# BIRATIONAL SUPERRIGIDITY AND SLOPE STABILITY OF FANO MANIFOLDS

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Abstract. We show a relation between the birational superrigidity of Fano manifold and its slope stability in the sense of Ross-Thomas [RT07].

Contents



## 1. Introduction

The *birational (super)rigidity* of Fano manifold (or of Mori fiber space, in general) is introduced to extend the work of Iskovskih-Manin [IM71] for quartic threefolds. The concept emerged in the study of rationality problem with a focus on the study of rational maps among them.

The purpose of this paper is to show a relation between the birational (super)rigidity and GIT stability, which seems to have been unexpected from their different natures of origins. More precisely, in this paper, we study the *slope stability* of polarized varieties, which was introduced by Ross-Thomas (cf. [RT07]) as an analogue of the Mumford-Takemoto's slope stability of vector bundles. It is also a weaker version of K-stability, which was firstly formulated by Tian in [Tia97] and later

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reformulated and generalized by Donaldson [Don02], of which Ross-Thomas [RT07] followed the definition.

Our main result is the following.

**Theorem 1.1.** *Let $X$ be a birationally superrigid Fano manifold of index 1. If $|-K_X|$ is base point free, $(X, \mathcal{O}_X(-K_X))$ is slope stable.*

We remark that the two assumptions in Theorem 1.1 on the index and the base point freeness of the anticanonical linear system $|-K_X|$ seems to be weak. As far as the authors know, every Fano manifold which has been known to be birationally superrigid satisfies both assumptions (see section 2.1 for examples of birationally superrigid Fano manifolds).

Actually we prove the following more general but technical result from which Theorem 1.1 follows.

**Theorem 1.2.** *Let $X$ be a Fano manifold of index 1 which is log maximal singularity free (see section 2.1 for the definition). If $|-K_X|$ is base point free, $(X, \mathcal{O}_X(-K_X))$ is slope stable.*

Recall that the motivation for introducing K-stability is to formulate the following conjectural relation with the existence of Kähler metrics. From the recent progress on the relation (in particular, [Tia97], [Don05], [CT08], [Stp08], [Mab08] and [Mab09]), the following is known.

**Fact 1.3.** *If a Fano manifold $X$ with discrete automorphism group $\mathrm{Aut}(X)$ admits a Kähler-Einstein metric, then $(X, -K_X)$ is K-stable. In particular, it is slope stable.*

Note that every birationally superrigid Fano manifold has discrete automorphism group indeed, as it should not be (birationally) ruled.

We remark that the following example shows that our result can *not* be a direct consequence of the main result of [OS10].

*Example* 1.4. Let $X$ be a smooth projective hypersurface of dimension $n$ and degree $n+1$ in $\mathbb{P}^{n+1}$. Due to [P98], a general $X$ is birationally superrigid for $n \geq 4$ and this is conjectured to hold for every nonsingular $X$. In fact, every smooth $X$ is proved to be birationally superrigid for $n = 3$ by [IM71] and for $4 \leq n \leq 12$ by [dFEM03]. On the other hand, it is known that the global log canonical threshold $\mathrm{lct}(X)$ is $\frac{n}{n+1}$ if $X$ contains some *generalized Eckardt points* (or equivalently, hyperplane sections of cone type) so that strict stability does *not* directly follow from [OS10]. We are grateful to Professor Constantin Shramov for pointing out this to us.

Our proof of Theorem 1.1 is similar to that of [OS10]. Recall that the two fundamental observations in [OS10] are that:

- Certain explicit upper bounds of the Seshadri constants imply K-stability of $\mathbb{Q}$-Fano variety (= [OS10, Corollary 4.4]).
- Mildness of singularities of pluri-anticanonical divisors gives upper bounds of the Seshadri constants.

We combine these observations to prove Theorem 1.1 which is possible since the birational superrigidity asserts certain mildness of singularities of pluri-anticanonical divisors as we will review in subsection 2.4.

In the next section, we will prepare some basic definitions and review the background. In section 3, we prove the stability along divisors and in section 4, we prove the stability along higher codimensional sublocus. The last section proposes a more general conjecture about stability of Fano manifolds.

We work over the field of complex numbers $\mathbb{C}$ throughout this paper.

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## 2. Preliminary

2.1. **Birational (super)rigidity.** In this subsection, we recall the definition of birational (super)rigidity and its basic property.

