

SYMMETRIC CHAIN DECOMPOSITION FOR CYCLIC QUOTIENTS OF BOOLEAN ALGEBRAS AND RELATION TO CYCLIC CRYSTALS

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ABSTRACT. The quotient of a Boolean algebra by a cyclic group is proven to have a symmetric chain decomposition. This generalizes earlier work of Griggs, Killian and Savage on the case of prime order, giving an explicit construction for any order, prime or composite. The combinatorial map specifying how to proceed downward in a symmetric chain is shown to be a natural cyclic analogue of the \mathfrak{sl}_2 lowering operator in the theory of crystal bases.

1. INTRODUCTION

Griggs, Killian, and Savage [5] have given a symmetric chain decomposition for the quotient of a Boolean algebra B_n by a cyclic group C_n in the case where n is prime. We generalize this to all n through the use of Lyndon words [9] and the theory of crystal bases [6].

Our construction is based on a completely explicit injective map ϕ which takes a quotient poset element to an element it covers, and which may be interpreted as a cyclic analogue of the lowering operator for \mathfrak{sl}_2 crystals. Two elements x, y in the quotient poset B_n/C_n belong to the same symmetrically placed chain in our symmetric chain decomposition if and only if either $x = \phi^r(y)$ or $y = \phi^r(x)$ for some r .

We use properties of Lyndon words (see for example [9]) to define ϕ directly on the quotient poset and prove it is injective. The definition of ϕ is then rephrased in a simpler way that makes it evident that ϕ is a cyclic analogue of the usual \mathfrak{sl}_2 lowering operator for crystals. This rephrasing involves a matching of 0's with 1's and then proceeding downward through a symmetrically placed chain by changing unmatched 1's to 0's in a Lyndon word from right to left. It was inspired by the approach of [5].

A different proof for all n of the existence of a symmetric chain decomposition was given prior to ours by Jordan in [7]. Specifically, she provided an algorithm to produce an SCD for any n . The resulting SCD

The authors were supported by NSF grants DMS-1002636 and DMS-0652641, DMS-0652652, DMS-1001256, respectively.

is not the same one we obtain. Jordan's result was subsequently generalized in [2] to show that B_n/G is a symmetric chain order whenever G is generated by powers of disjoint cycles in the symmetric group. Very recently, Dhand generalized the result in a different direction in [1], namely to an SCD for the quotient poset P^n/C_n , where P is any finite poset with symmetric chain decomposition, C_n is a cyclic group of order n , and P^n denotes the Cartesian product of n copies of P . Our work was done independently of [1], [2] and [7]; our approach is in fact quite different than is taken in these papers.

One reason for interest in finding symmetric chain decompositions stems back to work of Stanley (cf. [11, 12]) and Proctor (cf. [8]), among others, in the 1980's on unimodality questions for rank generating functions – since a symmetric chain decomposition implies unimodality. In 1977, Griggs gave quite general sufficient conditions in [4] for a poset to admit a symmetric chain decomposition by a clever argument involving the max flow-min cut theorem as well as Hall's marriage theorem.

Subsequently, a powerful approach to unimodality questions was developed by using the representation theory of \mathfrak{sl}_2 , since dimensions of weight spaces of the same parity in any \mathfrak{sl}_2 -representation necessarily form a unimodal sequence. In this paper we show that this is also possible in the case of the quotient of a Boolean algebra by a cyclic group using crystal bases.

Acknowledgments. We are indebted to Dennis Stanton for very helpful discussions. We would also like to thank Georgia Benkart for her insight on unimodality of crystals and the reduction to the \mathfrak{sl}_2 case. Thanks to Georgia Benkart, Monica Vazirani, Stephanie van Willigenburg, and the Banff International Research Station for providing a stimulating environment during the Algebraic Combinatorixx workshop, enabling the connection to be made there between symmetric chain decomposition and crystal lowering operators.

2. SYMMETRIC CHAIN DECOMPOSITION

We regard the elements of the quotient poset B_n/C_n as equivalence classes of words in $\{0, 1\}^n$, where two words are equivalent if they differ by a cyclic shift of the position indices. We often speak of words, when in fact we always mean equivalence classes of words. Our poset is graded with rank function being the number of 1's in a word. We have $u \leq v$ if and only if there exist cyclic rearrangements of u and v , denoted $\sigma_i(u)$ and $\sigma_j(v)$, such that the set of positions with value 1 in $\sigma_i(u)$ is a subset of that of $\sigma_j(v)$. Here σ_i denotes the rotation which increases indices by $i \bmod n$.

Example 2.1. We have $01001 \leq 11110$ because $\sigma_1(11110) = 01111$.

