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On epimorphisms of spherical Moufang buildings

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Abstract

In this paper we classify the the epimorphisms of irreducible spherical Moufang buildings (of rank ≥ 2) defined over a field. As an application we characterize indecomposable epimorphisms of these buildings as those epimorphisms arising from $\mathbb{R}$-buildings.

1 Introduction

The theory of buildings was introduced by Jacques Tits in the late 60's in order to better understand certain classes of (algebraic) groups. This theory certainly attained this goal and much more. The two most studied subclasses are the spherical and affine buildings.

The spherical buildings have been classified by Jacques Tits in 1974 ([22]) provided that the rank is at least three. Using the so-called 'spherical building at infinity' of an affine building, Tits also classified in the affine buildings of rank at least 4 ([23]). This classification also includes non-discrete generalizations of affine buildings, the $\mathbb{R}$-buildings.

Whereas this classification uses spherical buildings to say something about $\mathbb{R}$-buildings, in the current paper we will use $\mathbb{R}$-buildings to answer a problem concerning spherical buildings. The question is to classify or characterize epimorphisms of Moufang spherical buildings. We will show that these correspond to valuations, provided that the building is defined over a field.

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Epimorphisms arising from Moufang $\mathbb{R}$-buildings turn out to be the 'primitive' epimorphisms for this class, i.e. if one cannot decompose the epimorphism into two proper epimorphisms, then the epimorphism arises directly from an $\mathbb{R}$-building.

This extends known results for projective spaces (see Section 2.4). For a precise version of the main results and corollaries we refer to Section 3. The remaining open class, the one consisting of the polar spaces of pseudo-quadratic form type defined over a proper skew field is handled by the author and Petra N. Schwer in a forthcoming paper ([19]) using different, case-specific methods.

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2 Preliminaries

2.1 Buildings

Let (W, S) be a Coxeter system, then a *weak building of type* (W, S) is a pair $(\mathcal{C}, \delta)$ consisting of a nonempty set $\mathcal{C}$ (called *chambers*) and a map $\delta : \mathcal{C} \times \mathcal{C} \to W$ (called the *Weyl distance*), such that for every two chambers C and D the following holds.

(WD1) $\delta(C, D) = 1$ if and only if $C = D$.

(WD2) If $\delta(C, D) = w$ and $C' \in \mathcal{C}$ satisfies $\delta(C', C) = s \in S$, then $\delta(C', D) \in \{sw, w\}$. If moreover $l(sw) = l(w) + 1$ (where l is the word metric on W w.r.t. S), then $\delta(C', D) = sw$.

(WD3) If $\delta(C, D) = w$, then for any $s \in S$ there exists a chamber $C' \in \mathcal{C}$ such that $\delta'(C', C) = s$ and $\delta(C', D) = sw$.

This weak building is said to be *spherical* if the Coxeter group W is finite. The *rank* of a weak building is defined to be $|S|$. Two chambers are *s-equivalent* (with $s \in S$) if the Weyl distance between them is either s or the identity element 1 of W. Consider a subset $S' \subset S$. The connected components of $\mathcal{C}$ using only equivalences in S' are called the *S'-residues*, which are again buildings. The rank one residues (so $S' = \{s\}$) are also called *(s-)panels*. If each panel of the weak building has cardinality at least 3, then we say that $(\mathcal{C}, \delta)$ is a *building*.

A building is *irreducible* if it cannot be decomposed as a direct product of two (non-trivial) buildings.

A *morphism* ϕ between two (weak) buildings $(\mathcal{C}, \delta)$ and $(\mathcal{C}', \delta')$ of type (W, S) is a map from $\mathcal{C}$ to $\mathcal{C}'$ preserving s-equivalency for each $s \in S$. If in addition this map is respectively injective or surjective, then it is respectively called an *endomorphism* or an *epimorphism*. If it is both injective and surjective then it is an *isomorphism*. We say that an automorphism g of a building $(\mathcal{C}, \delta)$ *descends* under an epimorphism ϕ from $(\mathcal{C}, \delta)$ to $(\mathcal{C}', \delta')$ if there exists an automorphism g' of $(\mathcal{C}', \delta')$ such that $\phi \circ g = g' \circ \phi$. One easily verifies this is equivalent to the condition $\forall C, D \in \mathcal{C} : C^\phi = D^\phi \Leftrightarrow C^{g\phi} = D^{g\phi}$. Note that the automorphisms who descend form a subgroup of the full automorphism group.

Two chambers of a building of spherical building are *opposite* if the Weyl distance between them is maximal w.r.t. the word metric on W. An s-panel and s'-panel are *opposite* if s and s' are mapped to each other by the opposition involution of the Coxeter group (see [2, p. 61]) and contain opposite chambers. Opposite panels have the property that for each chamber in one of these panels there is a unique non-opposite chamber in the other one.

For more information on (spherical) buildings, we refer to [2] and [25].

Remark 2.1 If we speak about an epimorphism of a building, we assume that its image is not a weak building. Non-type preserving epimorphisms will not be considered in this paper.

Remark 2.2 Our main results only deal with irreducible buildings. As reducible buildings are direct products of irreducible buildings, the study of the epimorphisms of these can be brought back to their components.

2.2 Generalized polygons

For the spherical buildings of rank 2, the generalized polygons, we will take an incidence geometric point of view using the panels as basic objects. We define them as follows.

Let $\Gamma := (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a rank 2 geometry consisting of a *point set* $\mathcal{P}$, a *line set* $\mathcal{L}$ (with $\mathcal{P} \cap \mathcal{L} = \emptyset$), and *incidence relation* $\mathtt{I}$ between $\mathcal{P}$ and $\mathcal{L}$. An *element* of Γ is a point or line of it. An ordered sequence $(x_0, x_1, \ldots, x_k)$ of elements of Γ is called a *path* of *length k* if each two subsequent elements in it are incident. We say it *stammers* if there is an i such that x_i and x_{i+2} are identical.

The rank 2 geometry $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a generalized n-gon ($n \in \mathbb{N}$, $n \geq 2$) if it satisfies the following axioms.

(GP1) Every element is incident with at least three other elements.

(GP2) For every pair of elements $x, y \in \mathcal{P} \cup \mathcal{L}$, there exists a non-stammering path $(x_0 = x, x_1, \ldots, x_{k-1}, x_k = y)$ of length at most n

(GP3) The sequence in (GP2) is unique if its length is strictly smaller than n.

Note that this definition is self-dual in the notions point and line. The chambers here are incident point-line pairs. Panels are the sets of chambers containing a certain element. The corresponding building is irreducible if and only if $n \geq 3$. We define an *apartment* to be an ordinary n-gon. A *root* is a non-stammering path of length n.

The *distance* between two elements is the length of a shortest path between them. Two elements at maximal distance n are said to be *opposite*. If x and y are not opposite or equal, then the *projection of y on x* (denoted by $\mathrm{proj}_x y$) is the unique element incident with x closest to y.

Morphisms from this point of view are maps between generalized n-gons, mapping points to points, lines to lines, such that incident elements are mapped to incident elements. Endomorphisms and epimorphisms are then defined as usual.

If the image of a non-stammering path under a epimorphism of the generalized polygon becomes stammering, we say that the path *collapses* under the epimorphism. We will use the same notion for apartments, by considering a non-stammering path of length $2n$ defining the apartment.

2.2.1 The Moufang property

Let $\alpha := (x_0, x_1, \ldots, x_n)$ be a root of a generalized n-gon Γ with $n \geq 3$. A *root elation of α* is an automorphism of Γ fixing each element incident with an element of the subpath $(x_1, \ldots, x_{n-1})$. The *root group of α* is the group consisting of all root elations of α. We say that α is *Moufang* if this group acts transitively on the elements incident with x_0 different from x_1. One shows that if this is the case then the group acts sharply transitive on this set (see for instance [16, Def. 5.2.1]). The generalized polygon Γ is *Moufang* if all its roots are.

Remark 2.3 It is possible to generalize this definition to higher rank (spherical) buildings. An irreducible spherical building of rank at least 3 is automatically Moufang by a result of Tits ([22]).

2.3 Classifications of spherical buildings and the field of definition

The book [22] of Tits includes a classification of the irreducible spherical buildings of rank at least 3. Moufang generalized polygons have been classified by Tits and Weiss in [24].

The aim of this section is to briefly discuss this classification and clarify what we mean by 'defined over a field' and 'defining field' in the statement of the main results and corollaries (Section 3). These notions are not unambiguous and will be different than the point of view of [26, Rem 30.29].

2.3.1 Moufang generalized polygons

We start with the Moufang generalized polygons (in which we follow [24]). Let Σ be an apartment of a generalized Moufang n-gon Γ, and label the elements of it by x_i, with $i \in \mathbb{Z}$ such that $x_i \mathrm{I} x_{i+1}$ and $x_i = x_{i+2n}$. This apartment will be called the *hat-rack*. Let U_i be the root group of the root $(x_i, x_{i+1}, \ldots, x_{i+n})$. All of the U_i forms the *root group data* of Γ associated to Σ. We will often use subscripts to indicate to which root group an automorphism belongs.

Define $U_{[i,j]}$ to be the group generated by $U_i, U_{i+1}, \ldots, U_j$ (if $j < i$, then we let $U_{[i,j]}$ denote the group consisting only of the trivial automorphism).

The following lemmas express the commutation relations between U_i and U_j when the corresponding roots are not opposite (i.e. $i \not\equiv j \mod 2n$).

Lemma 2.4 ([24], Prop. 5.5) *If $i+1 \leq j \leq i+n-1$, then $[U_i, U_j] \leq U_{[i+1,j-1]}$.* □

Lemma 2.5 ([24], Prop. 5.6) *If $i \leq j \leq i+n-1$, then the product $U_i U_{i+1} \ldots U_j$ is the group $U_{[i,j]}$, and every element of this group has a unique decomposition as $u_i u_{i+1} \ldots u_j$ with $u_k \in U_k$.* □

The last lemma implies that by giving descriptions of the root groups U_1 up to U_n and the commutation relations between them, that one can completely describe the group $U_{[1,n]}$, moreover this information suffices to describe the Moufang generalized polygon up to isomorphism.

