

ON FINITENESS OF \mathbf{B} -REPRESENTATIONS AND SEMI-LOG CANONICAL ABUNDANCE

CHRISTOPHER D. HACON AND CHENYANG XU

Dedicated to Professor Shigefumi Mori on his 60th Birthday

ABSTRACT. We give a new proof of the finiteness of \mathbf{B} -representations. As a consequence of the finiteness of \mathbf{B} -representations and Kollár’s gluing theory on lc centers, we prove that the (relative) abundance conjecture for slc pairs is implied by the abundance conjecture for log canonical pairs.

CONTENTS

1.	Introduction	1
2.	Hodge theoretic construction	3
3.	Finiteness of \mathbf{B} -representations	6
4.	Abundance for slc pairs	9
	References	12

1. INTRODUCTION

Throughout this note, the ground field will be the field \mathbb{C} of complex numbers. It is well known that even though the log minimal model program is focused on the study of log pairs (X, Δ) where X is a normal variety, for technical reasons it is often necessary to deal with log pairs (X, Δ) where X is a semi-normal variety. This naturally occurs in proofs by induction on the dimension where, for example, we restrict to the reduced part of the boundary of a dlt pair (X, Δ) (cf. e.g. [Kolláretal92, KMM94, Birkar11, HX11, FG11]) or when we study moduli of pairs as normal varieties can degenerate to non-normal ones (cf. [Kollár11a, HX11]). In [Fujino00], O. Fujino first used the \mathbf{B} -representation to study semi-log canonical abundance conjecture and proved the conjecture in 3 dimensional case. Recently, J. Kollár has developed a useful technique for gluing log canonical centers (cf.

Date: July 22, 2011.

We are indebted to Y. Gongyo, J. McKernan and J. Kollár for helpful conversations. Especially, the main ideas in Section 4 follow [Kollár11b]. Part of this work was done while the second author was visiting RIMS, which he would like to thank for the hospitality. The first author was partially supported by NSF research grant no: 0757897, the second author was partially supported by NSF research grant no: 0969495.

[Kollár11a, Kollár11b]) that reduces many questions on semi-normal pairs to questions on their normalizations. An important result used in Kollár's theory and in Fujino's work (cf. [Fujino00]), is the finiteness of \mathbf{B} -representations, which was first proved by Ueno in the klt case, and then generalized to the log canonical case by Fujino-Gongyo (cf. [FG11]). Recall the following.

Definition 1.1. Let (X, Δ) be a projective dlt pair. We define the birational automorphism group $\text{Bir}(X, \Delta)$ of (X, Δ) to be the group of all birational maps g of X such that if we take a common resolution

$$\begin{array}{ccc} & Y & \\ & \swarrow p & \searrow q \\ X & \dashrightarrow g & X \end{array}$$

then $p^*(K_X + \Delta) = q^*(K_X + \Delta)$. We call the induced homomorphism

$$\rho_m : \text{Bir}(X, \Delta) \rightarrow \text{Aut}(H^0(X, \mathcal{O}_X(m(K_X + \Delta))))$$

the **B-representation** of the pair (X, Δ) . As far as we know, \mathbf{B} -birational maps and \mathbf{B} -representations for general log pairs were first explicitly introduced in [Fujino00].

In this note we first aim to give a new proof of the following result.

Theorem 1.2. *Given a projective dlt pair (X, Δ) such that $K_X + \Delta$ is a semi-ample \mathbb{Q} -divisor. There exists $m \in \mathbb{N}$, such that the image of the \mathbf{B} -representation*

$$\rho_M : \text{Bir}(X, \Delta) \rightarrow \text{Aut}(H^0(X, \mathcal{O}_X(M(K_X + \Delta))))$$

is finite for any positive integer M divisible by m .

Remark 1.3. Theorem (1.2) was proven by different methods in [FG11, 1.1]. The argument in the current note was originally contained in [HMX11]. During the preparation of [HMX11] we were informed of [FG11] and we decided to include (1.2) in a separate paper. Our proof uses the case when the Kodaira dimension is 0, which was proved by Gongyo (cf. [Gongyo10]) using ideas from [Fujino00].

The second part of our paper is focused on the study of semi-log canonical abundance. One of the main applications of the finiteness of \mathbf{B} -representations is to prove that abundance for semi-log canonical pairs, follows from abundance for log canonical pairs (cf. [Fujino00, FG11]). Using Kollár's gluing theory, as a consequence of (1.2), we prove the following.

Theorem 1.4. *Let (X, Δ) be a semi-log canonical pair, $f : X \rightarrow S$ a projective morphism, $n : \bar{X} \rightarrow X$ the normalization and write $n^*(K_X + \Delta) = K_{\bar{X}} + \bar{\Delta} + \bar{D}$, where \bar{D} is the double locus. If $K_{\bar{X}} + \bar{\Delta} + \bar{D}$ is semi-ample over S , then $K_X + \Delta$ is semi-ample over S .*

As a corollary we recover the following result conjectured by C. Birkar (cf. [Birkar11, 1.2]), which is known to be a natural step of log canonical minimal model program in the relative case (cf. [KMM94, Section 7]).

Corollary 1.5. *Let (X, Δ) be a \mathbb{Q} -factorial dlt pair which is projective over a variety S , and $T := \lfloor \Delta \rfloor$ where Δ is a \mathbb{Q} -divisor. Suppose that*

- (1) $K_X + \Delta$ is nef over S ,
- (2) $(K_X + \Delta)|_{T_i}$ is semi-ample over S for each component T_i of T ,
- (3) $K_X + \Delta - \epsilon P$ is semi-ample over S for some \mathbb{Q} -divisor $P \geq 0$ with $\text{Supp}(P) = T$ and for any sufficiently small rational number $\epsilon > 0$.

