TEST IDEALS VIA A SINGLE ALTERATION AND DISCRETENESS AND RATIONALITY OF F-JUMPING NUMBERS

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ABSTRACT. Suppose (X, Δ) is a log-$\mathbb{Q}$-Gorenstein pair. Recent work of M. Blickle and the first two authors gives a uniform description of the multiplier ideal $\mathcal{J}(X; \Delta)$ (in characteristic zero) and the test ideal $\tau(X; \Delta)$ (in characteristic $p > 0$) via regular alterations. While in general the alteration required depends heavily on Δ, for a fixed Cartier divisor D on X it is straightforward to find a single alteration (*e.g.* a log resolution) computing $\mathcal{J}(X; \Delta + \lambda D)$ for all $\lambda \geq 0$. In this paper, we show the analogous statement in positive characteristic: there exists a single regular alteration computing $\tau(X; \Delta + \lambda D)$ for all $\lambda \geq 0$. Along the way, we also prove the discreteness and rationality for the F-jumping numbers of $\tau(X; \Delta + \lambda D)$ for $\lambda \geq 0$ where the index of $K_X + \Delta$ is arbitrary (and may be divisible by the characteristic).

1. Introduction

Suppose that X is a normal algebraic variety over a perfect field k and $(X, \Delta \geq 0)$ is an effective log-$\mathbb{Q}$-Gorenstein pair. In [BST11], the authors showed

$$\operatorname{Image}\left(\pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta) \rceil) \xrightarrow{\operatorname{Tr}} \mathcal{O}_X\right) = \begin{cases} \mathcal{J}(X; \Delta) & \text{the multiplier ideal, if } \operatorname{char}(k) = 0 \\ \tau(X; \Delta) & \text{the test ideal, if } \operatorname{char}(k) = p > 0 \end{cases}$$

for a sufficiently large regular separable alteration $\pi : Y \longrightarrow X$, where $\operatorname{Tr} : K(Y) \longrightarrow K(X)$ is the corresponding trace map of function fields. In general the alteration required depends heavily on Δ; however, when working in characteristic zero, one can often perturb Δ while using the same alteration. More precisely, fix a Cartier divisor $D \geq 0$ on X. Any regular alteration $\pi : Y \longrightarrow X$ such that

$$\operatorname{Supp}(K_Y - \pi^*(K_X + \Delta)) \cup \operatorname{Supp}(\pi^* D)$$

is a divisor with simple normal crossings (*e.g.* a log resolution) will suffice to compute $\mathcal{J}(X; \Delta + \lambda D)$ for all $\lambda \geq 0$. Our principal result is to show the analogous statement for the test ideal in positive characteristic.

Main Theorem (Theorem 4.2)**.** *Let X be a normal algebraic variety over a perfect field of characteristic $p > 0$ and suppose $\Delta \geq 0$ is a $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. If $D \geq 0$ is a Cartier divisor on X, there exists a finite separable cover (alternatively, a regular separable alteration) $\pi : Y \longrightarrow X$ such that*

$$\tau(X; \Delta + \lambda D) = \operatorname{Image}\left(\pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta + \lambda D) \rceil) \xrightarrow{\operatorname{Tr}} K(X)\right)$$

for all real numbers $\lambda \geq 0$, where $\operatorname{Tr} : K(Y) \longrightarrow K(X)$ is the corresponding trace map of function fields.

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In fact, in producing the finite separable cover above, one may take X to be an arbitrary F-finite normal scheme (rather than just a variety over a perfect field).

Our method of proof relies, perhaps unsurprisingly, on the discreteness and rationality for the F-jumping numbers of $\tau(X; \Delta + \lambda D)$ – a result which is *a posteriori* equivalent to the theorem above. Nevertheless, in order to prove our main result, we must first generalize existing discreteness and rationality results to show the following.

