Matroid 3-connectivity and branch width*

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Abstract

We prove that, for each nonnegative integer k and each matroid N, if M is a 3-connected matroid containing N as a minor, and the the branch width of M is sufficiently large, then there is a k-element set $X \subseteq E(M)$ such that one of $M \setminus X$ and M/X is 3-connected and contains N as a minor.

1 Introduction

We prove the following theorem.

Theorem 1.1. *Let M be a matroid, let N be a minor of M, and let k be a nonnegative constant. If the branch width of M is at least $20k + 2|E(N)|$ then there is a set $X \subseteq E(M)$ that has at least k elements and is both independent and coindependent, such that one of $M \setminus X$ and M/X is 3-connected with N as minor.*

Our main result (Theorem 5.3) is a strengthening of Theorem 1.1 that involves tangles. Theorem 1.1 can be seen as a generalization of the Splitter Theorem, proven by Seymour [16] and, independently, by Tan [17]. In particular, consider the following formulation:

Theorem 1.2 (Splitter Theorem)**.** *Let M be a 3-connected matroid, and N a 3-connected proper minor of M. If M is not a wheel or a whirl, then there is an $e \in E(M)$ such that one of $M \setminus e$ and M/e is 3-connected with a minor isomorphic to N.*

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When the minor N is the empty matroid, this result is known as Tutte's Wheels and Whirls Theorem [18]. Several variants exist, such as [12, 13].

In the Splitter Theorem, the two obstructions to the existence of a removable element, the wheels and whirls, have branch width 2. The branch width of a matroid is minor-monotone, so an easy consequence of Theorem 1.2 is

Corollary 1.3. *Let M be a 3-connected matroid with* $\mathrm{bw}(M) \geq 3$*, and N a 3-connected proper minor of M. Then there is an $e \in E(M)$ such that one of $M \backslash e$ and M/e is 3-connected with a minor isomorphic to N.*

Sometimes deleting one element is insufficient. For instance, in papers on stabilizers or excluded minors the notion of a *deletion pair* is central [19, 5, 6, 8]. In those papers, 3-connectivity cannot be guaranteed when two elements are removed, but the 2-separations that are introduced can be handled at the cost of a more complicated analysis. Our result generalizes Corollary 1.3 by showing that, if the branch width is large enough, then we can either delete or contract any number of elements and preserve both 3-connectivity and a specified minor.

Note that, rather than preserving a matroid isomorphic to the minor N, we preserve N itself. Additionally, we impose fewer conditions on the connectivity of N. Oxley [14, Theorem 11.1.2] describes a version of the Splitter Theorem in which N is not 3-connected, but the conclusion of that theorem is significantly weaker than in the 3-connected case.

Notation. Our notation and terminology follows Oxley [14]. Additionally, if $\mathscr{X}$ is a collection of sets, $\cup\mathscr{X}$ denotes the union of all sets in $\mathscr{X}$, and $\cap\mathscr{X}$ the intersection. As has become customary in matroid theory papers, $\mathrm{si}(M)$ denotes the simplification of M and $\mathrm{co}(M)$ denotes the cosimplification.

2 Connectivity, branch width, and tangles

2.1 Closure

Definition 2.1. Let M be a matroid and $X \subseteq E(M)$. We say X is *closed* if $\mathrm{rk}_M(X \cup e) > \mathrm{rk}_M(X)$ for all $e \in E(M) - X$. We say X is *coclosed* if X is closed in M^*. We say X is *fully closed* if X is both closed and coclosed.

As usual we denote the smallest closed set containing X by $\mathrm{cl}_M(X)$, and the smallest coclosed set containing X by $\mathrm{cl}^*_M(X)$. Additionally, the smallest fully closed set containing X is denoted by $\mathrm{fcl}_M(X)$. Some more

terminology: a *line* is a closed set of rank two. A line is *long* if it has at least three elements.

The following elementary lemma is surprisingly useful. It has appeared in at least [10, 3].

Lemma 2.2. *Let M be a matroid, $e \in E(M)$, and (A,B) a partition of $E(M)-e$. Then $e \in \mathrm{cl}_M(A)$ if and only if $e \notin \mathrm{cl}^*_M(B)$.*

Proof. Note that $e \in \mathrm{cl}_M(A)$ if and only if e is a loop in M/A. Dually, $e \in \mathrm{cl}^*_M(B)$ if and only if e is a coloop in $M\backslash B$. Suppose $e \in \mathrm{cl}_M(A)$. Then there is a circuit containing e in $M\backslash B$, so $e \notin \mathrm{cl}^*_M(B)$. Conversely, suppose $e \notin \mathrm{cl}^*_M(B)$. Then there is a circuit containing e in $M\backslash B$, so $e \in \mathrm{cl}_M(A)$. □

2.2 Connectivity and separations

An unfortunate consequence of the graph-theoretic pedigree of matroid theory is that two definitions of the connectivity function coexist (differing from each other by an additive constant of 1). We will take the smaller of these definitions:

Definition 2.3. Let M be a matroid. The *connectivity function* $\lambda_M : 2^{E(M)} \to \mathbb{N}$ is defined by

$$\lambda_M(X) := \mathrm{rk}_M(X) + \mathrm{rk}_M(E(M)-X) - \mathrm{rk}(M).$$

We will use the following elementary properties of the connectivity function:

Lemma 2.4. *Let M be a matroid, and $X, Y \subseteq E(M)$. The connectivity function of M has the following properties.*

(i) $\lambda_M(X) = \mathrm{rk}_M(X) + \mathrm{rk}_{M^*}(X) - |X|$;

(ii) $\lambda_M(E(M)-X) = \lambda_M(X)$;

(iii) $\lambda_{M^*}(X) = \lambda_M(X)$;

(iv) *If $e \in E(M)-X$ then* $\lambda_{M\backslash e}(X) \le \lambda_M(X) \le \lambda_{M\backslash e}(X)+1$.

(v) $\lambda_M(X) + \lambda_M(Y) \ge \lambda_M(X\cap Y) + \lambda_M(X\cup Y)$;

For ease of reference we repeat the usual definitions of separations and connectivity before stating some less common results.

Definition 2.5. Let M be a matroid. A set $X \subseteq E(M)$ is *k-separating* if $\lambda_M(X) < k$. It is *exactly* k-separating if $\lambda_M(X) = k-1$.

Definition 2.6. Let M be a matroid, and let (X,Y) be a partition of $E(M)$. If $|X|, |Y| \ge k$ and $\lambda_M(X) < k$ then (X,Y) is a *k-separation* of M. If $\lambda_M(X) = k-1$ then (X,Y) is an *exact* k-separation of M.

Definition 2.7. A matroid M is *k-connected* if M has no k'-separations with $k' < k$.

Two partitions (X_1, Y_1) and (X_2, Y_2) *cross* if $X_1 \cap X_2$, $X_1 \cap Y_2$, $Y_1 \cap X_2$, $Y_1 \cap Y_2$ are all nonempty. An application of the following lemma (from [11]) is called an *uncrossing*.

Lemma 2.8. *Let M be a k-connected matroid, and let X_1, X_2 be k-separating sets.*

(i) If $|X_1 \cap X_2| \geq k-1$ then $X_1 \cup X_2$ is k-separating.

(ii) If $|E(M) - (X_1 \cup X_2)| \geq k-1$ then $X_1 \cap X_2$ is k-separating.