This reduces the classification to determining the possible root groups $U_1, \ldots, U_n$ and their commutation relations. Let us briefly list the possibilities (for a detailed description see [24, §16]).

- The triangles $\mathcal{T}(A)$.
- The quadrangles $\mathcal{Q}_{\mathcal{I}}(K, K_0, \sigma)$ of involutory type.
- The quadrangles $\mathcal{Q}_{\mathcal{Q}}(K, L_0, q)$ of quadratic form type.
- The quadrangles $\mathcal{Q}_{\mathcal{D}}(K, K_0, L_0)$ of indifferent type.
- The quadrangles $\mathcal{Q}_{\mathcal{P}}(K, K_0, \sigma, L_0, q)$ of pseudo-quadratic form type.
- The quadrangles $\mathcal{Q}_{\mathcal{E}}(K, L_0, q)$ of type E_i $(i = 6, 7, 8)$.
- The quadrangles $\mathcal{Q}_{\mathcal{F}}(K, L_0, q)$ of type F_4.
- The hexagons $\mathcal{H}(J, F, \#)$.
- The octagons $\mathcal{O}(K, \sigma)$.

For the remainder of this paper we will consider the quadrangles of involutory type to be a subclass of those of pseudo-quadratic form type. However we will need the class of *quadrangles of quadratic and honorary involutory type*. These are quadrangles of quadratic form type where the vector space L_0 over K with quadratic form q can be interpreted as a composition algebra over K with norm q. These can also interpreted as involutory quadrangles except when this composition algebra is an octonion algebra (in which case one calls them honorary).

The *underlying field skew field or octonion algebra* for all of these cases is A, K or J where appropriate. We consider quadrangles of quadratic and honorary involutory type to be of quadratic form type, so they are defined over the underlying field K, not over the composition algebra.

2.3.2 Higher rank

In order to describe the higher rank case one considers the following reduction. Let $(\mathcal{C}, \delta)$ be a spherical Moufang building of type (W, S). Choose a chamber C in $\mathcal{C}$. The rank 2 residues containing this chamber form a collection of Moufang generalized polygons, each of which can be described as in the previous section. The rank 1 residues containing C correspond to the 'extremal' root groups U_1 and U_n of the description of the rank 2 residues. This data completely determines the building by Tits' extension result [22, Th. 4.2.1].

Let us list, without much detail, the possibilities with rank at least three (after [25, 12.12-19]), with as modification considering involutory type as a subclass of pseudo-quadratic form type). We also list each time the different isomorphism classes of rank 2 residues which occur (apart from digons).

- $\mathbf{A}_l(K)$: $\mathcal{T}(K)$.
- $\mathbf{B}_l(K, L_0, q)$: $\mathcal{T}(K)$, $\mathcal{Q}_{\mathcal{Q}}(K, L_0, q)$.
- $\mathbf{C}_l(K, K_0, \sigma)$ of quadratic or honorary type: $\mathcal{T}(K)$, $\mathcal{Q}_{\mathcal{Q}}(K_0, K, N)$ (where N is the norm induced on the composition algebra K over K_0).
- $\mathbf{BC}_l(K, K_0, \sigma, L_0, q)$: $\mathcal{T}(K)$, $\mathcal{Q}_{\mathcal{P}}(K, K_0, \sigma, L_0, q)$.
- $\mathbf{E}_l(K)$ $(i = 6, 7, 8)$: $\mathcal{T}(K)$.
- $\mathbf{F}_4(K, F, \sigma)$: $\mathcal{T}(K)$, $\mathcal{T}(F)$, $\mathcal{Q}_{\mathcal{Q}}(F, K, N)$ (where N is the norm induced on the composition algebra K over K_0).

The first four classes can be considered as continuations of rank 2 cases (see the last rank 2 residue listed each time).

The *underlying skew field or octonion algebra* for all these cases is defined to be K, except for the third case (where we define it to be the field K_0) and sixth case (where it is F). Note that a spherical Moufang building defined over a field might have rank 2 residues not defined over a field.

With this convention the only spherical Moufang buildings not defined over field (and hence not covered by the results of this paper) are the projective spaces $\mathbf{A}_l(K)$ where K is a proper skew field or octonion algebra, and the polar spaces $\mathbf{BC}_l(K, K_0, \sigma, L_0, q)$, not of type $\mathbf{C}_l(K, K_0, \sigma)$, where K is a proper skew field.

2.4 Known results on epimorphisms of spherical buildings

Epimorphisms of generalized n-gons are well studied for generalized triangles (also known as *projective planes*). Skornyakov expressed in [20] epimorphisms in terms of the coordinatizing planar ternary rings as *places*. Subsequently the epimorphisms of projective Moufang planes and spaces have been classified (see [3], [8] and [14]).

For other generalized polygons much less is known. There is a result of Pasini ([18]) which says that the cardinalities of the preimages of an epimorphism between generalized n-gons are either always 1 or always infinite. This implies that epimorphisms between finite generalized n-gons are always isomorphisms. Epimorphisms from a generalized n-gon to a generalized m-gon with $m < n$ are studied by Gramlich and Van Maldeghem in [9] and [10].

For other Moufang spherical buildings the only result known to the author are constructions using the theory of affine buildings and their non-discrete generalizations $\mathbb{R}$-buildings (see [17] and [26]). One spherical building is then the 'building at infinity' of an $\mathbb{R}$-building and the other a residue of it. We will call such morphisms *affine epimorphisms*. The $\mathbb{R}$-buildings with an irreducible Moufang spherical building of rank at least 2 at infinity have been classified by Tits (see [5] and [23]). Without going in details, $\mathbb{R}$-buildings arise from valuations of the underlying (alternative) division algebra.

Remark 2.6 The trivial epimorphisms, i.e. isomorphisms, can and will be considered to be affine epimorphisms in this paper.

Remark 2.7 In the non-Moufang case a wild variety of epimorphisms is possible. One way to do this is by using free constructions. Another way is to slightly perturbate the constructions of $\mathbb{R}$-buildings in [21], giving rise to epimorphisms of translation planes which are not arising from $\mathbb{R}$-buildings.

3 Statement of the main results and corollaries

The first Main Result shows that being Moufang is preserved under epimorphisms (note that this is trivial for higher dimensions).

Main Result 1 *The epimorphic image of a Moufang generalized polygon is again a Moufang polygon.*

The second Main Result classifies the epimorphisms of a large class of spherical Moufang buildings.

Main Result 2 *Epimorphisms of an irreducible spherical Moufang building of rank at least 2 defined over a field, correspond to valuations over the defining field satisfying the compatibility conditions listed in Section 7.4 for a set of constants.*

The following corollaries indicate that the 'primitive' epimorphisms are the affine ones.

Main Corollary 1 *If moreover this valuation has finite rank (which is always the case if the defining field has finite transcendency degree), then the epimorphism can be realized by combining a finite number of affine epimorphisms.*

Main Corollary 2 *If an epimorphism of an irreducible spherical Moufang building of rank at least 2 defined over a field, is not decomposable in two proper epimorphisms (i.e. not isomorphisms), then it is an affine epimorphism.*

4 Reducing to the generalized polygon case

The aim of this section is to show how one can obtain epimorphisms between generalized polygons from higher rank spherical buildings. This will turn out to be useful when studying Moufang buildings via their rank 2 residues. Let ϕ be an epimorphism between spherical buildings $(\mathcal{C}, \delta)$ and $(\mathcal{C}', \delta')$ of type (W, S).

Lemma 4.1 *If C and D are two chambers of $(\mathcal{C}, \delta)$ such that $\delta'(C^\phi, D^\phi) = s \in S$, then there exists a chamber E in $\mathcal{C}$ such that $D^\phi = E^\phi$ and $\delta(C, E) = s$.*

Proof. We start by finding a chamber F of $(\mathcal{C}, \delta)$ such that F^ϕ is opposite to both C^ϕ and D^ϕ. As an epimorphism only can shorten the (numerical) distance between two chambers, one has that F is opposite to both C and D. If we project the chamber D on the s'-panel containing F (where s' is the image of s under the opposition involution) we obtain a chamber G which is the unique chamber in this panel not opposite D. Clearly, its image is the unique chamber of the s'-panel containing F^ϕ not opposite to D^ϕ. As D^ϕ is the unique chamber in the s-panel containing C^ϕ and D^ϕ not opposite to G^ϕ, we have that the projection of the chamber G back on the s-panel containing C yields a chamber E whose image has to be D^ϕ. As $\delta(C, E)$ has to be s by the definition of an s-panel, one has proven the lemma. □

Lemma 4.2 *Let C be a chamber of $(\mathcal{C}, \delta)$ and $S' \subset S$ a subset of size 2. Then ϕ induces an epimorphism from the S'-residue of $(\mathcal{C}, \delta)$ containing C to the S'-residue of $(\mathcal{C}', \delta')$ containing C^ϕ.*

Proof. The restriction of ϕ to the S'-residue of $(\mathcal{C}, \delta)$ containing C will map elements into the S'-residue of $(\mathcal{C}', \delta')$ containing C^ϕ by the definition of epimorphisms and residues. Surjectivity of this morphism is a consequence of the previous lemma. □

5 Proof of the first Main Result

5.1 Setting

Let $\Gamma := (\mathcal{P}, \mathcal{L}, \mathtt{I})$ and $\Gamma' = (\mathcal{P}', \mathcal{L}', \mathtt{I}')$ be two generalized n-gons, $\phi : \Gamma \to \Gamma'$ an epimorphism between them. Choose a root $\alpha := (x_0, x_1, \ldots, x_n)$ of Γ which does not collapse under ϕ. (To verify that these indeed exist pick x_0 and x_n to be two elements of Γ which are mapped to opposite elements, then each root beginning in x_0 and ending in x_n cannot collapse as epimorphisms only shorten distances.)

The main part of the proof is devoted to investigating under which conditions root elations of α descend under ϕ.

Let g be a root elation of α. It maps an element x_{-1} incident with x_0 but different from x_1 to an element x'_{-1} such that $x_{-1} \neq x_1 \neq x'_{-1}$. We will prove that a sufficient condition for the root elation to descend is that $x_{-1}^\phi \neq x_1^\phi \neq x'^\phi_{-1}$ (which is clearly a necessary condition as well).