**Definition 2.1.** Let $\pi\colon V \to S$ be a projective surjective morphism between normal projective varieties with connected fibers. We say that $\pi\colon V \to S$ is a *Mori fiber space* if

(i) $V$ is $\mathbb{Q}$-factorial and has at most terminal singularities,
(ii) $-K_V$ is $\pi$-ample,
(iii) $\dim S < \dim V$, and
(iv) the relative Picard number $\rho(V/S)$ is 1.

Throughout this subsection, let $X$ be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano variety with Picard number 1 and with at worst terminal singularities. Note that $X$, together with the structure morphism (to a point), can be seen as a Mori fiber space. Although birational (super)rigidity can be defined for any Mori fiber space, we only give the definition for $\mathbb{Q}$-Fano varieties with Picard number one.

**Definition 2.2.** We say that $X$ is *birationally rigid* if for any birational map $\varphi\colon X \dashrightarrow X'$ to a Mori fiber space there is a birational self-map $\tau\colon X \dashrightarrow X$ such that $\varphi \circ \tau\colon X \dashrightarrow X'$ can be extended to an isomorphism. We say that $X$ is *birationally superrigid* if in the above definition of birational rigidity one can always take $\tau = \mathrm{id}_X$.

It is easy to see that $X$ is birationally superrigid if and only if $X$ is birationally rigid and $\mathrm{Bir}(X) = \mathrm{Aut}(X)$. Let $\mathcal{H}$ be a movable linear system on $X$, that is, a linear system without fixed components. We define $\mu = \mu(X, \mathcal{H})$ to be the rational number for which $\mu K_X + \mathcal{H} \equiv 0$, where $\equiv$ denotes the numerical equivalence. Let $\lambda$ be a nonnegative rational number. We say that a pair $(X, \lambda\mathcal{H})$ is *terminal* (resp., *canonical*, resp., log canonical) if every rational number $a(X, \lambda\mathcal{H}, E)$ determined by

$$K_V + \lambda f_*^{-1}\mathcal{H} = f^*(K_X + \lambda\mathcal{H}) + \sum a(X, \lambda\mathcal{H}, E)E,$$

is positive (resp., non-negative, resp., $\geq -1$) for every birational morphism $f \colon V \to X$, where $E$ runs over the $f$-exceptional prime divisors.

The *canonical threshold* (resp., *log canonical threshold*) of the pair $(X, \mathcal{H})$ is defined to be the number

$$\mathrm{ct}(X, \mathcal{H}) := \sup\{\lambda \in \mathbb{Q}_{>0} \mid (X, \lambda\mathcal{H}) \text{ is canonical}\}$$
$$(\text{resp., } \mathrm{lct}(X, \mathcal{H}) := \sup\{\lambda \in \mathbb{Q}_{>0} \mid (X, \lambda\mathcal{H}) \text{ is log canonical}\}).$$

**Definition 2.3.** We say that $X$ has a *maximal singularity* (resp., *log maximal singularity*) if there is a movable linear system $\mathcal{H}$ on $X$ such that $(X, \frac{1}{\mu}\mathcal{H})$ is not canonical (resp., log canonical). We say that $X$ is *maximal singularity free* (resp., *log maximal singularity free*) if $X$ does not have a maximal singularity (resp., log maximal singularity).

The Noether-Fano-Iskovskikh inequality ([Co95, Theorem 4.2]) shows that if $X$ is maximal singularity free then it is birationally superrigid. We use the maximal singularity freeness to prove the stability in the following sections. The following shows that the maximal singularity freeness characterizes the birational superrigidity. This may be well known to specialists but we give a proof for the reader's convenience.

**Proposition 2.4.** *$X$ is birationally superrigid if and only if it is maximal singularity free.*

*Proof.* The if part follows from the Noether-Fano-Iskovskikh inequality as we have mentioned above. We shall prove the only if part. Assume that $X$ admits a maximal singularity, that is, there is a movable linear system $\mathcal{H}$ on $X$ such that $(X, \frac{1}{\mu}\mathcal{H})$ is not canonical, where $\mu = \mu(X, \mathcal{H})$. Let $V$ be the global sections which span the linear system $\mathcal{H}$. Possibly by replacing $\mathcal{H}$ with the linear system spanned by a symmetric power $\mathrm{Sym}^l V$ for some sufficiently large $l > 0$, we may assume that $\mu \geq 1$. We put $c = 1/\mathrm{ct}(X, \mathcal{H})$, where $\mathrm{ct}(X, \mathcal{H})$ is the canonical threshold of the pair $(X, \mathcal{H})$. By the assumption, we have $c > \mu$. We prove the following claim.

