

General perversities and L^2 de Rham and Hodge theorems for stratified pseudomanifolds

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Abstract

Given a compact stratified pseudomanifold with a Thom-Mather stratification and a class of riemannian metrics over its regular part, we study the relationships between the L^2 de Rham and Hodge cohomology and the intersection cohomology of X associated to some perversities. More precisely, to a kind of metric which we call **quasi edge with weights**, we associate two general perversities in the sense of G. Friedman, p_g and its dual q_g . We then show that:

1. The absolute L^2 Hodge cohomology is isomorphic to the maximal L^2 de Rham cohomology and this is in turn isomorphic to the intersection cohomology associated to the perversity q_g .
2. The relative L^2 Hodge cohomology is isomorphic to the minimal L^2 de Rham cohomology and this is in turn isomorphic to the intersection cohomology associated to the perversity p_g .

Introduction

Let X be a compact stratified pseudomanifold. The study of the relationship between the L^2 de Rham and Hodge cohomology associated to a given riemannian metric and the intersection cohomology of X has a long history initiated at the end of seventies with celebrated papers of J. Cheeger [6] and [7]. In [6] Cheeger established a Hodge theorem for manifolds with isolated conic singularities; in [7] he showed that given a riemannian metric on $reg(X)$, with X a closed PL stratified pseudomanifolds and $reg(X)$ the regular part of X , such that (X, g) is an admissible riemannian pseudomanifold (this means that X is a Witt space), cfr. [7] pag. 127, then the L^2 maximal Hodge cohomology is finite dimensional and isomorphic to the maximal L^2 de Rham cohomology and this is in turn isomorphic to the intersection cohomology of X associated to the lower middle perversities. Subsequently the L^2 de Rham theorem of Cheeger was generalized by M. Nagase which in [19] showed that given a perversity $p \leq \underline{m}$, where \underline{m} is the lower middle perversity, it is possible to construct over the regular part of X a riemannian metric g associated to the the perversity p such that the maximal L^2 de Rham cohomology is isomorphic to the intersection cohomology of X associated to the perversity p . In both these papers the proofs of the L^2 de Rham theorem were done by constructing a subcomplex of the complex of L^2 differential form with weak differential quasi-isomorphic to it and integrating the forms of this subcomplex over some PL -chains. Afterwards in the paper [20] Nagase presented a new proof of his L^2 de Rham Theorem that employed the sheaf-theoretic approach of Goresky-MacPherson [13]¹. We note that using some additional hypothesis about the calculation of the maximal L^2 cohomology of a cone over a riemannian manifold, see lemma 3.4 in [7] and lemma 3.12 in [19], the results in [7] and [19] holds for any compact PL stratified pseudomanifold without the Witt assumption.

Recently R. Mazzeo and E. Hunsicker proved [16] a L^2 de Rham and Hodge theorem on a manifold with edges. We recall that a manifolds with edges is a compact stratified pseudomanifold of depth one, $X \supset B$, $B = \bigcup_j B_j$. For each stratum B_j , which in this case is just a

¹In [7], [19] and [20] the simbol \overline{m} is used for the lower middle perversity.

closed manifold, there exists an open neighbourhood U_j of B_j in X which is diffeomorphic to a bundle of cones, that is, a bundle with basis B_j and fibers $C(F_j)$ with F_j a closed manifold that depends only on B_j . Over $X - B$ they consider an edge metric g , that is, a riemannian metric such that over each $U_j - B_j$ it is quasi-isometric to $dr \otimes dr + \pi_j^* h_j + r^2 k_j$ where k_j is a two symmetric tensor field which restricts to a metric on each fiber F_j , $\pi_j : U_j \rightarrow B_j$ is the projection and h_j is a riemannian metric on B_j . Then for the maximal and minimal L^2 de Rham cohomology and for the absolute and relative Hodge cohomology the following isomorphisms holds :

$$I^m H^i(X) \cong H_{2,max}^i(\text{reg}(X), g) \cong \mathcal{H}_{abs}^i(\text{reg}(X), g)$$

$$\bar{I}^m H^i(X) \cong H_{2,min}^i(\text{reg}(X), g) \cong \mathcal{H}_{rel}^i(\text{reg}(X), g)^2$$

This result was later generalized by Hunsicker [15]. Given a manifold with edges with only one singular stratum B , Hunsicker considers a riemannian metric g on $\text{reg}(X)$ such that over $U - B$ it is quasi-isomorphic to

$$dr \otimes dr + \pi^* h + r^{2c} k \text{ where } 0 < c \leq 1.$$

The isomorphisms between the L^2 de Rham, the Hodge and the intersection cohomology of X that she gets, for this kind of metrics, are the following:

$$\mathcal{H}_{abs}^i(\text{reg}(X), g) \cong H_{2,max}^i(\text{reg}(X), g) \cong \begin{cases} I^{m - \lceil \frac{1}{2c} \rceil} H^i(X) & f \text{ is even} \\ I^{m - \lceil \frac{1}{2} + \frac{1}{2c} \rceil} H^i(X) & f \text{ is odd} \end{cases}$$

$$\mathcal{H}_{rel}^i(\text{reg}(X), g) \cong H_{2,min}^i(\text{reg}(X), g) \cong \begin{cases} \bar{I}^{m + \lceil \frac{1}{2c} \rceil} H^i(X) & f \text{ is even} \\ \bar{I}^{m + \lceil \frac{1}{2} + \frac{1}{2c} \rceil} H^i(X) & f \text{ is odd} \end{cases}$$

where $\lceil x \rceil$ denotes the greatest integer stricly less than x .³ It is immediate to note that when $c = 1$ then $\lceil \frac{1}{2} \rceil = \lceil \frac{1}{2} + \frac{1}{2c} \rceil = 0$ and then this result reduces to the results in [16].

The main goal of this paper can be sketched as follows:

- generalize the result established by Hunsicker in the simple edge case to the case of any compact and oriented smoothly stratified pseudomanifold with a Thom-Mather stratification.

Given X , a compact and oriented smoothly stratified pseudomanifold with a Thom-Mather stratification, we consider a riemannian metric g over its regular part, $\text{reg}(X)$, that satisfies the following properties:

1. Take any stratum Y of X ; for each $q \in Y$ there exist an open neighbourhood U of q in Y such that $\phi : \pi_Y^{-1}(U) \rightarrow U \times C(L_Y)$ is a stratified isomorphism; in particular $\phi : \pi_Y^{-1}(U) \cap \text{reg}(X) \rightarrow U \times \text{reg}(C(L_Y))$ is a diffeomorphism. Then, for each $q \in Y$, there exists one of these trivializations (ϕ, U) such that g restricted on $\pi_Y^{-1}(U) \cap \text{reg}(X)$ satisfies the following properties:

$$(\phi^{-1})^*(g|_{\pi_Y^{-1}(U) \cap \text{reg}(X)}) \cong dr \otimes dr + h_U + r^{2c_Y} g_{L_Y}$$

where h_U is a riemannian metric defined over U , $c_Y \in \mathbb{R}$ and $c_Y > 0$, g_{L_Y} is a riemannian metric on $\text{reg}(L_Y)$, $dr \otimes dr + h_Y + r^{2c_Y} g_{L_Y}$ is a riemannian metric of product type on $U \times \text{reg}(C(L_Y))$ and with \cong we mean **quasi-isometric**.

2. The weight c_Y depends only on the stratum Y where the point q lie.

²In [16] the first isomorphism involves the upper middle perversity and the second involves the lower middle perversity. The reason is that in [16] the definitions of these perversities are reversed from the usual ones.

³Also in this case there is a switch of perversities from [15]. It is caused by the fact that in [15] the intersection cohomology for a perversity p is the cohomology of the Deligne sheaf for such perversity or equivalently the cohomology of the complex of the intersection chain sheaves for the perversity p .

We call such kind of riemannian metric **quasi edge metric with weights**. To these we associate a **general perversity** p_g in the sense of G. Friedman:

$$p_g(Y) := Y \mapsto \left\lfloor \left\lfloor \frac{l_Y}{2} + \frac{1}{2c_Y} \right\rfloor \right\rfloor = \begin{cases} \frac{l_Y}{2} + \left\lfloor \left\lfloor \frac{1}{2c_Y} \right\rfloor \right\rfloor & l_Y \text{ even} \\ \frac{l_Y-1}{2} + \left\lfloor \left\lfloor \frac{1}{2} + \frac{1}{2c_Y} \right\rfloor \right\rfloor & l_Y \text{ odd} \end{cases}$$

where $l_Y = \dim L_Y$ and, given any real and positive number x , $\lfloor \lfloor x \rfloor \rfloor$ is the greatest integer strictly less than x .

The isomorphisms between the L^2 de Rham, the Hodge and the intersection cohomology that we get are then:

$$I^{q_g} H^i(X, \mathcal{R}_0) \cong H_{2, \max}^i(\text{reg}(X), g) \cong \mathcal{H}_{\text{abs}}^i(\text{reg}(X), g) \quad (1)$$

$$I^{p_g} H^i(X, \mathcal{R}_0) \cong H_{2, \min}^i(\text{reg}(X), g) \cong \mathcal{H}_{\text{rel}}^i(\text{reg}(X), g) \quad (2)$$

where q_g is the complementary perversity of p_g , that is $q_g = t - p_g$ with t the usual top perversity. \mathcal{R}_0 is the stratified coefficient system made of the pair of coefficient systems given by $(X - X_{n-1}) \times \mathbb{R}$ over $X - X_{n-1}$ where the fibers \mathbb{R} have the discrete topology and the constant 0 system on X_{n-1} . In particular, for all $i = 0, \dots, n$ the groups

$$H_{2, \max}^i(\text{reg}(X), g), H_{2, \min}^i(\text{reg}(X), g), \mathcal{H}_{\text{abs}}^i(\text{reg}(X)), \mathcal{H}_{\text{rel}}^i(\text{reg}(X))$$

are all finite dimensional. Note that in this paper we allow for the existence of one codimensional strata; furthermore p_g and q_g are not classical perversities in the sense of Goresky-MacPherson. This is why we have to replace the coefficient \mathbb{R} with \mathcal{R}_0 . It will be shown in corollary 9 that if p_g and q_g are classical perversities in the sense of Goresky-MacPherson and $X_{n-1} = X_{n-2}$ then it is possible to replace \mathcal{R}_0 with \mathbb{R} . It is immediate to note that when X is a manifold with edges with only one singular stratum this result reduces to the one proved by Hunsicker in [15].

The paper is structured in the following way: in the first part we recall notions which are fundamental to the whole work such as Hilbert complexes, intersection homology, intersection homology with general perversity, as defined by G. Friedman [9] and [10] and stratified pseudomanifold with a Thom-Mather stratification. We also introduce the riemannian metrics which we will use for the rest of the paper and the general perversities associated to them. The second part contains some results needed in order to calculate the maximal L^2 de Rham cohomology of a cone over a riemannian manifold endowed with a conic metric. The third part contains the calculation of the maximal L^2 de Rham cohomology of a cone over a riemannian manifold endowed with a conic metric. Finally the last part contains the result that we have announced above, its proof and some corollaries. For the proof of the main result in the last section we use a sheaf-theoretic point of view as is [15], [16] and [19]. More precisely to show the isomorphism (1) we will construct a complex of fine sheaves whose hypercohomology is the maximal/minimal L^2 de Rham cohomology and we will show that such complexes satisfy the generalization given by Friedman of the theorem of Goresky and MacPherson in [13]. Finally using some duality results we will get the isomorphisms (2).

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1 Background

1.1 Hilbert complexes

In this first subsection we recall the notion of Hilbert complex following [16].

Definition 1. A Hilbert complex is a complex, (H_*, D_*) of the form:

$$0 \rightarrow H_0 \xrightarrow{D_0} H_1 \xrightarrow{D_1} H_2 \xrightarrow{D_2} \dots \xrightarrow{D_{n-1}} H_n \rightarrow 0, \quad (3)$$

where each H_i is a separable Hilbert space and each map D_i is a closed operator called the differential such that:

1. $\mathcal{D}(D_i)$, the domain of D_i , is dense in H_i .
2. $\text{ran}(D_i) \subset \mathcal{D}(D_{i+1})$.
3. $D_{i+1} \circ D_i = 0$ for all i .

The cohomology groups of the complex are $H^i(H_*, D_*) := \text{Ker}(D_i)/\text{ran}(D_{i-1})$. If the groups $H^i(H_*, D_*)$ are all finite dimensional we say that it is a *Fredholm complex*.

Given a Hilbert complex there is a dual Hilbert complex

$$0 \leftarrow H_0 \xleftarrow{D_0^*} H_1 \xleftarrow{D_1^*} H_2 \xleftarrow{D_2^*} \dots \xleftarrow{D_{n-1}^*} H_n \leftarrow 0, \quad (4)$$

defined using $D_i^* : H_{i+1} \rightarrow H_i$, the Hilbert space adjoints of the differentials $D_i : H_i \rightarrow H_{i+1}$. The cohomology groups of $(H_j, (D_j)^*)$, the dual Hilbert complex, are

$$H^i(H_j, (D_j)^*) := \text{Ker}(D_{n-i-1}^*)/\text{ran}(D_{n-i}^*).$$

For all i there is also a laplacian $\Delta_i = D_i^* D_i + D_{i-1} D_{i-1}^*$ which is a self-adjoint operator on H_i with domain

$$\mathcal{D}(\Delta_i) = \{v \in \mathcal{D}(D_i) \cap \mathcal{D}(D_{i-1}^*) : D_i v \in \mathcal{D}(D_i^*), D_{i-1}^* v \in \mathcal{D}(D_{i-1})\}$$

and nullspace:

$$\mathcal{H}_i(H_*, D_*) := \text{ker}(\Delta_i) = \text{Ker}(D_i) \cap \text{Ker}(D_{i-1}^*).$$

The following propositions are standard results for these complexes. The first result is a weak Koidara decomposition:

Proposition 1 ([5], Lemma 2.1). *Let (H_i, D_i) be a Hilbert complex and $(H_i, (D_i)^*)$ its dual complex, then:*

$$H_i = \mathcal{H}_i \oplus \overline{\text{ran}(D_{i-1})} \oplus \overline{\text{ran}(D_i^*)}.$$

Proposition 2 ([5], corollary 2.5). *If the cohomology of a Hilbert complex (H_*, D_*) is finite dimensional then, for all i , $\text{ran}(D_{i-1})$ is closed and $H^i(H_*, D_*) \cong \mathcal{H}^i(H_*, D_*)$.*

Proposition 3 ([5], corollary 2.6). *A Hilbert complex (H_j, D_j) , $j = 0, \dots, n$ is a Fredholm complex if and only if its dual complex, (H_j, D_j^*) , is Fredholm. If this is the case then*

$$\mathcal{H}_i(H_j, D_j) \cong H_i(H_j, D_j) \cong H_{n-i}(H_j, (D_j)^*) \cong \mathcal{H}_{n-i}(H_j, (D_j)^*) \quad (5)$$

Proposition 4 ([5], Corollary 2.15). *Let (H'_*, D'_*) and (H''_*, D''_*) be two Hilbert complexes. Form the completed tensor product Hilbert complex (H_*, D_*) where:*

$$H_j = \bigoplus_{i+l=j} H'_i \hat{\otimes} H''_l,$$

$$D_j = \bigoplus_{i+l=j} (D'_i \otimes id_{H''_l} + (-1)^i id_{H'_i} \otimes D''_l).$$

Suppose that D''_* has closed range in all degree. Then

$$H^j(H_*, D_*) = \bigoplus_{i+l=j} H^i(H'_*, D'_*) \otimes H^l(H''_*, D''_*).$$

The final result that we recall shows that is possible to compute these cohomology groups using a core subcomplex

$$\mathcal{D}^\infty(H_i) \subset H_i.$$

For all i $\mathcal{D}^\infty(H_i)$ consists of all elements η that are in the domain of Δ_i^l for all $l \geq 0$.

Proposition 5 ([5], Theorem 2.12). *The complex $(\mathcal{D}^\infty(H_i), D_i)$ is a subcomplex quasi-isomorphic to the complex (H_i, D_i)*

The main case of interest here is when (M, g) is a (not necessarily complete) riemannian manifolds, $H_i = L^2\Omega^i(M, g)$, and D_i is the exterior derivative operator. Consider the de Rham complex $(C_0^\infty\Omega^*(M), d_*)$ where each form $\omega \in (C_0^\infty\Omega^i(M))$ is a i -form with compact support. To turn this complex into a Hilbert complex we must specify a closed extension of d . With the two following propositions we will recall the two canonical closed extensions of d

Definition 2. *The maximal extension d_{max} ; this is the operator acting on the domain:*

$$\mathcal{D}(d_{max,i}) = \{\omega \in L^2\Omega^i(M, g) : \exists \eta \in L^2\Omega^{i+1}(M, g)\} \quad (6)$$

$$s.t. \langle \omega, \delta_i \zeta \rangle_{L^2(M, g)} = \langle \eta, \zeta \rangle_{L^2(M, g)} \quad \forall \zeta \in C_0^\infty\Omega^{i+1}(M)$$

In this case $d_{max,i}\omega = \eta$. In other words $\mathcal{D}(d_{max,i})$ is the largest set of forms $\omega \in L^2\Omega^i(M, g)$ such that $d_i\omega$, computed distributionally, is also in $L^2\Omega^{i+1}(M, g)$.