Proof. By Lemma 2.4(v),

$$2(k-1) \geq \lambda_M(X_1) + \lambda_M(X_2) \geq \lambda_M(X_1 \cap X_2) + \lambda_M(X_1 \cup X_2). \qquad (1)$$

Since M is k-connected, any set X with at least $k-1$ elements has $\lambda_M(X) \geq k-1$. Therefore, if one of the summands on the right is at least $k-1$ then the other is at most $k-1$, and the result follows. □

Since we wish to preserve 3-connectivity, we have to know how separations change when taking minors.

Lemma 2.9. *Let M be a k-connected matroid, (X, Y) an exact k-separation of M, and $e \in X$, not a loop. The following are equivalent.*

(i) $(X-e, Y)$ is a $(k-1)$-separation in M/e;

(ii) $e \in \mathrm{cl}_M(Y) \cap \mathrm{cl}_M(X-e)$;

(iii) $e \notin \mathrm{cl}^*_M(Y) \cup \mathrm{cl}^*_M(X-e)$.

Proof. The equivalence of (*ii*) and (*iii*) is an immediate consequence of Lemma 2.2. Assume (*i*) holds. Since e is not a loop, $\mathrm{rk}(M/e) < \mathrm{rk}(M)$. Hence

$$k-2 = \lambda_{M/e}(X-e) = \mathrm{rk}_{M/e}(X-e) + \mathrm{rk}_{M/e}(Y) - \mathrm{rk}(M) + 1.$$

It follows that $\mathrm{rk}_{M/e}(X-e) < \mathrm{rk}_M(X)$ and $\mathrm{rk}_{M/e}(Y) < \mathrm{rk}_M(Y)$. But this necessarily implies $e \in \mathrm{cl}_M(Y)$. If $e \notin \mathrm{cl}_M(X-e)$ then $\lambda_M(X-e) = k-1$, contradicting k-connectedness of M.

Conversely, assume (*ii*) holds. Then $\mathrm{rk}_{M/e}(X-e) < \mathrm{rk}_M(X-e) = \mathrm{rk}_M(X)$ and $\mathrm{rk}_{M/e}(Y) < \mathrm{rk}_M(Y)$. Since also $\mathrm{rk}(M/e) < \mathrm{rk}(M)$, the result follows. □

Lemma 2.10. *Let M be a k-connected matroid, (X, Y) a k-separation of M, and $e \in X$ such that M/e is k-connected. Then $e \notin \mathrm{cl}_M(Y)$.*

Proof. Let M, X, Y, e be as stated, and suppose that, contrary to the claim, $e \in \mathrm{cl}_M(Y)$. If $e \in \mathrm{cl}_M(X-e)$ then, by Lemma 2.9, M/e is not k-connected, a contradiction. If $e \notin \mathrm{cl}_M(X-e)$ then $\lambda_M(X-e) < \lambda_M(X)$. But M is k-connected, a contradiction. □

In some of our proofs we will require that a minor N of a matroid M has no loops or coloops. The following easy lemma implies that this assumption is not overly restrictive:

Lemma 2.11. *Let M be a connected matroid and let N be a minor of M. If N has l elements that are either a loop or a coloop, then M has a minor N' such that N is a minor of N', such that N' has no loops and coloops, and such that $|E(N')| \leq |E(N)| + l$.*

Proof. Let M be a connected matroid, let N be a minor of M, and let $C, D \subseteq E(M)$ be such that $N = M/C \setminus D$ with C independent and D coindependent. Let e be a loop of N. Since M is connected, e is not a loop of M. Hence there is a circuit $X \subseteq C \cup \{e\}$ using e with $|X| \geq 2$. Let $f \in X - \{e\}$, and consider $N'' := M/(C - \{f\}) \setminus D$. Since $\{e, f\}$ is a parallel pair in N'', the matroid N'' has strictly fewer loops than N. Moreover, $|E(N'')| = |E(N)| + 1$. The result now follows by duality and induction. □

We note that Lemos and Oxley proved that, if N has k components, then M has a *connected* minor N' on at most $|E(N)| + 2k - 2$ elements [7].

2.3 2-separations

In this subsection we consider preserving a minor in the presence of a 2-separation. We omit the straightforward proof of the following lemma.

Lemma 2.12. *Let (A, B) be a 2-separation of a connected matroid M. If $|B| = 2$, then B is a parallel or series pair.*

Lemma 2.13. *Let M be a matroid, N a minor of M, and (A, B) a 2-separation of M with $B \cap E(N) = \emptyset$. Then one of $M \setminus B$ and M/B has N as minor.*

Proof. Let $M, N, (A, B)$ be as stated. Since no element of B is in N, there are disjoint sets $C, D \subseteq B$ such that $B = C \cup D$ and $M/C \setminus D$ has N as minor. If $\lambda_{M \setminus D}(C) = 0$ then $M \setminus D/C = M \setminus D \setminus C$ and the result follows. Therefore $\lambda_{M \setminus D}(C) = \mathrm{rk}_M(A) + \mathrm{rk}_M(C) - \mathrm{rk}_M(A \cup C) = 1$. But

$$\begin{aligned}\lambda_{M/C}(A) &= \mathrm{rk}_M(A \cup C) - \mathrm{rk}_M(C) + \mathrm{rk}_M(B) - \mathrm{rk}_M(C) - (\mathrm{rk}(M) - \mathrm{rk}_M(C)) \\ &= \mathrm{rk}_M(A) + \mathrm{rk}_M(C) - 1 - \mathrm{rk}_M(C) + \mathrm{rk}_M(B) - \mathrm{rk}(M) \\ &= \lambda_M(A) - 1 = 0,\end{aligned}$$

so D is a separator of M/C. Hence $M/C \setminus D = M/C/D$, and the result follows. □

An easy consequence is this:

Corollary 2.14. *Let M be a matroid, N a minor of M, and (A,B) a 2-separation of M with $B \cap E(N) = \emptyset$. If $e \in B - \mathrm{cl}_M(A)$ then M/e has N as minor.*

Proof. Let $M, N, (A,B), e$ be as stated. If M/B has N as minor then we are done, so we may assume $M\backslash B$ has N as minor. Consider $M' := M\backslash(B-e)$. Since $e \notin \mathrm{cl}_{M'}(A)$, it is a coloop of M', and therefore $M' \setminus e = M'/e$. □

We immediately find the following:

Corollary 2.15. *Let M be a matroid, N a minor of M, and (A,B) a 2-separation of M with $B \cap E(N) = \emptyset$. If $e \in B - (\mathrm{cl}_M(A) \cup \mathrm{cl}^*_M(A))$ then both $M \setminus e$ and M/e have N as minor.*

Next we consider the case in which $E(N)$ intersects B in exactly one element. The following is [14, Lemma 8.3.3].

Lemma 2.16. *Let M be a matroid, and let (A,B) be a 2-separation of M. If C_1, C_2 are circuits of M, both of which meet both A and B, then $(C_1 \cap A) \cup (C_2 \cap B)$ is a circuit of M.*

Lemma 2.17. *Let M be a connected matroid, N a minor of M with no loops and coloops, and (A,B) a 2-separation of M with $B \cap E(N) = \{f\}$. If f is not in series or in parallel with any other element in M, then there exists an element $e \in B - \{f\}$ such that $M \setminus e$ and M/e both contain N as a minor.*

Proof. Suppose the lemma is false. Let $M, N, (A,B), f$ constitute a counterexample with $|B|$ minimal. Let C and D be disjoint subsets of $B - \{f\}$ such that $M/C\setminus D$ has N as a minor, and pick $e \in D$. If there is a circuit X of M/e using f and at least one element of A, then Lemma 2.16 implies that M/e has N as minor.