Then, $K_X + \Delta$ is semi-ample over S .

Another corollary is the following result, which answers a question raised by J. Kollár in the problem session in MSRI in March 2009.

Corollary 1.6. *Let (X, Δ) be a semi-log canonical pair and $f : X \rightarrow S$ a projective morphism, such that $K_X + \Delta \equiv_{\mathbb{Q}, S} 0$. Then $K_X + \Delta \sim_{\mathbb{Q}, S} 0$.*

Remark 1.7. A result due to O. Fujino and Y. Gongyo (cf. [FG11, 4.13]), implies (1.4) when the base S is projective. Recent work of J. Kollár on gluing lc centers provides us with a technique to prove the general case.

The absolute case of (1.6) or the case when S is projective is also known (cf. [Gongyo10]). However, the general relative case does not seem to be available anywhere in the literature (see the remark in [FG11, 4.16]).

2. HODGE THEORETIC CONSTRUCTION

Construction 2.1 (cf. [Kollár07, 8.4.6]). Let (X, Δ) be a log canonical pair, and $f : X \rightarrow Y$ a proper surjective morphism of normal varieties with connected fibers such that $n = \dim X - \dim Y$ and $K_X + \Delta \sim_{\mathbb{Q}, Y} 0$.

Let $p : W \rightarrow X$ be a log resolution of (X, Δ) . Write

$$p^*(K_X + \Delta) = K_W + E + F - G,$$

where E and G are integral effective divisors with no common components and $F = \{F\}$. Let a be an integer such that aF is an integral divisor. Denote by $\phi = f \circ p : W \rightarrow Y$.

Let $Y^0 \subset Y$ be a smooth open subset such that ϕ is smooth over Y^0 . We denote by \bullet^0 the base change over Y^0 . Replacing Y^0 by an open subset, we may assume that $(K_X + \Delta)|_{X^0} \sim_{\mathbb{Q}} 0$. We define a line bundle $V^0 = \mathcal{O}_{W^0}(G^0 - K_{W^0} - E^0)$ so that $V^{0 \otimes a} \cong \mathcal{O}_{W^0}(aF^0)$. This data defines a local system \mathbb{V}^0 on $W^0 \setminus \text{Supp}(E^0 \cup F^0)$ (cf. [Kollár07, 8.4.6], [EV92, 3.2] and its proof).

Consider the normalization of the corresponding μ_a -cover $\pi : W' \rightarrow W^0$, and denote by E' the reduced divisor supported on π^*E^0 . The push-forward $\pi_*(\mathbb{C}|_{W' \setminus E'})$ has a μ_a -action. If we decompose

$$\pi_*(\mathbb{C}|_{W' \setminus E'}) = \bigoplus_i \pi_*(\mathbb{C}|_{W' \setminus E'})^{(i)}$$

into the corresponding eigenspaces, then \mathbb{V}^0 is isomorphic to the restriction of $\pi_*(\mathbb{C}|_{W' \setminus E'})^{(1)}$ to $W^0 \setminus (E^0 \cup F^0)$, we denote $\pi_*(\mathbb{C}|_{W' \setminus E'})^{(1)}$ by \mathbb{V} . The choice of \mathbb{V} is determined up to the choice of a unit in \mathcal{O}_{Y^0} . However, $(R^n \phi_* \mathbb{V})^{\otimes a}$ (we will sloppily denote $\phi|_{W^0 \setminus E^0}$ by ϕ) is a well defined local system on Y^0 (cf. [Kollár07, 8.4.7]).

Denote by $\phi' : W' \rightarrow Y^0$ the composite morphism $\phi \circ \pi$ (and its restriction to open subsets). Then $R^n \phi_* \mathbb{V}$ is a direct summand of $R^n \phi'_* \mathbb{C}|_{W' \setminus E'}$ which carries a variation of mixed Hodge structure. We remark that even though W' has quotient singularities, locally (W', E') is a finite quotient of a log smooth pair and hence Hodge theoretically it behaves as a smooth variety with a simple normal crossing divisor, at least for \mathbb{Q} -coefficients (cf. [Steenbrink77, Section 1]).

The local system $R^n \phi_* \mathbb{V}$ gives a variation of mixed Hodge structure on Y^0 . It follows from [Steenbrink77, 1.18] that the bottom piece of the Hodge filtration gives an isomorphism

$$F^n R^n \phi'_*(\mathbb{C}|_{W' \setminus E'}) \cong \phi'_* \omega_{W'/Y}(E').$$

Considering the eigenspace, we conclude that $F^n R^n \phi_*(\mathbb{V}|_{W^0 \setminus E^0}) \cong \phi_* \mathcal{O}_{W^0}(G^0)$ is a line bundle over Y^0 (cf. [Kollár07, 8.4.5(7')]).

Denote by L this line bundle. Since L is 1 dimensional, there exists an integer i such that, $L \subset W_{n+i}(R^n \phi_* \mathbb{V})$ but $L \not\subset W_{n+i-1}(R^n \phi_* \mathbb{V})$. Let \mathbb{H} be the smallest pure sub- \mathbb{Q} -VHS of $Gr_{n+i}^W(R^n \phi_* \mathbb{V})$ which contains L .

Lemma 2.2. \mathbb{H} does not depend on the choice of the resolution of W .