**Claim 2.5.** *There exists a birational morphism $f\colon Y \to X$ such that the pair $(Y, \frac{1}{c}\mathcal{H}_Y)$ is terminal and $K_Y + \frac{1}{c}\mathcal{H}_Y = f^*(K_X + \frac{1}{c}\mathcal{H})$, where $\mathcal{H}_Y$ is the birational transform of $\mathcal{H}$.*

*proof of Claim 2.5.* Let $g\colon V \to X$ be a log resolution of the pair $(X, \mathcal{H})$ and $\mathcal{H}_V$ the birational transform of $\mathcal{H}$ on $V$. For a general member $H_V \in \mathcal{H}_V$, the pair $(V, \frac{1}{c}H_V)$ is klt since $c > \mu \geq 1$. By [BCHM10], we can run a $(K_V + \frac{1}{c}\mathcal{H}_V)$-MMP, which is equivalent to a $(K_V + \frac{1}{c}H_V)$-MMP, with a scaling. Thus we obtain a minimal model $(Y, \frac{1}{c}\mathcal{H}_Y)$ over $X$ and we can easily check that $(Y, \frac{1}{c}\mathcal{H}_Y)$ satisfies the stated properties. This ends the proof of Claim 2.5. □

Now we see that $K_Y + \frac{1}{c}\mathcal{H}_Y$ is $\mathbb{Q}$-linearly equivalent to $-(\frac{1}{\mu} - \frac{1}{c})f^*\mathcal{H}$ and it is not pseudoeffective. Again by [BCHM10], we can run a $(K_Y + \frac{1}{c}\mathcal{H}_Y)$-MMP (with a scaling) which gives a $(K_{X'} + \frac{1}{c}\mathcal{H}')$-negative Mori fiber space $X' \to S'$, where $\mathcal{H}'$ is the pushforward of $\mathcal{H}$ by the induced birational map $Y \dashrightarrow X'$. In each step of the above MMP, we are in the category of pairs $(Z_i, \frac{1}{c}\mathcal{H}_i)$, where $(Z_i, \frac{1}{c}\mathcal{H}_i)$ has only terminal singularities and $\mathcal{H}_i$ is movable. In particular, $X'$ has only terminal singularities and $X' \to S'$ is a $K_{X'}$-negative extremal contraction, that is, $X' \to S'$ is a Mori fiber space. If the induced birational map $X \dashrightarrow X'$ is an isomorphism then the pair $(X, \frac{1}{c}\mathcal{H}) \cong (X', \frac{1}{c}\mathcal{H}')$ is terminal. This contradicts to the choice of $c$. This shows that $X$ is not birationally superrigid, which completes the proof of Proposition 2.4. □

Let us see some examples. The followings are Fano threefolds which have been known to be birationally superrigid.

- A smooth quartic threefold ([IM71]).
- A sextic double solid, that is, a double cover $X \to \mathbb{P}^3$ ramified along a surface $S \subset \mathbb{P}^3$ of degree 6 ([I79]).

The followings are higher-dimensional examples.

- A general hypersurface $X_{n+1} \subset \mathbb{P}^{n+1}$ of degree $n + 1$, with $n \geq 4$ ([P98]).
- A general complete intersection $X_{d_1,\ldots,d_k} \subset \mathbb{P}^{n+k}$ of hypersurfaces of degree $d_i$ with $d_i \geq 2$, $\sum_{i=1}^{k} d_i = n + k > 3k$ and $n \geq 4$ ([P01]).
- A smooth complete intersection $X_{2,4} \subset \mathbb{P}^6$ of quadric and quartic which does not contain a plane ([C03]).
- A double cover $X \to \mathbb{P}^n$ ramified along a hypersurface $F \subset \mathbb{P}^n$ of degree $2n$, with $n \geq 4$ ([P97]).