Our plan is to define a completely explicit map ϕ that will tell us how to proceed downward through each chain in the symmetric chain decomposition.

Definition 2.2. Given an ordered alphabet \mathcal{A} and a word in $w \in \mathcal{A}^n$, define the *Lyndon rearrangement* of w to be the lexicographically smallest word obtained by a cyclic rotation of the letters in w . The resulting word is called a *Lyndon word*.

Example 2.3. We use the alphabet $\mathcal{A} = \{0, 1\}$ with ordering $1 \prec 0$. Then 00011101001100111001 has 11101001100111001000 as its Lyndon rearrangement.

Now let us define the map ϕ which has as its domain all elements of the top half of the quotient poset and those elements from the bottom half of the poset which do not turn out to be bottom elements of chains in the symmetric chain decomposition. The map ϕ sends any poset element x upon which it acts to an element $\phi(x)$ which x covers. We focus first on the case of words with more 1's than 0's, i.e. the top half of the quotient poset; we will handle the case of words with at least as many 0's as 1's in a different manner that will depend on our map on the top half of ranks in the quotient poset.

Given a word of 0's and 1's, cyclically rotate it into its Lyndon rearrangement. Now apply the following process repeatedly until all unmatched elements are 1's: take any 0 that is immediately followed by a 1 (cyclically) and match these pairs of letters, removing them from further consideration. Using parentheses to depict our pairing, here is an example of this algorithm: $1101100110 \rightarrow 1)1(01)10(01)1(0 \rightarrow 1)1(01)1(0(01)1)(0$.

Call each of these 01 pairs a *parenthesization pair*, or sometimes a *matching pair*. If there is at least one unpaired 1 at the end of this process, then ϕ maps the word to a word in which the rightmost unmatched 1 in the Lyndon expression is changed to a 0. For example, ϕ maps $1)1(01)1(0(01)1)(0$ to 1101000110 . Notice that the Lyndon rearrangement of this new word is 1101101000 and this has bracketing $1)1)(01)1)(01)(0(0(0$. In particular, changing the 1 to a 0 created a new parenthesization pair involving the letter changed from a 1 to a 0 together with another 1 which had been unmatched, provided there are at least two unmatched 1's just prior to the application of ϕ . We refer to this pair of letters the most recently created parenthesization pair. If there is only a single unmatched 1, this is still changed by ϕ to

Thus, all unmatched 1's are in the same connected component once noncrossing arcs are inserted connecting elements of parenthesization pairs (arranged around a circle).

Corollary 2.8. *The map ϕ applied to any element in the upper half of ranks will not change the collection of parenthesization pairs other than creating one additional pair comprised of the letter transformed by ϕ from a 1 to a 0 along with the leftmost 1 that is unmatched just prior to this application of ϕ .*

Proof. This follows from Lemma 2.7. □

In the next proof, we will say $b_1 \dots b_r \leq_{Lyndon} c_1 \dots c_r$ to denote the fact that $b_1 \dots b_r$ is at least as small lexicographically as $c_1 \dots c_r$.

Proposition 2.9. *The map ϕ is injective on the upper half of ranks.*

Proof. Suppose two different cyclic words $A = a_1 \dots a_n$ and $A' = a'_1 \dots a'_n$ satisfy $\phi(A) = \phi(A') = B$, with A, A' each having more 1's than 0's. Without loss of generality, we may assume A and A' agree except that $a_i = 1, a'_i = 0, a_j = 0, a'_j = 1$ for some $i < j$ where B agrees with both except that B has $b_i = b_j = 0$. That is, ϕ changes a_i from 1 to 0 in A , ϕ changes a'_j from 1 to 0 in A' . We may also assume that one of the words A, A' is written as a Lyndon word, so that either a_i or a'_j is the rightmost unmatched 1 in its Lyndon word. That is, we have written one of the words A or A' using its Lyndon rearrangement, and we have written the other in its rearrangement so that the words agree except in the two locations where one or the other word has a 1 switched to a 0 by ϕ . We do not know a priori which of the two words is written as a Lyndon word, but rather we will consider both cases.

Our plan is to show in both cases that all of the letters in the word $W(A) = W(A') = a_{i+1} \dots a_{j-1}$ belong to parenthesization pairs with other letters also in this segment, implying A' has a'_i belonging to a parenthesization pair together with a'_j , a contradiction to ϕ acting on A' by changing a'_j to a 0.

Suppose first that A is Lyndon. If there is an unmatched 1 in $W(A)$, this would contradict our definition of ϕ : ϕ turns the rightmost unmatched 1 into a 0 in the Lyndon rearrangement of A , since a_i would then not be rightmost. There cannot be a 1 in $W(A)$ that is matched with a 0 outside $W(A)$, since this 0 outside $W(A)$ would instead match with a_i . On the other hand, if there were a 0 in $W(A')$ that is not matched with any 1 in $W(A')$, this would match with a'_j , contradicting a'_j being unmatched. Thus, $W(A')$ must be fully matched, implying a'_i matches with a'_j , as desired.