Theorem (Theorem 3.3)**.** *Suppose that X is an F-finite normal integral scheme, D is a Cartier divisor, and Δ is a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier (with arbitrary index). Then the F-jumping numbers of $\tau(X; \Delta + \lambda D)$ for $\lambda \geq 0$ are discrete and rational.*

This result follows easily by combining the main result of [BSTZ10] (which in turn builds upon [KLZ09, BMS09]) together with [ST10]. As far as we are aware, this is also the first application of the discreteness and rationality of these F-jumping numbers.

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2. Preliminaries

Throughout this paper, all schemes (and rings) are Noetherian, separated, have characteristic $p > 0$, and are always F-finite. In particular, they are also excellent [Kun76] and are assumed to possess a dualizing complex (*cf.* [Gab04]). In fact, very little is lost if one restricts to the study of varieties of finite type over a perfect field (as in the introduction). For an integral scheme X, we use ω_X to denote the first non-zero cohomology of the dualizing complex $\omega_X^{\bullet}$.

On a normal integral scheme X, a *canonical divisor* K_X is any integral Weil divisor such that $\mathcal{O}_X(K_X) \cong \omega_X$. A $\mathbb{Q}$-divisor (respectively $\mathbb{R}$-divisor) Δ is a formal linear combination of prime Weil divisors with coefficients in $\mathbb{Q}$ (respectively $\mathbb{R}$). Writing $\Delta = \sum a_i D_i$ where the D_i are distinct prime divisors, we use $\lceil \Delta \rceil = \sum \lceil a_i \rceil D_i$ to denote the round-up of Δ. We say a $\mathbb{Q}$-divisor Δ is $\mathbb{Q}$-*Cartier* if there exists an integer $n > 0$ such that $n\Delta$ has integer coefficients and as such is also a Cartier divisor. In this case, the smallest such n is called the *index* of Δ.

Definition 2.1. Suppose X is an integral scheme.

(1) We say (X, Δ) is a *pair* if X is normal and Δ is a $\mathbb{R}$-divisor. When additionally $K_X + \Delta$ is $\mathbb{Q}$-Cartier, (X, Δ) is called *log-$\mathbb{Q}$-Gorenstein*.
(2) An *alteration* $\pi : Y \longrightarrow X$ is a generically finite proper dominant map from an integral scheme Y. A *finite cover* is an alteration which is also a finite map.

For an alteration $f : Y \longrightarrow X$, we have the Grothendieck trace map

$$\mathrm{Tr}_f : f_* \omega_Y \longrightarrow \omega_X$$

coming from the theory of Grothendieck-Serre duality; for a detailed construction of this map, please see [BST11]. In the case that $f = F^e$ is the e-iterated Frobenius, we use Φ^e to denote Tr_{F^e} and call this map the *canonical dual of Frobenius.* In contrast, if f is separable and $K_Y - \pi^* K_X = \mathrm{Ram}_\pi$ equals the ramification divisor wherever π is finite, then the corresponding map $\mathrm{Tr}_f : \mathcal{O}_Y(K_Y) \longrightarrow \mathcal{O}_X(K_X)$ is induced by the trace map of function fields $\mathrm{Tr} : K(Y) \longrightarrow K(X)$.

As in [BST11, Proposition 2.6], for any log-$\mathbb{Q}$-Gorenstein pair (X, Δ) and any alteration $f : Y \longrightarrow X$ from a normal scheme Y, the trace induces a natural map

$$\mathrm{Tr}_f : f_* \mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta) \rceil) \longrightarrow K(X).$$

In fact, the image of this map is contained in $\mathcal{O}_X(\lceil -\Delta \rceil)$, and so is inside $\mathcal{O}_X$ whenever $\Delta \geq 0$. Using also that the trace is compatible with composition in general, this readily implies

$$\begin{aligned} &\text{Image}(g_*\mathcal{O}_{Y'}(\lceil K_{Y'} - g^*(K_X + \Delta)\rceil) \xrightarrow{\text{Tr}_g} K(X)) \\ &\subseteq \quad \text{Image}(f_*\mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta)\rceil) \xrightarrow{\text{Tr}_f} K(X)) \end{aligned} \tag{2.1.1}$$

whenever another alteration $g : Y' \longrightarrow X$ from a normal scheme Y' factors through f.