- A cyclic triple cover $X \to \mathbb{P}^{2n}$ ramified along a hypersurface $F \subset \mathbb{P}^{2n}$ of degree $3n$ with $n \geq 2$ ([C04]).
- A general cyclic cover $X \to V \subset \mathbb{P}^n$ of degree $d \geq 2$ ramified along a smooth divisor $R \subset V$ such that $V$ is a hypersurface of degree $m \geq 2$, $m + (d-1)k = n$, where $k$ is a positive integer such that $\mathcal{O}_V(R) \cong \mathcal{O}_V(dk)$, $n \geq 5$ and either $d = 2$ or $n \geq 6$ ([P00], [P04]).
- A general weighted complete intersection in a weighted projective space

  $$\mathbb{P}(1^{l+1}, a_1, \dots, a_m) = \mathrm{Proj}(\mathbb{C}[x_0, \dots, x_l, y_1, \dots, y_m])$$

  of $m+k$ hypersurfaces $y_i^2 = g_i(x_0, \dots, x_l)$, $i = 1, \dots, m$, and $f_j(x_0, \dots, x_l) = 0$, $j = 1, \dots, k$, of degree $2l_i$ and $d_j$, respectively, such that

  $$\sum_{i=1}^{m} a_i + \sum_{i=1}^{k} d_i = l, l > 3k \text{ and } l - k \geq 4$$

  It is an iterated double cover of general complete intersection in projective space ([P03]).

In all the above examples, we assume the smoothness of the Fano variety in concern. In some examples, we can allow some mild singularities while keeping the property of birational superrigidity and we can also drop the generality assumptions. We refer the readers to [C05] for a detailed account of this subject. We see that every birationally superrigid Fano manifold in the above examples has index 1 and has an base point free anticanonical divisor.

2.2. **Seshadri constants.** Let $I \subset \mathcal{O}_X$ be a coherent ideal on $X$. The Seshadri constant of $I$ with respect to an ample $\mathbb{Q}$-line bundle $L$ is defined by

$$\mathrm{Sesh}(J; (X, L)) := \sup\{c > 0 \mid \pi^* L(-cE) \text{ is ample}\},$$

where $\pi \colon Bl_I(X) \to X$ is the blow up of $X$ along $I$ and $E$ is the associated exceptional Cartier divisor, i.e., $\mathcal{O}(-E) = \pi^{-1} I$. This invariant plays a key role in this paper as in [HKLP11], [OS10], [F11] and [F].

2.3. **Slope stability.** Consult [Don02, Chapter 2, especially 2.3], [RT07, especially Section 3] or [Od09, Definition 2.4] for more general background. We remark that our formulation below are formally different from the original presentation by Ross-Thomas [RT07], but they are equivalent as they proved in [RT07, Theorem 4.18]. See below for the more detailed explanation. Let $(X, L)$ be a $n$-dimensional polarized variety.

A *test configuration* (resp. a *semi test configuration*) for $(X, L)$ is a polarize scheme $(\mathcal{X}, \mathcal{M})$ with a $\mathbb{G}_m$-action on $(\mathcal{X}, \mathcal{M})$ and a proper flat morphism $\Pi \colon \mathcal{X} \to \mathbb{A}^1$ such that (i) $\Pi$ is $\mathbb{G}_m$-equivariant for the multiplicative action of $\mathbb{G}_m$ on $\mathbb{A}^1$, (ii) $\mathcal{M}$ is relatively ample (resp. relatively semi-ample), and (iii) $(\mathcal{X}, \mathcal{M}) \mid_{\Pi^{-1}(\mathbb{A}^1 \setminus \{0\})}$ is $\mathbb{G}_m$-equivariantly isomorphic to $(X, L^{\otimes r}) \times (\mathbb{A}^1 \setminus \{0\})$ for some positive integer $r$. If $\mathcal{X} \simeq X \times \mathbb{A}^1$, it is called $(\mathcal{X}, \mathcal{M})$ a *product test configuration.* Moreover, if $\mathbb{G}_m$ acts trivially, we call it a trivial test configuration.

The slope stability treats certain special semi test configurations, called *deformation to the normal cone.* The definition is as follows. Take a coherent ideal $I \subset \mathcal{O}_X$ and set $\mathcal{J} := I + (t) \subset \mathcal{O}_{X \times \mathbb{A}^1}$. Then, for $r \in \mathbb{Z}_{>0}$ with $r >$ (*resp.* $\geq$)$\frac{1}{\mathrm{Sesh}(I;(X,-K_X))}$, we set $f \colon \mathcal{B} := Bl_{\mathcal{J}}(X \times \mathbb{A}^1) \to X \times \mathbb{A}^1$, $\mathcal{L} := f^*(L \times \mathbb{A}^1)$ and $\mathcal{O}_{\mathcal{B}}(-E) = f^{-1}\mathcal{J}$ with the effective exceptional Cartier divisor $E$. We note that $(\mathcal{B}, \mathcal{L}(-E))$ naturally becomes a test configuration (resp. semi test configuration, if $\mathcal{L}(-E)$ is semiample). We call them the *deformation to the normal cone* as in [RT07].