Hence there is a separation (A', B') such that $E(N) - \{f\} \subseteq A'$ and $f \in B'$. By uncrossing with (A,B), it follows that $B' \subsetneq B$. But this contradicts minimality of B.

The only remaining possibility is that D is empty. But then, by duality, also $C = \emptyset$, a contradiction. □

2.4 3-connectivity and fans

Recall the following lemma by Bixby:

Lemma 2.18 (Bixby [1]; see also Oxley [14, Proposition 8.4.6])**.** *Let M be a 3-connected matroid, and $e \in E(M)$. Then at least one of* $\mathrm{si}(M/e)$ *and* $\mathrm{co}(M \setminus e)$ *is 3-connected.*

Recall that a set $T \subseteq E(M)$ is a *triangle* if $M|T \cong U_{2,3}$, and a *triad* if it is a triangle of M^*.

Lemma 2.19 (Tutte's Triangle Lemma; see Oxley [14, Lemma 8.4.9])**.** *Let M be a 3-connected matroid with $|E(M)| \geq 4$, and let $T = \{e, f, g\}$ be a triangle such that neither $M \setminus e$ nor $M \setminus f$ is 3-connected. Then M has a triad containing e and exactly one of f and g.*

Tutte's Triangle Lemma naturally leads to the notion of a *fan*:

Definition 2.20. Let M be a matroid, and $F = (x_1, x_2, \ldots, x_k)$, $k \geq 3$, an ordered set of distinct elements of $E(M)$. We say that F is a *fan* of M if $\{x_1, x_2, x_3\}$ is either a triangle or a triad, and for each $i \in \{1, \ldots, k-3\}$, if $\{x_i, x_{i+1}, x_{i+2}\}$ is a triangle then $\{x_{i+1}, x_{i+2}, x_{i+3}\}$ is a triad, and if $\{x_i, x_{i+1}, x_{i+2}\}$ is a triad then $\{x_{i+1}, x_{i+2}, x_{i+3}\}$ is a triangle.

A few trivial observations:

Lemma 2.21. *Let $F = (x_1, x_2, \ldots, x_k)$ be a fan of a matroid M.*

(i) F is a fan of M^*, with triangles and triads exchanged;

(ii) $(x_k, x_{k-1}, \ldots, x_1)$ is a fan of M;

(iii) If $X \subseteq E(M)$ is fully closed, and F is an inclusionwise maximal fan contained in X, then F is an inclusionwise maximal fan in $E(M)$;

(iv) If $k \geq 4$, and $1 < l < k$, then neither $M \setminus x_l$ nor M/x_l is 3-connected.

The following lemma is due to Oxley and Wu [9].

Lemma 2.22. *Let M be a 3-connected matroid that is not a wheel or a whirl, and let F be an inclusionwise maximal fan of M with $k \geq 3$ elements. Then the elements of F can be ordered $(x_1, x_2, \ldots, x_k)$ such that $(x_1, x_2, \ldots, x_k)$ is a fan, one of $M \setminus x_1, M/x_1$ is 3-connected, and one of $M \setminus x_k, M/x_k$ is 3-connected.*

Note that in a fan of length at least 4 the ends of the fan, x_1 and x_k, are the same for any ordering, and in a fan of length at least 5 the order is completely fixed. We will upgrade Oxley and Wu's result so that we can preserve a minor, at the cost of a slightly worse bound on the size:

Lemma 2.23. *Let M be a 3-connected matroid that is not a wheel or a whirl, let N be a minor of M without loops or coloops, and let $F = (x_1, x_2, \ldots, x_k)$ be an inclusionwise maximal fan of M with $k \geq 4$ elements. If $|E(N) \cap F| \leq 1$ then one of $M\backslash x_1, M/x_1, M\backslash x_k, M/x_k$ is 3-connected with N as minor.*

Proof. Let M, N, F, k be as stated, and suppose the theorem fails. To simplify notation we will assume k to be even, leaving the analogous case for odd k to the reader. By reversing the fan if necessary we may assume $x_1 \notin E(N)$. By dualizing M and N if necessary we may assume that $\{x_1, x_2, x_3\}$ is a triangle (and therefore that $\{x_{k-2}, x_{k-1}, x_k\}$ is a triad). Hence $M \setminus x_1$ is 3-connected. Suppose $M \setminus x_1$ does not have N as minor. Then M/x_1 has N as minor. Let $M' := M/x_1$. The set $F - x_1$ is 2-separating in M'.

Claim 2.23.1. $x_k \in E(N)$.

Proof. Suppose this is not the case. Note that $F - \{x_1, x_k\}$ is a separator of $M' \setminus x_k$. First, if $E(N) \cap (F - \{x_1, x_k\}) = \{f\}$, then $M' \setminus x_k$ cannot have N as minor, since in such a minor f would be either a loop or a coloop. Hence M'/x_k has N as minor. Next, if $E(N) \cap (F - x_1) = \emptyset$ then Corollary 2.14 implies that M'/x_k has N as minor. In both cases it follows that M/x_k has N as minor. But that matroid is 3-connected, and the result holds. □

Note that $\{x_2, x_3\}$ form a parallel pair in M'. If $k = 4$ then x_k is a coloop in $M'\backslash\{x_2, x_3\}$, so M'/x_2 has N as minor. If $k > 4$ then $\{x_2, x_3, x_4\}$ is a 2-separation in M' disjoint from $E(N)$. Moreover, we have $x_2 \in \mathrm{cl}^*_{M'}(\{x_3, x_4\})$, so by Lemma 2.2, $x_2 \notin \mathrm{cl}_{M'}(E(M') - \{x_2, x_3, x_4\})$. From Corollary 2.14 it then follows that M'/x_2 has N as minor.

Therefore M/x_2 has N as minor. In that matroid x_1 and x_3 are in parallel, from which it follows that $M\backslash x_1$ has N as minor, a contradiction. □

2.5 Tangles and their matroids

Instead of using branch width directly, we will use the notion of a *tangle*, first defined by Robertson and Seymour [15] for hypergraphs, and extended to matroids by Dharmatilake [2]. Our definitions follow Geelen, Gerards, Robertson, and Whittle [4].

Definition 2.24. Let M be a matroid, and $\mathscr{T}$ a collection of subsets of $E(M)$. Then $\mathscr{T}$ is a *tangle of order θ* of M if

(*i*) For all $X \in \mathscr{T}$, $\lambda_M(X) < \theta$;

(*ii*) For all $X \subseteq E(M)$ with $\lambda_M(X) < \theta$, either $X \in \mathscr{T}$ or $E(M) - X \in \mathscr{T}$;

(*iii*) If $X, Y, Z \in \mathscr{T}$ then $X \cup Y \cup Z \neq E(M)$;

(*iv*) For each $e \in E(M)$, $E(M) - \{e\} \notin \mathscr{T}$.

Hence the empty set is a tangle of order 0 of any nonempty matroid. The following theorem, which was implicit in Robertson and Seymour [15], shows that tangles and branch width are closely related. A proof using the definition of tangle given above can be found in Geelen et al. [4]. Note that they stated and proved the result for arbitrary connectivity functions.

Theorem 2.25. *Let M be a matroid. The branch width of M equals the maximum order θ of a tangle of M.*

Because of this result, there is no need to define branch width here. We continue with some basic tangle facts, which can easily be deduced from the definition:

Lemma 2.26. *Let M be a matroid, and $\mathscr{T}$ a tangle of M of order θ.*

(i) If $X \in \mathscr{T}$ and $X' \subseteq X$ is such that $\lambda_M(X') < \theta$ then $X' \in \mathscr{T}$;

(ii) If $\theta' < \theta$, and $\mathscr{T}' = \{X \in \mathscr{T} : \lambda_M(X) < \theta'\}$, then $\mathscr{T}'$ is a tangle of M of order θ';

(iii) $\mathscr{T}$ is a tangle of order θ of M^*.