Rather than reviewing the definitions of the test ideals and parameter test modules of pairs (involving $\mathbb{R}$-divisors) here, we refer the reader to [ST11, BST11, ST08]. However, a few words of warning are in order. First, whenever we speak of the test ideal throughout, we will always mean the big or non-finitistic test ideal; regardless, the classical or finitistic test ideal coincides with this notion in the log-$\mathbb{Q}$-Gorenstein setting, *i.e.* in (essentially) all situations considered in this paper. Furthermore, we also caution that we allow the $\mathbb{R}$-divisors Δ in our pairs (X, Δ) to have *negative coefficients*, as in [ST10, BST11]. In particular, this means that the test ideal $\tau(X, \Delta)$ may not be an honest ideal sheaf, but rather it is simply a fractional ideal sheaf (respectively, the parameter test module $\tau(\omega_X; \Gamma)$ may not be a submodule of ω_X). We briefly record the following list of relevant properties of test ideals and parameter test modules for pairs below.

Proposition 2.2. [Tak04, BSTZ10, BST11, ST10, ST11, ST08] *Suppose that (X, Δ) is a log-$\mathbb{Q}$-Gorenstein pair, Γ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, $t \geq 0$ is a real number, and $D \geq 0$ is a Cartier divisor. Then*

(1) $\tau(X; \Delta + D) = \tau(X; \Delta) \otimes \mathcal{O}_X(-D)$ *and* $\tau(\omega_X; \Gamma + D) = \tau(\omega_X; \Gamma) \otimes \mathcal{O}_X(-D)$.

(2) *For any sufficiently small real number $\varepsilon > 0$,*
$$\tau(X; \Delta + \varepsilon D) = \tau(X; \Delta) \text{ and } \tau(\omega_X; \Gamma + \varepsilon D) = \tau(\omega_X; \Gamma).$$

(3) *For any finite cover $\pi : Y \longrightarrow X$ with trace map Tr_π as above, we have*
$$\text{Tr}_\pi\left(\pi_*\tau(\omega_Y; \pi^*(\Gamma + tD))\right) = \tau(\omega_X; \Gamma + tD).$$

(4) *In particular, if Φ^e is the canonical dual of Frobenius, then*
$$\Phi^e\big(F^e_*\tau(\omega_X; p^e\Gamma + tp^eD)\big) = \tau(\omega_X; \Gamma + tD).$$

(5) *If $t' > t$, then*
$$\tau(X; \Delta + t'D) \subseteq \tau(X; \Delta + tD).$$

Proof. The first half of part (1) is [Tak04, Page 9, Basic Property (ii)], and the proof of the second is similar (or equivalent via [BST11, Lemma 2.29]). Part (2) is [BSTZ10, Lemma 3.23] and parts (3) and (4) are [BST11, Proposition 4.3] (*cf.* [ST10] for further generalizations). For (5), see [Tak04, Page 9, Basic Property (i)]. □

3. Discreteness and rationality

Definition 3.1. Suppose that X is a normal integral scheme, Δ and Γ are $\mathbb{Q}$-divisors, and D is a Cartier divisor. A real number $t \geq 0$ is an *F-jumping number of the test ideals $\tau(X; \Delta + \lambda D)$ for $\lambda \geq 0$* if for all $\varepsilon > 0$, we have

$$\tau(X; \Delta + (t - \varepsilon)D) \neq \tau(X; \Delta + tD).$$

Similarly, a real number $t \geq 0$ is an *F-jumping number of the parameter test modules $\tau(\omega_X; \Gamma + \lambda D)$ for $\lambda \geq 0$* if for all $\varepsilon > 0$ we have

$$\tau(\omega_X; \Gamma + (t - \varepsilon)D) \neq \tau(\omega_X; \Gamma + tD).$$

Furthermore, we say the the F-jumping numbers of $\tau(X, \Delta + \lambda D)$ (respectively $\tau(\omega_X; \Gamma + \lambda D)$) are *discrete and rational* if they are a set of rational numbers without any accumulation points (technically, this is a stronger condition than being discrete).