First, let us recall the general definition of the Donaldson-Futaki invariants of a test configuration $(\mathcal{X}, \mathcal{M})$. Let $P(k) := \dim H^0(X, L^{\otimes k})$, which is a polynomial in $k$ of degree $n$ due to the Riemann-Roch theorem. Since the $\mathbb{G}_m$-action preserves the central fibre $\mathcal{X}_0$ of $\mathcal{X}$, $\mathbb{G}_m$ acts also on $H^0(\mathcal{X}_0, \mathcal{M}^{\otimes K} \mid_{\mathcal{X}_0})$, where $K \in \mathbb{Z}_{>0}$. Let $w(Kr)$ be the weight of the induced action on the highest exterior power of $H^0(\mathcal{X}_0, \mathcal{M}^{\otimes K} \mid_{\mathcal{X}_0})$, which is a polynomial of $K$ of degree $n+1$ due to the Mumford's droll Lemma (cf. [Mum77, Lemma 2.14] and [Od09, Lemma 3.3]) and the Riemann-Roch theorem. Here, the *total weight* of an action of $\mathbb{G}_m$ on some finite-dimensional vector space is defined as the sum of all weights, where the *weights* mean the exponents of eigenvalues which should be powers of $t \in \mathbb{A}^1$. Let us take $rP(r)$-th power and SL-normalize the action of $\mathbb{G}_m$ on $(\Pi_*\mathcal{M})|_{\{0\}}$, then the corresponding normalized weight on $(\Pi_*\mathcal{M}^{\otimes K})|_{\{0\}}$ is $\tilde{w}_{r,Kr} := w(k)rP(r) - w(r)kP(k)$, where $k := Kr$. It is a polynomial of form $\sum_{i=0}^{n+1} e_i(r)k^i$ of degree $n+1$ in $k$ for $k \gg 0$, with coefficients which are also polynomial of degree $n+1$ in $r$ for $r \gg 0$ : $e_i(r) = \sum_{j=0}^{n+1} e_{i,j}r^j$ for $r \gg 0$. Since the weight is normalized, $e_{n+1,n+1} = 0$. The coefficient $e_{n+1,n}$ is called the *Donaldson-Futaki invariant* of the test configuration, which we denote by $\mathrm{DF}(\mathcal{X}, \mathcal{M})$. For an arbitrary *semi* test configuration $(\mathcal{X}, \mathcal{M})$ of order $r$, we can also define the Donaldson-Futaki invariant as well by setting $w(Kr)$ as the total weight of the induced action on $H^0(\mathcal{X}, \mathcal{M}^{\otimes K})/tH^0(\mathcal{X}, \mathcal{M}^{\otimes K})$ (cf. [RT07]). Now we can define the stability notions in concern.

**Definition 2.6.** We say that $(X, L)$ is *slope stable* (resp. *slope semistable*) if and only if the Donaldson-Futaki invariants is positive (resp., non-negative) for any non-trivial deformation to the normal cone.

**Definition 2.7.** We say that $(X, L)$ is *K-stable* (resp. *K-semistable*) if and only if the Donaldson-Futaki invariants is positive (resp., non-negative) for any non-trivial test configuration.

Let us recall that the original definition of the slope stability by Ross-Thomas [RT07] is of the following form;

**Definition 2.8** (Ross-Thomas [RT07, Definition 4.17])**.** $(X, L)$ is slope stable (resp. slope semistable) if and only if

$$\mu_c(I, L) < (\text{resp.}, \leq\ )\ \mu(X)$$

for all $c \in (0, \mathrm{Sesh}(I; (X, L)))$ and also for $c = \mathrm{Sesh}(I; (X, L))$ if $\mathrm{Sesh}(I; (X, L)) \in \mathbb{Q}$ and the global sections of $L^{\otimes k} \otimes I^{\mathrm{Sesh}(I;(X,L))k}$ saturate for suffieciently divisible positive integer $k$.