Once this is established, the first Main Result will follow quickly.

5.2 Proof

We start with an auxiliary lemma.

Lemma 5.1 *If $a^\phi \mathtt{I}' b^\phi$, then there exists an element b' such that $b^\phi = b'^\phi$ and $a \mathtt{I} b'$.*

Proof. This is a reformulation of Lemma 5.1 in the language of epimorphisms between generalized polygons. □

We say that an element x of Γ has Property (*) if for each two elements $a, b \mathtt{I} x$ one has that $a^\phi = b^\phi$ if and only if $a^{g\phi} = b^{g\phi}$.

Proposition 5.2 *If each element of Γ has Property (*), then the root elation descends.*

Proof. First of all note that if two elements a and b of Γ have opposite images under ϕ, that then they also have opposite images under $\phi \circ g$ (because if a path does not collapse under ϕ, then its image under g will neither by Property (*)).

Suppose that the root elation does not descend, or equivalently that there exist elements x and y in Γ such that $x^\phi = y^\phi$, but $x^{g\phi} \neq y^{g\phi}$. (One also needs to consider the reverse statement, but this follows from an analogous exposition for the root elation g^{-1}.) Choose a pair of points x and y minimizing the distance k between them. Note that k has to be bigger than zero and even, as ϕ does not map points to lines or vice versa. If k would be 2, then a and b are both incident with some element c. Property (*) for this element then gives rise to a contradiction.

Let $(y_0 := x, y_1, \ldots, y_k := y)$ be a path of shortest length between x and y. Remark that $x^\phi = y_i^\phi$ only if $i = 0$ or k, as otherwise it would contradict the way we choose the elements x and y (as it is impossible that $x^{g\phi} = y_i^{g\phi} = y^{g\phi}$). In particular this implies that $y_1^\phi = y_{k-1}^\phi$. Minimality of k yields $y_1^{g\phi} = y_{k-1}^{g\phi}$.

Using Lemma 5.1, one can find a path $(a_0, a_1, \ldots, a_{n-1} := x, a_n := y_1)$ of length n which does not collapse under ϕ. So a_0^ϕ is opposite y_1^ϕ. Combining this with $y_1^\phi = y_{k-1}^\phi$ gives that a_0 is opposite y_{k-1}. Let $(b_0 := a_0, b_1, \ldots, b_{n-1} := y, b_n := y_{k-1})$ be the unique shortest path from a_0 to y_{k-1} containing y (which cannot collapse either). As x^ϕ equals y^ϕ, and $y_1^\phi = y_{k-1}^\phi$ is opposite to a_0^ϕ, one has that $a_1^\phi = b_1^\phi$. As the distance between a_1 and b_1 is at most 2, this implies that $a_1^{g\phi} = b_1^{g\phi}$. Now because $y_1^{g\phi} = y_{k-1}^{g\phi}$ is opposite to $a_0^{g\phi}$, we have that the distance between $y_1^{g\phi}$ and $a_1^{g\phi}$ is $n-1$. In particular if follows that $x^{g\phi} = y^{g\phi}$, which contradicts the way we have chosen x and y. $\square$

Lemma 5.3 *Let $(y_0, \ldots, y_n)$ be a path of length n in Γ which does not collapse under ϕ and $\phi \circ g$. If Property (*) is satisfied for y_n, then it is also satisfied for y_0.*

Proof. Let a and b be two elements incident with y_0. Note that y_0^ϕ and $y_0^{g\phi}$ are opposite to respectively y_n^ϕ and $y_n^{g\phi}$. Because of this one has that $a^\phi = b^\phi$ if and only if $(\mathrm{proj}_{y_n} a)^\phi = (\mathrm{proj}_{y_n} b)^\phi$, and $a^{g\phi} = b^{g\phi}$ if and only if $(\mathrm{proj}_{y_n} a)^{g\phi} = (\mathrm{proj}_{y_n} b)^{g\phi}$. Property (*) for y_n now implies that the conditions $(\mathrm{proj}_{y_n} a)^\phi = (\mathrm{proj}_{y_n} b)^\phi$ and $(\mathrm{proj}_{y_n} a)^{g\phi} = (\mathrm{proj}_{y_n} b)^{g\phi}$ are equivalent. We conclude that $a^\phi = b^\phi$ if and only if $a^{g\phi} = b^{g\phi}$, so y_0 satisfies Property (*). $\square$

Corollary 5.4 *If all elements of a root of an apartment which does not collapse under ϕ satisfy Property (*), then all elements of that apartment do.*

Proof. The apartment cannot collapse under $\phi \circ g$ by Property (*), and hence we can apply the above lemma to obtain that all elements of it satisfy this property. $\square$

Let Σ be the unique apartment containing x_{-1}, x_0, $\ldots$, x_n; and Σ' the unique apartment containing x'_{-1}, x_0, $\ldots$, x_n. So g maps Σ to Σ'. Note that our assumption $x_{-1}^\phi, x_{-1}'^\phi \neq x_1^\phi$ implies that both apartments do not collapse under ϕ. Let x_{n+1} be the unique element of Σ opposite x_1.

Proposition 5.5 *All the elements of Γ satisfy Property (*).*

Proof. The elements $x_1, \ldots, x_{n-1}$ all satisfy Property (*) as all elements incident with one of them are fixed by g. Applying the above lemma, one then has that all elements of Σ, except from possibly x_0 and x_n, satisfy Property (*). Using Lemma 5.1 one can find an element $y_2 \mathtt{I} x_1$ such that $x_0^\phi \neq y_2^\phi \neq x_2^\phi$. Let $(x_1, y_2, y_3, \ldots, y_n, x_{n+1})$ be the unique shortest path from x_1 to x_{n+1} containing y_2. Note that the path obtained by adding x_0 or x_2 as first element cannot collapse under ϕ or $\phi \circ g$ by the oppositeness of x_1^ϕ and x_{n+1}^ϕ, and Property (*) for x_1. The above lemma applied to the path $(y_n, y_{n-1}, \ldots, y_2, x_1, x_2)$ implies that y_n satisfies Property (*), and applied to the path $(x_0, x_1, y_2, \ldots, y_n)$ it implies that x_0 satisfies Property (*). One concludes that all elements of Σ satisfy Property (*).

Choose an element z of Γ. Let $(z, z_1, \ldots, z_k)$ be a shortest path from z to an element z_k of Σ ('shortest' over all elements of Σ). There are exactly two apartments of Γ containing a root of Σ and the element z_{k-1}. As it is impossible that both apartments collapse under ϕ (this would imply that Σ collapses as well), let Σ'' be such an apartment which does not collapse. It is easily seen that this apartment will not collapse under $\phi \circ g$ as z_k satisfies Property (*). So by the above corollary all elements of it satisfy property (*). By repeating this algorithm (substituting the role of Σ by Σ'') a finite number of steps, one sees that z satisfies Property (*). Hence all elements of Γ satisfy Property (*). □

Corollary 5.6 *The root elation g descends.*

Proof. By combining the above proposition with Proposition 5.2. □

The first Main Result now follows easily.

Corollary 5.7 *The epimorphic image of a Moufang polygon is again a Moufang polygon.*

Proof. For every root α' in Γ' one can find a root α in Γ mapped to it using Lemma 5.1. Even stronger, one can find for each two apartments Ξ and Ξ' containing α two corresponding apartments Σ and Σ' in Γ. The unique root elation mapping Σ to Σ' descends as it has to satisfy the condition stated in Section 5.1. Hence there is a root elation of α' mapping Ξ to Ξ'. We conclude that Γ' is a Moufang polygon. □

6 Epimorphisms and root groups

In this section we study various general properties that the root groups of a generalized Moufang polygon with an epimorphism should have. The main goal is to develop tools to be used in the next section where we invoke the classification of Moufang polygons and separate into cases.

Let Σ be an apartment of a generalized Moufang n-gon Γ which does not collapse under an epimorphism $\phi : \Gamma \to \Gamma'$, and label the elements of it by x_i, with $i \in \mathbb{Z}$ such that $x_i \mathtt{I} x_{i+1}$ and $x_i = x_{i+2n}$. This apartment will be called the *hat-rack*. Let U_i be the root group of the root $(x_i, x_{i+1}, \ldots, x_{i+n})$. All of the U_i forms the *root group data* of Γ associated to Σ. We will often use subscripts to indicate to which root group an automorphism belongs. Also if we identify a root group with, for example, the additive group of a field, then an element of the field with a subscript i denotes the corresponding element in the root group U_i.

6.1 Root group labelings of epimorphisms

By Section 5, we have for each root group U_i two subgroups $W_i \lhd V_i < U_i$ such that V_i consists of all root elations of U_i who descend, and W_i consists of those who descend to the trivial automorphism. This implies that the root groups of the apartment Σ^ϕ of Γ' are the quotients $U_i' := V_i/W_i$.

Later on, in Section 6.5 we will see that the subgroup information of U_1 and U_n suffices to determine the epimorphism uniquely. We will call this information a *root group labeling of the epimorphism.* When this information is described using superlevel sets with respect to some norm (see Section 7.3) without assuring that these sets form subgroups we speak about a *weak root group labeling of the epimorphism.*

For higher rank buildings a similar (weak) root group labeling for epimorphisms can be defined (see Section 2.3.2). The rank 2 residues containing a chosen chamber C form generalized polygons, on which epimorphisms are induced (see Lemma 4.1). The (weak) root group labelings of these epimorphisms are linked together as the rank 1 residues correspond to the U_1 and U_n of the generalized n-gons forming the residues (this n may vary over the possible residues).

In Section 6.5 we will again show that this subgroup information of this set of root groups suffices.

6.2 Opposite root groups

In this section we investigate the behavior of two opposite root groups U_i and U_j in Σ (meaning that $j \equiv i+n \mod 2n$). Without loss of generality we can assume that these are the root groups U_0 and U_n, who both fix the element x_0. Especially we consider the action of them on the elements incident with x_0.

Remark 6.1 This kind of action is also known as a Moufang set, for a detailed discussion see [7].