Lemma 3.2. *Suppose $D \geq 0$ is a Cartier divisor on a normal integral scheme X, and let m be a positive integer. The following statements are equivalent:*

(1) *For all $\mathbb{Q}$-divisors Δ where $K_X+\Delta$ is $\mathbb{Q}$-Cartier with index m, the F-jumping numbers of $\tau(X; \Delta + \lambda D)$ for $\lambda \geq 0$ are discrete and rational.*
(2) *For all $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors Γ with index m, the F-jumping numbers of $\tau(\omega_X; \Gamma+\lambda D)$ for $\lambda \geq 0$ are discrete and rational.*

In particular, the main result of [BSTZ10] *yields both statements if m is not divisible by p.*

Proof. Setting $\Gamma = K_X + \Delta$, it follows from the argument of [BST11, Lemma 2.29] that we have

$$\tau(\omega_X; \Gamma + \lambda D) = \tau(X; \Delta + \lambda D).$$

for all real numbers $\lambda \geq 0$. In particular, the corresponding F-jumping numbers are the same, and the desired equivalence follows immediately. For the remaining assertion, as these statements are local, it is harmless to assume that X is affine and $\Delta \geq 0$. When p does not divide m, (1) is precisely the statement of [BSTZ10, Theorem 5.6] □

In order to prove our main result in the next section, we must first generalize the existing discreteness and rationality results to show the following.

Theorem 3.3. *Suppose (X, Δ) is a log-$\mathbb{Q}$-Gorenstein pair and $D \geq 0$ is a Cartier divisor. Then the F-jumping numbers of $\tau(X; \Delta + \lambda D)$ for $\lambda \geq 0$ are discrete and rational.*

Proof. By Lemma 3.2, it suffices to check that the F-jumping numbers of $\tau(\omega_X; \Gamma + \lambda D)$ for $\lambda \geq 0$ are discrete and rational for any $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor Γ. This statement is known if p does not divide the index of Γ, and one can use Frobenius to reduce to this case in general. Choose $e > 0$ such that $p^e\Gamma$ has index not divisible by p. If Φ^e is the canonical dual of the e-iterated Frobenius F^e, we know that

$$\Phi^e\big(F^e_* \tau(\omega_X; p^e\Gamma + \lambda p^e D)\big) = \tau(\omega_X; \Gamma + \lambda D)$$

for any real number $\lambda \geq 0$ by Proposition 2.2(4). Since [BSTZ10] gives that the F-jumping numbers of $\tau(\omega_X; p^e\Gamma + \lambda D)$ for $\lambda \geq 0$ are discrete and rational, those of $\tau(\omega_X; \Gamma + \lambda D)$ must be as well. □

4. The existence of a single alteration

In this section, we prove our main theorem. The crux of the argument lies in the following lemma.

Lemma 4.1. *Suppose that (X, Δ) is a log $\mathbb{Q}$-Gorenstein pair and that $D \geq 0$ is a Cartier divisor. Given two adjacent F-jumping numbers $t_0 < t_1$ of $\tau(X; \Delta + \lambda D)$ for $\lambda \geq 0$, there exists a finite separable cover $f : Y \longrightarrow X$ such that*

$$\tau(X; \Delta + tD)) = \mathrm{Image}\Big(f_*\mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta + tD)\rceil) \xrightarrow{\mathrm{Tr}_f} K(X)\Big)$$

for all t satisfying $t_0 \leq t < t_1$.