Remark 2.3. Let \mathbb{H}_1 and \mathbb{H}_2 be two local system defined over nonempty open subsets $U_1, U_2 \subset Y$ such that $\mathbb{H}_1|_{U_1 \cap U_2} \cong \mathbb{H}_2|_{U_1 \cap U_2}$, since Y is normal, then there is a unique local system \mathbb{H} (up to a unique isomorphism) defined over $U_1 \cup U_2$, such that its restriction to U_i is isomorphic to \mathbb{H}_i .

Proof of (2.2). From the above remark, it suffices to verify the lemma for any nonempty open set $Y^0 \subset Y$. For two resolutions W_1, W_2 , we can assume that there exists a morphism $\psi : W_2 \rightarrow W_1$. By shrinking Y^0 , we can also assume that W_1 and W_2 are both smooth over Y^0 (cf. (2.3)). If $p_1^*(K_X + \Delta) = K_{W_1} + E_1 + F_1 - G_1$, and aF_1 is integral, then

$$p_2^*(K_X + \Delta) = \psi^*(K_{W_1} + E_1 + F_1 - G_1) = K_{W_2} + E_2 + F_2 - G_2,$$

and aF_2 is easily seen to be integral. Applying the construction (2.1) for both W_i (here we choose the same unit in \mathcal{O}_{Y^0} for W_i), we have that W'_2 is the normalization of $W_2^0 \times_{W_1^0} W'_1$ and we denote by $\psi' : W'_2 \rightarrow W'_1$. Let $E^* = \text{Supp}(\psi'^{-1} E'_1)$,

then we know that $E'_2 \subset E^*$.

$$\begin{array}{ccc} E'_2 \subset E^* \subset W'_2 & \longrightarrow & E_2 \subset W_2 \\ \downarrow \psi' & & \downarrow \psi \\ E'_1 \subset W'_1 & \longrightarrow & E_1 \subset W_1 \end{array}$$

Therefore, there exist morphisms between mixed Hodge structures on Y^0

$$\begin{aligned} i &: R^n \phi'_{2*}(\mathbb{C}|_{W'_2 \setminus E'_2}) \rightarrow R^n \phi'_{2*}(\mathbb{C}|_{W'_2 \setminus E^*}) \quad \text{and} \\ j &: R^n \phi'_{1*}(\mathbb{C}|_{W'_1 \setminus E'_1}) \rightarrow R^n \phi'_{2*}(\mathbb{C}|_{W'_2 \setminus E^*}). \end{aligned}$$

Applying the Hodge filtration $F^n(\cdot)$, it follows from [Steenbrink77, 1.18] that

$$F^n R^n \phi'_{2*}(\mathbb{C}|_{W'_2 \setminus E'_2}) \cong \phi'_{2*} \omega_{W'_2/Y}(E'_2),$$

and similarly for other pairs. Thus we have morphisms

$$\begin{aligned} F^n i &: \phi'_{2*} \omega_{W'_2/Y}(E'_2) \rightarrow \phi'_{2*} \omega_{W'_2/Y}(E^*) \quad \text{and} \\ F^n j &: \phi'_{1*} \omega_{W'_1/Y}(E'_1) \rightarrow \phi'_{2*} \omega_{W'_2/Y}(E^*). \end{aligned}$$

which are isomorphism by (2.4).

Replacing Y^0 by a smaller open set, by the discussion in (2.1), there are morphisms between mixed Hodge structures,

$$R^n \phi_{1*} \mathbb{V}_1 \rightarrow R^n \phi_{2*}(\mathbb{C}|_{W'_2 \setminus E^*}) \leftarrow R^n \phi_{2*} \mathbb{V}_2.$$

We conclude that if we take $F^n(\cdot)$ of each term above, then we obtain an induced isomorphism $F^n(R^n \phi_{1*} \mathbb{V}_1) \rightarrow F^n(R^n \phi_{2*} \mathbb{V}_2)$. Since polarized-VHSs form a semi-simple category, considering $Gr_{n+i}^W R^n \phi_{j*} \mathbb{V}_j$, we see that the images of \mathbb{H}_1 and \mathbb{H}_2 are mapped to the same pure Hodge substructure of $Gr_{n+i}^W R^n \phi_{2*}(\mathbb{C}|_{W'_2 \setminus E^*})$. \square

Lemma 2.4. *Let $f : Y_2 \rightarrow Y_1$ be a birational morphism between normal projective varieties. Let Δ_1 be reduced divisor on Y_1 , such that (Y_1, Δ_1) is log canonical. Let $f_*^{-1}(\Delta_1) \leq \Delta_2 \leq f_*^{-1}(\Delta_1) + \text{Ex}(f)$ be an effective Weil divisor on Y_2 whose support contains all divisors of discrepancy -1 with respect to the pair (Y_1, Δ_1) . Then there is a natural morphism $f_* \mathcal{O}_{Y_2}(K_{Y_2} + \Delta_2) \rightarrow \mathcal{O}_{Y_1}(K_{Y_1} + \Delta_1)$ inducing an isomorphism*

$$H^0(Y_2, \mathcal{O}_{Y_2}(K_{Y_2} + \Delta_2)) \cong H^0(Y_1, \mathcal{O}_{Y_1}(K_{Y_1} + \Delta_1)).$$

Proof. By assumption, we can write

$$F + f^*(K_{Y_1} + \Delta_1) = K_{Y_2} + \Delta_2 + E,$$

where $F, E \geq 0$, F is exceptional and $[E] = 0$. Therefore,

$$\begin{aligned} H^0(Y_1, \mathcal{O}_{Y_1}(K_{Y_1} + \Delta_1)) &\cong H^0(Y_2, \mathcal{O}_{Y_2}([F + f^*(K_{Y_1} + \Delta_1)])) \\ &\cong H^0(Y_2, \mathcal{O}_{Y_2}(K_{Y_2} + \Delta_2 + [E])) \cong H^0(Y_2, \mathcal{O}_{Y_2}(K_{Y_2} + \Delta_2)). \end{aligned}$$