For an element $g \in U_n^*$ we define $\kappa_n(g)$ to be the unique element of U_0 which maps x_{-1} to x_1^g. This defines a bijection from U_n^* to U_0^*.

Lemma 6.2 *The bijection κ_n maps (bijectively)*

- W_n^* *to* $U_0 \setminus V_0$,
- $V_n \setminus W_n$ *to* $V_0 \setminus W_0$,
- $U_n \setminus V_n$ *to* W_0^*.

Proof. First assume that $g \in W_n^*$, or equivalently that $x_1^{g\phi} = x_1^\phi$. So if $\kappa(g)$ would descend, then it would map x_1^ϕ to x_{-1}^ϕ, which is impossible for an element of the root group U_0' of Γ'. Hence $\kappa_n(g) \in U_0 \setminus V_0$.

If one assumed that $g \in V_n \setminus W_n$, then the unique root elation $\kappa_n(g)$ in U_0 which maps $x_1^{g\phi}$ to x_{-1}^ϕ will descend by Section 5 (and not to the trivial one, as g descends and cannot map x_1^ϕ to x_{-1}^ϕ), so $\kappa_n(g) \in V_0 \setminus W_0$.

Lastly, assume that $g \in U_n \setminus V_n$. Because g does not descend, we have by Section 5 that $x_1^{g\phi} = x_{-1}^\phi$, which implies that $\kappa_n(g) \in W_0^*$.

Each of these maps has to be a bijection because κ_n is a bijection from U_n^* to U_0^*. □

Lemma 6.3 *Let v_n be an element of $V_n \setminus W_n$. Then the map $g \in U_n \setminus V_n \mapsto \kappa_n^{-1}(\kappa_n(v_n)\kappa_n(g))v_n^{-1}$ is a bijection from $U_n \setminus V_n$ to W_n^*.*

Proof. The orbit of $x_1^{v_n}$ under W_0 or W_n is the preimage of $x^{v_n\phi}$ under ϕ. In particular these orbits coincide. Also note that the groups W_0 and W_1 act regularly on the orbit. So we can conclude that we have bijection which maps a $w_0 \in W_0$ to the unique element $w_n \in W_n$ such that $x_1^{v_n w_0} = x_1^{w_n v_n}$ (where we made use of the fact that $W_n \lhd V_n$). By the definition of κ_n we now have that

$$\begin{aligned} x_1^{w_n v_n} &= x_1^{v_n w_0} \\ &= x_{-1}^{\kappa_n(v_n) w_0} \\ &= x_1^{\kappa_n^{-1}(\kappa_n(v_n) w_0)}. \end{aligned}$$

Hence $w_n = \kappa_n^{-1}(\kappa_n(v_n)w_0)v_n^{-1}$. The lemma is now proven because κ_n is a bijection from $U_n \setminus V_n$ to W_0^* by the previous lemma. □

6.3 Other pairs of root groups

We now investigate the behavior of non-opposite root groups of the hat-rack. In particular we want to study the interacting with the commutation relations between them (see Section 2.3.1).

Remind that $U_{[i,j]}$ is the group generated by $U_i, U_{i+1}, \ldots, U_j$ if $i \leq j$ and the trivial group otherwise. We use similar notations $V_{[i,j]}$ and $W_{[i,j]}$ to denote the subgroup of generated by the subgroups of the form V_k and W_k respectively.

Lemma 6.4 *If $i \leq j \leq i+n-1$, then the product $u_i u_{i+1} \ldots u_j$ (where $u_k \in U_k$) descends if and only if each of the factors descend.*

Proof. We prove this by induction. Assume that the product $g := u_i u_{i+1} \ldots u_j$ descends. If $i = j$, then it is trivial that the factors descend, so suppose $i < j$. Note that $x_{j-1}^g = x_{j-1}^{u_j}$, hence $x_{j-1}^{u_j \phi} = x_{j-1}^{g\phi} \neq x_{j+1}$ (the inequality holds as g descends and fixes x_{j+1}). The results from Section 5 imply that u_j descends. The product $g u_j^{-1} = u_i \ldots u_{j-1}$ descends as well, so by induction all factors descend. The other direction is trivial. □

Corollary 6.5 *If $i+1 \leq j \leq i+n-1$, then*

$$[V_i, V_j] \leq V_{[i+1,j-1]},$$
$$[V_i, W_j] \leq W_{[i+1,j-1]},$$
$$[W_i, V_j] \leq W_{[i+1,j-1]}.$$

Proof. Let $u_i \in U_i$, and $u_j \in U_j$. By the first two of the above lemmas, one can write $[u_i, u_j]$ in a unique way as a product $u_{i+1} u_{i+2} \ldots u_{j-1}$, with $u_k \in U_k$. Now suppose that $u_i \in V_i$ and $u_j \in V_j$, then the product $u_{i+1} u_{i+2} \ldots u_{j-1}$ descends, so the last lemma implies that $u_{i+1} u_{i+2} \ldots u_{j-1} \in V_{[i+1,j-1]}$. If moreover either $u_i \in W_i$ or $u_j \in W_j$, then their commutator descends to the trivial automorphism of Γ'. So the product $u_{i+1} \ldots u_{j-1}$ is an element of $W_{[i+1,j-1]}$ by applying Lemma 2.5 to Γ'. □

6.4 Action of μ-maps

The μ-maps form another type of interaction between the root groups, as the next lemma describes.

Lemma 6.6 ([24], Prop. 6.1-2) *Let $\kappa_i : U_i^* \to U_{i+n}^*$ be as in Section 6.2. The automorphism $\mu_i(u_i) := \kappa_i(u_i)u_i(\kappa(u_i^{-1}))^{-1}$ (with $u_i \in U_i^*$) fixes x_i and x_{i+n}, reflects Σ, and $U_j^{\mu_i(u_i)} = U_{2i+n-j}$ for each $j \in \mathbb{Z}$.* □

Applying Lemma 6.2 this yields the following direct corollary.

Corollary 6.7 *Let $v_i \in V_i \setminus W_i$, then*

$$V_j^{\mu_i(v_i)} = V_{2i+n-j},$$
$$W_j^{\mu_i(v_i)} = W_{2i+n-j}$$

for each $j \in \mathbb{Z}$. □

The action of various μ-maps can be found explicitly in [24, §32], and implicitly using [24, Lem. 6.4].

Lemma 6.8 *Choose a $u_1 \in U_1$ and a $u_n \in V_n \setminus W_n$. Let $[u_1, u_n^{-1}] = u_2 \dots u_{n-1}$ (with $u_i \in U_i$), then*

$$u_1 \in V_1 \Leftrightarrow u_2 \in V_2,$$
$$u_1 \in W_1 \Leftrightarrow u_2 \in W_2.$$

Proof. Corollary 6.5 states that the implications from left to right are true. So suppose that $u_1 \in U_1 \setminus V_1$. By [24, Lem. 6.2, 6.4] one has that $[u_2, \kappa_1(u_1^{-1})] = u_3 \dots u_{n-1}u_n$. As $U_1 \setminus V_1$ is stabilized under inversion as V_1 is a subgroup, it follows by Lemma 6.2 that $\kappa_1(u_1^{-1}) \in W_{n+1}$. Using Corollary 6.5 and the assumption that $u_n \in V_n \setminus W_n$ yields that $u_2 \in U_2 \setminus V_2$. This proves $u_1 \in V_1 \Leftrightarrow u_2 \in V_2$. The proof of the second part is analogous. □

6.5 Rigidity and factorizations

We end this section by stating results on how the epimorphism is determined when certain V_k and W_k are known, and how different epimorphisms are related.

Lemma 6.9 *Let $\omega := u_2 \dots u_n$ with $u_i \in U_i$ for $i \in \{2, \dots, n\}$. The image of the element x_1^ω under the epimorphism ϕ is opposite x_{n+1}^ϕ if and only if all the root elations u_i descend.*

Proof. If each of the factors $u_i \in U_i$ ($i \in \{2, \ldots, n\}$) descend then the product descends as well, so $(x_1^\omega)^\phi$ will be opposite $(x_{n+1}^\omega)^\phi = x_{n+1}^\phi$. Now suppose that one of the factors does not descend and let u_j be the one with maximal index j, then one has by the results of Section 5 that $(x_j^\omega)^\phi = (x_{j-1}^{u_j \ldots u_n})^\phi = (x_{j+1}^{u_{j+1} \ldots u_j})^\phi = (x_{j+1}^\omega)^\phi$, so the path $(x_1^\omega, x_2^\omega, \ldots, x_{n+1}^\omega)$ collapses, or equivalently $(x_1^\omega)^\phi$ is not opposite x_{n+1}^ϕ. □

Proposition 6.10 *Suppose we have two root group labelings (given respectively by subgroups $W_k \lhd V_k < U_k$ and $W'_k \lhd V'_k < U_k$, with $k = 1$ or n) of epimorphisms $\phi : \Gamma \to \Gamma_1$ and $\phi' : \Gamma \to \Gamma_2$ of the generalized n-gon Γ with respect to the same hat-rack. If $V'_k \leq V_k$ and $W_k \leq W'_k$ for $k = 1, n$, then there exists an epimorphism $\phi'' : \Gamma_1 \to \Gamma_2$ such that $\phi' = \phi'' \circ \phi$.*

Proof. Note that by Lemma 6.2 and Corollary 6.7 (using elements of $V'_k \setminus W'_k$ for an appropriate index k), similar inclusions hold for the other root groups corresponding with the hat-rack.

First we show that for each two elements y and z of Γ one has that $y^\phi = z^\phi$ implies $y^{\phi'} = z^{\phi'}$. The hat-rack in Γ contains at least one element x such that $x^{\phi'}$ is opposite to $y^{\phi'}$. Without loss of generality we may assume that this is the element x_{n+1}. In particular this element x_{n+1} is opposite to y. Let $\omega := u_2 \ldots u_n$ and $\omega' := u'_2 \ldots u'_n$ in $U_{[2,n]}$ (written as products of root elations with $u_i, u'_i \in U_i$) be the unique elements such that $y = x_1^\omega$ and $z = x_1^{\omega'}$. Note that we can do this because of Lemma 2.5 and as the group $U_{[2,n]}$ fixes the elements x_n and x_{n+1} of the hat-rack while acting regularly on elements opposite x_{n+1}. By the above lemma we have that $u_i \in V'_i$ as $y^{\phi'}$ is opposite to $x_{n+1}^{\phi'}$. As $V'_k \subset V_k$, it follows that y^ϕ is opposite to x_{n+1}^ϕ. Because $y^\phi = z^\phi$, we obtain that $u_i u_i'^{-1} \in W_i$ for $i \in \{2, \ldots, n\}$. Using $W_i \leq W'_i$, we conclude that $y^{\phi'} = z^{\phi'}$.