Definition 3. *The minimal extension $d_{min,i}$; this is given by the graph closure of d_i on $C_0^\infty\Omega^i(M)$ respect to the norm of $L^2\Omega^i(M, g)$, that is,*

$$\mathcal{D}(d_{min,i}) = \{\omega \in L^2\Omega^i(M, g) : \exists \{\omega_j\}_{j \in J} \subset C_0^\infty\Omega^i(M, g), \omega_j \rightarrow \omega, d_i\omega_j \rightarrow \eta \in L^2\Omega^{i+1}(M, g)\} \quad (7)$$

and in this case $d_{min,i}\omega = \eta$

Obviously $\mathcal{D}(d_{min,i}) \subset \mathcal{D}(d_{max,i})$. Furthermore, from these definitions, it follows immediately that

$$d_{min,i}(\mathcal{D}(d_{min,i})) \subset \mathcal{D}(d_{min,i+1}), \quad d_{min,i+1} \circ d_{min,i} = 0$$

and that

$$d_{max,i}(\mathcal{D}(d_{max,i})) \subset \mathcal{D}(d_{max,i+1}), \quad d_{max,i+1} \circ d_{max,i} = 0.$$

Therefore $(L^2\Omega^*(M, g), d_{max/min,*})$ are both Hilbert complexes and their cohomology groups are denoted by $H_{2,max/min}^*(M, g)$.

Another straightforward but important fact is that the Hilbert complex adjoint of $(L^2\Omega^*(M, g), d_{max/min,*})$ is $(L^2\Omega^*(M, g), \delta_{min/max,*})$ with δ_* the formal adjoint of d_* , that is

$$(d_{max,i})^* = \delta_{min,i}, \quad (d_{min,i})^* = \delta_{max,i}. \quad (8)$$

Using proposition 1 we obtain three weak Koidara decompositions:

$$L^2\Omega^i(M, g) = \mathcal{H}_{abs/rel/max}^i \oplus \overline{\text{ran}(d_{max/min/min,i-1})} \oplus \overline{\text{ran}(\delta_{min/max/min,i})} \quad (9)$$

with summands mutually orthogonal in each case. The first summand in the right, called the absolute, relative or maximal Hodge cohomology, respectively, is defined as the orthogonal complement of the other two summands. Since $(\text{ran}(d_{max,i-1}))^\perp = \text{Ker}(\delta_{min,i-1})$ and $(\text{ran}(d_{min,i-1}))^\perp = \text{Ker}(\delta_{max,i-1})$, we see that

$$\mathcal{H}_{abs/rel/max}^i = \text{Ker}(d_{max/min/min,i}) \cap \text{Ker}(\delta_{min/max/max,i-1}). \quad (10)$$

Now consider the following operators:

$$\Delta_{abs,i} = \delta_{min,i}d_{max,i} + d_{max,i-1}\delta_{min,i-1}, \quad \Delta_{rel,i} = \delta_{max,i}d_{min,i} + d_{min,i-1}\delta_{max,i-1} \quad (11)$$

These are selfadjoint and satisfy:

$$\mathcal{H}_{abs}^i(M, g) = \text{Ker}(\Delta_{abs,i}), \quad \mathcal{H}_{rel}^i(M, g) = \text{Ker}(\Delta_{rel,i}). \quad (12)$$

Furthermore, by proposition 2, if $H_{2,max/min}^i(M, g)$ is finite dimensional then the range of $d_{max/min,i-1}$ is closed and $\mathcal{H}_{abs/rel}^i(M, g) \cong H_{2,max/min}^i(M, g)$.

Now we recall a result that is a particular case of proposition 5.

Proposition 6 ([5], pag 110, [7] appendix). *Consider the smooth differential forms $\Omega^*(M)$ and the following complex:*

$$(\Omega_2^*(M, g), d_*) := 0 \rightarrow \Omega_2^0(M, g) \xrightarrow{d_0} \Omega_2^1(M, g) \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} \Omega_2^n(M, g) \xrightarrow{d_n} 0 \quad (13)$$

where $\Omega_2^i(M, g) = \{\omega \in \Omega^i(M) : \|\omega\|_{L^2(M, g)} < \infty \text{ and } \|d_i\omega\|_{L^2(M, g)} < \infty\}$.
Then $(\Omega_2^*(M, g), d_*)$ is a subcomplex quasi-isomorphic to the complex $(L^2\Omega^*(M, g), d_{max,*})$

Finally we conclude the section by stating a result about the product of Hilbert complexes.

Proposition 7 ([5], pag 110). *Let $(M_1, g_1), (M_2, g_2)$ be two riemannian manifolds; then we have that the following Hilbert complexes are isomorphic:*

$$(L^2\Omega^*(M_1 \times M_2, g_1 \times g_2), d_{max/min,*}) \cong (L^2\Omega^*(M_1, g_1), d_{max/min,*}) \hat{\otimes} (L^2\Omega^*(M_2, g_2), d_{max/min,*}) \quad (14)$$

where the expression at the right is the completed tensor product of Hilbert complexes (see prop. 4).

1.2 Stratified pseudomanifolds and intersection homology

We begin by recalling the concept of stratified pseudomanifold. The definition is given by induction on the dimension.

Definition 4. *A 0-dimensional stratified space is a countable set with the discrete topology. For $m > 0$ a m -dimensional topologically stratified space is paracompact Hausdorff topological space X equipped with a filtration*

$$X = X_m \supset X_{m-1} \supset \dots \supset X_1 \supset X_0 \quad (15)$$

of X by closed subsets X_j such that if $x \in X_j - X_{j-1}$ there exists a neighbourhood N_x of x in X , a compact $(m - j - 1)$ -dimensional topologically stratified space L with a filtration

$$L = L_m \supset L_{m-1} \supset \dots \supset L_1 \supset L_0 \quad (16)$$

and a homeomorphism

$$\phi : N_x \rightarrow \mathbb{R}^j \times C(L) \quad (17)$$

where $C(L) = L \times [0, 1)/L \times \{0\}$ is the open cone on L , such that ϕ takes $N_x \cap X_{j+i+1}$ homeomorphically onto

$$\mathbb{R}^j \times C(L_i) \subset \mathbb{R}^j \times C(L) \quad (18)$$

for $m - j - 1 \geq i \geq 0$ and ϕ takes $N_x \cap X_j$ homeomorphically onto

$$\mathbb{R}^j \times \{\text{vertex of } C(L)\} \quad (19)$$

This definition guaranties that, for each j , the subset $X_j - X_{j-1}$ is a topological manifold of dimension j . The **strata** of X are the connected components of these manifolds. If a stratum Y is a subset of $X - X_{n-1}$ it is called a **regular stratum**; otherwise it is called a **singular stratum**. The space L is referred as to the **link** of the stratum. In general it is not uniquely determined up to homeomorphism, though if X is a stratified pseudomanifold it is unique up to stratum preserving homotopy equivalence (see[10] pag 108).

Definition 5. *A topological pseudomanifolds of dimension m is a para-compact Hausdorff topological space X wich posses a topological stratification such that*

$$X_{m-1} = X_{m-2} \quad (20)$$

and $X - X_{m-2}$ is dense in X . (For more details see [2] or [18]).

Over these spaces, at the end of the seventies, Mark Goresky and Robert MacPherson have defined a new homological theory known as intersection homology. Here we recall briefly the main definitions and we refer to [2], [3], [12], [13] and [18] for a complete developement of the theory.

Definition 6. A perversity is a function $p : \{2, 3, 4, \dots, n\} \rightarrow \mathbb{N}$ such that

$$p(2) = 0 \text{ and } p(i) \leq p(i + 1) \leq p(i) + 1. \quad (21)$$

Let $\Delta_i \subset \mathbb{R}^{i+1}$ the standard i -simplex. The j -**skeleton** are of Δ_i is the set of j -subsimplices. We say a singular i -simplex in X , i.e. a continuous map $\sigma : \Delta_i \rightarrow X$, is p -**allowable** if

$$\sigma^{-1}(X_{m-k} - X_{m-k-1}) \subset \{(i - k + p(k)) - \text{skeleton of } \Delta_i\} \text{ for all } k \geq 2. \quad (22)$$

The elements of the space $IP S_i(X)$ are the finite linear combinations of singular i -simplex $\sigma : \Delta_i \rightarrow X$ such that σ and $\partial\sigma$ are p -allowable. Clearly $(IP S_i(X), \partial_i)$ is a complex, more precisely a subcomplex of $(S_i(X), \partial_i)$, and the **perversity p singular intersection homology groups**, $IP H_i(X)$, are the homology of this complex.

Remark 1. The above definition is not the original definition given by Goresky and MacPherson in [12]. In fact in their paper Goresky and MacPherson use a simplicial point of view and in particular the notion of p -allowable simplicial chains. The definition that we have recalled here was given in [17] by H. King. Over a PL-stratified pseudomanifold it is equivalent to the Goresky and MacPherson's definition but the advantage is that it holds even if X is only a stratified pseudomanifold.

However, for our goals we need a more general notion of perversity and associated intersection homology. A generalization of the theory of Goreski and MacPherson that is suited for our needs was made by Greg Friedman. As in the previous case we recall only the main definitions and results and we refer to the [9], [10] and [11] for a complete development of the theory.

First, we remember that the theory proposed by Friedman applies to a **wider class of spaces**: from now on a **stratified pseudomanifold will be simply a paracompact Hausdorff topological space X which posses a topological stratification and such that $X - X_{n-1}$ is dense in X . That is, we do not require that the condition $X_{m-1} = X_{m-2}$ apply.** In the following propositions each stratified pseudomanifolds will have a **fixed stratification**. We start by introducing the notion of **general perversity**:

Definition 7. A general perversity on a stratified pseudomanifold X is any function

$$p : \{\text{Singular Strata of } X\} \rightarrow \mathbb{Z}. \quad (23)$$

The notion of p -**allowable** singular simplex is modified in the following way: a singular i -simplex in X , i.e. a continuous map $\sigma : \Delta_i \rightarrow X$, is p -**allowable** if

$$\sigma^{-1}(Y) \subset \{(i - \text{cod}(Y) + p(Y)) - \text{skeleton of } \Delta_i\} \text{ for any singular stratum } Y \text{ of } X. \quad (24)$$

A key ingredient in this new theory is the notion of **homology with stratified coefficient system**. (The definition uses the notion homology with local coefficient system; for the definition of local coefficient system see [14], [21], [8])

Definition 8. Let X stratified pseudomanifold and let \mathcal{G} a local system on $X - X_{n-1}$. Then the stratified coefficient sistem \mathcal{G}_0 is defined to consist of the pair of coefficient systems given by \mathcal{G} on $X - X_{n-1}$ and the costant 0 system on X_{n-1} i.e. we think of \mathcal{G}_0 as consisting of a locally costant fiber bundle $\mathcal{G}_{X-X_{n-1}}$ over $X - X_{n-1}$ with fiber G with the discrete topology together with the trivial bundle on X_{n-1} with the stalk 0.

Then a **coefficient n** of a singular simplex σ can be described by a lift of $\sigma|_{\sigma^{-1}(X-X_{n-1})}$ to \mathcal{G} over $X - X_{n-1}$ together with the trivial lift of $\sigma|_{\sigma^{-1}(X_{n-1})}$ to the 0 system on X_{n-1} . A coefficient of a simplex σ is considered to be the 0 coefficient if it maps each points of Δ to the 0 section of one of the coefficient systems. Note that if $\sigma^{-1}(X - X_{n-1})$ is path-connected then a coefficient lift of σ to \mathcal{G}_0 is completely determined by the lift at a single point of $\sigma^{-1}(X - X_{n-1})$ by the lifting extension property for \mathcal{G} . The intersection homology chain complex $(IP S_*(X, \mathcal{G}_0), \partial_*)$ are defined in the same way as $IP S_*(X, G)$, where G is any

field, but replacing the coefficient of simplices with coefficient in \mathcal{G}_0 . If $n\sigma$ is a simplex σ with its coefficient n , its boundary is given by the usual formula $\partial(n\sigma) = \sum_j (-1)^j (n \circ i_j)(\sigma \circ i_j)$ where $i_j : \Delta_{i-1} \rightarrow \Delta_i$ is the j -face inclusion map. Here $n \circ i_j$ should be interpreted as the restriction of n to the j th face of σ , restricting the lift to \mathcal{G} where possible and restricting to 0 otherwise. The basic idea behind the definition is that when we consider allowability of chains with respect to a perversity, simplices with support entirely in X_{n-1} should vanish and thus not be counted for allowability considerations. (For more details see [9], [10] and [11]).

The next proposition shows that the Friedman's theory is an extension of the classical theory made by Goresky and MacPherson.

Proposition 8. (see [10] pag. 110, [11] pag. 1985) *If p is a traditional perversity, that is a perversity like those defined in definition 6, and $X_{n-1} = X_{n-2}$ then*

$$I^p S_*(X, \mathcal{G}) = I^p S_*(X, \mathcal{G}_0).$$

Example 1. *Let X be a stratified pseudomanifold and p a general perversity on X . Consider as stratified coefficient system \mathcal{R}_0 , that is the pair of coefficient systems given by $(X - X_{n-1}) \times \mathbb{R}$ over $X - X_{n-1}$ where the fibers \mathbb{R} have the discrete topology and the constant 0 system on X_{n-1} . Now suppose that X and p satisfy the assumptions of proposition 8; then*

$$I^p S_*(X, \mathbb{R}) = I^p S_*(X, \mathcal{R}_0)$$

where $I^p S_*(X, \mathbb{R})$ is the usual intersection homology chain complex with coefficient in the field \mathbb{R} .

We conclude this section recalling some fundamental results of this theory that generalize the previous results obtained by Goresky and MacPherson.

Let X a stratified pseudomanifold, \mathfrak{X} a fixed stratification on X , p a generalized perversity on X , \mathcal{G} a local system on $X - X_{n-1}$ and \mathcal{O} the orientation sheaf on $X - X_{n-1}$.

Consider now the following **set of axioms** $(AX1)_{p, \mathfrak{X}, \mathcal{G} \otimes \mathcal{O}}$ for a complex of sheaves (\mathcal{S}^*, d_*) :

1. \mathcal{S}^* is bounded, $\mathcal{S}^i = 0$ for $i < 0$ and $\mathcal{S}^*|_{X - X_{n-1}}$ is quasi-isomorphic to $\mathcal{G} \otimes \mathcal{O}$.
2. If $x \in Z$ for a stratum Z , then $H_i(\mathcal{S}_x^*) = 0$ for $i > p(Z)$.
3. Let $U_k = X - X_{n-k}$ and let $i_k : U_k \rightarrow U_{k+1}$ the natural inclusion. Then for $x \in Z \subset U_{k+1}$ the attachment map $\alpha_k : \mathcal{S}^*|_{U_{k+1}} \rightarrow Ri_{k*} i_k^* \mathcal{S}^*|_{U_{k+1}}$, given by the composition of natural morphism $\mathcal{S}^*|_{U_{k+1}} \rightarrow i_{k*} i_k^* \mathcal{S}^*|_{U_{k+1}} \rightarrow Ri_{k*} i_k^* \mathcal{S}^*|_{U_{k+1}}$, is a quasi-isomorphism at x up to $p(Z)$.

In almost all references the previous axioms are formulated in the derived category of sheaves on X . In that case the term quasi-isomorphism should be replaced with the term isomorphism.

Theorem 1. (see [9] pag 116) *Let X a compact stratified pseudomanifold of dimension n , p a general perversity on X and (\mathcal{S}^*, d_*) a complex of sheaves that satisfies the set of axioms $(AX1)_{p, \mathfrak{X}, \mathcal{G} \otimes \mathcal{O}}$. Then the following isomorphism holds:*

$$\mathbb{H}^i(X, \mathcal{S}^*) \cong I^p H_{n-i}(X, \mathcal{G}_0) \tag{25}$$

that is the i -th hypercohomology group of the complex (\mathcal{S}^*, d_*) is isomorphic to the $(n-i)$ -th intersection homology group with coefficient in the stratified system \mathcal{G}_0 and relative to the perversity p .

Corollary 1. *In the same hypothesis of the previous theorem if (\mathcal{S}^*, d_*) is a complex of fine or flabby or soft sheaves then the following isomorphism holds:*

$$H^i(\mathcal{S}^*(X), d_*) \cong I^p H_{n-i}(X, \mathcal{G}_0) \tag{26}$$

where $H^i(\mathcal{S}^*(X), d_*)$ are the cohomology groups of the complex

$$0 \dots \xrightarrow{d_{i-1}} \mathcal{S}^i(X) \xrightarrow{d_i} \mathcal{S}^{i+1}(X) \xrightarrow{d_{i+1}} \mathcal{S}^{i+2}(X) \xrightarrow{d_{i+2}} \dots$$

Theorem 2. (see [9] pag 122 or [10] pag 25.) Let F a field, X a compact and F -oriented stratified pseudomanifold of dimension n , p, q general perversities on X such that $p + q = t$ (that is for each stratum $Z \subset X$ $p(Z) + q(Z) = \text{codim}(Z) - 2$) and \mathcal{F}_0 a stratified coefficient system over X , consisting of the pair of coefficient systems given by $(X - X_{n-1}) \times F$ over $X - X_{n-1}$ where the fibers F have the discrete topology and the constant 0 system on X_{n-1} . Then the following isomorphism holds:

$$I^p H_i(X, \mathcal{F}_0) \cong \text{Hom}(I^q H_{n-i}(X, \mathcal{F}_0), F). \quad (27)$$

Remark 2. In this paper with the symbol $I^p H^i(X, \mathcal{G}_0)$ we mean the cohomology of the complex

$$(\text{Hom}(I^p \mathcal{S}_i(X), G), (\partial_i)^*).$$

We call it the i -th intersection cohomology group of X with respect to the perversity p and the stratified coefficient system \mathcal{G}_0 . When $G = F$ is a field then

$$I^p H^i(X, \mathcal{F}_0) \cong \text{Hom}(I^p H_i(X, \mathcal{F}_0), F).$$

Remark 3. Summarizing, by theorems 1 and 2, it follows that if (\mathcal{S}^*, d_*) is a complex of sheaves that satisfies the set of axioms $(AX1)_{p,x,\mathcal{F} \otimes \mathcal{O}}$ then

$$\mathbb{H}^i(X, \mathcal{S}^*) \cong I^q H^i(X, \mathcal{F}_0) \quad (28)$$

where $p + q = t$ and if (\mathcal{S}^*, d_*) is a complex of fine or flabby or soft sheaves then, by corollary 1,

$$H^i(\mathcal{S}^*(X), d_*) \cong I^q H^i(X, \mathcal{F}_0) \quad (29)$$

1.3 Thom-Mather stratification and quasi edge metrics with weights

We start this subsection by giving the definition of a smoothly stratified pseudomanifold with a Thom-Mather stratification. We follow [1].