□

3. FINITENESS OF \mathbf{B} -REPRESENTATIONS

In this section we prove (1.2). By assumption, there exists a positive integer m , such that $|m(K_X + \Delta)|$ is base point free and it induces an algebraic fibration structure $f : X \rightarrow Y$, so that

$$Y = \text{Proj} \bigoplus_{d \geq 0} H^0(X, \mathcal{O}_X(dm(K_X + \Delta))).$$

It follows from Kawamata's canonical bundle formula (cf. [Kawamata98]) that we can write

$$K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Y + B + J),$$

where B is the boundary part and J is the moduli part. Let $P \in Y$ be a codimension 1 point, then the coefficient of P in B is defined by

$$1 - \text{lct}(X, \Delta; f^{-1}(P))$$

where the log canonical threshold is computed over a neighborhood of the generic point of P . In particular, B is an effective \mathbb{Q} -divisor. The moduli part J is defined as an equivalence class of \mathbb{Q} -divisors, coming from Hodge theory.

For any $g \in \text{Bir}(X, \Delta)$ and any $d \geq 0$ we have homomorphisms $\rho_{dm} : \text{Bir}(X, \Delta) \rightarrow \text{Aut}_{\mathbb{C}} H^0(X, \mathcal{O}_X(dm(K_X + \Delta)))$ where

$$\rho_{dm}(g) = g^* : H^0(X, \mathcal{O}_X(dm(K_X + \Delta))) \rightarrow H^0(X, \mathcal{O}_X(dm(K_X + \Delta))).$$

Consider the induced homomorphism

$$\chi : \text{Bir}(X, \Delta) \rightarrow \text{Aut}(Y).$$

For any element $g \in \text{Bir}(X, \Delta)$, there is a commutative diagram

$$\begin{array}{ccc} X & \overset{g}{\dashrightarrow} & X \\ \downarrow f & & \downarrow f \\ Y & \xrightarrow{\chi(g)} & Y \end{array}$$

Let F be the generic geometric fiber of f and $n = \dim F = \dim X - \dim Y$ be the relative dimension. Then for any d , we have a short exact sequence

$$1 \rightarrow G \rightarrow \rho_{dm}(\text{Bir}(X, \Delta)) \rightarrow \chi(\text{Bir}(X, \Delta)) \rightarrow 1,$$

where $G \subset H^0(Y, \mathcal{O}_Y^*) = \mathbb{C}^*$. We know that G is finite (cf. [Gongyo10, 4.9]), so it suffices to show the following result.

Theorem 3.1. *The image $\chi(\text{Bir}(X, \Delta)) \subset \text{Aut}(Y)$ is finite.*

First it is easy to see the following result.

Lemma 3.2. *The image of $\chi(\text{Bir}(X, \Delta))$ is contained in $\text{Aut}(Y, B)$.*

Proof. Let $P \in Y$ be a codimension 1 point, as we noted, the coefficient of P in B is defined by $1 - \text{let}(X, \Delta; f^{-1}(P))$. This number is unchanged if we replace (X, Δ) by any log resolution. Thus the lemma easily follows from the above commutative diagram, and the fact that g is in $\text{Bir}(X, \Delta)$. \square

By the following result, we see that the birational maps also preserve \mathbb{H} , where \mathbb{H} is the local system defined in (2.1),

Proposition 3.3. (1) *Let $g \in \text{Bir}(X, \Delta)$. If we assume \mathbb{H} is defined over an open set $Y^0 \subset Y$, then over $Y^0 \cap \chi(g)^{-1}(Y^0)$, there exists an isomorphism $i_g : \chi(g)^*\mathbb{H} \cong \mathbb{H}$.*

(2) *Let $g_1, g_2 \in \text{Bir}(X, \Delta)$, then over $Y^0 \cap \chi(g_1)^{-1}(Y^0) \cap \chi(g_2)^{-1}(Y^0)$ we have $i_{g_1} \circ i_{g_2} = i_{g_1 \circ g_2}$.*

Proof. (1) Let W be a common log resolution,

$$\begin{array}{ccc}
 & W & \\
 p_1 \swarrow & & \searrow p_2 \\
 X & \overset{g}{\dashrightarrow} & X \\
 f \downarrow & & \downarrow f \\
 Y & \xrightarrow{\chi(g)} & Y
 \end{array}$$

Since $p_1^*(K_X + \Delta) = p_2^*(K_X + \Delta)$, we obtain the same local system on some open subset $W^0 \subset W$, whose image is an open subset $Y^0 \subset Y$. We can shrink Y^0 , so that $R^n\phi_{1*}(\mathbb{V})$ (resp. $R^n\phi_{2*}(\mathbb{V})$) is defined over $\chi(g)^{-1}(Y^0)$ (resp. Y^0), where we denote $f \circ p_i$ by ϕ_i .

Since $p_1^*(K_X + \Delta) = p_2^*(K_X + \Delta)$, we conclude $\chi(g)^*(L) \cong L$. Therefore, because of the semi-simplicity of VHS, the isomorphism

$$\chi(g)^* : R^n\phi_{1*}(\mathbb{V}) \rightarrow R^n\phi_{2*}(\mathbb{V})$$

which sends L to L , will send \mathbb{H} to \mathbb{H} .