Now suppose instead that A' is Lyndon. Again, there cannot be any 0's in $W(A')$ that are not matched with 1's in $W(A')$, since any such 0 would match with a'_j . Suppose there is some $a'_k = 1$ in $W(A')$ that is not matched within $W(A')$. Then the corresponding 1 in $W(A)$, i.e. $a_k = 1$, is also unmatched in A , hence must appear farther to the left than a_i in the Lyndon rearrangement of A . Thus, the Lyndon rearrangement of A would begin with a letter a_s to the right of a_i and to the left of a_{k+1} , whereas the Lyndon rearrangement of A' begins with a'_1 . But then we would have $a_s \dots a_j \leq_{\text{Lyndon}} a_1 \dots a_{j-s+1} \leq_{\text{Lyndon}} a'_1 \dots a_{j-s+1} \leq_{\text{Lyndon}} a'_s \dots a'_j = a_s \dots a_{j-1} a'_j \prec a_s \dots a_j$, a contradiction. Thus, each 1 in $W(A')$ is also matched within $W(A')$, so $W(A')$ is fully matched, implying that a'_i again comprises a parenthesization pair with a'_j , again a contradiction. \square

For n odd, the map at the middle pair of ranks is perhaps worth special discussion:

Proposition 2.10. *For n odd, the map ϕ is injective between the two middle ranks, i.e. when it turns the unique unmatched 1 into a 0.*

Proof. This is immediate from the fact that changing this 1 to a 0 cannot impact the collection of parenthesization pairs, so that we may look at an element of $\text{im}(\phi)$ having one more 0 than 1 and determine its inverse by seeing which 0 is unmatched. \square

Proposition 2.11. *The map ϕ is injective on the lower half of ranks.*

Proof. To prove injectivity on the bottom half, we show how to define the inverse map. Given an element u with at least as many 0's as 1's, we begin by finding the unique element v of the same chain that is exactly as many ranks above the middle as u is below the middle. This is done as follows. Obtain a parenthesization of u by pairing any 0 immediately followed by a 1 cyclically, and keep repeating until all unpaired letters are 0's. Now obtain v from u by changing all these unpaired letters from 0's to 1's. If the difference in ranks between v and u is r , then $\phi^r(v) = u$ and $\phi^{-1}(u) = \phi^{r-1}(v)$. By induction, we may assume ϕ is well-defined at all ranks strictly above u , enabling us thereby to use the uniqueness of v to prove that $\phi^{-1}(u)$ is unique. This also enables us to apply ϕ in a well-defined manner to u . \square

We now state our main result:

Theorem 2.12. *The quotient poset B_n/C_n has a symmetric chain decomposition such that $u, v \in B_n/C_n$ for $\text{rank}(u) < \text{rank}(v)$ belong to the same chain iff $u = \phi^r(v)$ for some r .*

Proof. The fact that the chains are symmetric is immediate from the construction. We use rank symmetry of the poset to deduce that we have fully covered the bottom half of the poset from the fact that we fully covered the top half. \square

3. ALTERNATIVE DESCRIPTION OF ϕ

Next we develop some properties of the map ϕ that yield a much more explicit alternative description of ϕ , enabling us to establish a connection in the next section to the theory of crystal bases. The upshot will be that in spite of our taking the lexicographically smallest representation of each element to which ϕ applies in a symmetric chain, we nonetheless end up proceeding down a symmetric chain simply by changing the unmatched 1's to 0's from right to left in the Lyndon expression for the highest element of the symmetric chain.

Proposition 3.1. *Each time the map ϕ is applied turning a 1 into a 0 in a Lyndon word $a_1 \dots a_n$ to obtain a word $a'_1 \dots a'_n$ with $a_i = 1, a'_i = 0$ and $a_j = a'_j$ for $j \neq i$, the Lyndon expression for $a'_1 \dots a'_n$ will either shift the letter a'_i to the right or leave its position unchanged.*

Proof. Suppose there is some $1 < j < i$ such that the Lyndon expression for $a'_1 \dots a'_n$ is $a'_j \dots a'_i \dots a'_n a'_1 \dots a'_{j-1}$. Then $a'_j \dots a'_i \leq_{\text{Lyndon}} a'_1 \dots a'_{i-j+1} = a_1 \dots a_{i-j+1} \leq_{\text{Lyndon}} a_j \dots a_i <_{\text{Lyndon}} a'_j \dots a'_i$, a contradiction. \square