Definition 9. A smoothly stratified pseudomanifold X with a Thom-Mather stratification is a metrizable, locally compact, second countable space which admits a locally finite decomposition into a union of locally closed strata $\mathfrak{S} = \{Y_\alpha\}$, where each Y_α is a smooth, open and connected manifold, with dimension depending on the index α . We assume the following:

1. If $Y_\alpha, Y_\beta \in \mathfrak{S}$ and $Y_\alpha \cap \overline{Y_\beta} \neq \emptyset$ then $Y_\alpha \subset \overline{Y_\beta}$
2. Each stratum Y is endowed with a set of control data T_Y, π_Y and ρ_Y ; here T_Y is a neighbourhood of Y in X which retracts onto Y , $\pi_Y : T_Y \rightarrow Y$ is a fixed continuous retraction and $\rho_Y : T_Y \rightarrow [0, 2)$ is a proper radial function in this tubular neighbourhood such that $\rho_Y^{-1}(0) = Y$. Furthermore, we require that if $Z \in \mathfrak{S}$ and $Z \cap T_Y \neq \emptyset$ then $(\pi_Y, \rho_Y) : T_Y \cap Z \rightarrow Y \times [0, 2)$ is a proper differentiable submersion.
3. If $W, Y, Z \in \mathfrak{S}$, and if $p \in T_Y \cap T_Z \cap W$ and $\pi_Z(p) \in T_Y \cap Z$ then $\pi_Y(\pi_Z(p)) = \pi_Y(p)$ and $\rho_Y(\pi_Z(p)) = \rho_Y(p)$.
4. If $Y, Z \in \mathfrak{S}$, then $Y \cap \overline{Z} \neq \emptyset \Leftrightarrow T_Y \cap Z \neq \emptyset$, $T_Y \cap T_Z \neq \emptyset \Leftrightarrow Y \subset \overline{Z}, Y = Z$ or $Z \subset \overline{Y}$.
5. For each $Y \in \mathfrak{S}$, the restriction $\pi_Y : T_Y \rightarrow Y$ is a locally trivial fibration with fibre the cone $C(L_Y)$ over some other stratified space L_Y (called the link over Y), with atlas $\mathcal{U}_Y = \{(\phi, \mathcal{U})\}$ where each ϕ is a trivialization $\pi_Y^{-1}(U) \rightarrow U \times C(L_Y)$, and the transition functions are stratified isomorphisms which preserve the rays of each conic fibre as well as the radial variable ρ_Y itself, hence are suspensions of isomorphisms of each link L_Y which vary smoothly with the variable $y \in U$.

6. For each j let X_j be the union of all strata of dimension less or equal than j , then

$$X - X_{n-1} \text{ is dense in } X$$

We make a few comments to the previous definition (for more details we refer to [1]):

1. The previous definition is more general than that given in [1]. In [1] a space that satisfies the definition 9 is only a smoothly stratified spaces (with a Thom-Mather stratification). To be a smoothly stratified pseudomanifold (with a Thom-Mather stratification) there is another requirement to satisfy: let X_j be the union of all strata of dimensions less or equal than j , then

$$X = X_n \supset X_{n-1} = X_{n-2} \supset X_{n-3} \supset \dots \supset X_0 \quad (30)$$

and $X - X_{n-2}$ is dense in X . For our goals, thanks to the results of Friedman, we can waive the requirement $X_{n-1} = X_{n-2}$ and therefore we will call smoothly stratified pseudomanifold with a Thom-Mather stratification each space X that satisfies the definition 9.

2. The link L_Y is uniquely determined, up to isomorphism (see point number 5 below for the notion of isomorphism), by the stratum Y .
3. The depth of a stratum Y is largest integer k such that there is a chain of strata $Y = Y_k, \dots, Y_0$ such that $Y_j \subset \overline{Y_{j-1}}$ for $i \leq j \leq k$. A stratum of maximal depth is always a closed subset of X . The maximal depth of any stratum in X is called the depth of X as stratified spaces.
4. Consider the filtration

$$X = X_n \supset X_{n-1} \supset X_{n-2} \supset X_{n-3} \supset \dots \supset X_0 \quad (31)$$

We refer to the open subset $X - X_{n-1}$ of a stratified pseudomanifold X as its regular set, and the union of all other strata as the singular set,

$$\text{reg}(X) := X - \text{sing}(X) \text{ where } \text{sing}(X) := \bigcup_{Y \in \mathfrak{S}, \text{depth} Y > 0} Y.$$

5. If X, X' are two stratified spaces a stratified isomorphism between them is a homeomorphism $F : X \rightarrow X'$ wich carries the strata of X to the strata of X' diffeomorphically, and such that $\pi'_{F(Y)} \circ F = F \circ \pi_Y$, $\rho_Y = \rho'_{(F(Y))} \circ F$ for all $Y \in \mathcal{G}(X)$.

Summarizing a smoothly stratified pseudomanifold with Thom-Mather stratification is a stratified pseudomanifold wich richer structure from a differentiable and topological point of view. Now we introduce an important class of riemannian metrics on the regular part of

a smoothly stratified pseudomanifold with a Thom-Mather stratification. Before giving the definition we recall that two riemannian metrics g, h on a smooth manifold M are **quasi-isometric** if there are constants c_1, c_2 such that $c_1 h \leq g \leq c_2 h$.

Definition 10. Let X be a smoothly stratified pseudomanifold with a Thom-Mather stratification and let g a riemannian metric on $\text{reg}(X)$. We call g a **quasi edge metric with weights** if it satisfies the following properties:

1. Take any stratum Y of X ; by definition 9 for each $q \in Y$ there exist an open neighbourhood U of q in Y such that $\phi : \pi_Y^{-1}(U) \rightarrow U \times C(L_Y)$ is a stratified isomorphism; in particular $\phi : \pi_Y^{-1}(U) \cap \text{reg}(X) \rightarrow U \times \text{reg}(C(L_Y))$ is a diffeomorphism. Then, for each $q \in Y$, there exists one of these trivializations (ϕ, U) such that g restricted on $\pi_Y^{-1}(U) \cap \text{reg}(X)$ satisfies the following properties:

$$(\phi^{-1})^*(g|_{\pi_Y^{-1}(U) \cap \text{reg}(X)}) \cong dr \otimes dr + h_U + r^{2c_Y} g_{L_Y} \quad (32)$$

where h_U is a riemannian metric defined over U , $c_Y \in \mathbb{R}$ and $c_Y > 0$, g_{L_Y} is a riemannian metric on $\text{reg}(L_Y)$, $dr \otimes dr + h_Y + r^{2c_Y} g_{L_Y}$ is a riemannian metric of product type on $U \times \text{reg}(C(L_Y))$ and with \cong we mean **quasi-isometric**.

2. The weight c_Y depends only on the stratum Y where the point q lie.

Before continuing we make some **remarks**:

1. In the first point of the previous definition the metric g_{L_Y} depends also on the open neighborhood U and the stratified isomorphism ϕ . However we prefer to use the notation g_{L_Y} instead of $g_{L_Y,U,\phi}$ for the sake of simplicity.
2. Let g and U be like in the first point of the previous definition and let $\psi : \pi_Y^{-1}(U) \rightarrow U \times C(L_Y)$ another stratified isomorphism that satisfies the requirements of definition 9. From the fifth point of definition 9 it follows that $\psi \circ \phi^{-1} : U \times C(L_Y) \rightarrow U \times C(L_Y)$ acts in this way: given $p = (y, [r, x]) \in U \times C(L_Y)$ $(\psi \circ \phi^{-1})(p) = (y, [r, f(y, x)])$ where the maps $x \mapsto f(y, x)$ are a family of smooth stratified isomorphisms of L_Y which vary smoothly with the variable $y \in U$. From this follows immediatly that if we fix a point $y_0 \in U$ and if we put $h_{L_Y} = (f(y_0, x)^{-1})^*(g_{L_Y})$ then there exists an open subset $V \subset U, y_0 \in V$ such that $(\psi^{-1})^*(g|_{\pi_Y^{-1}(V) \cap \text{reg}(X)}) \cong dr \otimes dr + h_U|_V + r^{2c_Y} h_{L_Y}$ where $h_U|_V$ is the metric h_U restricted to V .

Now we give a definition wich is a more refined version of the previous one. It is given by induction on $\text{depth}(X)$.

Definition 11. Let X be a stratified pseudomanifold with a Thom-Mather stratification and let g a riemannian metric on $\text{reg}(X)$. If $\text{depth}(X) = 0$, that is X is a closed manifold, a **quasi rigid iterated edge metric with weights** is any riemannian metric on X . Suppose now that $\text{depth}(X) = k$ and that the definition of quasi rigid iterated edge metric with weights is given in the case $\text{depth}(X) \leq k - 1$; then we call a riemannian metric g on $\text{reg}(X)$ a **quasi rigid iterated edge metric with weights** if it satisfies the following properties:

1. Take any stratum Y of X ; by definition 9 for each $q \in Y$ there exist an open neighbourhood U of q in Y such that $\phi : \pi_Y^{-1}(U) \rightarrow U \times C(L_Y)$ is a stratified isomorphism; in particular $\phi : \pi_Y^{-1}(U) \cap \text{reg}(X) \rightarrow U \times \text{reg}(C(L_Y))$ is a diffeomorphism. Then, for each $q \in Y$, there exists one of these trivializations (ϕ, U) such that g restricted on $\pi_Y^{-1}(U) \cap \text{reg}(X)$ satisfies the following properties:

$$(\phi^{-1})^*(g|_{\pi_Y^{-1}(U) \cap \text{reg}(X)}) \cong dr \otimes dr + h_U + r^{2c_Y} g_{L_Y} \quad (33)$$

where h_U is a riemannian metric defined over U , $c_Y \in \mathbb{R}$ and $c_Y > 0$, g_{L_Y} is a **quasi rigid iterated edge metric with weights** on $\text{reg}(L_Y)$, $dr \otimes dr + h_U + r^{2c_Y} g_{L_Y}$ is a riemannian metric of product type on $U \times \text{reg}(C(L_Y))$ and with \cong we mean **quasi-isometric**.

2. The weight c_Y depends only on the stratum Y where the point q lie.

Also in this case a remark to the previous definition is in order. Let $\psi : \pi_Y^{-1}(U) \rightarrow U \times C(L_Y)$ another stratified isomorphism that satisfies the requirements of definition 9. Using the same observations and notations of the second remark of definition 10 we can conclude that there exists an open subset $V \subset U$ and a quasi rigid iterated edge metric with weights h_{L_Y} on $\text{reg}(L_Y)$ such that $(\psi^{-1})^*(g|_{\pi_Y^{-1}(V) \cap \text{reg}(X)}) \cong dr \otimes dr + h_U|_V + r^{2c_Y} h_{L_Y}$. Furthermore, by the fact that $f(y_0, x)$ is a smooth stratified isomorphism between L_Y and L_Y such that $(f(y_0, x))^*(h_{L_Y}) = g_{L_Y}$, follows that g_{L_Y} and h_{L_Y} have the same weights and therefore, by proposition 10, g_{L_Y} and h_{L_Y} are quasi-isometric on $\text{reg}(L_Y)$.

Proposition 9. Let X be a smoothly stratified pseudomanifold with a Thom-Mather stratification. For any stratum $Y \subset X$ fix a positive real number c_Y . Then there exist a quasi rigid iterated edge metric with weights g on $\text{reg}(X)$ having the numbers $\{c_Y\}_{Y \in \mathfrak{X}}$ like weights.

Proof. The proof is given by induction on $\text{depth}(X)$. If $\text{depth}(X) = 0$ there is nothing to show. Now suppose that the proposition is true for $\text{depth}(X) = n - 1$ and that $\text{depth}(X) = n$. For any singular stratum $Y \subset X$ and for any point $p \in Y$ consider an open neighbourhood $U_{Y,p}$ of p in U such that $\pi_Y^{-1}(U_{Y,p}) \cong U_{Y,p} \times C(L_Y)$. We call this stratified isomorphism $\phi_{U_{Y,p}}$. In this way

we get a family of open subset of X , $\mathcal{A} = \{\pi_Y^{-1}(U_{Y,p})\}$, that is an open cover of $\text{sing}(X)$. From \mathcal{A} we can get another cover of $\text{sing}(X)$, that we call \mathcal{B} , with the following property: also \mathcal{B} is made of open subsets $\pi_Y^{-1}(U_{Y,p}) \cong U_{Y,p} \times C(L_Y)$; for each singular stratum Y and for any point $p \in Y$ there is an open neighbourhood $V_{Y,p}$ of p in Y such that $\pi_Y^{-1}(V_{Y,p}) \cap \pi_Y^{-1}(U_{Y,p}) \neq \emptyset$ only for a finite number of open subsets $\pi_Y^{-1}(U_{Y,p}) \in \mathcal{B}$. Now for any singular stratum Y fix a riemannian metric h_Y . By induction there is a rigid iterated edge metric g_{L_Y} on the link L_Y for any singular stratum Y . On $\pi_Y^{-1}(U_{Y,p}) \cong U_{Y,p} \times C(L_Y)$ consider the following metric $h_Y + dr^2 + r^{2c_Y} g_{L_Y}$ and let $g_{U_{Y,p}}$ be a riemannian metric on $\text{reg}(\pi_Y^{-1}(U_{Y,p}))$ defined as $(\phi_{U_{Y,p}})^*(h_Y + dr^2 + r^{2c_Y} g_{L_Y})$. Finally consider a partition of unity $\{\psi_{U_{Y,p}}\}$ subordinate to the open cover $\{\text{reg}(\pi_Y^{-1}(U_{Y,p}))\}$ which is an open submanifold of $\text{reg}(X)$ and where each $\pi_Y^{-1}(U_{Y,p}) \in \mathcal{B}$. Define

$$g := \sum \psi_{U_{Y,p}} g_{U_{Y,p}}.$$

Now we want to show that g satisfies definition 11. Let $p \in Y$ a singular stratum of X . Then there exists an open neighbourhood $V_{Y,p}$ such that $\pi_Y^{-1}(V_{Y,p}) \cong V_{Y,p} \times C(L_Y)$, such that $\psi_{V_{Y,p}}(q) \geq \epsilon > 0$ for each $q \in \pi_Y^{-1}(V_{Y,p})$ and for some $\psi_{V_{Y,p}}$ and such that $\pi_Y^{-1}(V_{Y,p}) \cap \pi_Y^{-1}(U_{Y,p}) \neq \emptyset$ only for a finite number of open subsets $\pi_Y^{-1}(U_{Y,p}) \in \mathcal{B}$. From this follows that

$$(\phi_{V_{Y,p}}^{-1})^*(g|_{\pi_Y^{-1}(V_{Y,p})}) = (\phi_{V_{Y,p}}^{-1})^*(\psi_{V_{Y,p}})(\phi_{V_{Y,p}}^{-1})^*(g_{V_{Y,p}}) + (\phi_{V_{Y,p}}^{-1})^*\left(\sum_{U_{Y,p} \neq V_{Y,p}} \psi_{U_{Y,p}} g_{U_{Y,p}}\right) \quad (34)$$

where the sum has a finite number of terms. Now let $h_Y + dr^2 + r^{2c_Y} g_{L_Y} = (\phi_{V_{Y,p}}^{-1})^*(g_{V_{Y,p}})$. Obviously there exists $c_1 > 0$ such that $c_1(h_Y + dr^2 + r^{2c_Y} g_{L_Y}) \leq (\phi_{V_{Y,p}}^{-1})^*(g|_{\pi_Y^{-1}(V_{Y,p})})$ on $V_{Y,p} \times C(L_Y)$. From the remark following definition 11 we also know that $g_{V_{Y,p}}$ and $g_{U_{Y,p}}$ are quasi-isometric on $\pi_Y^{-1}(V_{Y,p}) \cap \pi_Y^{-1}(U_{Y,p})$ and therefore they are quasi-isometric on $\{q \in \pi_Y^{-1}(V_{Y,p}) : \psi_{U_{Y,p}}(q) \neq 0\}$ for each $g_{U_{Y,p}}$ in (34). So, by the fact that the sum in (34) is made of a finite number of terms, we can conclude that there exist $c_2 > 0$ such that $(\phi_{V_{Y,p}}^{-1})^*(g|_{\pi_Y^{-1}(V_{Y,p})}) \leq c_2(h_Y + dr^2 + r^{2c_Y} g_{L_Y})$ on $V_{Y,p} \times C(L_Y)$. Therefore g satisfies definition 11. The problem is that g is defined only on $\bigcup \text{reg}(T_Y)$. Therefore we define $\tilde{T}_Y = \{p \in T_Y : \rho(p) < 1\}$, $\tilde{\pi}_Y := \pi_Y|_{\tilde{T}_Y}$ and $\tilde{\rho}_Y := \rho_Y|_{\tilde{T}_Y}$. Let $f_1 : \text{reg}(X) \rightarrow [0, 1]$ a smooth functions such that $f_1(p) = 1$ for each $p \in \bigcup \text{reg}(\tilde{T}_Y)$, $f_1(p) = 0$ for each $p \in X - \bigcup \text{reg}(T_Y)$ and let $f_2 = 1 - f_1$. Let k a riemannian metric on $\text{reg}(X)$. Define $g' := f_1 g + f_2 k$. Now this new metric satisfies definition 11 but respect to the new control data $\{\tilde{T}_Y, \tilde{\pi}_Y, \tilde{\rho}_Y\}$. Actually this would be sufficient to prove the main theorem of the paper. Nevertheless to complete the proof consider $\zeta : X \rightarrow X$ a stratified isomorphism of smoothly stratified spaces such that $\zeta^{-1}(\tilde{T}_Y) = T_Y$, $\zeta^*(\tilde{\pi}_Y) = \pi_Y$ and $\zeta^*(\tilde{\rho}_Y) = \rho_Y$. (This isomorphism exist: see for example [1]). Finally the metric $\zeta^*(g')$ satisfies definition 11. \square