This property enables us to construct a surjective map ϕ'' such that $\phi' = \phi'' \circ \phi$. The only thing left to prove is that ϕ'' preserves adjacency. Let a_1 and b_1 be two incident elements of Γ_1. By Lemma 5.1 there exist incident elements a and b of Γ such that $a_1 = a^\phi$ and $b_1 = b^\phi$. It follows that $a_1^{\phi''} = a^\phi$ and $b_1^{\phi''} = b^\phi$ are incident. □

Corollary 6.11 *If the subgroups V_1, V_n, W_1 and W_n are known, then the epimorphism ϕ is unique (up to combining it with isomorphisms).*

Proof. The above proposition implies that if there are two epimorphisms $\phi : \Gamma \to \Gamma_1$ and $\phi' : \Gamma \to \Gamma_2$ with the same subgroups V_1, V_n, W_1 and

W_n (with respect to the same hat-rack), that then there exist epimorphisms $\phi'' : \Gamma_1 \to \Gamma_2$, $\phi''' : \Gamma_2 \to \Gamma_1$ such that $\phi' = \phi'' \circ \phi$ and $\phi = \phi''' \circ \phi'$. One easily verifies that ϕ'' and ϕ''' are inverses of each other, hence we obtain that the epimorphism is unique up to an isomorphism. $\square$

Corollary 6.12 *The root group labeling of an epimorphism of a spherical Moufang building defines the epimorphism (up to isomorphisms).*

Proof. This follows from the previous corollary and Tits' extension theorem [22, Th. 4.2.1]. $\square$

7 Proof of the second Main Result and the corollaries

In contrast with the previous section we now invoke the classification of irreducible spherical Moufang buildings of rank at least 2 and study what the properties determined in the previous section imply. This will lead us to a classification of epimorphisms of those buildings defined over fields.

Sketch of proof. — We start with assuming the existence of an epomorphism ϕ. For deriving necessary conditions we look at the epimorphisms induced on the rank 2 residues (see Lemma 4.2). In Sections 7.1 up to 7.2, we study one pair of opposite root groups and show that the subgroups V_i and W_i arise from a valuation to an ordered abelian group of the underlying field of definition. We then use this information to study the other root groups (Section 7.3) and determine certain conditions that need to be satisfied (Section 7.4).

The second step is to show that this information suffices to construct an epimorphism, which is done in Section 7.5. Section 7.6 then concludes the proof of the second Main Result and the Main Corollaries.

Remark 7.1 From this point on we only work with the root groups, not with elements of generalized polygons. In particular notations of the form x_i will now denote parametrizations of the root groups, not elements of a hat-rack as in the previous section. These parametrizations are of the form $x_i : M \to U_i$, where M is some additive algebraic structure.

7.1 Projective lines

In this section we assume that U_0 and U_n are isomorphic to the additive group of an alternative division ring K (later on we will restrict to fields), by maps $x_i : K \to U_i$ ($i \in \{1, n\}$), and that the map κ_0 is given by $x_0(a) \mapsto x_n(a^{-1})$.

Applying Lemma 6.3 we obtain that the map $\phi_a : x_0(b) \mapsto x_0((a^{-1} + b^{-1})^{-1} - a)$ is an bijection from $U_0 \setminus V_0$ to W_0^* for every $x_0(a) \in V_0 \setminus W_0$. The expression $(x_0(b-a)^{\phi_a})^{-1}$ simplifies to $x_0(ab^{-1}a)$, which is also an involutory bijection from $U_0 \setminus V_0$ to W_0^*, as V_0 and W_0 are subgroups of U_0.

Choose an element $t \in K$ such that $x_0(t) \in V_0 \setminus W_0$. Define the following subsets of K:

$$\begin{aligned} A &:= \{yt^{-1} \in K | x_0(y) \in U_0 \setminus V_0\}, \\ B &:= \{yt^{-1} \in K | x_0(y) \in V_0 \setminus W_0\}, \\ C &:= \{yt^{-1} \in K | x_0(y) \in W_0^*\}. \end{aligned}$$

Lemma 7.2 *The subset $R := B \cup C \cup \{0\}$ forms a subring of K containing the identity element.*

Proof. Observe that $y \mapsto by^{-1}b$ interchanges A and C bijectively for every $b \in B$. This implies that this map stabilizes B. As the identity lies in B, one also has that the inverse is a map of this form and that squaring stabilizes B. Lastly remark that $y \mapsto byb$ stabilizes all three subsets A, B and C for every $b \in B$ (by combining the maps $y \mapsto by^{-1}b$ and $y \mapsto y^{-1}$).

The subset R is an additive subgroup of K as V_0 is a subgroup of U_0. So in order to show that R forms a subring, we only need to show that it is closed under multiplication. First suppose that b and c both lie in B. The maps $y \mapsto b^{-1}yb^{-1}$ and $y \mapsto c^{-1}yc^{-1}$ stabilize the sets A, B and C. The combination of both maps bc to $(cb)^{-1}$. If b and c commute this implies that $bc \in B$ (as taking the inverse interchanges A and C). If b and c do not commute and $bc \notin B$ then the sum $bc + cb$ lies in A (as bc and cb cannot both lie in A or C at the same time). If $b + c \in B$, then we know that the square $(b+c)^2 = b^2 + c^2 + bc + cb$ also lies in B, but this is a contradiction as b^2 and c^2 are elements of B while $bc + cb \in A$. If $b + c \in C$, then one can obtain a contradiction in a similar way considering $(1 + b + c)^2$. We conclude that $bc \in B$.

Now suppose that $b \in B$ and $c \in C$. So $1 + c \in B$, hence by the previous paragraph we have that $b(1+c) = b + bc \in B \subset R$. As R is closed additively, we have that $bc \in R$.

The last case is handled analogously. Suppose that $b, c \in C$, then $1 + c \in B$, so $b(1 + c) = b + bc \in R$, hence again $bc \in R$. □

Lemma 7.3 *The set of units of R is B, and $K = R \cup (R^*)^{-1}$.*

Proof. In order to prove this notice that taking the inverse stabilizes B, and interchanges A and C. □

Remark 7.4 A ring with these properties is also known as a *total subring.*

Corollary 7.5 *If k is a field, then there exists a valuation ν of K to an ordered abelian group Λ and the symbol ∞ such that*

$$\begin{aligned} A &= \{y \in K | \nu(y) < 0\}, \\ B &= \{y \in K | \nu(y) = 0\}, \\ C &= \{y \in K | \nu(y) > 0\}. \end{aligned}$$

Proof. The previous lemma implies that R is a valuation ring, and hence defines a valuation with the desired properties (see [15]). □

Returning to the root group U_0, we now have in the case that K is a field that

$$\begin{aligned} V_0 &= \{x_0(a) \in U_0 | \nu(a) \geq l\}, \\ W_0 &= \{x_0(a) \in U_0 | \nu(a) > l\}, \end{aligned}$$

where $l = \nu(t)$. Using Lemma 6.2 one can also describe the subgroups in U_n.

$$\begin{aligned} V_n &= \{x_n(a) \in U_n | \nu(a) \geq -l\}, \\ W_n &= \{x_n(a) \in U_n | \nu(a) > -l\}. \end{aligned}$$

Remark 7.6 Corollary 7.5 is not true for skew fields or octonion algebras, as a total subring is not necessarily stabilized by inner automorphisms, which is necessary for obtaining a valuation.

7.2 Orthogonal Moufang sets

The only case where there are no opposite root groups of the form discussed in the previous sections are the Moufang quadrangles of exceptional type and those of indifferent type (so $n = 4$). The method here is to consider a full subquadrangle of quadratic form type (*full* means that we do not have to restrict the root groups of even index). The epimorphism of the entire quadrangle implies one of the subquadrangle, but not necessarily to a thick generalized quadrangle. The 'full' property assures us at least some thickness, and due to the fact that the epimorphism arises by restricting root groups, one still can consider subgroups V_k and W_k and apply the results from Section 6.

Let us describe this subquadrangle. Let K be a field, L_0 a vector space over K equipped with an anisotropic quadratic form $q : L_0 \to K$. Let f be the bilinear form associated to q. Let the root groups U_0, U_2 and U_4 be parametrized by the additive group of L_0 via isomorphims x_0, x_2 and x_4. The root groups U_1, U_3 and U_5 are parametrized by the additive group of the field K via isomorphisms x_1, x_3 and x_5. The map $\kappa_0 : U_0 \to U_4$ is given by $x_0(u) \mapsto x_4(u/q(u))$. Because the subquadrangle is full, we have that $V_k \neq W_k$ for k even. This is however not guaranteed for those of odd index (and hence we cannot apply the results from Section 7.1 directly). We also list the non-trivial commutation relations between the root groups U_1, U_2, U_3 and U_4 (see [24, 16.3]):

$$[x_2(a), x_4(b)^{-1}] = x_3(f(a, b)),$$
$$[x_1(t), x_4(a)^{-1}] = x_2(ta)x_3(tq(a)).$$

The existence of such a subquadrangle (and with similar notations) of the Moufang quadrangles of exceptional type E_i ($i = 6, 7, 8$) follows from the description [24, 16.6-7], for those of indifferent type the notations from [24, 16.4] and our notations are related by the following table.