(2) This also follows by a similar argument since $(\chi(g_1 \circ g_2))^*L = L$. \square

Since $\chi(\text{Bir}(X, \Delta))$ preserves the polarization on Y , it is a subgroup of $\text{PGL}(N)$ for some N , i.e., it is an algebraic group. Thus, to show that $\text{Bir}(X, \Delta)$ is finite, we only need to verify that it does not contain \mathbb{G}_a or \mathbb{G}_m . We will show that $\chi(\text{Bir}(X, \Delta))$ does not contain \mathbb{G}_m . (The argument for \mathbb{G}_a is similar and we leave it to the reader.) If this is not the case, then for a general point $z \in Y$, we consider the closure of the orbit $o : \mathbb{P}^1 \rightarrow Y$ of $\mathbb{G}_m \cdot z$. Thus $o^*(\mathbb{H})$ gives a VHS which by (3.3) is defined over $\mathbb{G}_m \cong \mathbb{A}^1 \setminus \{0\} \subset \mathbb{P}^1$. There exists a ramified cover $d : \mathbb{P}^1 \rightarrow \mathbb{P}^1$, such that if we denote by $o_d = o \circ d$, then $o_d^*(\mathbb{H})$ is defined over

$\mathbb{A}^1 \setminus \{0\}$, and it has unipotent monodromies near 0 and ∞ . From our construction, \mathbb{H} is a polarized \mathbb{Q} -Hodge bundle of weight $n + i$. Its pull-back $o_d^*(\mathbb{H})$ has a canonical extension, whose bottom filtration satisfies $\bar{E}^{n+i,0}(o_d^*(\mathbb{H})) = \mathcal{O}_{\mathbb{P}^1}(J_d)$. By definition, the moduli part J in the Kawamata subadjunction formula satisfies that $o^*(J) = \frac{1}{d}d_*J_d$. From Theorem (3.4) below, it follows that if $\chi(\text{Bir}(X, \Delta))$ contains \mathbb{G}_m , then $J_d \cong \mathcal{O}_{\mathbb{P}^1}$. In fact, it is always true that $\deg J_d \geq 0$, but in our case the right hand side of the inequality in (3.4) is 0, thus we conclude that $\deg J_d = d \cdot \deg o^*(J) = 0$. On the other hand, by (3.5), we have

$$\deg(o^*(K_Y + B + J)) = \deg(o^*(K_Y + B)) \leq 0.$$

Since $K_Y + B + J$ is ample, this is a contradiction and so this completes the proof of (3.1).

Proposition 3.4 ([JZ02, Theorem 1]). *Suppose $(\oplus_{p+q=n} \bar{E}^{p,q}, \theta)$ is the quasi-canonical extension of the Hodge bundle corresponding to a real variation of polarized Hodge structures over a smooth quasi-projective curve $C \setminus S$ with $g(C) = g$. Let I_S be the cardinality of S . Denote by*

$$h_0^{p,q} := \text{rk}(\ker(\theta: \bar{E}^{p,q} \rightarrow \bar{E}^{p-1,q+1} \otimes \Omega_C^1(S))) \quad \text{and} \quad h^{p,q} := \text{rk}(\bar{E}^{p,q}).$$

If the weight is odd, $k = 2l + 1$, then

$$\deg(\bar{E}^{k,0}) \leq \left(\frac{1}{2}(h^{k-l,l} - h_0^{k-l,l}) + \sum_{j=0}^{l-1} (h^{k-j,j} - h_0^{k-j,j})\right) (2g - 2 + I_S).$$

If the weight is even $k = 2l$, then

$$\deg(\bar{E}^{k,0}) \leq \left(\sum_{j=0}^{l-1} (h^{k-j,j} - h_0^{k-j,j})\right) (2g - 2 + I_S).$$

Lemma 3.5. *Let (Y, B) be a log canonical pair and $\psi: \mathbb{G}_m \times (Y, B) \rightarrow (Y, B)$ a faithful action. For general $t \in Y$, if we denote by $\psi_t: \mathbb{P}^1 \times \{t\} \rightarrow Y$ the closure of the orbit, then $\deg \psi_t^*(K_X + B) \leq 0$.*

Proof. Let $\pi: Y' \rightarrow Y$ be a log resolution of (Y, B) and $B' = \text{Supp}(\pi_*^{-1}(B) + \text{Ex}(\pi))$. We may assume that $\pi(\text{Ex}(\pi))$ is contained in the union of $\text{Supp}(B)$ and the singular locus of X . Thus, for a general point $t \in Y$, we have that

$$\mathbb{G}_m \cdot t \cap (\text{Supp}(B) \cup \pi(\text{Ex}(\pi))) = \emptyset.$$

We then have the induced morphisms

$$(\mathbb{A}^1 - \{0\}) \rightarrow Y \setminus (\text{Supp}(B) \cup \text{Ex}(\pi)) \quad \text{and} \quad \phi_t: \mathbb{P}^1 \rightarrow Y'.$$

Since (Y, B) is log canonical, $\pi^*(K_Y + B) \leq K_{Y'} + B'$. We may assume that there is a smooth open set T and generically finite morphism, $\phi_T: \mathbb{P}^1 \times T \rightarrow Y'$ such that $\phi_T^{-1}(B') \subset \{0, \infty\} \times T$, and $\phi_T|_{t \times \mathbb{P}^1} = \phi_t$. Then it is easy to see by a local computation that

$$\phi_T^*(K_{X'} + B') \leq K_{T \times \mathbb{P}^1} + T \times \{0\} + T \times \{\infty\}.$$