Tangles can be helpful in dealing with crossing separations.

Lemma 2.27. *Let M be a matroid, $\mathscr{T}$ a tangle of order θ, and $X, Y \in \mathscr{T}$. If $\lambda_M(X \cup Y) < \theta$ then $X \cup Y \in \mathscr{T}$.*

Proof. Let $Z := E(M) - (X \cup Y)$. Either $X \cup Y \in \mathscr{T}$ or $Z \in \mathscr{T}$, by 2.24(*ii*). But if $Z \in \mathscr{T}$ then $X \cup Y \cup Z = E(M)$, contradicting 2.24(*iii*). □

We will apply this lemma regularly. In the case $Y = \{e\}$ we may do so without referring to it.

A useful means for studying tangles is the *tangle matroid*. The following result is from Geelen et al. [4]: *[Changed!]*

To Do!

Theorem 2.28. *Let M be a matroid, and $\mathscr{T}$ a tangle of M of order θ. Let $\rho : 2^{E(M)} \to \mathbb{N}$ be defined by*

$$\rho(X) := \begin{cases} \min\{\lambda_M(Y) : X \subseteq Y \in \mathscr{T}\} & \text{if } X \subseteq Y \in \mathscr{T} \\ \theta & \text{otherwise.} \end{cases}$$

Then ρ is the rank function of a matroid.

We will denote this matroid by $M(\mathscr{T})$, and write $\mathrm{rk}_{\mathscr{T}}, \mathrm{cl}_{\mathscr{T}}, \ldots$ as shorthand for $\mathrm{rk}_{M(\mathscr{T})}, \mathrm{cl}_{M(\mathscr{T})}, \ldots$. We will often work with independent sets in the tangle matroid.

Lemma 2.29. *Let M be a matroid, $\mathscr{T}$ a tangle of M of order θ, and X a set that is independent in $M(\mathscr{T})$. Then X is both independent and coindependent in M.*

Proof. Suppose X is not independent in M. Then $\lambda_M(X) \leq \mathrm{rk}_M(X) < |X| \leq \theta$. It follows that $E(M) - X \in \mathscr{T}$. But from repeated application of Lemma 2.27, starting from the singleton subsets of X, we conclude that $X \in \mathscr{T}$, a contradiction. The result now follows by duality. □

If N is a minor of M then we can derive a tangle of N from a tangle of M, as follows.

Lemma 2.30. *Let M be a matroid, and N a minor of M such that $E(M) - E(N) = S$. Let $\mathscr{T}$ be a tangle of M of order θ. Define*

$$\mathscr{T}' := \{X - S : X \in \mathscr{T}, \lambda_N(X) < \theta - |S|\}.$$

Then $\mathscr{T}'$ is a tangle of N of order $\theta - |S|$.

Proof. We give the proof if $S = \{e\}$. The result then follows by induction. The result is trivial if $\theta \leq 1$, since that implies $\mathscr{T}' = \emptyset$. Hence we may assume $\theta \geq 2$.

Note that 2.24(*i*) follows immediately from our definition. For 2.24(*ii*), if (X, Y) is a k-separation of N with $k < \theta$, then Lemma 2.4(*iv*) implies that $(X \cup e, Y)$ and $(X, Y \cup e)$ are $(k+1)$-separations of M, and hence either $X \in \mathscr{T}$ or $Y \in \mathscr{T}$ (using Lemma 2.27). We may assume the former. Then it follows immediately that $X \in \mathscr{T}'$. For 2.24(*iii*), note that $\lambda_M(X \cup e) \leq \lambda_M(X) + 1 < \theta$, so $(X \cup e) \cup Y \cup Z$ does not cover $E(M)$. Hence $X \cup Y \cup Z$ cannot cover $E(N)$. Finally, suppose $E(N) - \{f\} \in \mathscr{T}'$ for some $f \in E(N)$. Then we must have $E(M) - \{e, f\} \in \mathscr{T}$. But we also have $\{e\}, \{f\} \in \mathscr{T}$, contradicting 2.24(*iii*). □

We say $\mathscr{T}'$ is the tangle *inherited from* $\mathscr{T}$. We note some elementary properties of the corresponding tangle matroid:

Lemma 2.31. *Let M be a matroid, $\mathscr{T}$ a tangle of M of order θ, and N a minor of M with $E(M) - E(N) = \{e\}$. Let $\mathscr{T}'$ be the tangle of N inherited from $\mathscr{T}$, and let $Z \subseteq E(N)$.*

(i) $\mathrm{rk}_{\mathscr{T}}(Z) - 1 \leq \mathrm{rk}_{\mathscr{T}'}(Z) \leq \mathrm{rk}_{\mathscr{T}}(Z)$*;*

(ii) If $e \notin \mathrm{cl}_{\mathscr{T}}(Z)$ *and* $\mathrm{rk}_{\mathscr{T}}(Z) < \theta$ *then* $\mathrm{rk}_{\mathscr{T}'}(Z) = \mathrm{rk}_{\mathscr{T}}(Z)$.

Proof. Let $M, \mathscr{T}, N, e, \mathscr{T}', Z$ be as stated. The first claim is a straightforward consequence of 2.4(*iv*). Suppose the second claim is false. Let $Z' \supseteq Z$ be such that $Z' \in \mathscr{T}'$ and $k = \lambda_N(Z') < \mathrm{rk}_{\mathscr{T}}(Z)$. By dualizing M and N if necessary we may assume $N = M/e$. Since $\lambda_M(Z') > k$, we must have $e \in \mathrm{cl}_M(Z') \cap \mathrm{cl}_M(E(M) - (Z' \cup e))$. But then $\lambda_M(Z' \cup e) = k + 1$, and therefore $\mathrm{rk}_{\mathscr{T}}(Z \cup e) \leq k + 1 \leq \mathrm{rk}_{\mathscr{T}}(Z)$. But this implies $e \in \mathrm{cl}_{\mathscr{T}}(Z)$, a contradiction. □

An easy corollary is the following.

Lemma 2.32. *Let M be a matroid, let $\mathscr{T}$ be a tangle of M, let X be a $\mathscr{T}$-independent subset of $E(M)$, and let $e \in X$. Then $X - e$ is $\mathscr{T}'$-independent in $M \setminus e$, where $\mathscr{T}'$ is the tangle of $M \setminus e$ inherited from $\mathscr{T}$.*

Proof. Let $M, \mathscr{T}, X, e, \mathscr{T}'$ be as stated, and assume the result is false. Then there is a set $Z \supseteq X - e$ with $Z \in \mathscr{T}'$ and $\lambda_{M\setminus e}(Z) < |X - e|$. By definition of $\mathscr{T}'$, either $Z \in \mathscr{T}$ or $Z \cup e \in \mathscr{T}$. By Lemma 2.4(*iv*) we have that $\lambda_M(Z \cup e) \leq \lambda_{M\setminus e}(Z) + 1 < |X|$. It follows that $Z \cup e \in \mathscr{T}$, because otherwise its complement together with Z and $\{e\}$ would cover $E(M)$. But $X \subseteq Z$, a contradiction to X being $\mathscr{T}$-independent. □