Our notations	[24, 16.4]
K	K^2
L_0	L_0
q	$x \mapsto x^2$

Let $x_4(a)$ be an element of $V_4 \setminus W_4$, and b an element of L_0, linear independent of a. Denote by $\widehat{L}_0$ the two-dimensional subspace of L_0 spanned by both a and b. We parametrize this two-dimensional subspace by a quadratic extension F of K, using a map $\theta : F \to \widehat{L}_0$, such that the norm function $N : F \to K$ of this field extension agrees with q and that $\theta^{-1}(a)$ is an element of K (see for example [6, §2.6]). For $i = 0, 2$ and 4 this subspace implies a subgroup $\widehat{U}_i$ of U_i, parametrized by the map $x_i \circ \theta : F \to \widehat{U}_i$.

If the field extension F/K is separable, then we denote by σ the Galois involution of the extension. If it is inseparable, then σ will be the identity.

Invoking Section 7.1 on $\widehat{U}_0$ and $\widehat{U}_4$, we obtain a valuation ω of F such that (with $l := \omega(\theta^{-1}(a))$:

$$V_4 \cap \widehat{U}_4 = \{x_4(\theta(y)) \in \widehat{U}_4 | \omega(y) \geq l\},$$
$$W_4 \cap \widehat{U}_4 = \{x_4(\theta(y)) \in \widehat{U}_4 | \omega(y) > l\}.$$

Note that the restriction of ω to K does not depend of the choice of b. Also observe that each one-dimensional subspace of $\widehat{L}_0$ contains elements which are mapped to elements of V_4 by $x_4 \circ \theta$ (and analogously for V_0 and V_2).

We now claim that the automorphism σ arising from the field extension leaves the valuation ω invariant. Suppose that this is not the case, so there exists a $w \in F$ such that $\omega(w) < \omega(w^\sigma)$. Note that the field extension F/K must be separable and accordingly that the bilinear form f restricted to $\widehat{L}_0$ is non-trivial. Combined with the observation on one-dimensional subspaces of $\widehat{L}_0$, Corollary 6.5 and the commutation relation between U_2 and U_4 this yields that the subgroup $V_3 < U_3$ contains not only of the identity. Corollary 6.7 then implies that the subgroup $V_1 < U_1$ is non-trivial.

By the commutation relations and Lemma 6.8 we have that whenever $x_1(t) \in V_1$, then $\{x_2(\theta(y)) \in \widehat{U}_2 | \omega(y) \geq l + \omega(t)\} \subset V_2 \cap \widehat{U}_2$. A consequence of this is t cannot have arbitrary small valuations unless ω is the trivial valuation, as this would imply that $\widehat{U}_2 \subset V_2$ for every choice of b (and hence $U_2 = V_2$). A similar thing is true for choices of $x_3(t) \in V_3$ by Corollary 6.7.

Let $d := w^{-1+\sigma}$, so $d^\sigma = d^{-1}$ and $\omega(d) > 0$. Hence $x_4(\theta(d^m))$ will be an element of V_4 for high enough values of m. The following element is contained in V_3 by Corollary 6.5:

$$\begin{aligned}
[x_2(\theta(c)), x_4(\theta(d^m))^{-1}] &= x_3(f(\theta(c), \theta(d^m))) \\
&= x_3(q(\theta(c) + \theta(d^m)) - q(\theta(c)) - q(\theta(d^m))) \\
&= x_3(N(c + d^m) - N(c) - N(d^m)) \\
&= x_3(c^\sigma(d^m) + c(d^m)^\sigma) \\
&= x_3(cd^m + cd^{-m}) \\
&= x_3(c(d^m + d^{-m})).
\end{aligned}$$

The last factor has an arbitrary low valuation using arbitrary large m. This contradicts the earlier remark that one cannot choose a $x_3(t) \in V_3$ with $t \in K$ having arbitrary small valuations.

We conclude that σ leaves ω invariant, so for an element $x_4(\theta(f)) \in \widehat{U}_4$ we have that $\omega(q(\theta(f))) = \omega(N(f)) = 2\omega(f)$. As the valuation ω restricted to K is independent of the choice of b, we finally obtain:

$$V_4 = \{x_4(v) \in U_4 | \omega(q(v)) \geq 2l\},$$
$$W_4 = \{x_4(v) \in U_4 | \omega(q(v)) > 2l\}.$$

7.3 Implications on the root group sequence

Assume we have an epimorphism $\phi : \Gamma \to \Gamma'$ between Moufang polygons. We use the description of the root group sequence as found in [24, §16 and §32] (parametrizing the root group U_r by a map x_r, with $r \in \{1, \ldots, n\}$). One can define a 'norm' function on the algebraic structure defining the other root groups into the underlying field. We list the functions in question in the following table:

		i odd	i even
$n = 3$	$\mathcal{T}$:	id	id
$n = 4$	$\mathcal{Q}_\mathcal{Q}$:	id	q
	$\mathcal{Q}_\mathcal{D}$:	$a \mapsto a$	$a \mapsto a^2$
	$\mathcal{Q}_\mathcal{P}$:	$(a, t) \mapsto t$	id
	$\mathcal{Q}_\mathcal{E}$:	$(a, t) \mapsto q(\pi(a) + t)$	q
	$\mathcal{Q}_\mathcal{F}$:	$\hat{q}$	q
$n = 6$	$\mathcal{H}$:	N	id
$n = 8$	$\mathcal{O}$:	id	$(u, v) \mapsto R(u, v) := v^{\sigma+2} + uv + u^\sigma$

We will denote the 'norm' function on U_j by a generic η_j regardless of type. The involutory quadrangles $\mathcal{Q}_{\mathcal{I}}$ are not listed as we will treat them as a subcase of the pseudo-quadratic quadrangles $\mathcal{Q}_{\mathcal{P}}$. Fix i to be n if Γ is a quadrangle of quadratic form type or an octagon and 1 otherwise, this for the rest of this section. Also set j to be 2 when $i = n$, and $n-2$ when $i = 1$. The importance of the norm functions is illustrated by the following lemma.

Lemma 7.7 *Let $u_1 \in U_1$ and $u_n \in U_n$ be two root elations. If one writes $[u_1, u_n^{-1}]$ as a product $u_2 \dots u_{n-1}$ ($u_r \in U_r$), then $\eta_j(u_j) = \pm\eta_1(u_1)\eta_n(u_n)$.*

Proof. By straightforward calculations using the commutations relations found in [24, §16]. □

By applying the case studies made in Sections 7.1 and 7.2 to the explicit descriptions in [24, §16 and §32] one observes that if the generalized polygon is not defined over a (proper) skew field or alternative division algebra, that then there exists a valuation $\nu : K \twoheadrightarrow \Lambda \cup \{\infty\}$ and $l \in \Lambda$ such that

$$V_i = \{x_i(a) \in U_i | \nu(\eta_i(a)) \geq l\},$$
$$W_i = \{x_i(a) \in U_i | \nu(\eta_i(a)) > l\}.$$

Choose a $v_{n+1-i} \in V_{n+1-i} \backslash W_{n+1-i}$, and let $k := \nu(\eta_{n+1-i}(x_{n+1-i}^{-1}(v_{n+1-i})))$. One is now able to describe the groups V_j and W_j, and subsequently V_{n+1-i} and W_{n+1-i}.

Lemma 7.8

$$V_j = \{x_j(a) \in U_j | \nu(\eta_j(a)) \geq k + l\},$$
$$W_j = \{x_j(a) \in U_j | \nu(\eta_j(a)) > k + l\}.$$

Proof. We will prove this under the assumption that $i = n$ (so $j = 2$), the other case is symmetric. Let $u_2 \in V_2$, and $u_n := u_2^{(\mu_1(v_1)^{-1})}$. Using [24, Lem. 6.4] this implies that $[v_1, u_n^{-1}] = u_2 u_3 \dots u_{n-1}$ with $u_r \in U_r$ for $r \in \{3, \dots, n-1\}$. The previous lemma yields that $\nu(\eta_2(x_2^{-1}(u_2))) = \nu(\eta_1(x_1^{-1}(u_1))) + \nu(\eta_n(x_n^{-1}(u_n))) = k + \nu(\eta_n(x_n^{-1}(u_n)))$. The statement now follows from Corollary 6.7. □

Corollary 7.9

$$V_{n+1-i} = \{x_{n+1-i}(a) \in U_{n+1-i} | \nu(\eta_{n+1-i}(a)) \geq k\},$$
$$W_{n+1-i} = \{x_{n+1-i}(a) \in U_{n+1-i} | \nu(\eta_{n+1-i}(a)) > k\}.$$

Proof. From the above lemma and Lemmas 6.8 and 7.7. □

We now have a description of V_1, V_n, W_1 and W_n, which suffices to describe the epimorphism by Corollary 6.11.

The next goal is now to derive compatibility conditions. We start by describing the other subgroups of interest of U_r with $r \in \{2, \dots, n-1\}$, using Corollary 6.7 (and the relations given in [24, §16 and §32]) a finite number of times. We display this information schematically as a vector where the n coordinates correspond to respectively $U_1, \dots, U_n$, and the value at a coordinate r is the element of Λ which defines the subgroups V_r and W_r as a hyperlevel set and strict hyperlevel set respectively with respect to $\nu \circ \eta_r \circ x_r^{-1}$.

$\underline{n=3}$	$\mathcal{T}$:	$(l, l+k, k)$
$\underline{n=4}$	$\mathcal{Q_P}$:	$(k, l+k, l+l'+k, l)$
	$\mathcal{Q_Q}$:	$(l, 2l+k, l+k, k)$
	$\mathcal{Q_D}, \mathcal{Q_E}, \mathcal{Q_F}$:	$(k, l+k, 2l+k, l)$
$\underline{n=6}$	$\mathcal{H}$:	$(k, l+k, 3l+2k, 2l+k, 3l+k, l)$
$\underline{n=8}$	$\mathcal{O}$:	$(l, 2l+l'+k, l+l'+k, 2l+2l'+k+k',$ $l+l'+2k-k', 2l+l'+k+k', l+k, k)$

The l' for the quadrangle of pseudo-quadratic form type $\mathcal{Q_P}$ and the octagon case $\mathcal{O}$ is defined as follows. Let $x \in K$ such that $\nu(x) = l$, then we set $l' := \nu(x^\sigma)$. Note that l' has to be independent of the choice of x. The element k' is defined in a similar way.

7.4 Compatibility conditions

In this section we describe the extra conditions who arise from induced epimorphisms on the rank 2 residues. These conditions, which we call the *compatibility conditions*, involve the valuation ν and the underlying algebraic structures.

