

ASYMPTOTIC STABILITY OF DISSIPATED HAMILTON-POISSON SYSTEMS

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Abstract

We will further develop the study of the dissipation for a Hamilton-Poisson system introduced in [3]. We will give a tensorial form of this dissipation and show that it preserves the Hamiltonian function but not the Poisson geometry of the initial Hamilton-Poisson system. We will give precise results about asymptotic stabilizability of the stable equilibria of the initial Hamilton-Poisson system.

MSC: 37C10, 37C75, 70E50.

Keywords: dynamical systems, stability theory, rigid body.

1 Introduction

Let

$$\dot{x} = \Pi \nabla H(x)$$

be a Hamilton-Poisson system. If $C \in C^\infty(\mathbb{R}^n, \mathbb{R})$ is a Casimir function then we have the following obvious equality $\Pi \nabla C = 0$.

We will add to the Hamilton-Poisson system that we will denote by ξ_Π a dissipation term of the form $G \nabla C$, where G is a certain symmetric matrix that will be discussed below and C is a Casimir function. The dissipated system ξ will be

$$\dot{x} = \Pi \nabla H(x) + G \nabla C(x) \tag{1.1}$$

The notion of dissipative bracket was introduced by A. Kaufman [10] in his study of dissipative Hamilton-Poisson systems. In [15], P.J. Morrison introduced the notion of metriplectic systems which are Hamilton-Poisson systems with a dissipation of metric type. Metric type dissipation was also introduced in [15] and is given by a dissipative bracket constructed from a metric defined on the phase space. Dissipative terms and their implications for dynamics have been studied in connection with various dynamical systems derived from mathematical physics, see [8], [9], [11],[12], [13], [14].

The dissipation that we will study is the one introduced in [3]. Here we will write it in a tensorial form which enables us to better understand its geometry. It is a particular form of the metric dissipation introduced in [15]. More precisely, our system (1.1) is the same system with the one described by equations (25) in [15] under the condition $(g^{ij}) \nabla H = 0$. Like the dissipation from [7], our dissipation conserves the energy. A dissipation form that conserves the symplectic leaves and dissipates the Hamiltonian function have been studied in [5].

We will also illustrate the improvement of the stability result obtained in [3]. This improvement is obtained by the observation that the dissipation conserves the energy.

2 $G = \nabla H \otimes \nabla H - \|\nabla H\|^2 \mathbb{I}$

In this section we introduce the dissipation matrix

$$G = \nabla H \otimes \nabla H - \|\nabla H\|^2 \mathbb{I}.$$

This is the tensorial form of the dissipation constructed in [3] where the matrix \mathbb{I} is the identity matrix on \mathbb{R}^n . We denote by ξ the dissipated system (1.1) in the case when the matrix G is as above.

Lemma 2.1. *We have the following properties for G :*

- (i) $G\nabla H = 0$;
- (ii) $\nabla C \cdot G\nabla C \leq 0$, for any Casimir function C .

The equality holds in a point $x \in \mathbb{R}^n$ iff $\nabla C(x)$ and $\nabla H(x)$ are linear dependent.

Proof. For (i) we have

$$\begin{aligned} G\nabla H &= (\nabla H \otimes \nabla H)\nabla H - \|\nabla H\|^2 \mathbb{I}\nabla H \\ &= (\nabla H \cdot \nabla H)\nabla H - \|\nabla H\|^2 \nabla H = 0 \end{aligned}$$

Analogous for (ii) we have

$$\begin{aligned} \nabla C \cdot G\nabla C &= \nabla C \cdot [(\nabla H \otimes \nabla H)\nabla C - \|\nabla H\|^2 \mathbb{I}\nabla C] \\ &= \nabla C \cdot [(\nabla C \cdot \nabla H)\nabla H - \|\nabla H\|^2 \nabla C] \\ &= (\nabla C \cdot \nabla H)^2 - \|\nabla C\|^2 \cdot \|\nabla H\|^2 \leq 0. \end{aligned}$$

by C-B-S inequality. Equality holds iff ∇H and ∇C are linear dependent. \square

For the perturbed system ξ the initial Hamiltonian remains a conservation law but the Casimir function C does not.