Proposition 10. *Let X be a compact smoothly stratified pseudomanifold with a Thom-Mather stratification. For any stratum $Y \subset X$ fix a positive real number c_Y . Let g, g' two quasi edge metrics with weights on $\text{reg}(X)$ having both the numbers $\{c_Y\}_{Y \in \mathfrak{X}}$ like weights. Then g and g' are quasi-isometric.*

Proof. Let K a compact subset of X such that $K \subset \text{reg}(X)$. Obviously $g|_K$ is quasi-isometric to $g'|_K$. Now let Y be a stratum such that $Y \subset X_{n-1} - X_{n-2}$. Let $x \in Y$; consider $\pi_Y^{-1}(x)$ and let $V_{Y,x} := \pi_Y^{-1}(x) \cap \rho_Y^{-1}(1)$. Then there exist a compact subset of X , K such that $K \subset \text{reg}(X)$ and $\text{reg}(V_{Y,x}) \subset K$. Therefore $g|_{\text{reg}(V_{Y,x})}$ is quasi-isometric to $g'|_{\text{reg}(V_{Y,x})}$ and from this follows that, given an open neighbourhood U of x in Y sufficiently small such that $\pi_Y^{-1}(U) \cong U \times C(L_Y)$, $g|_{\text{reg}(\pi_Y^{-1}(U))}$ is quasi-isometric to $g'|_{\text{reg}(\pi_Y^{-1}(U))}$. This last assertion is a consequence of the fact that, by definition 10 and the relative remarks, there is an isomorphism $\phi : \pi_Y^{-1}(U) \rightarrow U \times C(L_Y)$ such that, by definition 10, $(\phi^{-1})^*(g|_{\text{reg}(\pi_Y^{-1}(U))})$ is quasi isometric to $h + dr^2 + r^{2c_Y} g_{L_Y}$ and analogously $(\phi^{-1})^*(g'|_{\text{reg}(\pi_Y^{-1}(U))})$ is quasi isometric to $h' + dr^2 + r^{2c_Y} g'_{L_Y}$. But from the fact that $g|_{\text{reg}(V_{Y,x})}$ is quasi-isometric to $g'|_{\text{reg}(V_{Y,x})}$ follows that g_{L_Y} is quasi-isometric to g'_{L_Y} and therefore for a sufficiently small U we get $g|_{\text{reg}(\pi_Y^{-1}(U))}$ is quasi-isometric to $g'|_{\text{reg}(\pi_Y^{-1}(U))}$. So we can conclude that if $K \subset (X - X_{n-2})$ is a compact subset then $g|_{\text{reg}(K)}$ is quasi-isometric to $g'|_{\text{reg}(K)}$. Now consider a stratum $Z \subset X_{n-2} - X_{n-3}$ and let $x \in Z$. As before consider $\pi_Z^{-1}(x)$ and let $V_{Z,x} = \pi_Z^{-1}(x) \cap \rho_Z^{-1}(1)$. Then there exist a compact subset

$K \subset (X - X_{n-2})$ such that $V_{Z,x} \subset K$. From this follows that $g|_{\text{reg}(V_{Z,x})}$ is quasi-isometric to $g'|_{\text{reg}(V_{Z,x})}$ and now, as before, we can conclude that given an open neighbourhood U of x in Z sufficiently small such that $g|_{\pi_Z^{-1}(U)} \cong U \times C(L_Z)$, $g|_{\text{reg}(\pi_Z^{-1}(U))}$ is quasi-isometric to $g'|_{\text{reg}(\pi_Z^{-1}(U))}$. As before from this follows that if $K \subset (X - X_{n-3})$ is a compact subset then $g|_{\text{reg}(K)}$ is quasi-isometric to $g'|_{\text{reg}(K)}$. Now it is obvious that iterating this procedure we obtain what was asserted. \square

Corollary 2. *Let X be a compact smoothly stratified pseudomanifold with a Thom-Mather stratification and let g a quasi edge metric with weights on $\text{reg}(X)$. Then there exist g' , a **quasi rigid iterated edge metric with weights** on $\text{reg}(X)$, that is quasi-isometric to g .*

We conclude this section introducing the notion of general perversity associated to a quasi edge metric with weights.

Definition 12. *Let X be a smoothly stratified pseudomanifold with a Thom-Mather stratification and let g a quasi edge metric with weights on $\text{reg}(X)$. Then the general perversity p_g associated to g is:*

$$p_g(Y) := Y \mapsto \left\lfloor \left\lfloor \frac{l_Y}{2} + \frac{1}{2c_Y} \right\rfloor \right\rfloor = \begin{cases} \frac{l_Y}{2} + \left\lfloor \left\lfloor \frac{1}{2c_Y} \right\rfloor \right\rfloor & l_Y \text{ even} \\ \frac{l_Y-1}{2} + \left\lfloor \left\lfloor \frac{1}{2} + \frac{1}{2c_Y} \right\rfloor \right\rfloor & l_Y \text{ odd} \end{cases} \quad (35)$$

where $l_Y = \dim L_Y$ and, given any real and positive number x , $\lfloor \lfloor x \rfloor \rfloor$ is the greatest integer strictly less than x .

2 Preliminary propositions

In this section we follow, with some modifications, [7]. Given an oriented riemannian manifold (F, g) of dimension f , $C^*(F)$ will be the regular part of $C(F)$, that is $C(F) - \{v\}$, and g_c will be the riemannian metric on $C^*(F)$ $g_c = dr \otimes dr + r^{2c} \pi^* g$ where $\pi : C^*(F) \rightarrow F$ is the projection over F and $c \in \mathbb{R}$, $c > 0$. With the symbol $d_F : \Omega^i(C^*(F)) \rightarrow \Omega^{i+1}(C^*(F))$ we mean the exterior differential obtained by ignoring the variable r .

Proposition 11. *Let $\phi \in L^2 \Omega^i(F, g)$, $\phi \neq 0$ and let $\pi : C^*(F) \rightarrow F$ be the projection. Then $\pi^*(\phi) \in L^2 \Omega^i(C^*(F), g_c)$ if and only if $i < \frac{f}{2} + \frac{1}{2c}$. In this case the pullback map is also bounded.*

Proof. If $\phi \in L^2 \Omega^i(F, g)$ then

$$\begin{aligned} \|\pi^*(\phi)\|_{L^2(C^*(F), g_c)}^2 &= \int_{C^*(F)} \|\pi^*(\phi)\|_{C^*(F)}^2 d\text{vol}_{C^*(F)} = \int_0^1 \int_F r^{c(f-2i)} \|\phi\|_F^2 d\text{vol}_F dr \\ &= \|\phi\|_{L^2(F, g)}^2 \int_0^1 r^{c(f-2i)} dr < \infty \end{aligned}$$

if and only if $i < \frac{f}{2} + \frac{1}{2c}$. Since $\int_0^1 r^{c(f-2i)} dr$ is independent of ϕ , the pullback map is bounded. \square

Proposition 12. *There exists a constant $K > 0$ such that for all $\alpha = \phi + dr \wedge \omega \in L^2 \Omega^i(C^*(F), g_c)$ and for any null set $S \subset (1/2, 1)$ there is an $a \in (1/2, 1) - S$ such that*

$$\|\phi(a)\|_{L^2(F, g)}^2 \leq K \|\alpha\|_{L^2(C^*(F), g_c)}^2 \leq K \|\alpha\|_{L^2(C^*(F), g_c)}^2.$$

Proof. Suppose that this proposition is false. Then for any $K > 0$ there is a form $\phi \in L^2 \Omega^i(C^*(F), g_c)$ such that

$$\begin{aligned} \|\phi\|_{L^2(C^*(F), g_c)}^2 &\geq \int_{1/2}^1 \int_F r^{c(f-2i)} \|\phi\|_F^2 d\text{vol}_F dr = \int_{1/2}^1 r^{c(f-2i)} \|\phi(r)\|_{L^2(F, g)}^2 dr > \\ &K \|\phi\|_{L^2(C^*(F), g_c)}^2 \int_{(1/2, 1) - S} r^{c(f-2i)} dr = K \|\phi\|_{L^2(C^*(F), g_c)}^2 \int_{(1/2, 1)} r^{c(f-2i)} dr. \end{aligned}$$

In this way by choosing $K > (\int_{(1/2, 1)} r^{c(f-2i)} dr)^{-1}$ we obtain a contradiction. \square

Proposition 13. *If $i < \frac{f}{2} + \frac{1}{2c} + 1$ and $\alpha = \phi + dr \wedge \omega \in L^2\Omega^i(C^*(F), g_c)$, then for any $a \in (1/2, 1)$*

$$K_a(\alpha) = \int_a^r \omega(s) ds \in L^2\Omega^{i-1}(C^*(F), g_c)$$

and K_a is a bounded operator uniformly in $a \in (1/2, 1)$.

Proof. By definition

$$\|K_a(\alpha)\|_{L^2(C^*(F), g_c)}^2 = \left\| \int_a^r \omega(s) ds \right\|_{L^2(C^*(F), g_c)}^2 = \int_0^1 \int_F \left\| \int_a^r \omega(s) ds \right\|_F^2 r^{c(f-2i+2)} d\text{vol}_F dr.$$

We consider the term $\left\| \int_a^r \omega(s) ds \right\|_F^2$. The following inequality holds :

$$\left\| \int_a^r \omega(s) ds \right\|_F^2 \leq \left(\int_a^r \|\omega(s)\|_F ds \right)^2$$

and using the Schwartz inequalities the right side of this becomes:

$$\begin{aligned} \left(\int_a^r \|\omega(s)\|_F ds \right)^2 &\leq \int_a^r ds \int_a^r \|\omega(s)\|_F^2 ds \\ &\leq \int_a^1 ds \int_a^r \|\omega(s)\|_F^2 ds = (1-a) \int_a^r \|\omega(s)\|_F^2 ds \leq (1-a) \int_a^1 \|\omega(s)\|_F^2 ds. \end{aligned}$$

So we have obtained that

$$\|K_a(\alpha)\|_{L^2(C^*(F), g_c)}^2 \leq (1-a) \int_0^1 \int_F \int_a^1 \|\omega(s)\|_F^2 ds r^{c(f-2i+2)} d\text{vol}_F dr.$$

Now consider the term $\int_0^1 \int_F \int_a^1 \|\omega(s)\|_F^2 ds r^{c(f-2i+2)} d\text{vol}_F dr$

$$= \int_0^1 \int_F \int_a^1 \|\omega(s)\|_F^2 (s^{c(f-2i+2)} + 1 - s^{c(f-2i+2)}) ds r^{c(f-2i+2)} d\text{vol}_F dr.$$

We can bound the term $\int_a^1 \|\omega(s)\|_F^2 s^{c(f-2i+2)} ds$ in the following way

$$\int_a^1 \|\omega(s)\|_F^2 s^{c(f-2i+2)} ds \leq \int_0^1 \|\omega(s)\|_F^2 s^{c(f-2i+2)} ds$$

and therefore

$$\int_F \int_a^1 \|\omega(s)\|_F^2 s^{c(f-2i+2)} ds d\text{vol}_F \leq \int_F \int_0^1 \|\omega(s)\|_F^2 s^{c(f-2i+2)} ds d\text{vol}_F = \|\omega\|_{L^2(C^*(F), g_c)}^2$$

while for the term $\int_a^1 \|\omega(s)\|_F^2 (1 - s^{c(f-2i+2)}) ds$ we can use the following observation: there exist $l > 0$ such that $1 - s^{c(f-2i+2)} \leq |1 - s^{c(f-2i+2)}| \leq l s^{c(f-2i+2)}$ for any $s \in (\frac{1}{2}, 1]$. Therefore:

$$\begin{aligned} \int_a^1 \|\omega(s)\|_F^2 (1 - s^{c(f-2i+2)}) ds &\leq \int_a^1 \|\omega(s)\|_F^2 (1 - s^{c(f-2i+2)}) ds \leq l \int_a^1 \|\omega(s)\|_F^2 s^{c(f-2i+2)} ds \leq \\ &l \int_0^1 \|\omega(s)\|_F^2 s^{c(f-2i+2)} ds \end{aligned}$$

and similarly to the previous case we get

$$\int_F \int_a^1 \|\omega(s)\|_F^2 (1 - s^{c(f-2i+2)}) ds d\text{vol}_F \leq l \|\omega\|_{L^2(C^*(F), g_c)}^2$$

and the constant l is independent of the choice of the form ω and of the choice of a . The fact that $i < \frac{f}{2} + \frac{1}{2c} + 1$ implies that $\int_0^1 r^{c(f-2i+2)} dr = \frac{1}{1+c(f-2i+2)} < \infty$ and so the following inequalities holds:

$$\|K_a(\alpha)\|_{L^2(C^*(F), g_c)}^2 \leq (1-a) \int_0^1 \int_F \int_a^1 \|\omega(s)\|_F^2 ds r^{c(f-2i+2)} d\text{vol}_F dr$$

$$\leq \int_0^1 r^{c(f-2i+2)} dr (1-a)(1+l) \|\omega\|_{L^2(C^*(F, g_c))}^2 \leq \frac{1+l}{2} \frac{1}{1+c(f-2i+2)} \|\alpha\|_{L^2(C^*(F, g_c))}^2.$$

Therefore we can conclude that for $i < \frac{f}{2} + \frac{1}{2c} + 1$

$$K_a : L^2\Omega^i(C^*(F), g_c) \longrightarrow L^2\Omega^{i-1}(C^*(F), g_c)$$

is a bounded operator uniformly in $a \in (\frac{1}{2}, 1)$. \square

Proposition 14. *Let $0 < \rho < 1$ and endow $(\rho, 1) \times F$ with the metric g_c restricted from $C^*(F)$. Let $\alpha = \phi + dr \wedge \omega \in L^2\Omega^i(C^*(F), g_c)$. If $i \geq \frac{f}{2} + \frac{1}{2c}$ then there exists a sequences $\epsilon_s \rightarrow 0$ such that*

$$\lim_{\epsilon_s \rightarrow 0} \|\phi(\epsilon_s)\|_{L^2((\rho, 1) \times F, g_c)}^2 = 0$$

Proof. By the fact that $\alpha \in L^2\Omega^i(C^*(F), g_c)$ follows that $\phi \in L^2\Omega^i(C^*(F), g_c)$, so we know that $\int_0^1 \int_F \|\phi(r)\|_F^2 r^{c(f-2i)} dvol_F dr < \infty$. This means that

$$\int_F \|\phi(r)\|_F^2 r^{c(f-2i)} dvol_F \in L^1(0, 1).$$

Thus by [7] lemma 1.2 there is a sequences $\epsilon_s \rightarrow 0$ for wich

$$\left| \int_F \|\phi(\epsilon_s)\|_F^2 \epsilon_s^{c(f-2i)} dvol_F \right| < \frac{C}{\epsilon_s |\ln(\epsilon_s)|}$$

for some constant $C > 0$. In this way we obtain

$$\left| \int_F \|\phi(\epsilon_s)\|_F^2 dvol_F \right| < \frac{C \epsilon_s^{c(f-2i)-1}}{|\ln(\epsilon_s)|}.$$

Since $i \geq \frac{f}{2} + \frac{1}{2c}$ the right side tends to zero as $\epsilon_s \rightarrow 0$. Thus we obtain:

$$\begin{aligned} \|\phi(\epsilon_s)\|_{L^2((\rho, 1) \times F, g_c)}^2 &= \int_\rho^1 \int_F \|\phi(\epsilon_s)\|_F^2 \epsilon_s^{c(f-2i)} dvol_F dr \\ &= \|\phi(\epsilon_s)\|_{L^2(F, g)}^2 \int_\rho^1 r^{c(f-2i)} dr \longrightarrow 0 \end{aligned}$$

when $\epsilon_s \rightarrow 0$. \square

Proposition 15. *If $i > \frac{f}{2} - \frac{1}{2c} + 1$ and $\alpha = \phi + dr \wedge \omega \in L^2\Omega^i(C^*(F), g_c)$, then*

$$K_0(\alpha) = \int_0^r \omega(s) ds \in L^2\Omega^{i-1}(C^*(F), g_c)$$

and $K_0 : L^2\Omega^i(C^*(F), g_c) \longrightarrow L^2\Omega^{i-1}(C^*(F), g_c)$ is a bounded operator.