Remark 7.10 We will not always derive the strongest conditions possible. This will not be a problem (and is even slightly beneficial) as we will see in Sections 7.5.1 and 7.5.2.

7.4.1 Digons, Triangles, Quadrangles of indifferent type

We impose no conditions here.

7.4.2 Quadrangles $\mathcal{Q}_{\mathcal{P}}(K, K_0, \sigma, L_0, q)$ of pseudo-quadratic form type

The first compatibility condition involves the involution σ. From the appearance of the l' in the last list, one has that if $x, y \in K$ and $\nu(x) = \nu(y) = l$, then $l' = \nu(x^\sigma) = \nu(y^\sigma)$. Note that there exists an $x \in K$ such that $\nu(x) = l$. Now suppose that y, z are elements of K such that $\nu(y) = \nu(z)$, then $\nu(xyz^{-1}) = l$. So $\nu((xyz^{-1})^\sigma) = \nu(x^\sigma)$, which implies that $\nu(y^\sigma) = \nu(z^\sigma)$. Suppose that there is an $y \in K$ such that $\nu(y) < \nu(y^\sigma)$, then $\nu(1 + y^{-1+\sigma}) = \nu(1) = 0$. Applying σ yields

$$\begin{aligned} 0 = \nu(1^\sigma) &= \nu((1 + y^{-1+\sigma})^\sigma) \\ &= \nu(1 + y^{1-\sigma}) \\ &= \nu(y^{1-\sigma}) < 0, \end{aligned}$$

which is a contradition. We conclude as first compatibility condition that

$$\forall t \in K : \nu(t) = \nu(t^\sigma).$$

Note that this implies that $l' = l$.

A second compatibility condition involves the skew-hermitian form f. By the commutation relations between U_1 and U_3 (and Corollary 6.5), we have that if $(u, t), (v, s) \in T$ with $\nu(t) \geq k$, $\nu(s) \geq 2l + k$, then $\nu(f(u, v)) \geq l + k$. One can simplify this by using substitutions with suitable scalar products to

$$\forall (u, t), (v, s) \in T : \nu(t), \nu(s) \geq k \Rightarrow \nu(f(u, v)) \geq k.$$

7.4.3 Quadrangles $\mathcal{Q}_{\mathcal{Q}}(K, L_0, q)$ of quadratic form type

In a similar way as for the second compatibility condition for pseudo-quadratic forms one obtains (using the commutation relations between U_2 and U_4) that

$$\forall u, v \in L_0 : \nu(q(u)), \nu(q(v)) \geq k \Rightarrow \nu(f(u, v)) \geq k.$$

Remark 7.11 Let us consider the special case that the quadrangle is also of quadratic or honorary involutory type. So L_0 is a composition algebra equipped over K equipped with norm q. The map $\nu' := \nu \circ q$ on this composition division algebra satisfies $\nu'(u.v) = \nu'(u) + \nu'(v)$. Observe that the compatibility condition, using scalar multiples in the composition algebra with an element w with $\nu'(w) = k$, together with ν being a valuation and $f(u, v) = q(u + v) - q(u) - q(v)$ implies that the subset $\{u \in L_0 | \nu(u)\}$ is a total subring. This total subring is closed under inner automorphisms and hence gives rise to a valuation on L_0.

7.4.4 Quadrangles $\mathcal{Q}_{\mathcal{E}}(K, L_0, q)$, $\mathcal{Q}_{\mathcal{F}}(K, L_0, q)$of types F_4, E_6, E_7 and E_8

A list of the compatibility conditions (ten in total) one needs for these cases is listed in Equation 3 in Section 7.5.3, where ϕ_r is the function $\nu \circ \eta_r$ and η_r as defined in Section 7.3.

The number of equations is much larger than in the other cases because the residues of the afffine buildings associated to the generalized Moufang quadrangles of these types are not fully described yet. When such a description becomes available (as announced in [26, Rem. 21.43 and p. 228]), one can expect to reduce the number of equations needed substantially.

7.4.5 Hexagons $\mathcal{H}(J, F, \#)$

Applying the same argument as for the second compatibility condition for pseudo-quadratic forms to the root groups U_1 and U_3 gives

$$\forall u, v \in J : \nu(N(u)) \geq k, \nu(N(v)) \geq -k \Rightarrow \nu(T(u, v)) \geq 0.$$

7.4.6 Octagons $\mathcal{O}(K, \sigma)$

The sole compatibility condition for octogonal systems involves the Tits-endomorphism σ. Analogously to the first compatibility condition for the pseudo-quadratic case one shows that if one has two elements x, y of K such that $\nu(x) = \nu(y)$, that then $\nu(x^\sigma) = \nu(y^\sigma)$. An equivalent way to state the condition is $\forall x \in K : \nu(x) = 0 \Rightarrow \nu(x^\sigma) = 0$ (to see this consider $\nu(xy^{-1})$ and $\nu(x^\sigma y^{-\sigma})$).

7.4.7 The higher rank case

As described with the cases listed in Section 2.3.2, there is at most one rank two residue containing a given chamber which gives rise to compatibility conditions. The other residues, different from digons, are triangles either over the field of definition, or over a composition algebra over this field. While this algebra is not necessary a field, a valuation on it is given by Remark 7.11, so we can still apply Section 7.3.

7.5 Constructing the epimorphism

In this section we assume that one is given a spherical building Δ defined over a field K, a valuation ν of the underlying field, a weak root group labelling, and that the compatibility conditions of the last section are satisfied for residues which are not generalized digons or triangles.

We will construct the epimorphism in multiple steps, generalizing the possible rank of the valuation at each step. The first two steps for generalized quadrangles of exceptional type will be handled seperately.

7.5.1 Valuations of rank one

A rank one valuation is a valuation to the real numbers. Because of this the compatibility conditions will imply stronger restrictions. For octogonal systems the compatibility condition is equivalent to $\nu(x^\sigma) = \sqrt{2}\nu(x)$ (see for instance [13, p. 1114]). We show what happens to the compatibility conditions consisting of inequalities using quadratic forms as example. The inequality there implies the following inequality (which can be shown by using appropriate scalar products):

$$\forall u, v \in L_0^* : \nu(f(u,v)) + C \geq (\nu(q(u)) + \nu(q(v)))/2,$$

where C is a constant. By [26, 19.4] this inequality with $C = 0$ is equivalent to a condition for the completion of the quadratic form with respect to ν. However upon taking a closer look at the proof of this proposition it turns out that you can still show the condition for the completion using the weaker inequality above. Hence the above inequality is equivalent to

$$\forall u, v \in L_0^* : \nu(f(u,v)) \geq (\nu(q(u)) + \nu(q(v)))/2.$$

A similar reasoning for the other inequalities occuring in Moufang polygons, which are not quadrangles of exceptional type, is possible using [26, Prop 24.9, 25.5 and 21.36].

With the extra conditions we derived here one satisfies exactly the conditions (which can be found in [13] and [26]) for the existence of an $\mathbb{R}$-building with the given spherical building at infinity, corresponding to the valuation ν. Using the theory of $\mathbb{R}$-buildings one can obtain a canonical epimorphism of the spherical building at infinity (being Δ) to a residue such that its weak root group labelling is exactly the one we started with.

The fact that our compatibility conditions only concern the rank 2 residues is reflected in the result [26, Th. 16.14] on the existence of $\mathbb{R}$-buildings.

Remark 7.12 The inequalities we derived in this section are not generally true if one leaves the rank one case. A consequence is that one cannot use the theory of Λ-buildings, which is the natural generalization of affne buildings for arbitrary valuations, to the construct the epimorphism in one step. More information on Λ-buildings can be found in [4] and [12].

7.5.2 Valuations of finite rank

An abelian ordered group Λ of rank t can be embedded as a subgroup in the lexicographically ordered group $\oplus_{j=1}^{t}\mathbb{R}$ by Hahn's embedding theorem (see [11]). Using this presentation one can define an epimorphism $e : \Lambda \to \mathbb{R} : (a_1, \ldots, a_t) \mapsto a_1$ of ordered abelian groups. Denote the kernel of this epimorphism by Λ_0. The function $\nu' := e \circ \nu$ is then a valuation of K of rank one.

The claim is now that the compatibility conditions are satisfied for the valuation ν' as well. We again illustrate this with the octogonal sets and quadratic forms as examples.

For octogonal sets we have to prove that for $x \in K$ one has that $\nu(x) \in \Lambda_0 \Rightarrow \nu(x^\sigma) \in \Lambda_0$ given $\nu(x) = 0 \Rightarrow \nu(x^\sigma) = 0$. Suppose that this is not the case. Without loss of generality one may additionally assume that $\nu(x) < \nu(x^\sigma)$ (otherwise one can consider x^{-1}). Note that $\nu(1 + x^\sigma x^{-1}) = 0$, so $\nu((1 + x^\sigma x^{-1})^\sigma) = \nu(1 + x^2 x^{-\sigma}) = 0$. But $\nu(x^2 x^{-\sigma}) < 0$ (note that this is true because $\nu(x) \in \Lambda_0$, $\nu(x^\sigma) \notin \Lambda_0$), hence $\nu(1 + x^2 x^{-\sigma}) < 0$, which is a contradiction. We conclude that $\nu(x) \in \Lambda_0 \Rightarrow \nu(x^\sigma) \in \Lambda_0$.

For quadratic forms, note that the inequality

$$\forall u, v \in L_0 : \nu'(q(u)), \nu'(q(v)) \geq e(k) \Rightarrow \nu'(f(u, v)) \geq e(k)$$

implies, by using scalar products with elements of the field with valuation in $\ker(e)$, that

$$\forall u, v \in L_0 : \nu(q(u)), \nu(q(v)) \geq k \Rightarrow \nu(f(u, v)) \geq k.$$

We can now apply the results of the previous section to the rank one valuation ν', and obtain an epimorphism of the spherical building to some other spherical building which is defined over the residue field of $K_{\nu'}$. The valuation ν of K allows us to define a rank $t-1$ valuation $\bar{\nu}$ of $K_{\nu'}$. On this new spherical building we can repeat the procedure until we have constructed the desired epimorphism, provided we can show the compatibility conditions for this new situation. We will be able to do this except for the Moufang quadrangles of exceptional type, for which there is no description (yet) of the possible residues. This is why we postpone this case to the next section.