Lemma 2.2. *We have the following behavior:*

- (i) H is conserved along the solutions of ξ .
- (ii) C decreases along the solutions of ξ .

Proof. For (i) we have

$$\begin{aligned} \frac{d}{dt}H &= \dot{x} \cdot \nabla H = (\Pi\nabla H + G\nabla C) \cdot \nabla H \\ &= \Pi\nabla H \cdot \nabla H + [(\nabla H \otimes \nabla H)\nabla C - \|\nabla H\|^2 \nabla C] \cdot \nabla H \\ &= -\nabla H \cdot \Pi\nabla H = 0 \end{aligned}$$

since Π is an antisymmetric matrix.

We do the same type of computation for (ii)

$$\begin{aligned} \frac{d}{dt}C &= \dot{x} \cdot \nabla C = (\Pi\nabla H + G\nabla C) \cdot \nabla C \\ &= -\nabla H \cdot \Pi\nabla C + \nabla C \cdot G\nabla C \\ &= \nabla C \cdot G\nabla C \leq 0 \end{aligned}$$

by Lemma 2.1 (ii). \square

Next, we will study the relation between the set of equilibria $E_{\xi_{\Pi}}$ for the unperturbed system ξ_{Π} and the set of equilibria E_{ξ} for the perturbed system ξ .

Proposition 2.3. *We have:*

- (i) If $\nabla C(x_e) \neq 0$ for a point $x_e \in \mathbb{R}^n$ then $x_e \in E_{\xi} \Leftrightarrow \nabla H(x_e)$ **and** $\nabla C(x_e)$ are linear dependent.
- (ii) $E_{\xi} \subset E_{\xi_{\Pi}}$.

Proof. By definition we have that

$$x_e \in E_{\xi_{\Pi}} \Leftrightarrow \Pi \nabla H(x_e) = 0$$

and

$$x_e \in E_{\xi} \Leftrightarrow \Pi \nabla H(x_e) + G \nabla C(x_e) = 0$$

(i) Let $x_e \in E_{\xi}$. If we multiply both sides of the above equality with $\nabla C(x_e)$ then

$$\nabla C(x_e) \cdot \Pi \nabla H(x_e) + \nabla C(x_e) \cdot G \nabla C(x_e) = 0$$

$$\Leftrightarrow -\Pi \nabla C(x_e) \cdot \nabla H(x_e) + \nabla C(x_e) \cdot G \nabla C(x_e) = 0 \Leftrightarrow \nabla C(x_e) \cdot G \nabla C(x_e) = 0$$

which by Lemma 2.1 (ii) implies that $\nabla H(x_e)$ and $\nabla C(x_e)$ are linear dependent.

For the converse, if $\nabla H(x_e)$ and $\nabla C(x_e)$ are linear dependent then there exists $\lambda \in \mathbb{R}$ such that $\nabla H(x_e) = \lambda \nabla C(x_e)$. Consequently,

$$\Pi \nabla H(x_e) = \lambda \Pi \nabla C(x_e) = 0.$$

In the case $\nabla H(x_e) = 0$ we obtain that $G(x_e) = 0$.

In the case $\nabla H(x_e) \neq 0$ we obtain that $\nabla C(x_e) = \frac{1}{\lambda} \nabla H(x_e)$ and using Lemma 2.1 (i) we have $G \nabla C(x_e) = \frac{1}{\lambda} G \nabla H(x_e) = 0$. In both cases we conclude that $x_e \in E_{\xi}$.

(ii) Let $x_e \in E_{\xi}$. If $\nabla C(x_e) = 0$ then $G \nabla C(x_e) = 0$ and we obtain that $\Pi \nabla H(x_e) = 0$ which implies $x_e \in E_{\xi_{\Pi}}$.