Proof. By definition

$$\|K_0(\alpha)\|_{L^2(C^*(F), g_c)}^2 = \int_0^1 \int_F \left\| \int_0^r \omega(s) ds \right\|_F^2 r^{c(f-2i+2)} dvol_F dr.$$

We consider the term $\left\| \int_0^r \omega(s) ds \right\|_F^2$. Then:

$$\left\| \int_0^r \omega(s) ds \right\|_F^2 \leq \left(\int_0^r \|\omega(s)\|_F ds \right)^2 = \left(\int_0^r s^{\frac{c}{2}(f-2i+2)} s^{\frac{c}{2}(2i-f-2)} \|\omega(s)\|_F ds \right)^2$$

and applying the Schwartz inequality we get that

$$\leq \int_0^r s^{c(2i-f-2)} ds \int_0^r s^{c(f-2i+2)} \|\omega(s)\|_F^2 ds = \frac{r^{1+c(f-2i+2)}}{1+c(f-2i+2)} \int_0^r s^{c(f-2i+2)} \|\omega(s)\|_F^2 ds.$$

The last equality is a consequence of the fact that $i > \frac{f}{2} - \frac{1}{2c} + 1$. Substituting the previous inequality in the definition of $\|K_0(\alpha)\|_{L^2(C^*(F), g_c)}^2$ we get:

$$\begin{aligned} \|K_0(\alpha)\|_{L^2(C^*(F), g_c)}^2 &\leq \int_0^1 \int_F \int_0^1 s^{c(2i-f-2)} ds \int_0^r s^{c(f-2i+2)} \|\omega(s)\|_F^2 ds dvol_F r^{c(f-2i+2)} dr \\ &\leq \int_0^1 \frac{r}{1+c(2i-f-2)} dr \int_F \int_0^1 s^{c(f-2i+2)} \|\omega(s)\|_F^2 ds dvol_F \\ &= \frac{1}{2+2c(2i-f-2)} \|\omega\|_{L^2(C^*(F), g_c)}^2 \leq \frac{1}{2+2c(2i-f-2)} \|\alpha\|_{L^2(C^*(F), g_c)}. \end{aligned}$$

Thus

$$K_0 : L^2\Omega^i(C^*(F), g_c) \longrightarrow L^2\Omega^{i-1}(C^*(F), g_c)$$

is a bounded operator. □

Proposition 16. *Let*

$$K_\epsilon(\alpha) = \int_\epsilon^r \omega(s) ds$$

and let $0 < \rho < 1$. If $i > \frac{f}{2} - \frac{1}{2c} + 1$ then on $(\rho, 1) \times F$ with the restricted metric g_c ,

$$K_\epsilon(\alpha) \longrightarrow K_0(\alpha)$$

in the $\|\cdot\|_{L^2((\rho, 1) \times F, g_c)}$ norm when $\epsilon \rightarrow 0$.

Proof. We have

$$\|K_\epsilon(\alpha) - K_0(\alpha)\| = \int_\rho^1 \int_F \left\| \int_0^\epsilon \omega(s) ds \right\|_F^2 r^{c(f-2i+2)} dvol_F dr.$$

Using the same techniques of the previous proof we obtain that the right hand side is at most

$$\frac{\epsilon^{1+c(2i-f-2)}}{1+c(2i-f-2)} \left(\int_\rho^1 r^{c(f-2i+2)} dr \right) \|\omega\|_{L^2(C^*(F), g_c)}^2.$$

Since $i > \frac{f}{2} - \frac{1}{2c} + 1$ the whole expression tends to 0 as $\epsilon \rightarrow 0$. □

Proposition 17. *Let (F, g) be an oriented riemannian manifold. Let $\phi \in \mathcal{D}(d_{max, i-1}) \subset L^2\Omega^{i-1}(F, g)$, $\eta \in L^2\Omega^i(F, g)$ such that $d_{max, i-1}\phi = \eta$. Then for all $\rho \in (0, 1)$ on $(\rho, 1) \times F$ with the restricted metric g_c :*

1. $\pi^*\phi \in L^2\Omega^{i-1}((\rho, 1) \times F)$
2. $\pi^*\eta \in L^2\Omega^i((\rho, 1) \times F)$
3. For all $\beta \in C_0^\infty\Omega^i((\rho, 1) \times F)$ we have

$$\langle \pi^*\phi, \delta_{i-1}\beta \rangle_{L^2((\rho, 1) \times F)} = \langle \pi^*\eta, \beta \rangle_{L^2((\rho, 1) \times F)}$$

that is on $(\rho, 1) \times F$ with the restricted metric g_c

$$d_{max, i-1}\pi^*\phi = \pi^*\eta$$

Proof.

$$\|\pi^*\phi\|_{L^2((\rho, 1) \times F)}^2 = \int_\rho^1 r^{c(f-2i+2)} dr \int_F \|\phi\|_F^2 dvol_F = \int_\rho^1 r^{c(f-2i+2)} dr \|\phi\|_{L^2(F, g)}^2 < \infty$$

so $\pi^*\phi \in L^2\Omega^{i-1}((1, \rho) \times F)$;

$$\|\pi^*\eta\|_{L^2((\rho, 1) \times F)}^2 = \int_\rho^1 r^{c(f-2i)} dr \int_F \|\eta\|_F^2 dvol_F = \int_\rho^1 r^{c(f-2i)} dr \|\eta\|_{L^2(F, g)}^2 < \infty$$

so $\pi^*\eta \in L^2\Omega^i((1, \rho) \times F)$.

By a result of Cheeger, [7] pag 93, $\langle \pi^*\phi, \delta_i\beta \rangle_{L^2((\rho,1)\times F)} = \langle \pi^*\eta, \beta \rangle_{L^2((\rho,1)\times F)}$ for all $\beta \in C_0^\infty\Omega^i((\rho, 1) \times F)$ if and only if there is a sequence of smooth forms $\alpha_j \in L^2\Omega^{i-1}((\rho, 1) \times F)$ such that $d_{i-1}\alpha_j \in L^2\Omega^i((\rho, 1) \times F)$, $\|\pi^*\phi - \alpha_j\|_{L^2((\rho,1)\times F)} \rightarrow 0$, $\|\pi^*\eta - d_{i-1}\alpha_j\|_{L^2((\rho,1)\times F)} \rightarrow 0$ for $j \rightarrow \infty$. Using this result of Cheeger, from the fact that $\phi \in \text{Dom}(d_{i-1, \max})$, it follows that there is a sequences of smooth forms $\phi_j \in L^2\Omega^{i-1}(F, g)$ such that $d_{i-1}\phi_j \in L^2\Omega^i(F, g)$, $\|\phi - \phi_j\|_{L^2(F, g)} \rightarrow 0$, $\|\eta - d_{i-1}\phi_j\|_{L^2(F, g)} \rightarrow 0$ for $j \rightarrow \infty$. Now if we put $\alpha_j = \pi^*(\phi_j)$ we obtain a sequence of smooth forms in $L^2\Omega^{i-1}((\rho, 1) \times F)$ that satisfies the assumptions of the same result of Cheeger cited above. Indeed for each j

$$d_i\alpha_j \in L^2\Omega^i((\rho, 1) \times F)$$

$$\|\alpha_j - \pi^*\phi\|_{L^2((\rho,1)\times F)} = \int_\rho^1 r^{c(f-2i+2)} dr \int_F \|\phi - \alpha_j\|_F^2 \text{vol}_F \rightarrow 0$$

for $j \rightarrow \infty$ and similarly

$$\|d\alpha_j - \pi^*\eta\|_{L^2((\rho,1)\times F)} \rightarrow 0$$

for $j \rightarrow \infty$. Therefore we can conclude that for all $\beta \in C_0^\infty\Omega^i((\rho, 1) \times F)$

$$\langle \pi^*\phi, \delta_i\beta \rangle_{L^2((\rho,1)\times F)} = \langle \pi^*\eta, \beta \rangle_{L^2((\rho,1)\times F)}$$

□

Proposition 18. *Let (F, g) be an oriented odd dimensional riemannian manifold such that $d_{\max, i-1} : \mathcal{D}(d_{\max, i-1}) \rightarrow L^2\Omega^i(F, g)$ has closed range, where $i = \frac{f+1}{2}$ and $f = \dim F$. Let $\alpha \in L^2\Omega^i(C^*(F), g_c)$ a smooth i -form such that $d_i\alpha \in L^2\Omega^{i+1}(C^*(F), g_c)$. Then:*

1. *For almost all $b \in (0, 1)$ there is an exact i -form $\eta_b \in \mathcal{D}(d_{\max, i}) \subset L^2\Omega^i(F, g)$, $\eta_b = d_{\max, i-1}\psi_b$, $\psi_b \in \mathcal{D}(d_{\max, i-1}) \subset L^2\Omega^{i-1}(F, g)$, such that for all $0 < \rho < 1$ on $(\rho, 1) \times F$ with the restricted metric g_c*

$$\|d_{i-1}(K_b\alpha) - (\alpha - K_0(d_i\alpha) - \pi^*(\eta_b))\|_{L^2((\rho,1)\times F)} = 0$$

2. *On $L^2\Omega^{i-1}(C^*(F), g_c)$ we have $d_{\max, i-1}(K_b\alpha + \pi^*(\psi_b)) + K_0(d_i\alpha) = \alpha$*

Proof. 1) Let $\alpha = \phi + dr \wedge \omega$. Consider $K_\epsilon(d_i\alpha) = \phi - \pi^*\phi(\epsilon) - \int_\epsilon^r d_F\omega ds$. Obviously for each $0 < \rho < 1$ $K_\epsilon(d_i\alpha) \in L^2\Omega^i((\rho, 1) \times F)$ with the restricted metric g_c . From the fact that α is an i -form and that $i+1 = \frac{f+1}{2} + 1 > \frac{f}{2} + 1 - \frac{1}{2c}$ follows that we can use prop. 16 to conclude that

$$K_0(d_i\alpha) \in L^2\Omega^i(C^*(F), g_c)$$

and

$$\|K_\epsilon(d_i\alpha) - K_0(d_i\alpha)\|_{L^2((\rho,1)\times F)} \rightarrow 0$$

for $\epsilon \rightarrow 0$. For the same reasons we can use prop. 14 to say that there is a sequence $\epsilon_j \rightarrow 0$ such that, on $(\rho, 1) \times F$ with the restricted metric g_c ,

$$\lim_{\epsilon_j \rightarrow 0} \|\pi^*\phi(\epsilon_j)\|_{L^2((\rho,1)\times F, g_c)}^2 = 0.$$

Therefore using these facts we can conclude that

$$\lim_{\epsilon_j \rightarrow 0} \int_{\epsilon_j}^r d_F\omega ds \text{ exist in } L^2\Omega^i((\rho, 1) \times F)$$

and, if we call this limit γ , we have

$$K_0(d_i(\alpha)) = \phi - \gamma$$

in $L^2\Omega^i((\rho, 1) \times F)$ with the restricted metric g_c .

From this fact follows that for almost all $b \in (0, 1)$ $\int_{\epsilon_j}^b d_F\omega ds \rightarrow \gamma(b)$ in $L^2\Omega^i(F, g)$ for $\epsilon_j \rightarrow 0$. But $\int_{\epsilon_j}^b d_F\omega ds$ is a smooth form in $L^2\Omega^i(F, g)$; $\int_{\epsilon_j}^b \omega ds$ is a smooth form in $L^2\Omega^{i-1}(F, g)$

and $d_{i-1}(\int_{\epsilon_j}^b \omega ds) = \int_{\epsilon_j}^b d_F \omega ds$. So we can conclude that $\int_{\epsilon_j}^b d_F \omega ds = d_{max,i-1}(\int_{\epsilon_j}^b \omega ds)$ with $d_{max,i-1} : \mathcal{D}(d_{max,i-1}) \rightarrow L^2 \Omega^i(F, g)$. From this follows that $\gamma(b)$ is in the closure of the image of $d_{max,i-1} : \mathcal{D}(d_{max,i-1}) \rightarrow L^2 \Omega^i(F, g)$ and so follows from the assumptions that there is $\psi_b \in \mathcal{D}(d_{max,i-1}) \subset L^2 \Omega^{i-1}(F, g)$ such that

$$d_{max,i-1} \psi_b = \gamma(b).$$

We choose one of these b and ϵ such that $b > \epsilon$.

Now we consider $d_{i-1}(K_b(\alpha)) = dr \wedge \omega + \int_b^r d_F \omega$. Adding $d_{i-1}(K_b(\alpha))$ and $K_\epsilon(d_i \alpha)$ we obtain

$$d_{i-1}(K_b(\alpha)) = \alpha - K_\epsilon(d_i \alpha) - \pi^* \phi(\epsilon) - \pi^* \left(\int_\epsilon^b d_F \omega ds \right) \in L^2 \Omega^i((\rho, 1) \times F)$$

with the restricted metric g_c for all $\rho \in (0, 1)$.

We analyze in detail the terms on the right of equality. As noted above from the prop. 14 we know that there is a sequence $\epsilon_j \rightarrow 0$ such that

$$\lim_{\epsilon_j \rightarrow 0} \|\pi^* \phi(\epsilon_j)\|_{L^2((\rho, 1) \times F, g_c)}^2 = 0.$$

Similarly from the proposition 16 we know that

$$\|K_{\epsilon_j}(d_i \alpha) - K_0(d_i \alpha)\|_{L^2((\rho, 1) \times F)} \rightarrow 0$$

for $\epsilon_j \rightarrow 0$. For the term $\pi^* \left(\int_{\epsilon_j}^b d_F \omega ds \right)$ we know, by the observations made at the beginning of the proof, that there is an $(i-1)$ -form $\psi_b \in \text{Dom}(d_{max,i-1}) \subset L^2 \Omega^{i-1}(F, g)$ such that

$$\|\pi^* \left(\int_{\epsilon_j}^b d_F \omega ds \right) - \pi^*(d_{max,i-1}(\psi_b))\|_{L^2((\rho, 1) \times F)} \rightarrow 0$$

for $\epsilon_j \rightarrow 0$. Summarizing, for all $\rho \in (0, 1)$, we have on $(\rho, 1) \times F$ with the restricted metric g_c

$$\lim_{\epsilon_j \rightarrow 0} \|\alpha - K_{\epsilon_j}(d_i \alpha) - \phi(\epsilon_j) - \pi^* \left(\int_{\epsilon_j}^b d_F \omega ds \right) - (\alpha - K_0(d_i \alpha) - \pi^*(d_{i-1,max}(\psi_b)))\|_{L^2((\rho, 1) \times F)} = 0.$$

Therefore, if we put $\eta_b = \gamma(b)$, by the fact that

$$d_{i-1}(K_b(\alpha)) = \alpha - K_{\epsilon_j}(d_i \alpha) - \pi^* \phi(\epsilon_j) - \pi^* \left(\int_{\epsilon_j}^b d_F \omega ds \right)$$

for all j , we can conclude that

$$\|d_{i-1}(K_b \alpha) - (\alpha - K_0(d_i \alpha) - \pi^*(\eta_b))\|_{L^2((\rho, 1) \times F)} = 0$$

2) Before proving the statement we observe that from that fact that $i = \frac{f+1}{2}$ follows that we can use prop 13 to conclude that $K_b \alpha \in L^2 \Omega^{i-1}(C^*(F), g_c)$. Analogously we can use prop 11 to conclude that $\pi^* \psi_b \in L^2 \Omega^{i-1}(C^*(F), g_c)$. Let $\phi \in C_0^\infty \Omega^i(C^*(F))$. Then there is $\rho \in (0, 1)$ such that $\text{supp}(\phi) \subset (\rho, 1) \times F$.

We consider now:

$$\langle K_b \alpha, \delta_{i-1} \phi \rangle_{L^2(C^*(F), g_c)} = \langle K_b \alpha, \delta_{i-1} \phi \rangle_{L^2((\rho, 1) \times F)}.$$

By the fact that $K_b(\alpha)$ is a smooth $(i-1)$ -form such that $\|K_b(\alpha)\|_{L^2((1, \rho) \times F)} < \infty$, $\|d_{i-1}(K_b \alpha)\|_{L^2((1, \rho) \times F)} < \infty$ and that ϕ is a smooth form with compact support follows that:

$$\langle K_b \alpha, \delta_{i-1} \phi \rangle_{L^2((\rho, 1) \times F)} = \langle d_{i-1}(K_b(\alpha)), \phi \rangle_{L^2((\rho, 1) \times F)}$$

$$\begin{aligned}
 &= \langle \alpha - K_0(d_i\alpha) - \pi^*(\eta_b), \phi \rangle_{L^2((\rho,1) \times F)} = \\
 &= \langle \alpha, \phi \rangle_{L^2((\rho,1) \times F)} - \langle K_0(d_i\alpha), \phi \rangle_{L^2((\rho,1) \times F)} - \langle \pi^*(\eta_b), \phi \rangle_{L^2((\rho,1) \times F)} = \\
 &= \langle \alpha, \phi \rangle_{L^2((\rho,1) \times F)} - \langle K_0(d_i\alpha), \phi \rangle_{L^2((\rho,1) \times F)} - \langle \pi^*(\psi_b), \delta_{i-1}\phi \rangle_{L^2((\rho,1) \times F)} = \\
 &= \langle \alpha, \phi \rangle_{L^2(C^*(F), g_c)} - \langle K_0(d_i\alpha), \phi \rangle_{L^2(C^*(F), g_c)} - \langle \pi^*(\psi_b), \delta_{i-1}\phi \rangle_{L^2(C^*(F), g_c)}.
 \end{aligned}$$

In particular the equality $\langle \pi^*(\psi_b), \delta_{i-1}\phi \rangle_{L^2((\rho,1) \times F)} = \langle \pi^*(\eta_b), \phi \rangle_{L^2((\rho,1) \times F)}$ follows from prop. 17. We have obtained that for all $\phi \in C_0^\infty \Omega^i(C^*(F))$

$$\langle K_b\alpha + \pi^*\psi_b, \delta_{i-1}\phi \rangle_{L^2(C^*(F), g_c)} = \langle \alpha - K_0(d_i\alpha), \phi \rangle_{L^2(C^*(F), g_c)}.$$

So we can conclude that

$$d_{max, i-1}(K_b\alpha + \pi^*(\psi_b)) + K_0(d_i\alpha) = \alpha.$$

□

3 L^2 cohomology of a cone over a riemannian manifold

In this section we continue to use the notations of the previous section.