For compatibility conditions involving an involution or Tits-endomorphism σ it is clear that the conditions stay true for a residue field. We will describe the behavior of conditions involving inequalities with the example of quadratic forms. Note that the previous section applied to ν' implies the stronger inequality

$$\forall u, v \in L_0^* : \nu'(f(u,v)) \geq (\nu'(q(u)) + \nu'(q(v)))/2.$$

The residue will be again a quadrangle of quadratic form type, where the quadratic space $\bar{L}$ is the quotient $\{v \in L_0 | \nu(q(v)) \geq e(k)\}/\{v \in L_0 | \nu(q(v)) > e(k)\}$ on which the residue field $K_{\nu'}$ acts naturally and for which the function $\bar{q} : \bar{L} \to K_{\nu'} : \bar{v} \mapsto \overline{q(v)/t}$ (where $\bar{.}$ indicated the natural map into $\bar{L}$ or $K_{\nu'}$, and $t \in K$ is such that $\nu'(t) = e(k)$) is an anisotropic quadratic function. See [26, Def. 19.33] for more details to this construction. The original compatibility condition applied to $\bar{L}$ yields (keeping in mind the previous inequality to show independence of choice of representants)

$$\forall u, v \in \bar{L} : \bar{\nu}(\bar{q}(\bar{u})), \nu(\bar{q}(\bar{v})) > k - \nu(t) \Rightarrow \nu(\bar{f}(\bar{u}, \bar{v})) \geq k - \nu(t),$$

where $\bar{f}$ is the bilinear form associated to $\bar{q}$. Hence we obtained a compatibility condition for the residue and we can continue with the construction of the epimorphism. For other types an analogous treatment is possible (see [26, Def. 24.50 and 25.28] for detailed descriptions of the residues).

7.5.3 Valuations of finite rank and quadrangles of exceptional type

In this section we handle Moufang quadrangles of exceptional type F_4 or E_i $(i = 6, 7, 8)$. Combining the valuation ν of finite rank on the underlying field to the ordered abelian group Λ with the norm functions η_r listed in 7.3, we obtain maps $\phi_r : U_r \to \Lambda$ $(r \in \{1, 2, 3, 4\})$. We are now interested in the interaction between these functions and the action of the μ-maps of elements of U_1 and U_4. One observes using the relations in [24, 32.10-11] and [26, Prop. 21.10 and 22.4] that (with $u_r \in U_r$)

$$\begin{aligned} \phi_4(u_2^{\mu_1(u_1)}) &= \phi_2(u_2) - \phi_1(u_1), \\ \phi_2(u_4^{\mu_1(u_1)}) &= \phi_4(u_4) + \phi_1(u_1), \\ \phi_3(u_1^{\mu_4(u_4)}) &= \phi_1(u_1) + 2\phi_4(u_4), \\ \phi_1(u_3^{\mu_4(u_4)}) &= \phi_3(u_3) - 2\phi_4(u_4). \end{aligned} \tag{1}$$

Other identities are not straightforward to obtain, One can also derive that if $[u_1, u_4] = u_2 u_3$, that

$$\begin{aligned} \phi_2(u_2) &= \phi_1(u_1) + \phi_4(u_4), \\ \phi_3(u_3) &= \phi_1(u_1) + 2\phi_4(u_4). \end{aligned}$$

One can use this in a reasoning similar to [26, Prop 15.25] (which makes use of the fact that double μ-actions maps the root groups to themselves) obtaining (where $u_r, v_r, w_r \in U_r$)

$$\begin{aligned} \phi_1(u_1^{\mu_1(v_1)\mu_1(w_1)}) &= \phi_1(u_1) - 2\phi_1(v_1) + 2\phi_1(w_1), \\ \phi_4(u_4^{\mu_4(v_4)\mu_4(w_4)}) &= \phi_4(u_4) - 2\phi_4(v_4) + 2\phi_4(w_4), \\ \phi_3(u_3^{\mu_1(v_1)}) &= \phi_3(u_3), \\ \phi_2(u_2^{\mu_4(v_4)}) &= \phi_2(u_2). \end{aligned} \tag{2}$$

The compatibility conditions are now be written as

$$\begin{gathered} \{u_1 \in U_1 | \phi(u_1) \geq k\} \text{ is a subgroup,} \\ \{u_1 \in U_1 | \phi(u_1) > k\} \text{ is a subgroup,} \\ \{u_4 \in U_4 | \phi(u_4) \geq l\} \text{ is a subgroup,} \\ \{u_4 \in U_4 | \phi(u_4) > l\} \text{ is a subgroup,} \\ \phi_1(u_1) = k, \phi_3(u_3) > k + 2l, [u_1, u_3] = u_2 \Rightarrow \phi_2(u_2) > k + l, \\ \phi_1(u_1) > k, \phi_3(u_3) = k + 2l, [u_1, u_3] = u_2 \Rightarrow \phi_2(u_2) > k + l, \\ \phi_1(u_1) = k, \phi_3(u_3) = k + 2l, [u_1, u_3] = u_2 \Rightarrow \phi_2(u_2) \geq k + l, \\ \phi_2(u_2) = k + l, \phi_4(u_4) > l, [u_2, u_4] = u_3 \Rightarrow \phi_3(u_3) > k + 2l, \\ \phi_2(u_2) > k + l, \phi_4(u_4) = l, [u_2, u_4] = u_3 \Rightarrow \phi_3(u_3) > k + 2l, \\ \phi_2(u_2) = k + l, \phi_4(u_4) = l, [u_2, u_4] = u_3 \Rightarrow \phi_3(u_3) \geq k + 2l. \end{gathered} \tag{3}$$

From now on we keep in mind only the derived identities and inequalities and 'forget' that we were dealing with an exceptional type case. The approach for constructing the epimorphism resembles the one from the previous two sections, but works in a more implicit way. As Λ is of finite rank, one can find an ordered abelian group epimorphism $e : \Lambda \to \mathbb{R}$, which we compose with the maps ϕ_r to obtain maps $\phi'_r : U_r \to \mathbb{R}$ ($r \in \{1, 2, 3, 4\}$). We claim that these maps satisfy the inequalities

$$[u_1, u_3] = u_2 \Rightarrow \phi'_2(u_2) \geq (\phi'_1(u_1) + \phi'_3(u_3))/2,$$
$$[u_2, u_4] = u_3 \Rightarrow \phi'_3(u_3) \geq (\phi'_2(u_2) + \phi'_4(u_4))/2,$$

and one has that

$$\{u_1 \in U_1 | \phi'_1(u_1) \geq t\},$$
$$\{u_4 \in U_4 | \phi'_4(u_4) \geq t\},$$

with $t \in \mathbb{R}$ are subgroups. One can prove this by using Equations 1 and 2 which allow us to add multiples of two to the constants k and l in the equalities of 3, . All of this implies that the maps ϕ'_r form a viable partial valuation in the sense of [26, Def. 15.4], and hence give rise to an $\mathbb{R}$-building (by adapting [26, Th. 15.21] to the non-discrete case) with the desired first epimorphism to a certain residue. The root groups $\overline{U}_r$ with $r \in \{1, 2, 3, 4\}$ are given by the quotient $\{u_r \in U_r | \phi'_r(u_r) \geq 0\}/\{u_r \in U_r | \phi'_r(u_r) > 0\}$. On these root groups one can define in a natural way functions $\overline{\phi}_r : \overline{U}_r \to \ker e$, which inherit the same identities and inequalities as derived for the functions ϕ_r. Hence we are back at our starting point and can apply recursion to obtain the desired epimorphism.

7.5.4 Valuations of arbitrary rank

In this final case we do not pose any conditions on the rank of the valuation anymore. If K' is a finitely generated subfield of the field with definition K, then one can define a subbuilding $\Delta(K')$ of Δ (in certain cases one need to extend the subfield K' to make it closed under an involution or Tits endomorphism σ or to make sure that certains forms are defined, however one can still keep the subfield finitely generated).

If C and C' are two chambers of Δ, then there exists a finitely generated subfield K' of K such that $\Delta(K')$ contains C and C'. If one restricts the valuation ν to this subfield then it is of finite rank (because a finitely generated field is of finite transcendency and Abhyankar's inequality ([1, Lem. 1])). So if one restricts the (weak) root group labeling of the epimorphism to K' one can apply the previous section and obtain an epimorphism $\phi_{K'}$ of $\Delta(K')$. We now define the new Weyl distance $\bar{\delta}$ between C and C' to be the distance between the images of these chambers under $\phi_{K'}$. This is independent of the choice of K' by Corollary 6.12.

We now define the building Δ' to have as chambers the equivalence sets of chambers having trivial Weyl distance $\bar{\delta}$ from each other, and the distance between two chambers of Δ' to be the distance $\bar{\delta}$ of two representants. One can also construct an epimorphism $\phi : \Delta \to \Delta'$ which maps each chamber of Δ into its equivalence set. If this definition is not well-defined, if Δ' is not a building or if the map ϕ not an epimorphism, then one can find a finite set Ω of chambers of Δ where a problem with this occurs. By restricting the field to a certain finitely generated subfield K'' one can make sure that $\Delta(K'')$ contains Ω. But at this point there cannot occur a problem because of our discussion of valuations of finite rank and the uniquely definedness of $\bar{\delta}$. Hence ϕ will be the desired epimorphism.

7.6 Conclusion

In Section 7.3 we determined, given an epimorphism ϕ, what the possible the (weak) root group labelings of the epimorphism are, and we derived some compatibility conditions which need to be fulfilled. We then continued in Section 7.5 starting from a weak root group labeling and these conditions to show that for the cases under consideration one could construct an epimorphism. Corollary 6.12 now implies that we obtain the original epimorphism ϕ again (up to an isomorphism).

This concludes the proof of the second Main Result. The first Main Corollary follows from Sections 7.5.2 and 7.5.3, the second from Sections 7.5.1 and 7.5.3 combined with Proposition 6.10.

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