Corollary 3.2. *The series of parentheses added by successively applying the map ϕ in the top half of a Boolean algebra must be nested with respect to each other, with each successive step adding a new innermost parenthesis.*

Proof. Proposition 3.1 implies that each time we turn a 1 into a 0 in the top half, creating a new parenthesized pair, the Lyndon rearrangement may only move this pair farther to the right. In particular, any remaining unmatched 1's must still be to the left of this new 0 and to the right of its partner 1, yielding the desired nesting property. \square

Corollary 3.3. *The map ϕ proceeds from right to left through the initial Lyndon word, successively turning each 1 that is not initially paired with a 0 into a 0.*

Proof. This follows from Proposition 3.1. In particular, the letters that comprise the left parentheses of the pairs created while proceeding downward through the top half of the Boolean algebra only move to the right at least until after any such letter is switched from a 1 to a 0. Therefore, a letter that is the rightmost unmatched 1 will continue to

be this under the Lyndon rearrangement in the top half of ranks. In addition, the nesting of parenthesis pairs as justified in Corollary 3.2 ensures the result for the bottom half of ranks. \square

4. RELATION TO CRYSTALS

We now point out the resemblance of the parenthesization procedure of the previous section and the signature rule for crystal bases, see for example [6]. Combinatorially, \mathfrak{sl}_2 crystals can be viewed as the graph on words of finite length in the letters $\{1, 2\}$ with an arrow from word w to word w' if w' is obtained from w by changing the rightmost unbracketed 1 into a 2, where now we successively bracket pairs 21. Identifying 2 with 0 from the previous section, this is precisely the parenthesization (up to the cyclic shift). If there is an arrow from w to w' in the crystal graph, the Kashiwara lowering operator f acts as $f(w) = w'$. If there is no arrow from w , f annihilates w . Similarly, the Kashiwara raising operator e acts as $e(w') = w$ or annihilates w' if there is no incoming arrow to w' .

An element x (resp. y) in the crystal is called highest (resp. lowest) weight if $e(x) = 0$ (resp. $f(y) = 0$). The weight of an element is the number of 1's in the word minus the number of 2's. If the weight of the highest weight element is L , then the crystal is $L + 1$ dimensional.

There exists an involution on the crystal, called the Schützenberger involution in type A or Lusztig involution for general types, which interchanges highest and lowest weight vectors x and y and the lowering and raising operators f and e (in our \mathfrak{sl}_2 setting).

Interpreting the map ϕ from the previous section as the Kashiwara lowering operator of a cyclic crystal, we can characterize the highest weight elements in the cyclic crystal.

Lemma 4.1. *Highest weight elements in the cyclic \mathfrak{sl}_2 crystal are Lyndon words where, if we do the cyclic bracketing, only the last string of consecutive 0's is bracketed cyclically, i.e. with 1's at the beginning of the word. Furthermore, removing a cyclic bracket by turning the corresponding 0 to 1 would break the Lyndon condition.*

Proof. By Proposition 3.1 and Corollary 3.2, the letters after application of ϕ move to the right after the rearrangement into a Lyndon word, and the parentheses that are added are nested. This proves the claim. \square

Example 4.2. The word 1111001110001110 is highest weight in the cyclic crystal, since the parenthesization yields:

$$\begin{array}{cccccccccccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ & & & & & & (& (&) &) & (& (& (&) &) &) & (\end{array}$$

Changing the last 0 to a 1 would break the Lyndon condition.

The word 11100010110 on the other hand is not highest weight since in

$$\begin{array}{cccccccccccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ & & &) &) & (& (& (&) & (&) &) & (\end{array}$$

the open bracket in position 4 and the closed bracket in position 2 are matched, which go across the cycle where the 0 is not at the end. In fact, this word would belong to the string of highest weight element

$$\begin{array}{cccccccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ & & &) & & (& (&) & (&) &) & (\end{array}$$

The following word is not highest weight, since one can remove a cyclic bracket without breaking the Lyndon property

$$\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ & & &) &) & (&) & (& (\end{array}$$

The corresponding highest weight element is 11110110.

Corollary 4.3.

- (1) *The map ϕ is a natural cyclic analogue of the lowering operators from the theory of crystal bases.*
- (2) *Lusztig’s involution sends elements in the upper (resp. lower) half of a symmetrically placed chain to elements in the lower (resp. upper) half of a chain in the dual symmetric chain decomposition. Notice that these two symmetric chain decompositions which are obtained from each other via the Lusztig involution are typically different from each other.*

Our symmetric chain decomposition from the first part now yields the following:

Theorem 4.4. *The \mathfrak{sl}_2 strings in the cyclic crystal yield a symmetric chain decomposition.*

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