Proof. Note that t_0 is rational by Theorem 3.3. Using the main result of [BST11], we can find a finite separable cover $f : Y \longrightarrow X$ such that $f^*(K_X + \Delta + t_0 D)$ is Cartier and

$$\tau(X; \Delta + t_0 D) = \mathrm{Image}\Big(f_*\mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta + t_0 D)\rceil) \xrightarrow{\mathrm{Tr}_f} K(X)\Big).$$

We need to prove that the same cover works for every real number t satisfying $t_0 \leq t < t_1$.

Choose rational number t' such that $t_0 \leq t \leq t' < t_1$. Applying [BST11] once again, we can find another cover $g : Y' \longrightarrow X$, factoring through f, such that $g^*(K_X + \Delta + t'D)$ is Cartier and

$$\tau(X; \Delta + t'D) = \operatorname{Image}\Big(g_*\mathcal{O}_{Y'}(\lceil K_{Y'} - g^*(K_X + \Delta + t'D)\rceil) \xrightarrow{\operatorname{Tr}_g} K(X)\Big).$$

Now, simply observe that

$$\begin{array}{rl} & \tau(X; \Delta + t_0 D) \\ = & \operatorname{Image}\Big(f_*\mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta + t_0 D)\rceil) \xrightarrow{\operatorname{Tr}_f} K(X)\Big) \\ \supseteq & \operatorname{Image}\Big(f_*\mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta + tD)\rceil) \xrightarrow{\operatorname{Tr}_f} K(X)\Big) \\ \supseteq & \operatorname{Image}\Big(f_*\mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta + t'D)\rceil) \xrightarrow{\operatorname{Tr}_f} K(X)\Big) \\ \supseteq & \operatorname{Image}\Big(g_*\mathcal{O}_{Y'}(\lceil K_{Y'} - g^*(K_X + \Delta + t'D)\rceil) \xrightarrow{\operatorname{Tr}_g} K(X)\Big) \\ = & \tau(X; \Delta + t'D). \end{array}$$

The first two containments $\supseteq$ are valid simply because $t' > t > t_0$, while the last occurs because g factors through f as in (2.1.1). Since $\tau(X; \Delta + t_0 D) = \tau(X; \Delta + t'D)$ by assumption, each containment is in fact an equality, and so

$$\tau(X; \Delta + tD) = \operatorname{Image}\Big(f_*\mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta + tD)\rceil) \xrightarrow{\operatorname{Tr}_f} K(X)\Big)$$

as claimed. □

We now prove our main result.

Theorem 4.2. *Suppose that (X, Δ) is a log $\mathbb{Q}$-Gorenstein pair and $D \geq 0$ is a Cartier divisor. Then there exists a single finite separable cover $\pi : Y \longrightarrow X$ such that*

$$\tau(X; \Delta + \lambda D) = \operatorname{Image}\Big(\pi_*\mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta + \lambda D)\rceil) \xrightarrow{\operatorname{Tr}_\pi} K(X)\Big) \tag{4.2.1}$$

for all real numbers $\lambda \geq 0$. Alternatively, if X is of finite type over a F-finite (respectively perfect) field, one may take $\pi : Y \longrightarrow X$ to be a regular (separable) alteration.

Proof. There are only finitely many adjacent pairs of F-jumping numbers of $\tau(X; \Delta + \lambda D)$ that are between zero and one by Theorem 3.3, and so we can find a single finite separable cover $\pi : Y \longrightarrow X$ dominating all of the covers produced by Lemma 4.1 for each individual pair. Thus, we have that (4.2.1) is valid for all $\lambda \in [0, 1)$; using the projection formula and Proposition 2.2(1) gives that it is valid for all $\lambda \geq 0$ as desired. The remaining statements when X is of finite type over an F-finite or perfect field follow immediately from the arguments of [BST11] utilizing [dJ96]. □

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