Theorem 3. *Let (F, g) be an oriented riemannian manifold. Then for the riemannian manifold $(C^*(F), g_c)$ the following isomorphism holds:*

$$H_{2, max}^i(C^*(F), g_c) = \begin{cases} H_{2, max}^i(F, g) & i < \frac{f}{2} + \frac{1}{2c} \\ 0 & i > \frac{f}{2} + 1 - \frac{1}{2c} \end{cases} \quad (36)$$

Proof. For the first part of the proof we use the complex $(\Omega_2^*(C^*(F), g_c), d_*)$ of prop. 6. Let $\alpha \in \Omega_2^i(C^*(F), g_c)$, $\alpha = \phi + dr \wedge \omega$, $i = 0, \dots, f+1$. Let $a \in (\frac{1}{2}, 1)$. Consider the following map

$$v_a : \Omega_2^i(C^*(F), g_c) \rightarrow \Omega_2^i(F, g), \quad v_a(\alpha) = \phi(a).$$

By prop.12 $v_a(\alpha) \in L^2\Omega^i(F, g)$. Furthermore this map satisfies $v_a \circ d_i = d_i \circ v_a$ where on the left of the equality d_i is the i -th differential of the complex $(\Omega_2^*(C^*(F), g_c), d_*)$ while on the right of the equality the operator d_i is the i -th differential of the complex $(\Omega_2^*(F, g), d_*)$. Therefore v_a is a morphism between the complex $(\Omega_2^*(C^*(F), g_c), d_*)$ and the complex $(\Omega_2^*(F, g), d_*)$ so it induces a map between the cohomology groups

$$v_a^* : H_2^i(C^*(F), g_c) \rightarrow H_2^i(F, g)$$

where $H_2^i(F, g)$ is the i -th cohomology group of the complex $(\Omega_2^*(F, g), d_*)$.

Now in the case $i < \frac{f}{2} + \frac{1}{2c}$, by proposition 13, we know that $K_a(\alpha)$ and $K_a(d_i\alpha)$ are two smooth form such that

$\|K_a(d_i\alpha)\|_{L^2(C^*(F), g_c)} < \infty$ and $\|K_a\alpha\|_{L^2(C^*(F), g_c)} < \infty$. If we add the two following terms, $d_{i-1}(K_a(\alpha))$ and $K_a(d_i(\alpha))$ we obtain:

$$d_{i-1}(K_a\alpha) + K_a d_i(\alpha) = dr \wedge \omega(s)ds + \int_a^r d_F(s)ds\omega + \phi - \phi(a) - \int_a^r d_F(s)ds\omega = \alpha - \pi^*(v_a(\alpha)). \quad (37)$$

So we have obtained that $\|d_{i-1}(K_a\alpha)\|_{L^2(C^*(F), g_c)} < \infty$ and from this and (37) follows that

$$(\pi^*)^* \circ v_a^* : H_2^i(C^*(F), g_c) \rightarrow H_2^i(C^*(F), g_c)$$

is an isomorphism for $i < \frac{f}{2} + \frac{1}{2c}$. Now from this fact follows that for the same i :

$$v_a^* : H_2^i(C^*(F), g_c) \rightarrow H_2^i(F, g)$$

is injective and that

$$(\pi^*)^* : H_2^i(F, g) \rightarrow H_2^i(C^*(F), g_c)$$

is surjective. But from prop. 11 we know that $v_a^i : H_2^i(C^*(F), g_c) \rightarrow H_2^i(F, g)$ is surjective. So for $i < \frac{f}{2} + \frac{1}{2c}$ $H_2^i(C^*(F), g_c)$ and $H_2^i(F, g)$ are isomorphic and therefore by proposition 6 for the same i we have

$$H_{2,max}^i(C^*(F), g_c) \cong H_{2,max}^i(F, g).$$

Now we start the second part of the proof. We know that for each i every cohomology class $[\alpha] \in H_{2,max}^i(C^*(F))$ has a smooth representative. So let $\alpha \in L^2\Omega^i(C^*(F), g_c)$, $i > \frac{f}{2} + 1 - \frac{1}{2c}$, a smooth form such that $d_i\alpha = 0$. Observe that from the fact that α is closed follows that $\phi' = d_F\omega$ and therefore, given $\epsilon \in (0, 1)$ we have $d_{i-1}(K_\epsilon\alpha) = d_{i-1}(\int_\epsilon^r \omega(s)ds) = dr \wedge \omega + \int_\epsilon^r d_F\omega(s)ds = dr \wedge \omega + \int_\epsilon^r \phi'(s)ds = dr \wedge \omega + \phi - \phi(\epsilon) = \alpha - \phi(\epsilon)$. Consider $K_0(\alpha)$; by proposition 15 we know that $K_0(\alpha) \in L^2\Omega^i(C^*(F), g_c)$. We want to show that $d_{max,i-1}(K_0(\alpha)) = \alpha$. Let $\beta \in C_0^\infty\Omega^i(C^*(F))$. Then there is $\rho > 0$ such that $supp(\beta) \subset (\rho, 1) \times F$. Therefore:

$$\begin{aligned} \langle K_0\alpha, \delta_{i-1}\beta \rangle_{L^2(C^*(F))} &= \langle K_0\alpha, \delta_{i-1}\beta \rangle_{L^2((\rho,1)\times F)} \text{ (by prop 16)} \\ &= \lim_{\epsilon \rightarrow 0} \langle K_\epsilon\alpha, \delta_{i-1}\beta \rangle_{L^2((\rho,1)\times F)}. \end{aligned}$$

By the fact that $K_\epsilon(\alpha)$ is a smooth form such that $\|K_\epsilon(\alpha)\|_{L^2((1,\rho)\times F)} < \infty$, $\|d_{i-1}(K_\epsilon\alpha)\|_{L^2((1,\rho)\times F)} < \infty$ and that ϕ is a smooth form with compact support follows that:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \langle K_\epsilon\alpha, \delta_{i-1}\beta \rangle_{L^2((\rho,1)\times F)} &= \lim_{\epsilon \rightarrow 0} \langle d_{i-1}(K_\epsilon\alpha), \beta \rangle_{L^2((\rho,1)\times F)} \\ &= \lim_{\epsilon \rightarrow 0} \langle \alpha - \phi(\epsilon), \beta \rangle_{L^2((\rho,1)\times F)} = \langle \alpha, \beta \rangle_{L^2((\rho,1)\times F)} - \lim_{\epsilon \rightarrow 0} \langle \phi(\epsilon), \beta \rangle_{L^2((\rho,1)\times F)}. \end{aligned}$$

In particular the limit

$$\lim_{\epsilon \rightarrow 0} \langle \phi(\epsilon), \beta \rangle_{L^2((\rho,1)\times F)}$$

exist. But from prop. 14 we know that there is a sequence $\epsilon_j \rightarrow 0$ such that

$$\lim_{\epsilon_j \rightarrow 0} \langle \phi(\epsilon_j), \beta \rangle_{L^2((\rho,1)\times F)} = 0.$$

Therefore

$$\langle K_0\alpha, \delta_{i-1}\beta \rangle_{L^2((\rho,1)\times F)} = \langle \alpha, \delta_{i-1}\beta \rangle_{L^2((\rho,1)\times F)} = \langle \alpha, \delta_{i-1}\beta \rangle_{L^2(C^*(F), g_c)}.$$

Thus we can conclude that $d_{max,i-1}(K_0(\alpha)) = 0$ and hence that $H_{2,max}^i(C^*(F), g_c) = 0$ for $i > \frac{f}{2} + 1 - \frac{1}{2c}$. \square

Corollary 3. *Suppose that one of three following hypotheses applies:*

1. $0 < c < 1$.
2. $c \geq 1$ and $f = \dim F$ is even.
3. $c \geq 1$, f is odd and $d_{max,i-1} : \mathcal{D}(d_{max,i-1}) \rightarrow L^2\Omega^i(F, g)$ has close range where $i = \frac{f+1}{2}$.
(By prop 2 this happen for example when $H_{2,max}^i(F, g)$ is finite dimensional.)

Then for the riemannian manifold $(C^*(F), g_c)$ the following isomorphism holds:

$$H_{2,max}^i(C^*(F), g_c) = \begin{cases} H_{2,max}^i(F, g) & i < \frac{f}{2} + \frac{1}{2c} \\ 0 & i \geq \frac{f}{2} + \frac{1}{2c} \end{cases} \quad (38)$$

Proof. If $0 < 1 < c$ then $\frac{f}{2} + \frac{1}{2c} > \frac{f}{2} + 1 - \frac{1}{2c}$.

If $c \geq 1$ and f is even then $i > \frac{f}{2} + 1 - \frac{1}{2c}$ if and only if $i \geq \frac{f}{2} + \frac{1}{2c}$.

Finally if $c \geq 1$, f is odd and $d_{max,i-1} : \mathcal{D}(d_{max,i-1}) \rightarrow L^2\Omega^i(F, g)$ has close range then the thesis immediatly follows from prop. 18. \square

Remark 4. *Theorem 3 also holds in the following cases:*

1. If we replace $C(F)$ with $C_\epsilon(F)$ with $C_\epsilon(F) = F \times [0, \epsilon]/F \times \{0\}$ and where ϵ is any real positive number. Furthermore, if $\epsilon < \delta$, it is immediate to note that

$$i^* : (L^2\Omega^*(C_\delta^*(F), g_c), d_{max,*}) \rightarrow (L^2\Omega^*(C_\epsilon^*(F), g_c), d_{max,*})$$

where i^* is the morphism of complexes induced by the inclusion $i : C_\epsilon(F) \rightarrow C_\delta(F)$, induces an isomorphism between the cohomology groups $H_{2,max}^i(C_\epsilon(F), g_c)$ and $H_{2,max}^i(C_\delta(F), g_c)$ for each i .

2. When (F, g) is a disconnected riemannian manifold made of a finite number of connected components all having the same dimension, that is $(F, g) = \bigcup_{j \in J} (F_j, g_j)$, $\dim F_i = \dim F_j$ for each $i, j \in J$ and J is finite. Indeed in this case:

$$H_{2,max}^i(C^*(F), g_c) = H_{2,max}^i(C^*(\bigcup_{j \in J} F_j), g_c) = \bigoplus_{j \in J} H_{2,max}^i(C^*(F_j), g_{c,j}) \quad (39)$$

$$= \bigoplus_{j \in J} \begin{cases} H_{2,max}^i(F_j, g_j) & i < \frac{f}{2} + \frac{1}{2c} \\ 0 & i > \frac{f}{2} + 1 - \frac{1}{2c} \end{cases} = \begin{cases} H_{2,max}^i(F, g) & i < \frac{f}{2} + \frac{1}{2c} \\ 0 & i > \frac{f}{2} + 1 - \frac{1}{2c} \end{cases} \quad (40)$$

Obviously if each (F_j, g_j) satisfies the assumptions of corollary 3 then also corollary 3 holds for $(C^*(F), g_c)$. This situation could happen in theorem 4 of the next section. In that case the manifold F will be the regular part of a link and it could happen that it is disconnected.

We conclude the section recalling a result from [7] that we will use in the proof of theorem 4.

Proposition 19. *Let (M, g) be a Riemannian manifold. Then for the riemannian manifold $((0, 1) \times M, dr \otimes dr + g)$ the following isomorphism holds:*

$$H_{2,max}^i((0, 1) \times M, dr \otimes dr + g) \cong H_{2,max}^i(M, g) \text{ for all } i = 0, \dots, \dim M + 1 \quad (41)$$

Proof. See [7] pag 115. □

From this proposition we get the following immediate corollary:

Corollary 4. *Consider the following riemannian manifold $((0, 1)^n, dr_1 \otimes dr_1 + \dots + dr_n \otimes dr_n)$. Then the following isomorphism holds:*

$$H_{2,max}^i((0, 1)^n, dr_1 \otimes dr_1 + \dots + dr_n \otimes dr_n) \cong \begin{cases} \mathbb{R} & i = 0 \\ 0 & i > 0 \end{cases} \quad (42)$$

4 L^2 Hodge and de Rham theorems

Before starting the section we make a remark about the notation. Given an open subset $U \subset X$ with $\mathcal{D}(U, d_{max/min,i})$ we mean the domain of $d_{max/min,i}$ in $L^2\Omega^i(\text{reg}(U), g|_{\text{reg}(U)})$. Given a complex of sheaves (\mathcal{L}^*, d_*) over X and an open subset U of X with the symbol $H^i(\mathcal{L}^*(U), d_*)$ we mean the i -th cohomology group of the complex

$$\dots \xrightarrow{d_{i-2}} \mathcal{L}^{i-1}(U) \xrightarrow{d_{i-1}} \mathcal{L}^i(U) \xrightarrow{d_i} \mathcal{L}^{i+1}(U) \xrightarrow{d_{i+1}} \dots$$

Theorem 4. *Let X be a compact and oriented smoothly stratified pseudomanifold of dimension n with a Thom-Mather stratification \mathfrak{X} . Let g be a quasi edge metric with weights on $\text{reg}(X)$, see definition 10. Let \mathcal{R}_0 be the stratified coefficient system made of the pair of coefficient systems given by $(X - X_{n-1}) \times \mathbb{R}$ over $X - X_{n-1}$ where the fibers \mathbb{R} have the discrete topology and the constant 0 system on X_{n-1} . Let p_g be the general perversity associated to the metric g , see definition 12. Then, for all $i = 0, \dots, n$, the following isomorphisms holds:*

$$I^{q_g} H^i(X, \mathcal{R}_0) \cong H_{2,max}^i(\text{reg}(X), g) \cong \mathcal{H}_{abs}^i(\text{reg}(X), g) \quad (43)$$

$$I^{p_g} H^i(X, \mathcal{R}_0) \cong H_{2, \min}^i(\text{reg}(X), g) \cong \mathcal{H}_{\text{rel}}^i(\text{reg}(X), g) \quad (44)$$

where q_g is the complementary perversity of p_g , that is, $q_g = t - p_g$ and t is the usual top perversity. In particular, for all $i = 0, \dots, n$ the groups

$$H_{2, \max}^i(\text{reg}(X), g), H_{2, \min}^i(\text{reg}(X), g), \mathcal{H}_{\text{abs}}^i(\text{reg}(X), g), \mathcal{H}_{\text{rel}}^i(\text{reg}(X), g)$$

are all finite dimensional.

Before proving the theorem we need some preliminary results.

Proposition 20. *Let X be an oriented smoothly stratified pseudomanifold of dimension n with a Thom-Mather stratification and let g a riemannian metric on $\text{reg}(X)$. Consider, for every $i = 0, \dots, n$, the following presheaves:*

$$U \longmapsto \mathcal{D}(U, d_{\max/\min, i}) = \begin{cases} \mathcal{D}(U, d_{\max/\min, i}) & U \cap X_{n-1} = \emptyset \\ \mathcal{D}(U - (U \cap X_{n-1}), d_{\max/\min, i}) & U \cap X_{n-1} \neq \emptyset \end{cases} \quad (45)$$

or

$$U \longmapsto \begin{cases} \omega \in \Omega_2^i(U, g|_U) & U \cap X_{n-1} = \emptyset \\ \omega \in \Omega_2^i(\text{reg}(U), g|_{\text{reg}(U)}) & U \cap X_{n-1} \neq \emptyset \end{cases} \quad (46)$$

Let $\mathcal{L}_{2, \max/\min}^i$ and \mathcal{L}_2^i be the sheaves associated to the previous presheaves; then for these sheaves we have the following explicit descriptions:

1. let U an open subset of X then: $\mathcal{L}_{2, \max/\min}^i(U) \cong \{\omega \in L_{\text{Loc}}^2 \Omega^i(\text{reg}(U), g|_{\text{reg}(U)}) : \forall p \in U \exists V \text{ open neighbourhood of } p \text{ in } U \text{ such that } \omega|_{\text{reg}(V)} \in \mathcal{D}(\text{reg}(V), d_{\max/\min, i})\}$.
2. $\mathcal{L}_2^i(U) \cong \{\omega \in \Omega^i(\text{reg}(U), g|_{\text{reg}(U)}) : \forall p \in U \exists V \text{ open neighbourhood of } p \text{ in } U \text{ such that } \omega|_{\text{reg}(V)} \in \Omega_2^i(\text{reg}(V), g|_{\text{reg}(V)})\}$.
3. If X is compact $\mathcal{L}_{2, \max/\min}^i(X) = \mathcal{D}(\text{reg}(X), d_{\max/\min, i})$.
4. $\mathcal{L}_2^i(X) = \{\omega \in \Omega^i(\text{reg}(X)) : \omega \in L^2 \Omega^i(\text{reg}(X), g), d_i \omega \in L^2 \Omega^i(\text{reg}(X), g)\}$.
5. The complexes $\mathcal{L}_{2, \max}^i$ and \mathcal{L}_2^i are quasi isomorphic.