Therefore, we conclude that

$$\deg \psi_t^*(K_Y + B) \leq \deg \phi_t^*(K_{Y'} + B') \leq \mathbb{P}^1 \cdot (K_{T \times \mathbb{P}^1} + T \times \{0\} + T \times \{\infty\}),$$

which is computed by $\deg(K_{\mathbb{P}^1} + \{0\} + \{\infty\}) = 0$. \square

4. ABUNDANCE FOR SLC PAIRS

In this section, we will study the semi-log canonical abundance. By [Fujino00], it is known that if S is a point, then (1.2) implies (1.4). By [FG11], (1.4) is also known when S is projective. The gluing theory for log canonical centers, developed by J. Kollár (cf. [Kollár11a, Chapter 3]), provides a very powerful tool to study semi-log canonical varieties and allows us to prove (1.4) in full generality.

More specifically, Kollár's recent theory of giving any lc center a source and a spring provides a useful tool for checking that the pro-finite equivalence relation is finite (cf. [Kollár11b]). We note that, this is not true for general pro-finite equivalence relations. The fact that we are gluing lc centers for relatively projective pairs plays an essential role here.

First, the main theorem for abstract gluing theory is the following.

Theorem 4.1 ([Kollár11a, 3.40]). *Let (X, S_*) be an excellent scheme or algebraic space over a field of characteristic 0 with a stratification. Assume that (X, S_*) satisfies the conditions (HN) and (HSN). Let $R \rightrightarrows X$ be a finite, set theoretic, stratified equivalence relation. Then*

- (1) *the geometric quotient X/R exists,*
- (2) *$\pi : X \rightarrow X/R$ is stratifiable and*
- (3) *$(X/R, \pi_* S_*)$ also satisfies the conditions (HN) and (HSN).*

We now recall the following notation, which was first introduced by F. Ambro (cf. [Ambro03]).

Definition 4.2. We call $f : (X, \Delta) \rightarrow Y$ a *crepant log structure* if

- (1) (X, Δ) is log canonical,
- (2) f is projective, surjective, with connected fibers,
- (3) $K_X + \Delta \sim_{\mathbb{Q}, f} 0$.

We note that, by taking a dlt modification of (X, Δ) (cf. [KK10, 3.1]), we can assume that (X, Δ) is dlt.

As in [HX11, Section 3], we let $\bar{g} : \bar{X} \rightarrow \bar{Y}$ be the morphism over S induced by the relatively semiample \mathbb{Q} -divisor $K_{\bar{X}} + \bar{\Delta} + \bar{D}$ and we consider the pro-finite equivalence relation

$$(\sigma_1, \sigma_2) : \bar{T} \rightrightarrows \bar{Y}$$

given by the image of the pro-finite relation $D^n \rightrightarrows \bar{X}$. Recall that D^n denotes the normalization of \bar{D} on \bar{X} . Note that \bar{T} and \bar{Y} are normal. We may assume that \bar{g} is induced by $|m(K_{\bar{X}} + \bar{\Delta} + \bar{D})|$ for some sufficiently divisible positive integer m . If we write $K_{D^n} + \Theta = (K_{\bar{X}} + \bar{\Delta} + \bar{D})|_{D^n}$, then (D^n, Θ) is also log canonical.

As in [HX11, Section 3], we consider the minimal qlc stratifications $S_*\bar{T}$ and $S_*\bar{Y}$ given by the crepant log structures $(D^n, \Theta) \rightarrow \bar{T}$ and $(\bar{X}, \bar{\Delta} + \bar{D}) \rightarrow \bar{Y}$. We note that if we let (X^d, Δ^d) be a dlt modification of $(\bar{X}, \bar{\Delta} + \bar{D})$ (cf. [KK10, 3.1]), then

$$(X^d, \Delta^d) \rightarrow (\bar{X}, \bar{\Delta} + \bar{D}) \rightarrow \bar{Y}$$

gives the same minimal qlc stratification on \bar{Y} .

Proposition 4.3. *We have the following facts.*

- (1) *The stratifications $S_*\bar{T}$ and $S_*\bar{Y}$ satisfy conditions (HN) and (HSN).*
- (2) *The pro-finite relation $(\sigma_1, \sigma_2) : \bar{T} \rightrightarrows \bar{Y}$ is stratified.*

Proof. The first statement follows from [KK10, 5.7] and the second one follows from [HX11, 3.11] which is a consequence of [KK10, 1.7]. \square

By (4.1), to prove that the quotient \bar{Y}/\bar{T} exists, we only need to verify that $\bar{T} \rightrightarrows \bar{Y}$ generates a finite relation $R \rightarrow \bar{Y}$. By [Kollár11a, 3.61], it suffices to check this over the generic point of each stratum V^0 of $S_i\bar{Y}$. We work over the generic point of $\bar{f}(V^0)$, where $\bar{f} : \bar{Y} \rightarrow S$ is the induced morphism. Let $\eta \in S$ be one such point and $\bar{\eta}$ its algebraic closure. We must verify that the equivalence relation $R \times_S \bar{\eta} \subset \bar{Y} \times \bar{Y} \times_S \bar{\eta}$ is finite over $V^0 \times \bar{\eta}$.

Let Z be one of the minimal lc centers of (X^d, Δ^d) which dominates the closure V of V^0 , then $(Z, \text{Diff}_Z^* \Delta^d)$ is dlt and its restriction over V^0 is klt. We take the Stein factorization $Z \rightarrow \tilde{V} \rightarrow V$. We will need the following results from [Kollár11b].