If $\nabla C(x_e) \neq 0$ then there exists $\lambda \in \mathbb{R}$ such that $\nabla H(x_e) = \lambda \nabla C(x_e)$. We observe that $\Pi \nabla H(x_e) = \lambda \Pi \nabla C(x_e) = 0$ and consequently, $x_e \in E_{\xi_{\Pi}}$. \square

3 Asymptotic stability

We will briefly recall some definitions of stability for a dynamical system on \mathbb{R}^n that will be used later

$$\dot{x} = f(x), \tag{3.1}$$

where $f \in C^{\infty}(\mathbb{R}^n, \mathbb{R}^n)$.

Definition 3.1. *An equilibrium point x_e is stable if for any small neighborhood U of x_e there is a neighborhood V of x_e , $V \subset U$ such that if initially x_0 is in V , then $\phi(t, x_0) \in U$ for all $t > 0$. If in addition we have*

$$\lim_{t \rightarrow \infty} \phi(t, x_0) = x_e$$

then x_e is called asymptotically stable.

For studying more complicated asymptotic behavior we need to introduce the notion of ω -limit set. Let ϕ^t be the flow defined by equation (3.1). The ω -limit set of x is $\omega(x) := \{y \in \mathbb{R}^n \mid \exists t_1, t_2 \dots \rightarrow \infty \text{ s.t. } \phi(t_k, x) \rightarrow y \text{ as } k \rightarrow \infty\}$. The ω -limit sets have the following properties that we will use later. For more details, see [16].

- (i) If $\phi(t, x) = y$ for some $t \in \mathbb{R}$, then $\omega(x) = \omega(y)$.
- (ii) $\omega(x)$ is a closed subset and both positively and negatively invariant (contains complete orbits).

The above properties of ω -limit sets have been used in the proof of the following version of LaSalle theorem, see [3].

Theorem 3.2. *Let x_e be an equilibrium point of a dynamical system*

$$\dot{x} = f(x)$$

and U a small compact neighborhood of x_e . Suppose there exists $L : U \rightarrow \mathbb{R}$ a C^1 differentiable function with $L(x) > 0$ for all $x \in U \setminus \{x_e\}$, $L(x_e) = 0$ and $\dot{L}(x) \leq 0$. Let E be the set of all points in U where $\dot{L}(x) = 0$. Let M be the largest invariant set in E . Then there exists a small neighborhood V of x_e with $V \subset U$ such that $\omega(x) \subset M$ for all $x \in V$.

The next result describes the asymptotic behavior for the solutions of the dissipated system ξ . We introduce the following set $C_* = \{x \in \mathbb{R}^n \mid \nabla C(x) = 0\}$.

Theorem 3.3. *Let $x_e \in E_\xi$ be an equilibrium point for the dissipated system ξ . Suppose there exists a function $\psi(H, C) \in C^\infty(\mathbb{R}^2, \mathbb{R})$ such that $\frac{\partial \psi}{\partial C}(H(x_e), C(x_e)) > 0$ and x_e is a strict relative minimum for $L(x) = \psi(H(x), C(x)) - \psi(H(x_e), C(x_e))$. Then there exists a small compact neighborhood U of x_e and another neighborhood V of x_e with $V \subset U$ such that every solution of ξ starting in V approaches the largest invariant set M in $U \cap (E_\xi \cup C_*)$ as $t \rightarrow \infty$ (M is an attracting set).*

Proof. We have $L(x_e) = 0$ with x_e being a strict local minimum for L . Then there exists a small compact neighborhood U of x_e such that $L(x) > 0$ and $\frac{\partial \psi}{\partial C}(H(x), C(x)) > 0$ on this neighborhood. We have the following computation