Proof. The first and the second statement follow from the fact that the sheaves $\mathcal{L}_{2, \max/\min}^i$, \mathcal{L}_2^i and the respective sheaves at the right of \cong have isomorphic stalks. The third and fourth statement are an immediate consequences of the compactness of X . The fifth statement follows immediately from proposition 6. \square

Proposition 21. *Let X be an oriented smoothly stratified pseudomanifold with a Thom-Mather stratification of dimension n and g a quasi rigid iterated edge metric with weights on $\text{reg}(X)$. Then, for each $i = 0, \dots, n$, $\mathcal{L}_{2, \max/\min}^i$ and \mathcal{L}_2^i are fine sheaves.*

Proof. From the description of the sheaves $\mathcal{L}_{2, \max/\min}^i$, \mathcal{L}_2^i given in prop. 20 it follows that in order to prove this proposition it is sufficient to show that on X , given an open cover $\mathcal{U}_A = \{U_\alpha\}_{\alpha \in A}$, there is a bounded partition of unity with bounded differential subordinate to \mathcal{U}_A , that is a family of functions $\lambda_\alpha : X \rightarrow [0, 1]$, $\alpha \in A$ such that

1. Each λ_α is continous and $\lambda_\alpha|_{\text{reg}(X)}$ is smooth.
2. $\text{supp}(\lambda_\alpha) \subset U_\alpha$ for some $\alpha \in A$.
3. $\{\text{supp}(\lambda_\alpha)\}_{\alpha \in A}$ is a locally finite cover of X .
4. For each $x \in X$ $\sum_{\alpha \in A} \lambda_\alpha(x) = 1$.
5. There are constants $C_\alpha > 0$ such that each λ_α satisfies $\|d(\lambda_\alpha|_{\text{reg}(X)})\|_{L^2(\text{reg}(X), g)} \leq C_\alpha$.

The proof is given by the induction on the depth of X . If $\text{depth}(X) = 0$ the statement is immediate because in this case X is a closed manifold. Suppose now that the statement is true if $\text{depth}(X) \leq k - 1$ and that $\text{depth}(X) = k$. Let $\mathcal{U}_J = \{U_j\}_{j \in J}$ be a locally finite refinement of \mathcal{U}_A such that for each U_j there is a diffeomorphism $\phi_j : U_j \rightarrow \mathbb{R}^n$ if $U_j \cap X_{n-1} = \emptyset$ or, in the case $U_j \cap X_{n-1} \neq \emptyset$, an isomorphism $\phi_j : U_j \rightarrow W_j \subset \mathbb{R}^k \times C(L_j)$ between U_j and an open subset, W_j , of the product $\mathbb{R}^k \times C(L_j)$ for some $k < n$ and stratified space L_j .

Let $\mathcal{V}_J = \{V_j\}_{j \in J}$ a shrinking of \mathcal{U}_J ; this means that \mathcal{V}_J is a refinement of \mathcal{U}_J such that if $V_j \subset U_j$ then $\bar{V}_j \subset U_j$. Now let $V_j \in \mathcal{V}_J, U_j \in \mathcal{U}_J$ such that $V_j \subset U_j$ and $U_j \cap X_{n-1} = \emptyset$. Let $\psi_j : \mathbb{R}^n \rightarrow [0, 1]$ be a smooth function such that $\psi_j|_{\bar{\phi_j(V_j)}} = 1$ and $\text{supp}(\psi_j) \subset \phi_j(U_j)$. Define $\lambda_j : X \rightarrow [0, 1], \lambda_j := \psi_j \circ \phi_j$. Now let $V_j \in \mathcal{V}_J, U_j \in \mathcal{U}_J$ such that $V_j \subset U_j$ and $U_j \cap X_{n-1} \neq \emptyset$. We can take two functions $\eta : \mathbb{R}^k \rightarrow [0, 1], \xi : [0, 1] \rightarrow [0, 1]$ and, using the inductive hypothesis, a third function $\tau_j : L_j \rightarrow [0, 1]$ smooth on $\text{reg}(L_j)$ and with bounded differential such that $\psi_j := \eta_j \xi_j \tau_j$ is a continuous function on $\mathbb{R}^k \times C(L_j) \rightarrow [0, 1]$ smooth on the regular part and with bounded differential such that $\psi_j|_{\bar{\phi_j(V_j)}} = 1$ and $\text{supp}(\psi_j) \subset \phi_j(U_j)$. Also in this case define $\lambda_j : X \rightarrow [0, 1], \lambda_j := \psi_j \circ \phi_j$. Finally define

$$\mu_j : X \rightarrow [0, 1], \mu_j = \frac{\lambda_j}{\sum_{j \in J} \lambda_j} \quad (47)$$

$\{\mu_j\}_{j \in J}$ is a partition of unity with bounded differential subordinated to the cover \mathcal{U}_J and therefore from this follows immediately that there exist a partition of unity with bounded differential subordinated to the cover \mathcal{U}_A . Now the statement of the proposition is an immediate consequence. \square

Now we state the last proposition that we will use in the proof of theorem 4.

Proposition 22. *Let L be a compact smoothly stratified pseudomanifold with a Thom-Mather stratification and let g_L be a riemannian metric on $\text{reg}(L)$. Let $C(L)$ be the cone over L and on $\text{reg}(C(L))$ consider the metric $dr \otimes dr + r^{2c} g_L$. Finally consider on $C(L)$ the complex of sheaves $(\mathcal{L}_{2, \max}^*, d_{\max, *})$. Then the canonical inclusion*

$$i_v : C(L) - \{v\} \longrightarrow C(L),$$

where v is the vertex of the cone, induces a quasi-isomorphism between the complexes

$$(\mathcal{L}_{2, \max}^*, d_{\max, *}) \text{ and } (i_{v*} i_v^* \mathcal{L}_{2, \max}^*, d_{\max, *})$$

for $i \leq \lceil [\frac{\dim L}{2} + \frac{1}{2c}] \rceil$.

Proof. We start the proof showing that the complexes $(\mathcal{L}_{2, \max}^*, d_{\max, *})$ and $(i_{v*} i_v^* \mathcal{L}_{2, \max}^*, d_{\max, *})$ are quasi isomorphic for $i \leq \lceil [\frac{\dim L}{2} + \frac{1}{2c}] \rceil$. This is equivalent to show that for each $x \in C(L)$

$$(H^i(\mathcal{L}_{2, \max}^*, d_{\max, *}))_x \cong (H^i(i_{v*} i_v^* \mathcal{L}_{2, \max}^*, d_{\max, *}))_x$$

where each term in the previous isomorphism is the stalk at the point x of the i -th cohomology sheaf associated to $(\mathcal{L}_{2, \max}^*, d_{\max, *})$ and $(i_{v*} i_v^* \mathcal{L}_{2, \max}^*, d_{\max, *})$ respectively. For every $i = 0, \dots, \dim L + 1$ the sheaf $i_{v*} i_v^* \mathcal{L}_{2, \max}^i$ is isomorphic to the following sheaf; let $U \subset C(L)$ an open subset then:

$$i_{v*} i_v^* \mathcal{L}_{2, \max}^i(U) \cong \{\omega \in L_{Loc}^2 \Omega^i(\text{reg}(U), dr \otimes dr + r^{2c} g_L|_{\text{reg}(U)}) : \forall p \in U - \{v\} \exists V \text{ open neighbourhood of } p \text{ in } U \text{ such that } \omega|_{\text{reg}(V)} \in \mathcal{D}(\text{reg}(V), d_{\max, i})\}.$$

From this fact and prop. 20 it follows that for every $x \in C(L) - \{v\}$

$$(H^i(\mathcal{L}_{2, \max}^*, d_{\max, *}))_x \cong (H^i(i_{v*} i_v^* \mathcal{L}_{2, \max}^*, d_{\max, *}))_x. \quad (48)$$

Obviously for each open neighborhood U of v there exists an $\delta > 0$ such that $C_\delta(L) \subset U$. Therefore, by the first point of remark 4, to show that $(H^i(i_{v*} i_v^* \mathcal{L}_{2, \max}^*, d_{\max, *}))_v \cong (H^i(\mathcal{L}_{2, \max}^*, d_{\max, *}))_v$ for $i \leq \lceil [\frac{\dim L}{2} + \frac{1}{2c}] \rceil$ is sufficient to show that for the same i

$$H^i(i_{v*} i_v^* \mathcal{L}_{2, \max}^*(C(L)), d_{\max, *}) \cong H^i(\mathcal{L}_{2, \max}^*(C(L)), d_{\max, *}).$$

On the whole cone $C(L)$ the main difference between the complexes $(\mathcal{L}_{2,max}^*, d_{max,*})$ and $(i_{v*}i_v^*\mathcal{L}_{2,max}^*, d_{max,*})$ is that for each $\omega \in \mathcal{L}_{2,max}^i(L)$, by prop. 11,

$$\pi^*\omega \in \mathcal{L}_{2,max}^i(C(L)) \text{ if and only if } i < \frac{\dim L}{2} + \frac{1}{2c}.$$

Instead

$$\pi^*\omega \in i_{v*}i_v^*\mathcal{L}_{2,max}^i(C(L)) \text{ for every } i = 0, \dots, \dim L.$$

Therefore by the proof of the first part of theorem 3 and in particular from (37) follows that

$$H^i(i_{v*}i_v^*\mathcal{L}_{2,max}^*(C(L)), d_{max,*}) \cong H_{2,max}^i(\text{reg}(L), g_L) \text{ for every } i = 0, \dots, \dim L + 1. \quad (49)$$

But from theorem 3 we know that

$$H^i(\mathcal{L}_{2,max}^*(C(L)), d_{*,max}) \cong H_{2,max}^i(\text{reg}(L), g_L) \text{ for } i \leq \left\lceil \left[\frac{\dim L}{2} + \frac{1}{2c} \right] \right\rceil. \quad (50)$$

So for $i \leq \left\lceil \left[\frac{\dim L}{2} + \frac{1}{2c} \right] \right\rceil$

$$(H^i(i_{v*}i_v^*\mathcal{L}_{2,max}^*, d_{max,*}))_v \cong (H^i(\mathcal{L}_{2,max}^*, d_{max,*}))_v$$

and therefore we can conclude that for the same i the complexes $(\mathcal{L}_{2,max}^*, d_{max,*})$ and $(i_{v*}i_v^*\mathcal{L}_{2,max}^*, d_{max,*})$ are quasi-isomorphic.

Now let j be the morphism between $(\mathcal{L}_{2,max}^*, d_{max,*})$ and $(i_{v*}i_v^*\mathcal{L}_{2,max}^*, d_{max,*})$ induced from $i_v : C(L) - \{v\} \rightarrow C(L)$. It is immediate to note that for each open subset $U \subset C(L)$ j_U is just the inclusion of $\mathcal{L}_{2,max}^*(C(L))$ in $i_{v*}i_v^*\mathcal{L}_{2,max}^*(C(L))$. Let j^* be the morphism induced from j between the cohomology sheaves $H^i(\mathcal{L}_{2,max}^*, d_{max,*})$ and $H^i(i_{v*}i_v^*\mathcal{L}_{2,max}^*, d_{max,*})$. It is immediate to note that j^* induces the isomorphism (48). Finally if we call ϕ and ψ respectively the isomorphisms (49) and (50) we have that

$$\phi \circ j^* = \psi.$$

Therefore we can conclude that

$$j : (\mathcal{L}_{2,max}^*, d_{max,*}) \rightarrow (i_{v*}i_v^*\mathcal{L}_{2,max}^*, d_{max,*})$$

is a quasi-isomorphism for $i \leq \left\lceil \left[\frac{\dim L}{2} + \frac{1}{2c} \right] \right\rceil$. \square

Corollary 5. *Let (M, h) be an oriented riemannian manifold, let L be a compact smoothly stratified pseudomanifold with a Thom-Mather stratification and let g_L be a riemannian metric on $\text{reg}(L)$. Consider now $M \times C(L)$ and on $\text{reg}(M \times C(L))$ consider the metric $h + dr \otimes dr + r^{2c}g_L$. Then the canonical inclusion*

$$i_M : M \times C(L) - (M \times \{v\}) \longrightarrow M \times C(L)$$

induces a quasi-isomorphism between the complexes

$$(\mathcal{L}_{2,max}^*, d_{max,*}) \text{ and } (i_{M*}i_M^*\mathcal{L}_{2,max}^*, d_{max,*})$$

for $i \leq \left\lceil \left[\frac{\dim L}{2} + \frac{1}{2c} \right] \right\rceil$.

Proof. The proof is completely analogous to the proof of proposition 22. For every $i = 0, \dots, \dim M + \dim L + 1$ the sheaf $i_{M*}i_M^*\mathcal{L}_{2,max}^i$ is isomorphic to the following sheaf; let $U \subset M \times C(L)$ an open subset then:

$$i_{M*}i_M^*\mathcal{L}_{2,max}^i(U) \cong \{\omega \in L_{Loc}^2\Omega^i(\text{reg}(U), h + dr \otimes dr + r^{2c}g_L|_{\text{reg}(U)}) : \forall p \in U - (U \cap (M \times \{v\})) \exists V \text{ open neighbourhood of } p \text{ in } U \text{ such that } \omega|_{\text{reg}(V)} \in \mathcal{D}(\text{reg}(V), d_{max,i})\}.$$

From this follows that for every $x \in M \times C(L) - (M \times \{v\})$

$$(H^i(\mathcal{L}_{2,max}^*, d_{max,*}))_x \cong (H^i(i_{M*}i_M^*\mathcal{L}_{2,max}^*, d_{max,*}))_x.$$

Now let $p = (m, v) \in M \times \{v\}$. Obviously for any open neighborhood W of p there exists an open neighborhood U of m in M and $\delta > 0$ such that $U \times C_\delta(L) \subset W$. Therefore, like in the proof of prop. 22, in order to show that

$$(H^i(i_{M*}i_M^*\mathcal{L}_{2,max}^*, d_{max,*}))_p \cong (H^i(\mathcal{L}_{2,max}^*, d_{max,*}))_p$$

for $i \leq \lceil \frac{\dim L}{2} + \frac{1}{2c} \rceil$ is sufficient to show that for the same i

$$H^i(i_{M*}i_M^*\mathcal{L}_{2,max}^*(U \times C(L)), d_{max,*}) \cong H^i(\mathcal{L}_{2,max}^*(U \times C(L)), d_{max,*})$$

where U is an open neighborhood of m in M diffeomorphic to an open ball in \mathbb{R}^s where $s = \dim M$. Now using the same observations of the proof of prop. 22 and prop. 19 it is easy to see that

$$H^i(i_{M*}i_M^*\mathcal{L}_{2,max}^*(U \times C(L)), d_{max,*}) \cong H_{2,max}^i(\text{reg}(L), g_L) \text{ for each } i$$

and that

$$H^i(\mathcal{L}_{2,max}^*(U \times C(L)), d_{max,*}) \cong H_{2,max}^i(\text{reg}(L), g_L) \text{ for } i \leq \lceil \frac{\dim L}{2} + \frac{1}{2c} \rceil.$$

So for $i \leq \lceil \frac{\dim L}{2} + \frac{1}{2c} \rceil$

$$(H^i(i_{M*}i_M^*\mathcal{L}_{2,max}^*, d_{max,*}))_p \cong (H^i(\mathcal{L}_{2,max}^*, d_{max,*}))_p$$

and therefore we can conclude that for the same i the complexes $(\mathcal{L}_{2,max}^*, d_{max,*})$ and $(i_{M*}i_M^*\mathcal{L}_{2,max}^*, d_{max,*})$ are quasi-isomorphic. Now using the same final considerations of the previous proof we get the conclusion. \square

Finally we can give the proof of the theorem announced at the beginning of the section:

Proof. (of theorem 4). Using proposition 9 we know that there is a quasi rigid iterated edge metric on $\text{reg}(X)$, g' , having the same weights of g and therefore, by corollary 2, quasi-isometric to g . So, without loss of generality, we can suppose that g is a quasi rigid iterated edge metric with weights. We start by proving the isomorphism 43. The proof is given by induction on the depth of X . If $\text{depth}(X) = 0$ there is nothing to show because, in this case, X is a closed manifold and therefore the isomorphisms 43 are the well know theorems of Hodge and de Rham. Suppose now that the theorem is true if $\text{depth}(X) \leq k - 1$ and that $\text{depth}(X) = k$. We will show that the theorem is also true in this case. We begin showing the first isomorphism, $H_{2,max}^i(\text{reg}(X), g) \cong I^{q_g} H^i(X, \mathcal{R}_0)$; to do this we will use theorem 1, corollary 1 and remark 3. More precisely we will show that the complex $(\mathcal{L}_{2,max}^i, d_{max,i})$ satisfies the three axioms of theorem 1 respect to the perversity p_g , the stratification \mathfrak{X} and the local system over $\text{reg}(X)$ given by $\mathcal{R} \otimes \mathcal{O}$ where \mathcal{R} is $(X - X_{n-1}) \times \mathbb{R}$ with \mathbb{R} endowed of the discrete topology and \mathcal{O} is the orientation sheaf (see example 1). By proposition 21 we know that $(\mathcal{L}_{2,max}^i, d_{max,i})$ is a complex of fine sheaves. The first two requirements of axiom 1 are obviously satisfied. The third requirement of the same axiom follows by the fact that for each $x \in \text{reg}(X)$ $(H^i(\mathcal{L}_{2,max}^*, d_{max,*}))_x$, that is the stalk at the point x of the i -th cohomology sheaf associated to the complex $(\mathcal{L}_{2,max}^*, d_{max,*})$, satisfies:

$$(H^i(\mathcal{L}_{2,max}^*, d_{max,*}))_x = \begin{cases} \mathbb{R} & i = 0 \\ 0 & i > 0 \end{cases} \quad (51)$$