Note that the slope "$\mu$" s, which we refer to [RT07] for the precise definitions, are defined in terms of intersection numbers on $X$ and $B := Bl_I(X)$. Our definition 2.6 above is proved in [RT07, proof of Theorem 4.18] to be equivalent to Ross-Thomas' definition. Of course, there are no essential differences but we will follow our formulation 2.6 just because we are more accustomed to treat the stability in such a way. We end this subsection with a small remark on an extension of the framework above. If we take a test configuration (resp. semi test configuration) $(\mathcal{X}, \mathcal{M})$, we can think of a new test configuration (resp. semi test configuration) $(\mathcal{X}, \mathcal{M}^{\otimes a})$ with $a \in \mathbb{Z}_{>0}$. From the definition of Donaldson-Futaki invariant above, we easily see that $\mathrm{DF}((\mathcal{X}, \mathcal{M}^{\otimes a})) = a^n\,\mathrm{DF}((\mathcal{X}, \mathcal{M}))$. Therefore, we can define K-stability (also K-polystability and K-semistability) of a pair $(X, L)$ of a projective scheme $X$ and an ample $\mathbb{Q}$**-line** bundle $L$.

A key for our study is the following formula, proved in [Od09], to estimate the Donaldson-Futaki invariant for a deformation to the normal cone $(\mathcal{B}, \mathcal{L}(-E))$ derived from the (flag) ideal $\mathcal{J} := I + (t) \subset \mathcal{O}_{X\times\mathbb{A}^1}$.

**Theorem 2.9** (cf. [Od09, Theorem 3.2])**.** *Let $X, L, I \subset \mathcal{O}_X$ and the corresponding notions as above. Assume $X$ is a Fano $n$-fold and $L = \mathcal{O}_X(-rK_X)$ with some $r \in \mathbb{Z}_{>0}$. Moreover, we assume that $(\mathcal{B}, \mathcal{L}(-E))$ is semi test configuration (a deformation to the normal cone) and let $(\overline{\mathcal{B}} := Bl_{\mathcal{J}}(X \times \mathbb{P}^1), \bar{\mathcal{L}}(-E))$ be its natural compactification. Let $p_i$ $(i = 1, 2)$ be the $i$-th projection from $X \times \mathbb{P}^1$ to the $i$-th factor. Suppose that $\mathcal{L}(-E)$ on $\mathcal{B}$ is semi-ample. Let us denote the normalization of $\mathcal{B}$*

*as $\tilde{\mathcal{B}}$ and use the same symbols for the pullbacks to $\tilde{\mathcal{B}}$ of the original polarization $\bar{\mathcal{L}}$ and the Cartier divisor $E$. Then, the corresponding Donaldson-Futaki invariant has a following lower bound.*

$$
\begin{aligned}
&2(n!)((n+1)!)\,\mathrm{DF}(\mathcal{B},\mathcal{L}(-E)) \\
&\geq -\big((\overline{\mathcal{L}}-E)^n.\overline{\mathcal{L}}+nE\big)+(n+1)r\big((\overline{\mathcal{L}}-E)^n.K_{\tilde{\mathcal{B}}/X\times\mathbb{A}^1}\big) \\
&= -\big((\overline{\mathcal{L}}-E)^n.\overline{\mathcal{L}}\big)+\big((\overline{\mathcal{L}}-E)^n.((n+1)rK_{\tilde{\mathcal{B}}/X\times\mathbb{A}^1}-nE)\big). \qquad (1)
\end{aligned}
$$

The right hand side is just the Donaldson-Futaki invariant of $(\tilde{\mathcal{B}},\mathcal{L}(-E))$ by [Od09, Theorem3.2] so that the inequality follows from [RT07, Proposition 5.1, Remark 5.2]. See [RT07] and [Od09] for more general statements.

We note that $\frac{1}{r}\leq \mathrm{Sesh}(I,(X,-K_X))$ from the assumption of the semiampleness of $\mathcal{L}(-E)$ on $\mathcal{B}$.

Recall that

**Proposition 2.10** ([OS10, Proposition4.3])**.** $-\big((\overline{\mathcal{L}}-E)^n.\overline{\mathcal{L}}\big)\geq 0$ *for any ideal* $I\subset\mathcal{O}_X$. *The equality holds if and only if* $\dim(\mathrm{Supp}(\mathcal{O}_X/I))=0$.

Therefore, to show that $\mathrm{DF}(\mathcal{B},\mathcal{L}(-E))>0$, it suffices to show the following claim.