$$\begin{aligned} \dot{L}(x) &= \frac{\partial \psi}{\partial H} \nabla H \cdot \dot{x} + \frac{\partial \psi}{\partial C} \nabla C \cdot \dot{x} = \left(\frac{\partial \psi}{\partial H} \nabla H + \frac{\partial \psi}{\partial C} \nabla C \right) (\Pi \nabla H + G \nabla C) \\ &= \frac{\partial \psi}{\partial H} \nabla H \cdot \Pi \nabla H + \frac{\partial \psi}{\partial H} \nabla H \cdot G \nabla C + \frac{\partial \psi}{\partial C} \nabla C \cdot \Pi \nabla H + \frac{\partial \psi}{\partial C} \nabla C \cdot G \nabla C \\ &= \frac{\partial \psi}{\partial H} \nabla H \cdot \Pi \nabla H + \frac{\partial \psi}{\partial H} G \nabla H \cdot \nabla C - \frac{\partial \psi}{\partial C} \Pi \nabla C \cdot \nabla H + \frac{\partial \psi}{\partial C} \nabla C \cdot G \nabla C \end{aligned}$$

Using Lemma 2.1 (i), antisymmetry of Π and the fact that C is a Casimir function for Π we obtain that

$$\dot{L}(x) = \frac{\partial \psi}{\partial C} \nabla C \cdot G \nabla C$$

From the hypothesis $\frac{\partial \psi}{\partial C} > 0$ on U and by Lemma 2.1 (ii) we have

$$\dot{L}(x) \leq 0.$$

Using again Lemma 2.1 (ii) we obtain that E from the Theorem 3.2 equals $E_\xi \cup C_*$ which give us the result. \square

Remark 3.4. Observe that L is also a Lyapunov function for the unperturbed system ξ_Π . Consequently, by adding the dissipation we render the stable points of ξ_Π into asymptotically stable points for ξ . Also the Casimir function C can be considered as a Lyapunov function.

Corollary 3.5. *Let $x_e \in E_\xi$ and a function $\psi(H, C) \in C^\infty(\mathbb{R}^2, \mathbb{R})$ such that $\frac{\partial \psi}{\partial C}(H(x_e), C(x_e)) > 0$. Suppose that the function $L(x) = \psi(H(x), C(x)) - \psi(H(x_e), C(x_e))$ has the properties:*

- i) $\delta L(x_e) = 0$;*
- ii) $\delta^2 L(x_e)$ is positive definite.*

Then there exists a small compact neighborhood U of x_e and another neighborhood V of x_e with $V \subset U$ such that every solution of ξ starting in V approaches the largest invariant set M in $U \cap (E_\xi \cup C_)$ as $t \rightarrow \infty$. Moreover H remains constant along these solutions and C decreases along these solutions.*

Proof. It is easy to observe that the point x_e is a strict relative minimum of L . All the condition of the Theorem 3.3 are satisfied and we obtain the desired result. \square

Remark 3.6. If we consider, in the Corollary 3.5, the function $\psi(H, C) = H + C$ we obtain the stability result of [3].

Corollary 3.7. *In the hypotheses of the Theorem 3.3 we consider a point $x_0 \in V$. Suppose that the set $H^{-1}(\{h\}) \cap U \cap (E_\xi \cup C_*)$ has a unique point x_h ($h = H(x_0)$). The solution $x(t, x_0)$ of the system ξ which verifies the initial condition $x(0, x_0) = x_0$ satisfies the property*

$$\lim_{t \rightarrow \infty} x(t, x_0) = x_h.$$

We can interpret the above result as follows, using the Lyapunov function of Theorem 3.3 we obtain that every equilibrium point in the neighborhood V is asymptotically stable for the dynamical system on the corresponding level set.

4 Applications to the rigid body dynamics

The motion of a rigid body can be reduced to the translation of center of mass and rotation about it. Rotation is conveniently described, in a coordinate system with the origin at the center of mass and the axes along principal central axes of inertia, by Euler's equations. This equations can be written in the following form

$$\begin{cases} \dot{x}_1 = (\frac{1}{I_3} - \frac{1}{I_2})x_2x_3 + u_1 \\ \dot{x}_2 = (\frac{1}{I_1} - \frac{1}{I_3})x_1x_3 + u_2 \\ \dot{x}_3 = (\frac{1}{I_2} - \frac{1}{I_1})x_1x_2 + u_3 \end{cases}$$

where $x_1 = I_1\omega_1$, $x_2 = I_2\omega_2$, $x_3 = I_3\omega_3$ are the components of \mathbf{x} , I_1, I_2, I_3 are the principal moments of inertia, $\omega_1, \omega_2, \omega_3$ are the components of the angular velocity and u_1, u_2, u_3 are the components of applied torques \mathbf{u} . In this paper we suppose that $I_1 > I_2 > I_3$.

