: A hierarchic structure of self-similar spaces. $h^i, i = 1, 2, \cdots$ are homeomorphisms. $f_j, f_j^1, f_j^2, \cdots$ are weak contractions such that $\bigcup_{j=1}^{m} f_j(S) = S, \bigcup_{j=1}^{m} f_j^1(\mathcal{D}^1) = \mathcal{D}^1, \bigcup_{j=1}^{m} f_j^2(\mathcal{D}^2) = \mathcal{D}^2, \cdots$, respectively.

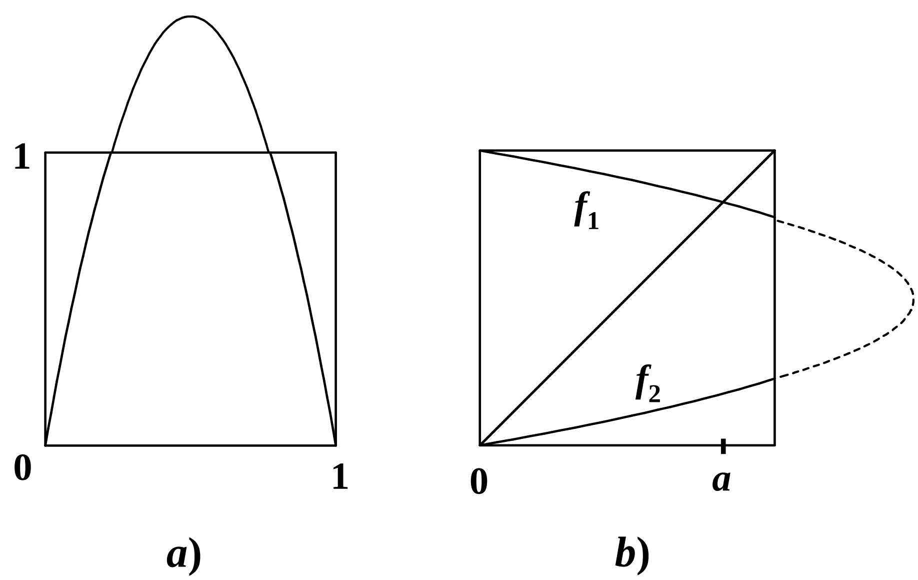

Figure 2: a) $F_\mu(x) = \mu x(1-x), \quad \mu > 4, \quad x \in [0,1]$. b) The quadratic dynamics $F_\mu(x)$ defines a system of contractions $\{f_j : [0,1] \to [0,1], j = 1,2\}$ which satisfies three conditions *i), ii), iii)* in **Statement** in the text. In fact, $\bigcup_{j=1,2} \{x \in [0,1]; f_j(x) = x\} = \{0, a\}$.

dendrite

$$S \xrightarrow{\ k\ } \delta \Longrightarrow \delta_S = \{k^{-1}(x) \subset S; x \in \delta\}$$

$$\mathcal{D}^1 \xrightarrow{\ k^1\ } \delta \Longrightarrow \delta_{\mathcal{D}^1} = \{(k^1)^{-1}(x) \subset \mathcal{D}^1; x \in \delta\}$$

$$\mathcal{D}^2 \xrightarrow{\ k^2\ } \delta \Longrightarrow \delta_{\mathcal{D}^2} = \{(k^2)^{-1}(x) \subset \mathcal{D}^2; x \in \delta\}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Figure 3: Generation of dendrites from each step of the sequence $S, \mathcal{D}^1, \mathcal{D}^2, \cdots$. $\delta, \delta_S, \delta_{\mathcal{D}^1}, \delta_{\mathcal{D}^2}, \cdots$ are dendrites.