**Claim 2.11.** $(n+1)K_{\tilde{\mathcal{B}}/X\times\mathbb{A}^1}-n\,\mathrm{Sesh}(I;(X,-K_X))E$ *is effective. In other word, the coefficient of each exceptional prime divisor $E_i$ of the divisor above is non-negative, i.e.,*

$$\mathrm{Sesh}(I;(X,-K_X))\leq\left(\frac{n+1}{n}\right)\min_i\left\{\frac{a_i}{c_i}\right\},$$

*where*

$$
\begin{aligned}
K_{\tilde{\mathcal{B}}/X\times\mathbb{A}^1} &= \sum_i a_iE_i, \\
\Big(\tilde{\Pi}^*(X\times\{0\}) &= \tilde{\Pi}_*^{-1}(X\times\{0\})+\sum_i b_iE_i,\Big) \\
\tilde{\Pi}^{-1}\mathcal{J} &= \mathcal{O}_{\tilde{\mathcal{B}}}(-\sum_i c_iE_i),
\end{aligned}
$$

*with exceptional prime divisors $E_i$ (as in* [OS10]*). Moreover, if* $\dim\mathrm{Supp}(\mathcal{O}_{X\times\mathbb{A}^1}/\mathcal{J})=0$*, the divisor is nonzero and effective, i.e.,*

$$\mathrm{Sesh}(I;(X,-K_X))<\frac{(n+1)a_i}{nc_i}$$

*for some $i$.*

Before proving Claim 2.11, we deform $I$ slightly as follows. Let $I = \mathcal{O}_X(-F)I'$ where $F \in |-mK_X|$ with some $m \in \mathbb{Z}_{>0}$, and $I'$ is an ideal with $\mathrm{codim}(\mathrm{Supp}(\mathcal{O}/I')) \geq 2$. Then, take a deformation of $F$ in $|-mK_X|$ so that we obtain the following property of (new) $I$.

**Condition 2.12.** *For the blow up of $\mathcal{J} := I + (t)$, $\Pi\colon \mathcal{B} \to X \times \mathbb{A}^1$, if* $\mathrm{codim}(\Pi(E_i) \subset X \times \{0\}) \geq 2$ *for* $\Pi$*-exceptional divisor $E_i$, $\Pi(E_i) \not\subseteq$* $\mathrm{Supp}(F)$*.*

This is possible since $|-mK_X|$ is base point free by our assumption. Note that the Seshadri constant $\mathrm{Sesh}(I; (X, -K_X))$ does not change by this deformation. Thus, we can take the corresponding semi test configuration $\mathcal{B} := Bl_{I+(t)}(X \times \mathbb{A}^1)$ for that perturbed $I$. It has the same Donaldson-Futaki invariant, which follows e.g., from the description via slope ([RT07]).

Thus, we can assume the above Condition 2.12 for $I$ from now on, to estimate the Donaldson-Futaki invariants.

## 3. Exceptional divisors with divisorial center

In this section, under the assumptions on $X$ as in Theorem 1.2, we will prove Claim 2.11 for $E_i$ in the case where $\Pi(E_i)$ is a divisor in $X \times \{0\}$, in this section. Let us recall that we denoted $\mathcal{J} = I + (t)$. As in the previous section, write $I = I'\mathcal{O}(-F)$ with coherent ideals $I' \subset \mathcal{O}_X$ satisfying $\mathrm{codim}(\mathrm{Supp}(\mathcal{O}/I')) \geq 2$ and divisor $F = \sum_i d_i D_i$, where each $D_i$ are prime divisors.

Firstly, we have

$$\mathrm{Sesh}(I, (X, -K_X)) \leq \frac{\mathrm{index}(X)}{d_i}.$$

The last inequality can be easily seen if we take into account that $H^0((X \setminus \mathrm{Supp}(\mathcal{O}/I')), (I\mathcal{O}(-K_X))^m) = H^0(X, \mathcal{O}_X(-mF))$, due to the normality of $X$ and $\mathrm{codim}((\mathrm{Supp}(\mathcal{O}/I')) \geq 2$.