The system of free rotations, denoted by ξ_Π , has the property that $\mathbf{u} = \mathbf{0}$. It has the following Hamilton-Poisson realization

$$((so(3))^* \approx \mathbb{R}^3, \{\cdot, \cdot\}_-, H)$$

where $\{\cdot, \cdot\}_-$ is minus-Lie-Poisson structure on $(so(3))^* \approx \mathbb{R}^3$ generated by the matrix

$$\Pi_- = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

and the Hamiltonian H is given by

$$H(x_1, x_2, x_3) = \frac{1}{2} \left(\frac{x_1^2}{I_1} + \frac{x_2^2}{I_2} + \frac{x_3^2}{I_3} \right).$$

It is easy to see that the function $C_0 \in C^\infty(\mathbb{R}^3, \mathbb{R})$ given by

$$C_0(x_1, x_2, x_3) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$$

is a Casimir of our configuration $((so(3))^* \approx \mathbb{R}^3, \{\cdot, \cdot\}_-)$, called the *standard Casimir*, i.e.

$$\{C_0, f\} = 0$$

for each $f \in C^\infty(\mathbb{R}^3, \mathbb{R})$ where

$$\{f, g\} = (\nabla f)^T \Pi_- \nabla g.$$

The set of the Casimir functions is given by

$$\{C = \varphi(C_0) \mid \varphi \in C^\infty(\mathbb{R}, \mathbb{R})\}.$$

It is known that the set of equilibria is given by

$$E_{\xi_\Pi} = \{(M_0, 0, 0) \mid M_0 \in \mathbb{R}\} \cup \{(0, M_0, 0) \mid M_0 \in \mathbb{R}\} \cup \{(0, 0, M_0) \mid M_0 \in \mathbb{R}\}$$

and the equilibrium points of the form $(M_0, 0, 0)$ or $(0, 0, M_0)$ are stable and the equilibrium points of the form $(0, M_0, 0)$ with $M_0 \neq 0$ are not stable.

In this paper we consider a torque of the form

$$\mathbf{u} = G \nabla C,$$

with C a Casimir function and G the matrix

$$G = \nabla H \otimes \nabla H - \|\nabla H\|^2 \mathbb{I} = \begin{pmatrix} -\frac{x_2^2}{I_2^2} - \frac{x_3^2}{I_3^2} & \frac{x_1 x_2}{I_1 I_2} & \frac{x_1 x_3}{I_1 I_3} \\ \frac{x_1 x_2}{I_1 I_2} & -\frac{x_1^2}{I_1^2} - \frac{x_3^2}{I_3^2} & \frac{x_2 x_3}{I_2 I_3} \\ \frac{x_1 x_3}{I_1 I_3} & \frac{x_2 x_3}{I_2 I_3} & -\frac{x_1^2}{I_1^2} - \frac{x_2^2}{I_2^2} \end{pmatrix}.$$

For the case of the free rigid body the above matrix is the one also used in [15].

Case I. We first consider the standard Casimir C_0 in order to construct the perturbation. We obtain a torque \mathbf{u}_0 with the components:

$$\begin{cases} u_1^0 = x_1 \left[\left(\frac{1}{I_1} - \frac{1}{I_2} \right) \frac{x_2^2}{I_2} + \left(\frac{1}{I_1} - \frac{1}{I_3} \right) \frac{x_3^2}{I_3} \right] \\ u_2^0 = x_2 \left[\left(\frac{1}{I_2} - \frac{1}{I_1} \right) \frac{x_1^2}{I_1} + \left(\frac{1}{I_2} - \frac{1}{I_3} \right) \frac{x_3^2}{I_3} \right] \\ u_3^0 = x_3 \left[\left(\frac{1}{I_3} - \frac{1}{I_1} \right) \frac{x_1^2}{I_1} + \left(\frac{1}{I_3} - \frac{1}{I_2} \right) \frac{x_2^2}{I_2} \right] \end{cases}$$