Proposition 4.4 ([Kollár11b, 1]). *For different choices of the minimal non-klt centers Z and Z' , we have*

- (1) *$(Z, \text{Diff}_Z^* \Delta^d)$ is \mathbf{B} -birational to $(Z', \text{Diff}_{Z'}^* \Delta^d)$ over V , and*
- (2) *\tilde{V} is isomorphic to \tilde{V}' over V .*

Definition 4.5 ([Kollár11b, 19]). We denote by $\text{Src}(V, X^d, \Delta^d) := (Z, \text{Diff}_Z^* \Delta^d)$ a *source* of the lc center V and $\text{Spr}(V, X^d, \Delta^d) := \tilde{V}$ its *spring*. As in (4.4), $\text{Src}(V, X^d, \Delta^d)$ is determined up to \mathbf{B} -equivalence.

Let V_{ij}^0 be an i -dimensional stratum, and \tilde{V}_{ij}^0 the pre-image of V_{ij}^0 under the morphism $\tilde{V}_{ij} \rightarrow V_{ij}$. We define $\text{Spr}_i(\bar{Y}, X^d, \Delta^d) = \coprod_j \tilde{V}_{ij}^0$, where the disjoint union runs over all i -dimensional strata V_{ij}^0 . Then the main structural result is the following.

Theorem 4.6 ([Kollár11b, 33]). *Let $R \subset \bar{Y} \times \bar{Y}$ be the relation generated by $\bar{T} \rightrightarrows \bar{Y}$ as above. Let $p_i : \text{Spr}_i(\bar{Y}, X^d, \Delta^d) \rightarrow S_i\bar{Y}$ be the induced finite morphisms. Let $\bar{\eta}_{ij}$ be the algebraic closure of the generic point of $\bar{f}(V_{ij})$. Then*

$$((p_i \times p_i)^{-1}(R \cap (S_i\bar{Y} \times S_i\bar{Y})) \times_S \bar{\eta}_{ij}) \cap (\tilde{V}_{ij}^0 \times \tilde{V}_{ij}^0 \times_S \bar{\eta}_{ij})$$

is a subset of the graph $\cup_g \Gamma(\chi(g))$ for all $g \in \text{Bir}(Z_{\bar{\eta}_{ij}}, \text{Diff}_{Z_{\bar{\eta}_{ij}}}^ \Delta^d)$.*

Proof of (1.4). By (3.1), we have that $\cup_g \Gamma(\chi(g))$ is finite over $\tilde{V}_{ij}^0 \times \bar{\eta}_{ij}$, and so the hypotheses of (4.1) are satisfied. Thus there exists a quotient Y of $\bar{T} \rightrightarrows \bar{Y}$. Following the proof of [HX11, 3.1], we see that there exists a line bundle L on Y whose pull back to X is isomorphic to $\mathcal{O}_X(m(K_X + \Delta))$ for some integer $m > 0$. Then it is easy to see that L is relatively ample over S , which means that $K_X + \Delta$ is relatively semi-ample over S . \square

Proof of (1.5). The argument is the same as in [KMM94, 7.4]. We include it here for reader's convenience. It follows from (1.4) that $m(K_T + \Delta_T) := m(K_X + \Delta)|_T$ is base point free over S for any integer $m > 0$ sufficiently divisible. By assumption (3), we may assume that the relative base locus of $m(K_X + \Delta)$ is contained in the support of T . We write

$$m(K_X + \Delta) - T = (m - 1)(K_X + \Delta - \epsilon P) + K_X + \Delta - (T - (m - 1)\epsilon P).$$

Since $K_X + \Delta - \epsilon P$ is semi-ample over S , and $(X, \Delta - (T - (m - 1)\epsilon P))$ is klt for $0 < \epsilon \ll 1$, by Kollár's injectivity theorem (cf. [Kollár86]), we have that

$$R^1 f_* \mathcal{O}_X(m(K_X + \Delta) - T) \xrightarrow{T} R^1 f_* \mathcal{O}_X(m(K_X + \Delta))$$

is an injection and hence $f_* \mathcal{O}_X(m(K_X + \Delta)) \rightarrow f_* \mathcal{O}_T(m(K_T + \Delta_T))$ is surjective. We have a commutative diagram.

$$\begin{array}{ccc} f^* f_* \mathcal{O}_X(m(K_X + \Delta)) & \longrightarrow & f^* f_* \mathcal{O}_T(m(K_T + \Delta_T)) \\ \downarrow & & \downarrow \\ \mathcal{O}_X(m(K_X + \Delta)) & \longrightarrow & \mathcal{O}_T(m(K_T + \Delta_T)) \end{array}$$

Since the upper arrow and the right arrow are surjective, $m(K_X + \Delta)$ is relatively globally generated along T over S and so $m(K_X + \Delta)$ is relatively globally generated over S . \square

Proof of (1.6). By (1.4), we may assume that X is irreducible, i.e., we only need to treat the case that (X, Δ) is log canonical. We may assume that S is normal and dominated by X . By induction, we may assume that (1.6) is known for lower dimensions. Replacing (X, Δ) by a dlt modification, we may assume that (X, Δ) is \mathbb{Q} -factorial and dlt. Let $T = \lfloor \Delta \rfloor$. Write $(K_X + \Delta)|_T = K_T + \Delta_T$, then (T, Δ_T) is a dslt pair.

We run a $(K_X + \Delta - \epsilon T)$ -MMP with scaling by an ample divisor H for a small positive integer ϵ . If T has a component which dominates S , then the MMP ends with a Fano contraction, and we can apply the same argument as in [Gongyo10]. Otherwise, all components of T are vertical over S . Since over the generic point of S , we have that $K_X + \Delta$ is klt, then we know that it is \mathbb{Q} -linearly equivalent to 0 (cf. [Nakayama04, V, 4.9]).