Secondly, since $(X \times \mathbb{A}^1, D_i \times \mathbb{A}^1)$ is canonical around the generic point of $D_i \times \{0\}$, it follows that

$$\left(\frac{\mathrm{index}(X)}{d_i} =\right) \frac{1}{d_i} \leq \frac{a_i}{c_i} < \frac{(n+1)a_i}{nc_i}$$

(The inequality can also be proved by using birational geometry of surfaces, by iterating cutting by hypersurface sections. ) As a conclusion, we proved Claim 2.11 for the case with $\dim(\Pi(E_i)) = n - 1$ follows. We note that the conditions $\rho = \mathrm{index}(X) = 1$ were sufficient for the above arguments in this section.

## 4. Exceptional divisors with higher codimensional center

In this section, under the same assumptions on $X$, we will prove Claim 2.11 for $E_i$ in the case where $\mathrm{codim}(\Pi(E_i) \subset X \times \{0\}) \geq 2$. Recall that we have Condition 2.12 for $I$.

Take a positive rational number $(0 <)c < \mathrm{Sesh}(I, (X, -K_X))$. For sufficiently divisible positive interger $l$, set the linear system $\Sigma_{I',l}^{(c)} \subset |-lK_X|$ which corresponds to $H^0(X, I^{cl}\mathcal{O}_X(-lK_X)) \subset H^0(X, \mathcal{O}_X(-lK_X))$. By the assumption of the log maximal singularity freeness, we have $\mathrm{lct}(X, \Sigma_{I',l}^{(c)}) \geq \frac{1}{l}$. Since this holds for any such $c$, as in the argument of [OS10, proof of Proposition 3.1, in particular (10)], we have

$$\mathrm{Sesh}(I', (X, -K_X)) \leq \min_{E_i \subset \mathrm{Exc}(\Pi)} \left\{ \frac{a_i}{c'_i} \right\}, \tag{2}$$

where $c'_i := \mathrm{val}_{E_i}(I')$, the algebraic valuation of $I'$ with respect to $E_i$, and $\mathrm{Exc}(\Pi)$ is the exceptional locus of $\Pi$. If $F \equiv -mK_X$ with $m \in \mathbb{Z}_{>0}$, the inequality (2) is equivalent to say that

$$\mathrm{Sesh}(I, (X, -K_X)) \leq \frac{\min\{\frac{a_i}{c'_i}\}}{1 + m \cdot \min\{\frac{a_i}{c'_i}\}}. \tag{3}$$

From Condition 2.12, which was obtained by perturbation of the ideal, we have $c_i = \mathrm{val}_{E_i}(\mathcal{J}) \leq \mathrm{val}_{E_i}(I) = \mathrm{val}_{E_i}(I') =: c'_i$.

Summarizing up,

$$\mathrm{Sesh}(I, (X, -K_X)) \leq \frac{\min\{\frac{a_i}{c'_i}\}}{1 + m \cdot \min\{\frac{a_i}{c'_i}\}} \leq \frac{a_i}{c'_i} \leq \frac{a_i}{c_i} < \frac{(n+1)a_i}{nc_i}. \tag{4}$$

We note that the idea of the perturbation of divisorial part of $\mathcal{J}$ (which made use of the assumption of base point freeness of $|-K_X|$) realizes the last inequality. This (4) completes the proof of Claim 2.11. Hence, the proof of Theorem 1.2 is also completed. □

*Remark* 4.1. A diffuculty would rise up if we try to strengthen Theorem 1.1, 1.2 to state K-stability, rather than slope stability. The problem is it is in general hard to deform"flag" ideal $\mathcal{J} = I_0 + I_1 + \cdots I_{N-1}t^{N-1} + (t^N)$ of length $N > 1$, which should have monotonicity of each pieces $I_i$ to obtain similar condition as Condition 2.12.

## 5. A conjecture

Recall that we treated in this paper special class of Fano manifolds of Picard rank 1 (i.e., $\mathrm{Pic}(X) \cong \mathbb{Z}$). More generally, we expect the following.

**Conjecture 5.1.** *For an arbitrary Fano manifold $X$ of Picard rank 1, $(X, -K_X)$ is K-semistable.*

We note some supporting evidences here. First, it is proved that $(X, -K_X)$ is slope stable with respect to divisors ([F]). We proved a stronger statement in section 3 under the additional assumption of index 1. Moreover, in the results by Hwang, Kim, Lee, Park and Fujita's results ([HKLP11, Theorem1.3], [F11, Theorem 1.1]), it is shown to be slope semistable along smooth curves too.

We also remark that we can not expect the strict (poly)stability as Tian proved in [Tia97] that small deformations of the Mukai-Umemura 3-fold are *not* K-polystable but K-semistable.

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