Let ξ^0 be the rotation system with the torque \mathbf{u}_0 . It is easy to see that the set of the equilibrium points of the dissipated system is

$$E_{\xi^0} = E_{\xi_{\text{II}}}.$$

Also we have

$$C_* = \{(0, 0, 0)\}.$$

We consider $M_0 \in \mathbb{R}^*$. The point $x_e = (0, 0, M_0)$ is an equilibrium point of the rotation system ξ^0 . We define the function $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\psi(H, C_0) = \left(C_0 - \frac{M_0^2}{2} \right)^2 + C_0 - I_3 H.$$

This function has the properties:

$$\frac{\partial \psi}{\partial C_0}(H, C_0) = 2 \left(C_0 - \frac{M_0^2}{2} \right) + 1$$

and

$$\frac{\partial \psi}{\partial C_0}(H(x_e), C_0(x_e)) = 1 > 0.$$

Using our notations we introduce the Lyapunov function

$$\begin{aligned} L(x_1, x_2, x_3) &= \left(C_0(x_1, x_2, x_3) - \frac{M_0^2}{2} \right)^2 + C_0(x_1, x_2, x_3) - I_3 H(x_1, x_2, x_3) \\ &= \left(\frac{1}{2}(x_1^2 + x_2^2 + x_3^2) - \frac{M_0^2}{2} \right)^2 + \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) - \frac{I_3}{2} \left(\frac{x_1^2}{I_1} + \frac{x_2^2}{I_2} + \frac{x_3^2}{I_3} \right). \end{aligned}$$

It is easy to see that

$$\delta L(x_e) = 0$$

and

$$\delta^2 L(x_e) = \begin{pmatrix} 1 - \frac{I_3}{I_1} & 0 & 0 \\ 0 & 1 - \frac{I_3}{I_2} & 0 \\ 0 & 0 & 2M_0^2 \end{pmatrix}.$$

Using our hypotheses we observe that $\delta^2 L(x_e)$ is positive definite. Consequently, the hypotheses of the Corollary 3.5 are satisfied and we have the following stability result.

Theorem 4.1. *There exists a small compact neighborhood U of x_e and an other $V \subset U$ such that every solution of ξ^0 starting in V approaches $U \cap \{(0, 0, x) \mid x \in \mathbb{R}\}$ as $t \rightarrow \infty$. Moreover H remains constant along solutions and C_0 decreases along this solutions.*

If U is sufficiently small then $U \cap E_{\xi^0} \subset \{(0, 0, x) \mid \text{sgn}(x) = \text{constant}\}$. Suppose that $\text{sgn}(x) = 1$. Using the Corollary 3.7 we have the following asymptotic stability result.

Theorem 4.2. *If $x_0 \in V$ then the solution $x(t, x_0)$ of the system ξ_0 which verifies the initial condition $x(0, x_0) = x_0$ satisfies the property*

$$\lim_{t \rightarrow \infty} x(t, x_0) = (0, 0, \sqrt{2I_3 H(x_0)}).$$

We present the simulation of the rotation of the rigid body in the case when the principal moments of inertia are $I_1 = 4$, $I_2 = 1.5$, $I_3 = 1$ and the initial conditions are $(-0.1, 0.2, 0.175)$.