Therefore, $(X, \Delta - \epsilon T)$ is a klt pair, whose generic fiber has a good model. By [HX11, 1.1 and 2.9], we conclude that the MMP with scaling terminates with a good model X' of $(X, \Delta - \epsilon T)$ over S . Since X' is indeed a minimal model for all $(X, \Delta - \epsilon' T)$ with $0 < \epsilon' < \epsilon$, which is good again by [HX11, 1.1].

The MMP sequence is $(K_X + \Delta)$ -trivial, so (X', Δ') is a lc pair. We let $\mu : (X^d, \Delta^d) \rightarrow (X', \Delta')$ be a dlt model. Let T' be the strict transform of T on X' , then $(X', \Delta' - \epsilon T')$ is klt and the non-klt locus of (X', Δ') is contained in $\text{Supp}(T')$. Since each μ -exceptional divisor has coefficient 1 in Δ^d , it follows that $\lfloor \Delta^d \rfloor = \text{Supp}(\mu^* T')$. Thus if $T^d = \mu^* T'$, then $K_{X^d} + \Delta^d - \epsilon T^d = \mu^*(K_{X'} + \Delta' - \epsilon T')$ is semiample over S . By induction on the dimension, $K_\Sigma + \Delta_\Sigma$ is semiample over S for all components Σ of $\lfloor \Delta^d \rfloor = \text{Supp}(T^d)$. By (1.5), $K_{X^d} + \Delta^d$ is semi-ample over S and hence so is $K_{X'} + \Delta'$. Since each step of the MMP is $(K_X + \Delta)$ -trivial, this implies that $K_X + \Delta$ is semi-ample over S , i.e., $K_X + \Delta \sim_{\mathbb{Q}, S} 0$. \square

REFERENCES

- [Ambro03] F. Ambro; *Quasi-log varieties*. Tr. Mat. Inst. Steklova **240** (2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 220-239; translation in Proc. Steklov Inst. Math. 2003, no. 1 **240**, 214-233.
- [Birkar11] C. Birkar; *Existence of log canonical flips and a special LMMP*. arXiv:1104.4981.
- [EV92] H. Esnault, E. Viehweg; *Lectures on Vanishing Theorems*. DMV Seminar, Band 20, Birkhuser Verlag Basel, 1992
- [FG11] O. Fujino, Y. Gongyo; *Log pluricanonical representations and abundance conjecture*. arXiv:1104.0361.
- [Fujino00] O. Fujino; *Abundance theorem for semi log canonical threefolds*. Duke Math. J. **102** (2000), no. 3, 513–532.
- [Gongyo10] Y. Gongyo; *Abundance theorem for numerical trivial log canonical divisors of semi-log canonical pairs*. arXiv:1005.2796.
- [HMX11] C. Hacon, J. McKernan, C. Xu; *Boundedness of the moduli functor*. Preprint.
- [HX11] C. Hacon, C. Xu; *Existence of log canonical closures*. arXiv:1105.1169v2.
- [JZ02] J. Jost, K. Zuo; *Arakelov type inequalities for Hodge bundles over algebraic varieties. I. Hodge bundles over algebraic curves*. J. Algebraic Geom. **11** (2002), no. 3, 535–546.
- [Kawamata98] Y. Kawamata; *Subadjunction of log canonical divisors. II*. Amer. J. Math. **120** (1998), no. 5, 893–899.
- [KK10] J. Kollár, S. Kovács; *Log canonical singularities are Du Bois*. J. Amer. Math. Soc. **23** (2010), no. 3, 791-813.
- [KMM94] S. Keel; K. Matsuki; J. McKernan; *Log abundance theorem for threefolds*. Duke Math. J. **75** (1994), no. 1, 99–119.
- [Kollár86] J. Kollár; *Higher direct images of dualizing sheaves. I*. Ann. of Math. (2) **123** (1986), no. 1, 11–42.
- [Kollár07] J. Kollár; *Kodaira’s canonical bundle formula and adjunction*. Flips for 3-folds and 4-folds, 134–162, Oxford Lecture Ser. Math. Appl., 35, Oxford Univ. Press, Oxford, 2007.
- [Kollár11a] J. Kollár; *Moduli of higher dimensional varieties*. Book to appear. See <http://www.math.princeton.edu/~kollar/>.
- [Kollár11b] J. Kollár; *Sources of log canonical centers*. arXiv:1107.2863.

- [Kolláretal92] J. Kollár et al.; Flips and abundance for algebraic threefolds. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991. Astérisque No. **211** (1992). *Société Mathématique de France, Paris*, 1992.
- [Nakayama04] N. Nakayama; *Zariski-decomposition and abundance*. MSJ Memoirs, **14**. Mathematical Society of Japan, Tokyo, 2004.
- [Steenbrink77] J. Steenbrink; *Mixed Hodge structure on the vanishing cohomology*. Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976, 525–563. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, 155 SOUTH 1400 EAST, JWB
233, SALT LAKE CITY, UT 84112, USA
E-mail address: `hacon@math.utah.edu`

BELJING INTERNATIONAL CENTER OF MATHEMATICS RESEARCH, 5 YIHEYUAN ROAD,
HAIDIAN DISTRICT, BEIJING 100871, CHINA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, 155 SOUTH 1400 EAST, SALT
LAKE CITY, UT 84112, USA
E-mail address: `dr.chenyang.xu@gmail.com`