Consider now a stratum $Y \subset X$ and a point $x \in Y$. Let $U \in Y$ an open neighbourhood of p in Y such that $\pi_Y^{-1}(U) \cong U \times C(L_Y)$. By definition 11 we know that there exists an open subset $V \subset U$ such that V is diffeomorphic to $(0, 1)^l$ where $l = \dim Y$ and such that

$$\phi : (\pi_Y^{-1}(V) \cap \text{reg}(X), g|_{\pi_Y^{-1}(V) \cap \text{reg}(X)}) \rightarrow (V \times \text{reg}(C(L_Y)), dr^2 + h + r^{2c_Y} g_{L_Y})$$

is a quasi-isometry. Therefore by the invariance of L^2 -cohomology under quasi-isometry we can use $(V \times \text{reg}(C(L_Y)), dr^2 + h + r^{2c_Y} g_{L_Y})$ to calculate the L^2 -cohomology of $\pi_Y^{-1}(V) \cap \text{reg}(X)$. Now by corollary 4 and the invariance of L^2 -cohomology under quasi-isometry we know that:

$$H_{2,max}^i(V, h) \cong \begin{cases} \mathbb{R} & i = 0 \\ 0 & i > 0 \end{cases} \quad (52)$$

Therefore by prop. 7 and the fact that $\dim H_{2,max}^i(V, h) < \infty$, which implies that the range of $d_{max,i}$ is closed in the complex $(L^2\Omega^i(V, h), d_{max,i})$, we can use prop. 4 to get that

$$\begin{aligned} H_{2,max}^i(\pi_Y^{-1}(V) \cap \text{reg}(X), g|_{\pi_Y^{-1}(V) \cap \text{reg}(X)}) &\cong \bigoplus_{l=0}^i H_{2,max}^l(V, h) \otimes H_{2,max}^{i-l}(\text{reg}(C(L_Y)), dr^2 + r^{2c_Y} g_{L_Y}) \\ &= H_{2,max}^i(\text{reg}(C(L_Y)), dr^2 + r^{2c_Y} g_{L_Y}). \end{aligned} \quad (53)$$

From the first point of remark 4 follows that

$$(H^i(\mathcal{L}_{2,max}^*, d_{max,*}))_x \cong H_{2,max}^i(\text{reg}(\pi_Y^{-1}(V)), g|_{\text{reg}(\pi_Y^{-1}(V))}).$$

Therefore from this and (53) we get that

$$(H^i(\mathcal{L}_{2,max}^*, d_{max,*}))_x \cong H_{2,max}^i(\text{reg}(C(L_Y)), dr \otimes dr + r^{2c_Y} g_{L_Y}) \quad (54)$$

Now, using the inductive hypothesis we know that this theorem is true for (L_Y, g_{L_Y}) that is $H_{2,max}^i(\text{reg}(L_Y), g_{L_Y}) \cong I^{q_{g_{L_Y}}} H^i(L_Y, \mathcal{R}_0)$ where $q_{g_{L_Y}} = t - p_{g_{L_Y}}$ and $p_{g_{L_Y}}$ is the general perversity associated to g_{L_Y} . This implies that $\dim H_{2,max}^i(\text{reg}(L_Y), g_{L_Y}) < \infty$ for each $i = 0, \dots, \dim L_Y$. From this it follows that at least one of the three hypotheses of corollary 3 is always satisfied. So we can use the same corollary to get:

$$H_{2,max}^i(\text{reg}(C(L_Y)), g_c) = \begin{cases} H_{2,max}^i(\text{reg}(L_Y), g_{L_Y}) & i < \frac{\dim L_Y}{2} + \frac{1}{2c_Y} \\ 0 & i \geq \frac{\dim L_Y}{2} + \frac{1}{2c_Y} \end{cases} \quad (55)$$

Now we can conclude that for each $x \in Y$

$$(H^i(\mathcal{L}_{2,max}^*, d_{max,*}))_x = 0 \text{ for } i > p_g(Y)$$

and therefore the complex $(\mathcal{L}_{2,max}^*, d_{max,*})$ satisfies the second axiom of theorem 1.

To conclude the first part of the proof we have to show that given any stratum $Z \subset X_{n-k-1} - X_{n-k}$ and any point $x \in Z$ the attaching map, that is the morphism given by the composition of

$$\mathcal{L}_{2,max}^*|_{U_{k+1}} \rightarrow i_{k*} \mathcal{L}_{2,max}^*|_{U_k} \rightarrow Ri_{k*} \mathcal{L}_{2,max}^*|_{U_k}$$

where the first morphism is induced by the inclusion $i_k : U_k \rightarrow U_{k+1}$, is a quasi-isomorphism at x up to $p_g(Z)$. By the fact that $(\mathcal{L}_{2,max}^*, d_{max,*})$ is a complex of fine sheaves follows that $i_{k*} \mathcal{L}_{2,max}^*|_{U_k} \rightarrow Ri_{k*} \mathcal{L}_{2,max}^*|_{U_k}$ is a quasi-isomorphism (for example see [3] pag. 32 or [5] pag. 222). Therefore, to conclude, we have only to show that the morphism $\mathcal{L}_{2,max}^*|_{U_{k+1}} \rightarrow i_{k*} \mathcal{L}_{2,max}^*|_{U_k}$ is a quasi-isomorphism at x up to $p_g(Z)$, that is, for each $x \in Z$ it induces an isomorphism

$$(H^i(\mathcal{L}_{2,max}^*|_{U_{k+1}}, d_{max,*}))_x \cong (H^i(i_{k*} \mathcal{L}_{2,max}^*|_{U_k}, d_{max,*}))_x \text{ for } i \leq p_g(Z).$$

Now we give an explicit description of the complexes $(\mathcal{L}_{2,max}^*|_{U_{k+1}}, d_{max,*})$ and $(i_{k*} \mathcal{L}_{2,max}^*|_{U_k}, d_{max,*})$. With this description it will be clear that we can apply corollary 5 to get the conclusion. For every $i = 0, \dots, n$ $\mathcal{L}_{2,max}^i|_{U_{k+1}}$ is the sheaf associated to the following presheaf: let $U \subset X - X_{n-k-1}$ an open subset

$$U \longmapsto \mathcal{D}(U, d_{max,i}) = \begin{cases} \mathcal{D}(U, d_{max,i}) & U \cap (X_{n-1} - X_{n-k-1}) = \emptyset \\ \mathcal{D}(U - (U \cap (X_{n-1} - X_{n-k-1})), d_{max,i}) & U \cap (X_{n-1} - X_{n-k-1}) \neq \emptyset \end{cases} \quad (56)$$

Therefore from proposition 20 follows that

$$\begin{aligned} \mathcal{L}_{2,max}^i|_{U_{k+1}}(U) &\cong \{\omega \in L_{Loc}^2\Omega^i(\text{reg}(U), g|_{\text{reg}(U)}) : \forall p \in U \exists V \text{ open neighbourhood of } p \text{ in } U \\ &\text{ such that } \omega|_{(V - V \cap (X_{n-1} - X_{n-k-1}))} \in \mathcal{D}((V - V \cap (X_{n-1} - X_{n-k-1})), d_{max,i})\}. \end{aligned}$$

We note that in this case $V - V \cap (X_{n-1} - X_{n-k-1}) = \text{reg}(V)$. Analogously for every $i = 0, \dots, n$ $\mathcal{L}_{2,max}^i|_{U_k}$ is the sheaf associated to the following presheaf: let $U \subset X - X_{n-k}$ an open subset

$$U \longmapsto \mathcal{D}(U, d_{max,i}) = \begin{cases} \mathcal{D}(U, d_{max,i}) & U \cap (X_{n-1} - X_{n-k}) = \emptyset \\ \mathcal{D}(U - (U \cap (X_{n-1} - X_{n-k})), d_{max,i}) & U \cap (X_{n-1} - X_{n-k}) \neq \emptyset \end{cases} \quad (57)$$

Therefore from proposition 20 follows that

$$\mathcal{L}_{2,max}^i|_{U_k}(U) \cong \{\omega \in L_{Loc}^2 \Omega^i(\text{reg}(U), g|_{\text{reg}(U)}) : \forall p \in U \exists V \text{ open neighbourhood of } p \text{ in } U \\ \text{such that } \omega|_{(V-V \cap (X_{n-1}-X_{n-k}))} \in \mathcal{D}((V-V \cap (X_{n-1}-X_{n-k})), d_{max,i})\}.$$

We note that in this case $V - V \cap (X_{n-1} - X_{n-k}) = \text{reg}(V)$. Finally we get that given an open subset U of $X - X_{n-k-1}$

$$i_{k*} \mathcal{L}_{2,max}^i|_{U_k}(U) \cong \{\omega \in L_{Loc}^2 \Omega^i(\text{reg}(U), g|_{\text{reg}(U)}) : \forall p \in U - (U \cap (X_{n-k} - X_{n-k-1})) \exists V \text{ open \\ neighbourhood of } p \text{ in } U \text{ such that } \omega|_{(V-V \cap (X_{n-1}-X_{n-k-1}))} \in \mathcal{D}(V-V \cap (X_{n-1}-(X_{n-k-1})), d_{max,i})\}.$$

Also in this case $V - V \cap (X_{n-1} - X_{n-k-1}) = \text{reg}(V)$.

Now, like in the previous case to prove the validity of the second axiom, to show that for each $x \in Z$

$$(H^i(\mathcal{L}_{2,max}^*|_{U_{k+1}}, d_{max,*})_x \cong (H^i(i_{k*} \mathcal{L}_{2,max}^*|_{U_k}, d_{max,*})_x \text{ for } i \leq p_g(Z)$$

is sufficient to show that there exists an open neighbourhood U of $x \in Z$ such that $\pi_Z^{-1}(U) \cong U \times C(L_Z)$ and such that

$$H^i(\mathcal{L}_{2,max}^*|_{U_{k+1}}(\pi_Z^{-1}(U)), d_{max,*}) \cong H^i(i_{k*} \mathcal{L}_{2,max}^*|_{U_k}(\pi_Z^{-1}(U)), d_{max,*}) \text{ for } i \leq p_g(Z)$$

where the isomorphism is induced by the inclusion $i_k : U_k \rightarrow U_{k+1}$. But as we mentioned earlier this last statement follows immediatly by the description given above of the complexes $(\mathcal{L}_{2,max}^*|_{U_{k+1}}, d_{max,*})$, $(i_{k*} \mathcal{L}_{2,max}^*|_{U_k}, d_{max,*})$ and from corollary 5. So given a stratum $Z \subset X_{n-k} - X_{n-k-1}$ and a point $x \in Z$ we can conclude that for $i \leq p_g(Z)$ the natural maps induced by the inclusion of U_k in U_{k+1} induces a quasi isomorphism between

$$\mathcal{L}_{2,max}^*|_{U_{k+1}} \rightarrow i_{k*} \mathcal{L}_{2,max}^*|_{U_k}.$$

So also the third axiom of theorem 1 is satisfied.

Therefore for all $i = 0, \dots, n$ $H^i(\mathcal{L}_{2,max}(\text{reg}(X)), d_{max,*}) \cong I^{q_g} H^i(X, \mathcal{R}_0)$. Finally by the compactness of X , see the third point of porposition 20, we get, for each $i = 0, \dots, n$, the desidered isomorphisms:

$$H_{2,max}^i(\text{reg}(X), g) \cong I^{q_g} H^i(X, \mathcal{R}_0).$$

From the isomorphism $H_{2,max}^i(\text{reg}(X), g) \cong I^{q_g} H^i(X, \mathcal{R}_0)$ it follows that $H_{2,max}^i(\text{reg}(X), g)$ is finite dimensional and then the isomorphism $\mathcal{H}_{abs}^i(\text{reg}(X)) \cong H_{2,max}^i(\text{reg}(X), g)$ is an immediate consequence of proposition 2 and formula 9. The first part of the proof is completed.

To prove the second part of the theorem it is sufficient observe that the finite dimension of $H_{2,max}^i(\text{reg}(X), g)$ for all $i = 0, \dots, n$ implies that the complex $(L^2 \Omega^*(\text{reg}(X), g), d_{max,*})$ is a Fredholm complex. Now, using the isomorphism induced by the Hodge star operator $*$ between the Hilbert complexes $(L^2 \Omega^*(\text{reg}(X), g), d_{min,*})$ and the adjoint complex of $(L^2 \Omega^*(\text{reg}(X), g), d_{max,*})$ and proposition 3, follows that $H_{2,max}^i(\text{reg}(X), g) \cong H_{2,min}^{n-i}(\text{reg}(X), g)$. Finally, using Poincaré duality for intersection homology, theorem 2, we get the isomorphism

$$H_{2,min}^i(\text{reg}(X), g) \cong I^{p_g} H^i(X, \mathcal{R}_0).$$

Now, like in the previous case, we know that $H_{2,min}^i(\text{reg}(X), g)$ is finite dimensional and then the isomorphism $\mathcal{H}_{rel}^i(\text{reg}(X)) \cong H_{2,min}^i(\text{reg}(X), g)$ is an immediate consequences of proposition 2 and formula 9. \square

In the same hypothesis of the theorem 4 we have the following immediate corollaries:

Corollary 6. *Consider the following complex $(C_0^\infty \Omega^i(\text{reg}(X)), d_i)$. Then a necessary condition to have the minimal exstension equal to the maximal one is that the perversities p_g and q_g gives isomorphic intersection cohomology groups.*

Corollary 7. *If every weights is greater or equal than 1, that is for every stratum Y $c_Y \geq 1$, then, for all i , we obtain the following isomorphisms:*

$$\mathcal{H}_{abs}^i(\text{reg}(X), g) \cong H_{2, \max}^i(\text{reg}(X), g) \cong I^{\underline{m}} H^i(X, \mathcal{R}_0) \quad (58)$$

$$\mathcal{H}_{rel}^i(\text{reg}(X), g) \cong H_{2, \min}^i(\text{reg}(X), g) \cong I^{\overline{m}} H^i(X, \mathcal{R}_0) \quad (59)$$

where \underline{m} is the lower middle perversity and \overline{m} is the upper middle perversity.

Corollary 8. *Suppose that the general perversity associated to the quasi edge metric with weights g satisfies $p_g(Z) \geq \text{cod}(Z) - 1$ for each singular stratum Z . Then, for all i , we have the following isomorphisms:*

$$\mathcal{H}_{abs}^i(\text{reg}(X), g) \cong H_{2, \max}^i(\text{reg}(X), g) \cong H^i(X - X_{n-1}, \mathbb{R}) \quad (60)$$

$$\mathcal{H}_{rel}^i(\text{reg}(X), g) \cong H_{2, \min}^i(\text{reg}(X), g) \cong H^i(X, \mathcal{R}_0). \quad (61)$$

Corollary 9. *If p_g is classical perversity in the sense of Goresky-MacPherson and $X_{n-1} = X_{n-2}$ then, for all i , we have the following isomorphisms:*

$$\mathcal{H}_{abs}^i(\text{reg}(X), g) \cong H_{2, \max}^i(\text{reg}(X), g) \cong I^{q_g} H^i(X, \mathbb{R}) \quad (62)$$

$$\mathcal{H}_{rel}^i(\text{reg}(X), g) \cong H_{2, \min}^i(\text{reg}(X), g) \cong I^{p_g} H^i(X, \mathbb{R}) \quad (63)$$

Corollary 10. *Let g, h be two quasi edge metrics with weights on $\text{reg}(X)$ such that $p_g = p_h$. Then for all i*

$$\mathcal{H}_{abs}^i(\text{reg}(x), g) \cong H_{2, \max}^i(\text{reg}(X), g) \cong H_{2, \max}^i(\text{reg}(X), h) \cong \mathcal{H}_{abs}^i(\text{reg}(X), h) \quad (64)$$

and

$$\mathcal{H}_{rel}^i(\text{reg}(x), g) \cong H_{2, \min}^i(\text{reg}(X), g) \cong H_{2, \min}^i(\text{reg}(X), h) \cong \mathcal{H}_{rel}^i(\text{reg}(X), h) \quad (65)$$

In particular a necessary condition for two quasi edge metrics with weights are quasi-isometric is that they induce perversities with isomorphic intersection cohomology groups.

Corollary 11. *Let X' be another compact and oriented smoothly stratified pseudomanifold with a Thom-Mather stratification and h a quasi edge metric with weights on $\text{reg}(X')$. Let $f : X \rightarrow X'$ a stratum preserving homotopy equivalence, see [18] pag 62 for the definition. Suppose that both p_g and p_h depend only on the codimension of the strata and that $p_g = p_h$. Then for all i*

$$\mathcal{H}_{abs}^i(\text{reg}(x), g) \cong H_{2, \max}^i(\text{reg}(X), g) \cong H_{2, \max}^i(\text{reg}(X'), h) \cong \mathcal{H}_{abs}^i(\text{reg}(X'), h) \quad (66)$$

and

$$\mathcal{H}_{rel}^i(\text{reg}(x), g) \cong H_{2, \min}^i(\text{reg}(X), g) \cong H_{2, \min}^i(\text{reg}(X'), h) \cong \mathcal{H}_{rel}^i(\text{reg}(X'), h) \quad (67)$$

Corollary 12. *Let $p : \{\text{singular strata of } X\} \rightarrow \mathbb{Z}$ a general perversity such that for each singular stratum Z of X we have $p(Z) \geq \overline{m}(Z)$. Then there exists a quasi edge metric with weights on $\text{reg}(X)$ such that, for all i ,*

$$\mathcal{H}_{abs}^i(\text{reg}(X), g) \cong H_{2, \max}^i(\text{reg}(X), g) \cong I^{q_g} H^i(X, \mathcal{R}_0) \quad (68)$$

$$\mathcal{H}_{rel}^i(\text{reg}(X), g) \cong H_{2, \min}^i(\text{reg}(X), g) \cong I^{p_g} H^i(X, \mathcal{R}_0) \quad (69)$$

where q_g is the complementary perversity of p_g .

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