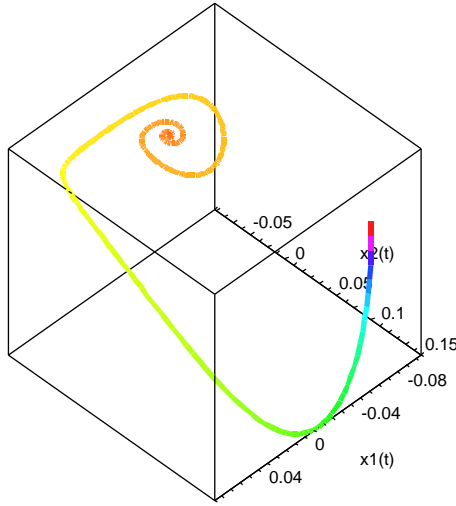


Figure 1: The rotation of the rigid body

Remark 4.3. We see that the function $C_1 = C_0 + (C_0 - \frac{M_0^2}{2})^2$ is also a Casimir function. If we consider $\psi_1(H, C_1) = C_1 - I_3 H$ we have $\frac{\partial \psi_1}{\partial C_1} = 1 > 0$ and $\psi(H, C_0) = \psi_1(H, C_1)$. It is possible to apply Corollary 3.5 for the system ξ_1 of the form (1.1) with the Casimir function C_1 and the equilibrium point $x_e = (0, 0, M_0)$.

Case II. We study the case of the Casimir $C_2 = -C_0$. We obtain a torque \mathbf{u}_2 with the components:

$$\begin{cases} u_1^2 = -x_1[(\frac{1}{I_1} - \frac{1}{I_2})\frac{x_2^2}{I_2} + (\frac{1}{I_1} - \frac{1}{I_3})\frac{x_3^2}{I_3}] \\ u_2^2 = -x_2[(\frac{1}{I_2} - \frac{1}{I_1})\frac{x_1^2}{I_1} + (\frac{1}{I_2} - \frac{1}{I_3})\frac{x_3^2}{I_3}] \\ u_3^2 = -x_3[(\frac{1}{I_3} - \frac{1}{I_1})\frac{x_1^2}{I_1} + (\frac{1}{I_3} - \frac{1}{I_2})\frac{x_2^2}{I_2}] \end{cases}$$

Let ξ^2 the rotation system with the torque \mathbf{u}_2 . It is easy to see that the set of the equilibrium points of the dissipated system is

$$E_{\xi^2} = E_{\xi_{\Pi}}.$$

Also we have

$$C_{2*} = \{(0, 0, 0)\}.$$

We consider the equilibrium point $x_e = (M_0, 0, 0)$ with $M_0 \in \mathbb{R}^*$.

If we define

$$\psi_2(H, C_2) := H + (C_2 + \frac{M_0^2}{2})^2 + \frac{C_2}{I_1},$$

we have that

$$\frac{\partial \psi_2}{\partial C_2}(H, C_2) = 2(C_2 + \frac{M_0^2}{2}) + \frac{1}{I_1}$$

and

$$\frac{\partial \psi_2}{\partial C_2}(H(x_e), C_2(x_e)) = \frac{1}{I_1} > 0.$$

In this situation we can apply Corollary 3.5 and we obtain the stability result.

Theorem 4.4. *There exists a small compact neighborhood U of x_e and an other $V \subset U$ such that every solution of ξ^2 starting in V approaches $U \cap \{(x, 0, 0) \mid x \in \mathbb{R}\}$ as $t \rightarrow \infty$. Moreover H remains constant along solutions and $C_2 = -C_0$ decreases along this solutions.*

If U is sufficiently small then $U \cap E_{\xi^2} \subset \{(x, 0, 0) \mid \text{sgn}(x) = \text{constant}\}$. Suppose that $\text{sgn}(x) = 1$. Using the Corollary 3.7 we obtain the asymptotic stability result.