Lemma 2.33. *Let M be a matroid, let $\mathscr{T}$ be a tangle of M, and let $X \subseteq E(M)$ be $\mathscr{T}$-independent. Let $Y := \mathrm{cl}_{\mathscr{T}}(X)$. If $e \in Y - X$ then $e \in \mathrm{cl}_M(Y - e)$ or $e \in \mathrm{cl}^*_M(Y - e)$.*

Proof. Let $M, \mathscr{T}, X, Y, e$ be as stated, and assume $\mathrm{rk}_{\mathscr{T}}(X) = t$. Then $\lambda_M(Y) = t$, and $\lambda_M(Y - e) \geq t$, since $X \subseteq Y - e$. Suppose e is neither in the closure nor in the coclosure of $Y - e$. Then

$$\begin{aligned}\lambda_M(Y) &= \mathrm{rk}_M(Y) + \mathrm{rk}^*_M(Y) - |Y| \\ &= \mathrm{rk}_M(Y - e) + 1 + \mathrm{rk}^*_M(Y - e) + 1 - (|Y - e| + 1) \\ &= \lambda_M(Y - e) + 1 > t,\end{aligned}$$

a contradiction. □

Lemma 2.34. *Let M be a 3-connected matroid, $\mathscr{T}$ a tangle of M, and X, X' long lines of $M(\mathscr{T})$ such that $\mathrm{rk}_{\mathscr{T}}(X \cup X') = 4$. Let $e \in X$ and $M' \in \{M \setminus e, M/e\}$ be such that M' is 3-connected. Let $\mathscr{T}'$ be the tangle of M' inherited from $\mathscr{T}$. Then X' is closed in $M(\mathscr{T}')$.*

Proof. Let $M, \mathscr{T}, X, X', e, M'$ be as stated. Since $\mathscr{T}$ is a tangle of M^* we may dualize as necessary and assume $M' = M/e$. Suppose there is a $Z \in \mathscr{T}'$ with $X' \subsetneq Z$, and $\mathrm{rk}_{\mathscr{T}'}(Z) = 2$. Define $Y := E(M) - Z$. Then $(Z, Y \cup e)$ and $(Z \cup e, Y)$ are 4-separations of M, with $e \in \mathrm{cl}_M(Z) \cap \mathrm{cl}_M(Y)$, by Lemma 2.9. Since $e \notin \mathrm{cl}_M(E(M) - X)$ by Lemma 2.10, $Z \cap X$ and $Y \cap X$ are both nonempty. Let $e' \in Z \cap X$. Then $\mathrm{rk}_{\mathscr{T}}(X' \cup \{e, e'\}) \leq 3$, since $Z \cup e$ is 4-separating. But $\mathrm{cl}_{\mathscr{T}}(\{e, e'\}) = X$, so $3 \geq \mathrm{rk}_{\mathscr{T}}(X \cup \{e, e'\}) = \mathrm{rk}_{\mathscr{T}}(X \cup X') = 4$, a contradiction. □

3 Finding elements to remove

As a first step towards our result we show that, if the branch width is high enough, we can remove a single element and preserve 3-connectivity and N as minor.

Theorem 3.1. *Let M be a 3-connected matroid, let N be a minor of M without loops or coloops, let $\mathscr{T}$ be a tangle of M of order at least 3, let X be a long line in $M(\mathscr{T})$, and let $f \in X$. If $(X - f) \cap E(N) = \emptyset$, then there exists an $e \in X - f$ such that either $M \setminus e$ or M/e is 3-connected with N as minor.*

Proof. Let $M, N, \mathscr{T}, X, f$ be as stated. If X contains a fan of length 4 or more then the result follows from Lemma 2.23. Therefore we can assume that X contains no fans of length at least 4.

Next, assume that there is an element $e \in X$ such that both $M \setminus e$ and M/e have N as minor. By Lemma 2.18, either $\mathrm{si}(M/e)$ or $\mathrm{co}(M \setminus e)$ is 3-connected. By duality we may assume the former. If M/e is simple then the result follows, so e is on a triangle, say with elements g and h. Assume $g \neq f$. Lemma 2.19 implies that at least two of $\{e, g, h\}$ can be deleted keeping 3-connectivity. If e is one of them the result follows. Otherwise, since $\{g, h\}$ is a parallel pair in M/e, it follows that $M/e\backslash g$ has N as minor. But then $M\backslash g$ has N as minor and is 3-connected. Hence we may assume that for all $e \in X - f$, exactly one of $M \setminus e$ and M/e has N as minor.

Claim 3.1.1. *There is an element $e \in X - f$ such that $M\backslash e$ is cosimple with N as minor, or M/e is simple with N as minor.*

Proof. Pick, possibly after dualizing, an element $e \in X - f$ such that M/e has N as minor. If M/e were simple, we would be done, so we can assume that e is in a triangle $\{e, g, h\}$ of M. Assume $g \neq f$. Since $\{g, h\}$ is a parallel pair, it follows as before that $M \setminus g$ has N as minor, and is simple. □

Now let e be an element such that M/e is simple and has N as minor. Clearly M/e is also cosimple. If M/e is 3-connected, then the result follows. Otherwise Corollary 2.15 and Lemma 2.17 imply the existence of an element e' such that both M/e' and $M \setminus e'$ have N as minor, a case we already dealt with. Duality now completes the proof. □

Next, we find a set of deletions and contractions:

Theorem 3.2. *Let s be an integer, let M be a 3-connected matroid, let $\mathscr{T}$ be a tangle of M of order $\theta \geq 6$, let N be a minor of M with no loops and coloops, and let $t := \mathrm{rk}_{\mathscr{T}}(E(N))$. If $\theta \geq 2s + t + 1$, then there are disjoint sets $C, D \subseteq E(M) - E(N)$ such that $M/C\backslash D$ is 3-connected with N as minor, such that $\mathrm{rk}_{\mathscr{T}}(E(N) \cup C \cup D) = t + |C \cup D|$, and such that $|C \cup D| \geq s$.*

To achieve this we use the following lemma:

Lemma 3.3. *Let M be a 3-connected matroid, let $\mathscr{T}$ be a tangle of M of order $\theta \geq 3$, let N be a minor of M with no loops and coloops, let*

$t := \mathrm{rk}_{\mathscr{T}}(E(N))$, and let H be a closed set of $M(\mathscr{T})$ containing $E(N)$. If $\theta > \mathrm{rk}_{\mathscr{T}}(H)$ then there is an element $e \in E(M) - H$ such that one of $M \setminus e$ and M/e is 3-connected with N as minor.

Proof. Let $M, \mathscr{T}, \theta, N$, and H be as stated. Suppose there is an element $e \in E(M) - H$ such that e is on no long line of $M(\mathscr{T})$. Then both M/e and $M \setminus e$ are 3-connected. One of these has N as minor, and the result follows.

Now pick $e \in E(M) - H$, and let X be a long line containing e. Note that X intersects H, and therefore $E(N)$, in at most one element. The result now follows from Theorem 3.1. □

With this in hand the proof of Theorem 3.2 is no longer difficult.

Proof of Theorem 3.2. Let $s, M, \mathscr{T}, \theta, N, t$ be as stated, and let $C, D \subseteq E(M) - E(N)$ be disjoint, such that $M/C \setminus D$ is 3-connected with N as minor, such that $\mathrm{rk}_{\mathscr{T}}(E(N) \cup C \cup D) = t + |C \cup D|$, and such that $|C \cup D|$ is maximal. Suppose $|C \cup D| < s$. Define $H := \mathrm{cl}_{\mathscr{T}}(E(N) \cup C \cup D)$.