Theorem 4.5. *If $x_0 \in V$ then the solution $x(t, x_0)$ of the system ξ_2 which verifies the initial condition $x(0, x_0) = x_0$ satisfies the property*

$$\lim_{t \rightarrow \infty} x(t, x_0) = (\sqrt{2I_1 H(x_0)}, 0, 0).$$

Case III. In [3] were introduced the functions

$$C_3 = (C_0 - \frac{M_0^2}{2})^2 - \frac{C_0}{I_1} = (C_2 + \frac{M_0^2}{2})^2 + \frac{C_2}{I_1}$$

and

$$L_3(x) = H(x) + C_3(x) - H(x_e) - C_3(x_e).$$

It is easy to see that:

- i) $\delta L_3(x_e) = 0$;
- ii) $\delta^2 L_3(x_e)$ is positive definite because the Hessian matrix of L_3 in x_e is

$$\begin{pmatrix} 8M_0^2 & 0 & 0 \\ 0 & \frac{1}{I_2} - \frac{1}{I_1} & 0 \\ 0 & 0 & \frac{1}{I_3} - \frac{1}{I_1} \end{pmatrix}.$$

We denote by ξ_3 the system of the form (1.1) with the Casimir C_3 . In [3] has been proved the following result.

Theorem 4.6. *There exists a small compact neighborhood U of x_e and an other $V \subset U$ such that every solution of ξ^3 starting in V approaches $U \cap \{(x, 0, 0) \mid x \in \mathbb{R}\}$ as $t \rightarrow \infty$.*

Our results applies to equilibria that are already stable. By adding the dissipation we can make them asymptotically stable if certain conditions are satisfied. Going from asymptotic stability on the level set to stability in the whole space have been studied in [2]. By adding other type of controls one can stabilize the unstable equilibria. This subject is studied for example in [1], [4], [6], [17].

References

- [1] **D. Aeyels**, *On stabilization by means of the Energy-Casimir method*, Systems Control Letters 18 (1992), 325-328.
- [2] **D. Aeyels, R. Sepulchre**, *Stability for dynamical systems with first integrals: A topological criterion*, Systems Control Letters 19 (1992), 461-465.
- [3] **P. Birtea, M. Boleantu, M. Puta, R. M. Tudoran**, *Asymptotic stability for a class of metriplectic systems*, J. Math. Phys., 48 (2007).
- [4] **A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, G. Sanchez de Alvarez**, *Stabilization of rigid body dynamics by internal and external torques*, Automatica 28 (1992), 745-756.
- [5] **A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, T. S. Ratiu**, *The Euler-Poincaré equations and double bracket dissipation*, Comm. Math. Phys. 175 (1996), 1-42.
- [6] **A. M. Bloch, J. Marsden**, *Stabilization of rigid body dynamics by the Energy-Casimir method*, Systems Control Letters 14 (1990), 341-346.
- [7] **A. M. Bloch, J. E. Marsden, T. S. Ratiu**, *Dissipation induces instability*, Ann. Inst. H. Poincaré Anal. Non Linéaire 11 (1994), 37-90.
- [8] **D. Fish**, *Dissipative perturbations of 3d Hamiltonian systems*, arXiv: math-ph/0506047, V1, 18 June 2005.
- [9] **M. Grmela**, *Bracket formulation of diffusion-convection equations*, Physica D 21 (1986), 179-212.
- [10] **A. Kaufman**, *Dissipative Hamiltonian systems: A Unifying principle*, Physics Letters 100A (1984), 419-422.
- [11] **A. Kaufman**, *Lorentz-covariant dissipative Lagrangian systems*, Physics Letters 109A (1985), 87-89.
- [12] **A. Kaufman, P. J. Morrison**, *Algebraic structure of the plasma quasilinear equations*, Physics Letters 88A (1982), 405-406.
- [13] **A. Kaufman, L. Turski**, *Canonical-dissipative formulation of relativistic plasma kinetic theory with self-consistent Maxwell field*, Physics Letters 120A (1987), 331-333.
- [14] **P.J. Morrison**, *Bracket formulation for irreversible classical fields*, Physics Letters 100A (1984), 423-427.
- [15] **P.J. Morrison**, *A paradigm for joined Hamiltonian and dissipative systems*, Physica D 18 (1986), 410-419.
- [16] **C. Robinson**, *Dynamical systems, Stability, Symbolic Dynamics, and Chaos*, CRC Press, 1995.
- [17] **E. D. Sontag, H. J. Sussmann**, *Further comments on the stabilizability of the angular velocity of a rigid body*, Systems Control Letters 12 (1989), 213-217.