Let $M' := M/C \setminus D$, let $\theta' := \theta - |C \cup D|$, let $\mathscr{T}'$ be the tangle of M' of order θ' inherited from $\mathscr{T}$, and let $H' := \mathrm{cl}_{\mathscr{T}'}(H)$. Then

$$\begin{aligned} \theta' &= \theta - |C \cup D| \\ &\geq 2s + t + 1 - (s-1) \\ &= s + t + 2 \\ &\geq (\mathrm{rk}_{\mathscr{T}'}(H') + 1) + 2. \end{aligned}$$

Clearly $\theta' \geq 3$. But then Lemma 3.3 implies we can find an element $e \in E(M') - H'$ such that one of $M' \setminus e, M'/e$ is 3-connected with N as minor. Since $e \notin \mathrm{cl}_{\mathscr{T}'}(H')$, certainly $e \notin \mathrm{cl}_{\mathscr{T}}(H)$, contradicting maximality of $|C \cup D|$. □

The final lemma of this section deals with a rather specific case in which elements need to be removed simultaneously.

Lemma 3.4. *Let M be a 3-connected matroid, let $\mathscr{T}$ be a tangle of M, let N be a minor of M, and let $X_1, \ldots, X_r$ be long lines of $M(\mathscr{T})$ with $\mathrm{rk}_{\mathscr{T}}(X_1 \cup \cdots \cup X_r) = 2r$. Suppose the following properties hold for all $i \in \{1, \ldots, r\}$:*

- *(i) $X_i \cap E(N) = \emptyset$;*
- *(ii) X_i contains a maximal fan F_i of length at least four;*
- *(iii) there is an element $e_i \in F_i$ such that $M \setminus e_i$ is 3-connected with N as minor.*

Then $M \setminus \{e_1, \ldots, e_r\}$ is 3-connected with N as minor.

Proof. We prove the result by induction on r, the case $r = 1$ being trivial. Let $r > 1$, and assume the result holds for all $r' < r$. Let $M, \mathscr{T}, N, X_1, \ldots, X_r, F_1, \ldots, F_r$, and $e_1, \ldots, e_r$ be as stated. Consider $M' := M \setminus e_r$, and let $\mathscr{T}'$ be the tangle of M' inherited from $\mathscr{T}$. Pick any $i \in \{1, \ldots, r-1\}$. By Lemma 2.34 we have that $\mathrm{cl}_{\mathscr{T}'}(X_i) = X_i$. Moreover, since e_r is not in the coclosure of X_i, the fan F_i is still maximal in M'. Clearly e_i is one of the ends of F_i, and then Lemma 2.23 implies that $M' \setminus e_i$ is 3-connected with N as minor.

It follows that $M', \mathscr{T}', N, X_1, \ldots, X_{r-1}$ satisfy all conditions of the lemma, and hence $M' \setminus \{e_1, \ldots, e_{r-1}\}$ is 3-connected with N as minor, by induction. But that matroid equals $M \setminus \{e_1, \ldots, e_r\}$, and the result follows. □

4 The restoration graph

We know now that we can find sets C and D such that $|C \cup D|$ is large, but for our main result we require that either all elements are deleted or all elements are contracted. In the remainder of the paper we will achieve this by studying subsets of $C \cup D$.

Lemma 4.1. *Let M be a matroid, $\mathscr{T}$ a tangle of M, and $\{c, d\}$ a $\mathscr{T}$-independent subset of $E(M)$ such that $M/c \setminus d$ is 3-connected but M/c is not. If d is not in a parallel pair in M then M is 3-connected. Moreover, either $M \setminus d$ is 3-connected or c and d are internal elements of a fan with size at least 4.*

Proof. Let $M, \mathscr{T}, c, d$ be as stated, and suppose that M is not 3-connected. Let (A, B) be a 2-separation of M, with $|A - \{c, d\}| \leq |B - \{c, d\}|$. Then $|A - \{c, d\}| \leq 1$, because otherwise (A, B) would be a 2-separation of $M/c \setminus d$. It follows that $A \in \mathscr{T}$. But then $|A \cap \{c, d\}| \leq 1$, since $\lambda_M(A) = 1 < \mathrm{rk}_{\mathscr{T}}(\{c, d\})$. It follows that A is a series pair or a parallel pair.

Since M/c is not 3-connected, d has to be in parallel with some element e in that matroid. In M we find no parallel pair containing d, so $\{c, d, e\}$ must be a triangle of M. The element c cannot be in any parallel pair of M, so c must be in a series pair. But then $\{c, f\}$ is a series pair for some $f \in \{d, e\}$. Since $d, e \in \mathrm{cl}_M(\{c, f\})$, it follows that $\{c, d, e\}$ is 2-separating, contradicting the assumption that $\{c, d\}$ is $\mathscr{T}$-independent.

For the second statement, suppose that $M \setminus d$ is not 3-connected. Then c must be in a series pair, say $\{c, f\}$. Since M is 3-connected, we must have that $\{c, d, f\}$ is a triad of M. This implies that $e \neq f$ or $M \cong U_{2,4}$, and the result follows. □

Lemma 4.2. *Let M be a 3-connected matroid, $\mathscr{T}$ a tangle of M, and C, D disjoint subsets of $E(M)$ such that $C \cup D$ is $\mathscr{T}$-independent and $M/C \setminus D$ is*

3-connected. For each $d \in D$, either $M/C\backslash(D-d)$ is 3-connected or there is an element $c \in C$ such that $M/(C-c)\backslash(D-d)$ is 3-connected.

Proof. Let $M, \mathscr{T}, C, D$ be as stated, and pick $d \in D$ such that $M/C\backslash(D-d)$ is not 3-connected. Call the resulting matroid M'. It follows from Lemma 2.32 that d is neither a loop nor a coloop of M'. It is impossible for d to be in a series pair, so d must be in a parallel pair, say with an element e.

The set $\{d,e\}$ is not 2-separating in M, so there must be a circuit Y with $\{d,e\} \subsetneq Y \subseteq C \cup \{d,e\}$. Pick $c \in Y \cap C$. In $M/(C-c)\backslash(D-d)$, we must have that $\{c,d,e\}$ is a triangle. Lemma 4.1 now implies the result. □

It is convenient to keep track of deletions and contractions using a certain bipartite graph. Let us fix some notation. If $G = (V,E)$ is a graph, and $S \subseteq V$ then $G[S]$ is the *induced subgraph* on S. For a vertex $v \in V$ we denote the set of vertices adjacent to v but not equal to v by $N(v)$.

Definition 4.3. Let M be a 3-connected matroid, and C, D disjoint subsets of $E(M)$ such that $M/C\backslash D$ is 3-connected. The *restoration graph* of M with respect to C and D, denoted by $R(M,C,D)$, is a bipartite graph with vertex set $C \cup D$ and edge set

$$\{cd : c \in C, d \in D, \text{ and } M/(C-c)\backslash(D-d) \text{ is 3-connected}\}.$$

Some more terminology: if $N = M/C\backslash D$, and $Z \subseteq C \cup D$, then we say that $M/(C-Z)\backslash(D-Z)$ was obtained from N by *restoring* Z. We say that an element $e \in C \cup D$ is *privileged* if restoring e yields a 3-connected matroid.

If the set of vertices of a restoration graph is $\mathscr{T}$-independent for a tangle $\mathscr{T}$ of M then it has many attractive properties. We list a few.

Lemma 4.4. *Let M be a 3-connected matroid, let $\mathscr{T}$ be a tangle of M, and let C, D be disjoint subsets of $E(M)$ such that $C \cup D$ is $\mathscr{T}$-independent and $M/C\backslash D$ is 3-connected. Then the restoration graph $R(M,C,D)$ has no isolated non-privileged vertices.*

Proof. This is an immediate consequence of Lemma 4.2. □

Lemma 4.5. *Let M be a 3-connected matroid, let $\mathscr{T}$ be a tangle of M, and let C, D be disjoint subsets of $E(M)$ such that $C \cup D$ is $\mathscr{T}$-independent and $M/C\backslash D$ is 3-connected. Let $G = R(M,C,D)$. Let $S \subseteq C \cup D$. Restoring S yields a 3-connected matroid if and only if $G[S]$ has no isolated non-privileged vertices.*

Proof. Let $M, \mathscr{T}, C$, and D be as stated, define $N := M/C\backslash D$, and define $G := R(M,C,D)$. Assume first that there is a set S such that $G[S]$ has

an isolated non-privileged vertex d, yet the matroid M' obtained from N by restoring S is 3-connected. Using duality if necessary we may assume $d \in D$. The matroid obtained from N by restoring d is not 3-connected, so d must be in a parallel pair with some element e in that matroid. Clearly $\{d,e\}$ is not a parallel pair in M', so there must be a circuit containing d,e, and at least one element $c \in C \cap S$. But then c and d satisfy all conditions of Lemma 4.1, and hence cd is an edge of G, a contradiction.

We will prove the converse by induction on the size of the set S to be restored. The case $S = \emptyset$ is trivial, so we may assume $|S| \geq 1$. Pick $d \in S$ such that d has minimum degree in the graph $G[S]$. If there is a choice, pick d to be non-privileged, and consider $G[S-d]$. Using duality if necessary we may assume $d \in D$. Let M' be the matroid obtained from N by restoring S, and let $\mathscr{T}'$ be the tangle inherited from $\mathscr{T}$.

First we assume that $G[S-d]$ has no isolated non-privileged vertices. By induction, restoring $S-d$ yields a 3-connected matroid. If M' does have a 2-separation, then d must be a loop or in parallel with another element in $E(M')$. The former cannot happen since S is $\mathscr{T}'$-independent. Hence d must be in parallel with an element f of $E(M')$. Note that $f \notin C \cup D$, because this again contradicts $\mathscr{T}'$-independence. Let c be a neighbour of d in $G[S]$, and let N' be the matroid obtained from N by restoring $\{c,d\}$. Then N' is 3-connected. But $d,f \in E(N')$ and N' is a minor of M', so $\mathrm{rk}_{N'}(\{d,f\}) \leq \mathrm{rk}_{M'}(\{d,f\}) = 1$, a contradiction. It follows that restoring S yields a 3-connected matroid.

Therefore $G[S-d]$ has an isolated non-privileged vertex $c \in C$. In $G[S]$, there must be an edge cd, and both c and d have degree one. By induction, then, restoring $S-\{c,d\}$ yields a 3-connected graph. Suppose that M' has a 2-separation (A,B). $M'/c\backslash d$ is 3-connected, so we must have $|A-\{c,d\}| \leq 1$ or $|B-\{c,d\}| \leq 1$. Assume, by relabelling if necessary, the former. Obviously $A \in \mathscr{T}$. Therefore $|A \cap S| \leq 1$. If M'' is the matroid obtained from N by restoring $\{c,d\}$, then $A \subseteq E(M'')$, and M'' is a minor of M'. Hence $\lambda_{M''}(A) \leq 1$, contradicting the definition of the restoration graph. □

5 The main result

Before proving the main theorem we find two structures in the restoration graph that will lead to the desired result. The first such structure, an unbalance between the sides, will be instrumental in our proof.

Lemma 5.1. *Let M be a matroid, let $\mathscr{T}$ be a tangle of M, and let $C,D \subseteq E(M)$ be such that $C \cup D$ is $\mathscr{T}$-independent, $M/C\backslash D$ is 3-connected, and $|C|-|D| \geq k$. Then there is a subset $C' \subseteq C$ such that $|C'| \geq k$ and M/C' is 3-connected.*

Proof. Let $M, \mathscr{T}, C, D$ be as stated, and let $G := R(M, C, D)$. Let $C'' \subseteq C$ be a minimal set such that each non-privileged $d \in D$ has a neighbour in C''. Clearly $|C''| \leq |D|$, and $G[C'' \cup D]$ has no isolated non-privileged vertices. Define $C' := C - C''$. By Lemma 4.5, restoring $C'' \cup D$ yields a 3-connected matroid. This matroid is M/C', and $|C'| = |C| - |C''| \geq |C| - |D| \geq k$. □

A *matching* of a graph is a subgraph in which each vertex has degree 1. We can use an *induced* matching in the restoration graph to increase the unbalance between the sides, through the following lemma:

Lemma 5.2. *Let k be an integer, let M be a matroid, let $\mathscr{T}$ be a tangle of M, let N be a minor of M, let $t := \mathrm{rk}_{\mathscr{T}}(E(N))$, and let C, D be disjoint subsets of $E(M)$ such that $\mathrm{rk}_{\mathscr{T}}(E(N) \cup C \cup D) = t + |C \cup D|$. If $R(M, C, D)$ contains an induced matching with at least $2k$ edges and no privileged vertices, then at least one of the following holds:*

- *(i) There is a set $C' \subseteq E(M)$ such that $\mathrm{rk}_{\mathscr{T}}(E(N) \cup C') = t + |C'|$, such that $|C'| \geq k$, and such that M/C' is 3-connected with N as minor;*
- *(ii) There is a set $D' \subseteq E(M)$ such that $\mathrm{rk}_{\mathscr{T}}(E(N) \cup D') = t + |D'|$, such that $|D'| \geq k$, and such that $M \setminus D'$ is 3-connected with N as minor.*

Proof. Let $k, M, \mathscr{T}, N, t, C, D$ be as stated, and define $G := R(M, C, D)$. By dualizing M and N, and swapping C and D if necessary, assume $|D| \leq |C|$. If $|C| - |D| \geq k$ then the result follows from Lemma 5.1, so assume $|C| - |D| = r < k$. Let H be a maximum-sized induced matching of G with at least $2k$ edges and no privileged vertices, and let M' be the matroid obtained from M by restoring $V(H)$. By Lemma 4.5, M' is 3-connected, $M' \setminus d$ is not 3-connected for each $d \in D \cap V(H)$, and M'/c is not 3-connected for each $c \in C \cap V(H)$. By Lemma 4.1, each pair $\{c, d\}$ of vertices connected by an edge in the graph H are internal elements of a fan F of M' of length at least 4.

Let $\mathscr{T}'$ be the tangle of M' inherited from $\mathscr{T}$, and for each edge $cd \in E(H)$, let X_{cd} be the long line of $M(\mathscr{T}')$ containing c and d. Since $rank_{\mathscr{T}'}(E(N) \cup \{c, d\}) = t + 2$, we have $X_{cd} \cap E(N) = \emptyset$. Moreover, $\mathrm{rk}_{\mathscr{T}'}(\bigcup_{cd \in E(H)} X_{cd}) = 2|E(H)|$. For each $cd \in E(H)$, let F_{cd} be the maximal fan of M' containing c and d, and let x_{cd} be an end of the fan. By Lemma 2.23, one of $M' \setminus x_{cd}$ and M'/x_{cd} is 3-connected with N as minor.

Consider the set $S := \{x_{cd} : cd \in E(H)\}$. Let $S' \subseteq S$ be such that $M' \setminus s$ is 3-connected for all $s \in S'$. Suppose $|S'| \geq k + r$. Define $C' := C - V(H)$, and $D' := (D - V(H)) \cup S'$. Then Lemma 3.4 implies that $M/C' \setminus D'$ is 3-connected with N as minor, and

$$|D'| - |C'| = |S'| + (|D| - |S'|) - (|C| - |S'|) = |S'| - r \geq k,$$

so the result follows from Lemma 5.1. Similarly, if $S' \subseteq S$ is such that M/s is 3-connected for all $s \in S'$, then the result follows if $|S'| \geq k - r$. But since $|S| \geq 2k$, one of these situations must hold, which completes our proof. □

Now we can state our main result. As mentioned in the introduction it depends on the rank of $E(N)$ in $M(\mathscr{T})$, rather than on the size of N.

Theorem 5.3. *Let k be a nonnegative integer, let M be a 3-connected matroid, let $\mathscr{T}$ be a tangle of M, let N be a minor of M with no loops and coloops, and let $t := \mathrm{rk}_{\mathscr{T}}(E(N))$. If the order of $\mathscr{T}$ is at least $20k + t - 13$, then there is a set $X \subseteq E(M)$ of size k such that $\mathrm{rk}_{\mathscr{T}}(E(N) \cup X) = t + k$, and such that one of $M \backslash X$ and M/X is 3-connected with N as minor.*

Proof. Let $k, M, \mathscr{T}, N$, and t be as stated. By applying Theorem 3.2 with $s = 10k - 7$ we can find sets $C, D \subseteq E(M)$ such that $\mathrm{rk}_{\mathscr{T}}(E(N) \cup C \cup D) = t + |C \cup D|$, such that $M/C \backslash D$ is 3-connected with N as minor, and such that $|C \cup D| \geq 10k - 7$. Let $G := R(M, C, D)$ be the restoration graph, and let $|C| - |D| = r$. We will call r the *balance* of the restoration graph. If $|r| \geq k$ then we are done by Lemma 5.1, so we may assume this is not the case. We partition the vertices of G into disjoint subsets $P_1, P_2, Q_1, Q_2, T_1, T_2, U_1, U_2$, with sizes $p_1, p_2, q_1, q_2, t_1, t_2, u_1, u_2$ respectively, as follows.

Let P_1 be the set of privileged vertices in C, and let P_2 be the set of privileged vertices in D. Let Q_1 be the vertices of C that only have neighbours in P_2, and let Q_2 be the set of vertices of D that only have neighbours in P_1. Let $C' := C - (P_1 \cup Q_1)$, let $D' := D - (P_2 \cup Q_2)$, and let $G' := G[C' \cup D']$. Let R be the vertex set of a maximal matching in G'. Note that, by our choice of Q_1 and Q_2, no vertex of G' is isolated, so all vertices in $V(G') - R$ have a neighbour in R.

Let $S_1 \subseteq R \cap C'$ be a minimal set such that the set of neighbours $N(S_1)$ includes all vertices in $D' - R$. Clearly $|S_1| \leq |D' - R|$, and $|N(S_1) \cap R| \geq |S_1|$ because R is a matching. Hence $|N(S_1)| \geq 2|S_1|$. Now let S_1' be a maximal set containing S_1 such that $|N(S_1')| \geq 2|S_1'|$, and define $U_2 := N(S_1')$. Let $T_2 := R - U_2$.

Symmetrically, let $S_2 \subseteq R \cap D'$ be a minimal set such that $N(S_2)$ includes all vertices in $C' - R$. Let S_2' be a maximal set containing S_2 such that $|N(S_2')| \geq 2|S_2'|$, and define $U_1 := N(S_2')$. Let $T_1 := R - U_1$. From the definitions it follows immediately that P_1, Q_1, U_1, T_1 partition C, and that P_2, Q_2, U_2, T_2 partition D. We will now bound the sizes of these sets.

If $p_1 \geq k + r$ then restoring P_1 yields a 3-connected matroid having a restoration graph with balance $|C| - p_1 - |D| \leq |C| - |D| - (k + r) = -k$, and the result follows from Lemma 5.1. Similarly, if $p_2 \geq k - r$ then restoring P_2 yields a restoration graph with balance k. It follows that we

may assume

$$p_1 + p_2 \leq 2k - 2. \tag{2}$$

Let $s_1 := |S_1'|$ and $s_2 := |S_2'|$. If $s_1 \geq k - r$ then restoring $S_1' \cup U_2$ yields a restoration graph with balance

$$|C| - s_1 - (|D| - u_2) = r + u_2 - s_1 \geq r + s_1 \geq r + k - r = k, \tag{3}$$

and we can apply Lemma 5.1 again. Likewise, if $s_2 \geq k + r$ then we can apply Lemma 5.1 to the restoration graph obtained by restoring $S_2' \cup U_1$. It follows that we may assume

$$s_1 + s_2 \leq 2k - 2. \tag{4}$$

Finally, if $u_2 - s_1 + q_2 - p_1 + p_2 \geq k - r$ then restoring $U_2 \cup S_1' \cup Q_2 \cup P_1 \cup P_2$ yields a restoration graph with balance

$$|C| - s_1 - p_1 - (|D| - u_2 - q_2 - p_2) \geq k, \tag{5}$$

and we can apply Lemma 5.1 again. Likewise, if $u_1 - s_2 + q_1 - p_2 + p_1 \geq k + r$ then we can apply Lemma 5.1 to the restoration graph obtained by restoring $U_1 \cup S_2' \cup Q_2 \cup P_2 \cup P_1$. It follows that

$$u_1 - s_2 + q_1 + u_2 - s_1 + q_2 \leq 2k - 2. \tag{6}$$

Next we direct our attention to T_1 and T_2. Let H_1 be the subgraph of the matching R containing all edges that meet T_1. Let H_2 be the subgraph of the matching R containing all edges that meet T_2.

Claim 5.3.1. *The matchings H_1 and H_2 are induced subgraphs of G.*

Proof. If some vertex $c \in V(H_1) \cap C'$ has degree at least 2, then c can be added to S_1', a contradiction. Hence all vertices in $V(H_1) \cap C'$ have degree exactly 1, and necessarily all vertices in $V(H_1) \cap D'$ have degree exactly 1. We omit the identical proof for H_2. □

If $t_1 \geq 2k$ or $t_2 \geq 2k$ then our result follows from Lemma 5.2. Hence we may assume that

$$t_1 + t_2 \leq 4k - 2. \tag{7}$$

Adding (2), (4), (6), and (7) we find

$$|C| + |D| = p_1 + u_1 + q_1 + t_1 + p_2 + u_2 + q_2 + t_2 \leq 10k - 8. \tag{8}$$

But $|C| + |D| \geq 10k - 7$ by assumption, a contradiction. □

The theorem from the introduction is now easy to prove:

Proof of Theorem 1.1. Let l be the number of elements of N that is either a loop or coloop. By Lemma 2.11, M has a minor N' such that N' has N as minor, N' has no loops and no coloops, and $|E(N')| \leq |E(N)| + l$. Clearly

$$\mathrm{bw}(N') \leq \mathrm{bw}(N) + l \leq 2|E(N)|. \tag{9}$$

The result now follows from Theorem 5.3 applied to M and N'. □

As a possible direction for future research, one could hope for a bound of a different nature, namely one that is a function of k and $\mathrm{rk}_{\mathscr{J}}(E(M) - E(N))$. Presumably such a bound would necessitate keeping only a minor isomorphic to N. However, the ideas from this paper do not seem to be suitable for proving such a result, and it is unclear if such a result has applications.

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